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Abstract: In this paper, we focus on the problem of a single contact with Coulomb friction, which we reduce to a root-finding problem on a degree 4 polynomial. This formulation give us the exact number of solutions, as well as their analytical form when they exist.

Key-words: coulomb friction, mechanics, contact

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Formulation quartique du contact avec frottement de Coulomb tridimensionnel

Résumé : Dans ce document, nous reformulons le problème unitaire (1 contact frottant) en la recherche de zéro d'un polynôme du quatrième degré. Si elles existent, cette formulation donne la forme analytique des solutions.

Mots-clés : frottement de coulomb, mécanique, contact

1 Introduction

Numerical simulation of three dimensional frictionnal contact with impacts has been thoroughly studied in the last two decade [RAD05], [Bro00], [AB08], [Cad09]. In this paper, we use the framework introduced by Moreau [Mor88] and we focus on the solution of the problem for a single contact, which we reduce to a root-finding problem on a degree 4 polynomial.

The idea consists in doing an enumerative exploration of the Coulomb friction's cases.

1.1 The Coulomb friction model

We first define the second order cone, $K_\mu \subset \mathbb{R}^3$, of parameter $\mu > 0$ in the unitary direction $e \in \mathbb{R}^3$:

$$K_\mu = \{x \in \mathbb{R}^3 : \| x - x.e\ e \| \leq \mu\ x.e\} \tag{1}$$

Where $a.b$ is the usual dot product of two vectors.
Coulomb's frictional law is a relation between the reaction force $R \in \mathbb{R}^3$ and the relative velocity $U \in \mathbb{R}^3$. In this case, e is a unitary normal to the surface at the contact point.

The relation can be written as a disjonctive formulation:

$$\left\{ \begin{array}{ll} -\text{ Take-off:} & R = 0 \text{ and } U.e \geq 0 \\ -\text{ Sticking:} & U = 0 \text{ and } R \in \mathbf{int} K_\mu \\ -\text{ Sliding:} & R \in \partial K_\mu \text{ and U.e=0 and } \exists \alpha > 0, R - \ R.e\ e = -\alpha U \end{array} \right. \tag{2}$$

We note $C(\mu, e)$ the set of all $(U, R) \in \mathbb{R}^3 \times \mathbb{R}^3$ satisfying this law.

1.2 Discretised problem

The discretization of the mechanical problem is deeply described in [Mor88], [AB08]. In the case of a single contact, it consists in finding $U \in \mathbb{R}^3$ and $R \in \mathbb{R}^3$ subject to:

$$\begin{array}{c} U = MR + q, \\ (U, R) \in C(\mu, e) \end{array} \tag{3}$$

Where $M \in \mathbb{R}^{3,3}$ a regular symetric positive matrix and $q \in \mathbb{R}^3$. The problem is written in a orthonormal frame (e, e_{t1}, e_{t2}). We distinguish between the tangential part and the normal part of R. The normal part is defined as $R_N := R.e$, that is the first coordinate of R.The tangential part is defined as $R_T \in \mathbb{R}^2$, the coordinates of the vector $R - e.R$ in the frame (e_{t1}, e_{t2}).

2 Enumerative method for solving

It consists in successively looking for a solution in each of the cases defined in (2).

2.1 Take-off case

In this case, R equals to zero. The system (3) leads to $U = q$ which is a solution if and only if $(q, 0) \in C(\mu, e)$. It consists in checking that $e.q \quad > 0$.

2.2 Sticking case

Here, U must be equal to zero, and f M is invertible (3) leads to $R = -M^{-1}q$ which is a solution if and only if $\emptyset \neq \{R \in \mathbb{R}^3 \mid MR = -q\} \cap \mathbf{int}K_\mu$.

(If M is invertible, this amounts to checking that $-M^{-1}q \in \mathbf{int}K_\mu$.)

2.3 Sliding case

When $\mu > 0$, This is the only non-trivial case. The main contribution of this document is the reformulation of this problem as a quartic root-finding problem.

In the two following sections, we derive formulations of such polynomials.

3 Quartic polynomial in $\frac{tan(\theta)}{2}$

It consists in finding $\alpha > 0$ and $R \in \partial K_\mu$ such that $-\alpha \begin{pmatrix} 0 \\ R_T \end{pmatrix} = MR + q$. That is :

$$\left[M + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \right] R + q = 0 \tag{4}$$

3.1 R_T is on a conic

The first line of the system (4) and the $R \in \partial K_\mu$ equation define the intersection between a plan and a cone in $\mathbb{R}^3$:

$$\begin{array}{l} \mu R_N = \| R_T \| \\ \frac{M_{11}}{\mu} \| R_T \| = -q_1 - M_{12}R_{T1} - M_{13}R_{T2} \end{array} \tag{5}$$

That is:

$$\begin{array}{l} \mu^2 R_N^2 = (R_{T1}^2 + R_{T1}^2) \\ \frac{M_{11}^2}{\mu^2}(R_{T1}^2 + R_{T1}^2) = (-q_1 - M_{12}R_{T1} - M_{13}R_{T2})^2 \end{array} \tag{6}$$

This means that R_T is contained into a conic, whose focus and directrice are:

$$\begin{array}{l} \mathcal{D} : q_1 + M_{12}R_{T1} + M_{13}R_{T2} = 0 \\ focus : \mathcal{O} \\ \frac{M_{11}^2}{\mu^2} Dist(\mathcal{O}, R_T)^2 = Dist(\mathcal{D}, R_T)^2 (M_{12}^2 + M_{13}^2) \\ \frac{Dist(\mathcal{O},R_T)}{Dist(\mathcal{D},R_T)} = \frac{\mu\sqrt{(M_{12}^2+M_{13}^2)}}{M_{11}} = e \end{array} \tag{7}$$

The parametric equation is:

$$\begin{array}{l} R_{T1} = rcos(\theta) \\ R_{T2} = rsin(\theta) \\ r = \frac{p}{1+ecos(\theta-\phi)} \end{array} \tag{8}$$

With p a simple expression of M_{11}, M_{12}, M_{13}, and ϕ a constant angle between $\mathcal{D}$ and (O, R_{T1})

3.2 The last two lines of the system (4)

$$\frac{\| R_T \|}{\mu}\tilde{M}_{1.} + \left(\tilde{M} + \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\right) R_T + \tilde{q} = 0 \tag{9}$$

$\tilde{M}$ is symetric, so there exists a unitary matrix V such that $V\tilde{M}V^T = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}$.
One can get:

$$\frac{\| R_T \|}{\mu}V\tilde{M}_{1.} + V\left(\tilde{M} + \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix}\right) V^T V R_T + V\tilde{q} = 0 \tag{10}$$

Rename:

$$\frac{\| \bar{R}_T \|}{\mu}\bar{M}_{1.} + \begin{pmatrix} d_1+\alpha & 0 \\ 0 & d_2+\alpha \end{pmatrix} \overline{R}_T + \bar{q} = 0 \tag{11}$$

In the plan, V is either a rotation or a symmetry. So $\bar{R}_T = VR_T$ is a conic with the same focus and a rotated directrice, which means that there exists ϕ_1 such that :

$$\begin{array}{l} \bar{R}_{T1} = rcos(\theta) \\ \bar{R}_{T2} = rsin(\theta) \\ r = \frac{p}{1+ecos(\theta-\phi_1)} \end{array} \tag{12}$$

The equation (11) is :

$$\begin{array}{l} (d_1+\alpha)\bar{R}_{T1} = -\bar{q}_1 + a_1 \| R_T \| \\ (d_2+\alpha)\bar{R}_{T2} = -\bar{q}_2 + a_2 \| R_T \| \end{array} \tag{13}$$

The case ($\bar{R}_{T1} = 0$ or $\bar{R}_{T2} = 0$) has to be examined. We try to eliminate *alpha*:

$$\begin{array}{l} d_1\bar{R}_{T1}\bar{R}_{T2} + \alpha\bar{R}_{T1}\bar{R}_{T2} = -\bar{q}_1\bar{R}_{T2} + a_1\bar{R}_{T2} \| R_T \| \\ d_2\bar{R}_{T1}\bar{R}_{T2} + \alpha\bar{R}_{T1}\bar{R}_{T2} = -\bar{q}_2\bar{R}_{T1} + a_2\bar{R}_{T1} \| R_T \| \end{array} \tag{14}$$

that leads to:

$$(d_1-d_2)\bar{R}_{T1}\bar{R}_{T2} = -\bar{q}_1\bar{R}_{T2} + \bar{q}_2\bar{R}_{T1} + (a_1\bar{R}_{T2} - a_2\bar{R}_{T1}) \| R_T \| \tag{15}$$

The parametric expression of $\bar{R}_T$ leads to:

$$\begin{array}{l} (d_1-d_2)r^2cos(\theta)sin(\theta) = -\bar{q}_1rsin(\theta) + \bar{q}_2rcos(\theta) + r(a_1rsin(\theta) - a_2rcos(\theta)) \\ \text{ie: } (d_1-d_2)rcos(\theta)sin(\theta) = -\bar{q}_1sin(\theta) + \bar{q}_2cos(\theta) + r(a_1sin(\theta) - a_2cos(\theta)) \end{array} \tag{16}$$

with the expression of r:

$$\begin{array}{ll} & (d_1-d_2)\frac{p}{1+ecos(\theta-\phi_1)}cos(\theta)sin(\theta) = \\ & -\bar{q}_1sin(\theta) + \bar{q}_2cos(\theta) + \frac{p}{1+ecos(\theta-\phi_1)}(a_1sin(\theta) - a_2cos(\theta)) \\ \\ \text{ie:} & (d_1-d_2)pcos(\theta)sin(\theta) = \\ & (1+ecos(\theta-\phi_1))(-\bar{q}_1sin(\theta) + \bar{q}_2cos(\theta)) + p(a_1sin(\theta) - a_2cos(\theta)) \\ \\ \text{ie:} & (d_1-d_2)pcos(\theta)sin(\theta) = p(a_1sin(\theta) - a_2cos(\theta)) \\ & +(1+e(cos(\theta)cos(\phi_1) + sin(\theta)sin(\phi_1)))(-\bar{q}_1sin(\theta) + \bar{q}_2cos(\theta)) \\ \\ \text{ie: } 0 = & (d_1-d_2)pcos(\theta)sin(\theta) + p(-a_1sin(\theta) + a_2cos(\theta)) \\ & +(1+ecos(\theta)cos(\phi_1) + esin(\theta)sin(\phi_1))(\bar{q}_1sin(\theta) - \bar{q}_2cos(\theta)) \end{array} \tag{17}$$

rename:

$$Acos(\theta)^2 + Bsin(\theta)^2 + Csin(\theta)cos(\theta) + Dsin(\theta) + Ecos(\theta) = 0 \tag{18}$$

with

$$\begin{array}{l} A = -e\bar{q}_2cos(\phi_1) \\ B = e\bar{q}_1sin(\phi_1) \\ C = (d_1 - d_2)p + ecos(\phi_1)\bar{q}_1 - esin(\phi_1)\bar{q}_2 \\ D = \bar{q}_1 - pa_1 \\ E = -\bar{q}_2 + pa_2 \end{array} \tag{19}$$

Performing the following change of variables:

$$\begin{array}{l} t = tan(\theta/2) \\ sin(\theta) = \frac{2t}{1+t^2} \\ cos(\theta) = \frac{1-t^2}{1+t^2} \end{array} \tag{20}$$

leads to:

$$\begin{array}{l} A\frac{(1-t^2)^2}{1+t^2} + B\frac{4t^2}{1+t^2} + C\frac{2t(1-t^2)}{1+t^2} + D2t + E(1-t^2) = 0 \\ \text{ie: } A(1-t^2)^2 + 4Bt^2 + C2t(1-t^2) + 2Dt(1+t^2) + E(1-t^2)(1+t^2) = 0 \\ \\ \text{ie: } P_4 = A - E \qquad P_3 = -2C + 2D \qquad P_2 = 4B - 2A \\ \qquad P_1 = 2C + 2D \qquad P_0 = A + E \end{array} \tag{21}$$

Finally, we get 4 possible values for R_T, checking the sign of α and R_N selects the solutions.

3.3 Case $R_{T12} = 0$

From (13), R_{T1} leads to:

$$\begin{array}{l} \| R_T \| = |\bar{R}_{T2}| = \frac{\bar{q}_1}{a_1} \\ \bar{R}_T = \begin{pmatrix} 0 \\ \pm\frac{\bar{q}_1}{a_1} \end{pmatrix} \end{array} \tag{22}$$

From (13), R_{T2} leads to:

$$\begin{array}{l} \| R_T \| = |\bar{R}_{T1}| = \frac{\bar{q}_2}{a_2} \\ \bar{R}_T = \begin{pmatrix} \pm\frac{\bar{q}_2}{a_2} \\ 0 \end{pmatrix} \end{array} \tag{23}$$

From $\bar{R}_T$, we have to check the coherence with the equation (12). If it is on the conic, we compute R, and the sign condition of the equation (4) must be checked.

4 Quartic polynomial in α

Just like in previous formulation, we want to find $\alpha > 0$ and $R \in \partial K_\mu$ such that $U_N = 0$ and $-\alpha \begin{pmatrix} 0 \\ R_T \end{pmatrix} = MR + q$.

That is :

$$\left\{\begin{array}{rcl} 0 & = & \left[M + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{pmatrix} \right] R + q \\ \| R_T \| & = & \mu R_N \end{array}\right. \tag{24}$$

Let's decompose M:

$$M := \begin{pmatrix} m_0 & m_{01} & m_{02} \\ m_{01} & & \\ m_{02} & & M_T \end{pmatrix}$$

If $m_{00} = 0$, since M is positive-semidefinite, $m_{01} = m_{02} = 0$. Then :

- if $q_N \neq 0$, there is no solution to the problem
- if $q_N = 0$, we can chose an arbitrary α and take any solution of the linear system on R_T then set $R_N = \frac{1}{\mu} \| R_T \|$.

We suppose now that $m_{00} > 0$. We can rewrite (24) as:

$$\left\{\begin{array}{rcl} 0 & = & (m_{00}, m_{01}, m_{02}) \cdot R + q_N \\ 0 & = & \left[M_T + \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} \right] R_T + \begin{pmatrix} m_{01} \\ m_{02} \end{pmatrix} R_N + q_T \\ \| R_T \| & = & \mu R_N \end{array}\right.$$

and then, substituting the first line into the second one :

$$\left\{\begin{array}{rcl} 0 & = & (m_{00}, m_{01}, m_{02}) \cdot R + q_N \\ 0 & = & \Lambda(\alpha) R_T + w \\ \| R_T \| & = & \mu R_N \end{array}\right. \tag{25}$$

with

$$\Lambda(\alpha) := \begin{pmatrix} \lambda_1 + \alpha & \lambda_{12} \\ \lambda_{12} & \lambda_2 + \alpha \end{pmatrix} = M_T + \begin{pmatrix} \alpha & 0 \\ 0 & \alpha \end{pmatrix} - \frac{1}{m_{00}} \begin{pmatrix} m_{01} \\ m_{02} \end{pmatrix} (m_{01}, m_{02})$$

and

$$w := q_T - \frac{q_N}{m_{00}} \begin{pmatrix} m_{01} \\ m_{02} \end{pmatrix}$$

To study $\Lambda(\alpha)$, we define:

$$L := m_{00} \Lambda(0) = m_{00} M_T - \begin{pmatrix} m_{01} \\ m_{02} \end{pmatrix} (m_{01}, m_{02})$$

Since M is positive definite, we have $\forall i \neq j |m_{ij}| \leq \sqrt{m_{ii} m_{jj}}$. That means that $\forall i = 0, 1 L_{ii} = m_{00} m_{(i+1)(i+1)} - m_{0(i+1)}^2 \geq 0$. We conclude that $\mathbf{tr} L \geq 0$.

Moreover, deriving the determinant of L, we remark that

$$\det L = m_{00} \det M \geq 0$$

.

L is therefore a 2×2 symmetric matrix with a positive trace and determinant, which means L is positive-semidefinite. So is Λ (since $m_{00} > 0$), and $\forall \alpha > 0$, $\Lambda(\alpha)$ is positive-definite.

We get

$$R_T = -\frac{1}{\det \Lambda(\alpha)} \begin{pmatrix} \lambda_2 + \alpha & -\lambda_{12} \\ -\lambda_{12} & \lambda_1 + \alpha \end{pmatrix} w \tag{26}$$

Now, let's return to equation (25) and combine its first line into the last one, then square the result:

$$R_{T1}^2 + R_{T2}^2 = \frac{\mu^2}{m_{00}^2} \left(m_{01} R_{T1} + m_{02} R_{T2} + q_N\right)^2$$

Multiplying both sides by $(\det \Lambda(\alpha))^2$:

$$\begin{aligned} 0 \quad = \quad & (\det \Lambda(\alpha) R_{T1})^2 + (\det \Lambda(\alpha) R_{T2})^2 \\ & - \frac{\mu^2}{m_{00}^2} \left[m_{01} \det \Lambda(\alpha) R_{T1} + m_{02} \det \Lambda(\alpha) R_{T2} + \det \Lambda(\alpha) q_N\right]^2 \end{aligned} \tag{27}$$

Now remember from (26) that $\det \Lambda(\alpha) R_{Ti}$ are linear functions in α, and $\det \Lambda(\alpha)$ is a degree 2 polynomial in α.

The expression in (27) is therefore a degree 4 polynomial (a quartic) in α, whose roots in $\mathbb{R}^{+*}$ are the solution of our frictionnal contact problem.

5 Numerical example

We consider the following problem, which is derived from a ??? system:

$$M = \begin{pmatrix} 0.01344 & -9.421e-07 & 0.001486 \\ -9.421e-07 & 0.1061 & 0.0001733 \\ 0.001486 & 0.0001733 & 0.001442 \end{pmatrix} \text{ and } q = \begin{pmatrix} -0.1458 \\ -0.2484 \\ -0.1515 \end{pmatrix} \text{ and } \mu = 0.6$$

The formulation, using $t = \frac{tan(\theta)}{2}$, described in the section 3, leads to the following polynomial:

$$P(t) = -1.349578\times 10^{-1} t^4 - 1.879711 t^3 - 6.600166\times 10^{-2} t^2 + 8.851747\times 10^{-1} t + 1.349783\times 10^{-1}$$

The formulation, using α, described in 4, leads to the following polynomial:

$$P(\alpha) = -7.65275\alpha^4 - 1.62238\alpha^3 - 7.13689\times 10^{-2}\alpha^2 + 7.43684\times 10^{-4}\alpha + 3.70595\times 10^{-5}$$

Both of thems yield the solution:

$$R = \begin{pmatrix} 10.2059 \\ 1.93189 \\ 5.8108 \end{pmatrix}$$

However, we found that in this particular case we could reach the machine precision using the formulation based on $t = \frac{tan(\theta)}{2}$ of the section 3, while the worse-conditioned polynomial of section 4 gave us an error of 10^{-8}. Precision measurements were done using the Alart-Curnier [AC91] function.

6 Conclusion

We presented two distinct approaches for expressing the sliding case of Coulomb's frictional law as a degree 4 polynomial, from which we can deduce the analytic solutions. While in itself restricted to systems with a single contact, this method can be embedded as a local solver in a Gauss-Seidel like algorithm [JAJ98], allowing us to deal with real-world problems. These quartic formulations are implemented in the Siconos Platform [AP07], [AB08], and are used as well in the MECHE framework for simulating fibrous materials.

References

[AB08] V. Acary and B. Brogliato. *Numerical Methods for Nonsmooth Dynamical Systems: Applications in Mechanics and Electronics*, volume 35 of *Lecture Notes in Applied and Computational Mechanics*. Springer Verlag, 2008.

[AC91] P. Alart and A. Curnier. A mixed formulation for frictional contact problems prone to Newton like solution methods. *Comput. Methods Appl. Mech. Eng.*, 92(3):353–375, 1991.

[AP07] V. Acary and F. Pérignon. An introduction to Siconos. Technical Report TR-0340, INRIA, http://hal.inria.fr/inria-00162911/en/, 2007.

[Bro00] B. Brogliato, editor. *Impact in Mechanical systems: Analysis and Modelling*, volume 551 of *Lecture Notes in Physics*. Springer, 2000.

[Cad09] F. Cadoux. *Méthodes d'optimisation pour la dynamique non-régulière*. PhD thesis, Université Joseph Fourier, November 2009.

[JAJ98] Franck Jourdan, Pierre Alart, and Michel Jean. A Gauss-Seidel like algorithm to solve frictional contact problems. *Comput. Methods Appl. Mech. Eng.*, 155(1-2):31–47, 1998.

[Mor88] J.-J. Moreau. Unilateral contact and dry friction in finite freedom dynamics. Nonsmooth mechanics and applications, CISM Courses Lect. 302, 1-82 (1988)., 1988.

[RAD05] M. Renouf, V. Acary, and G. Dumont. 3d frictional contact and impact multibody dynamics. a comparison of algorithms suitable for real-time applications. In *ECCOMAS*, 2005.

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