

# Smart Energy-Aware Sensors for Event-Based Control (with appendix)

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# Complementary notes on Smart Energy-Aware Sensors for Event-Based Control

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**Abstract**—This document completes the paper “Smart Energy-Aware Sensors for Event-Based Control” submitted to the 51<sup>st</sup> IEEE Conference on Decision and Control by the same authors. It is not intended to be self contained; it only gives the proof of Lemma 2.

## I. APPENDIX

We recall from [25] the following elements.

The closed loop system (the system (5) with the policy (18) and the initial conditions  $(z_0, m_0)$ ), that we note  $z_k(z_0, m_0)$ , evolves as follows:

$$\begin{cases} z_{k+1}(z_0, m_0) = f_{v_k^*}(z_k(z_0, m_0), u_k^*) \\ m_{k+1} = v_k^* = \eta(z_k, m_k) \\ u_k^* = \mu(z_k, m_k). \end{cases} \quad (19)$$

**Definition 1** The closed loop system (19) is said to be Input-to-State practically Stable (ISpS) if there exist a  $\mathcal{KL}$ -function  $\gamma$ , and a constant  $c \geq 0$ , such that, for all  $z_0 \in \mathbb{R}^{n_z}$  and for all  $m_0 \in \mathbb{M}$ :

$$\|z_k(z_0, m_0)\| \leq \gamma(\|z_0\|, k) + c, \quad k \in \mathbb{Z}_{\geq 0}. \quad (20)$$

**Definition 2**  $V : \mathbb{R}^{n_z} \times \mathbb{M} \rightarrow \mathbb{R}_{\geq 0}$  is called a ISpS-Lyapunov function for the closed loop system (19) if:

- there exist a pair of  $\mathcal{K}_{\infty}$ -functions  $\alpha_1$ ,  $\alpha_2$ , and a constant  $c_1 \geq 0$  such that, for all  $z \in \mathbb{R}^{n_z}$  and for all  $m \in \mathbb{M}$ :

$$\alpha_1(\|z\|) \leq V(z, m) \leq \alpha_2(\|z\|) + c_1, \quad (21)$$

- there exist a suitable  $\mathcal{K}_{\infty}$ -function  $\alpha_3$  and a constant  $c_2 \geq 0$  such that, for all  $z \in \mathbb{R}^{n_z}$  and for all  $m \in \mathbb{M}$ :

$$\begin{aligned} \Delta V(z, m) &\triangleq V(f_{v^*}(z, u^*), v^*) - V(z, m) \\ &\leq -\alpha_3(\|z\|) + c_2. \end{aligned} \quad (22)$$

**Lemma 2** If the closed loop system (19) admits an ISpS-Lyapunov function, then it is ISpS.

*Proof:* This proof is based on the proofs of ISS and ISpS from [15], [22].

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We assume that Eq.s (21)-(22) hold, *i.e.* that the closed loop system (19) admits an ISpS-Lyapunov function, denoted  $V(z, m)$  hereafter. Let’s prove that the closed loop system is ISpS, *i.e.* that Eq. (20) holds.

*Step 1:* First, we prove that the closed loop system (19) admits an invariant set  $\Omega \subset \mathbb{R}^{n_z} \times \mathbb{M}$ , *i.e.*, for all  $(z, m) \in \Omega$ ,  $f_{v^*}(z, u^*) \in \Omega$ .

We define  $\bar{\alpha}_2(s) \triangleq \alpha_2(s) + s$ , then, noting that  $c_1 \geq 0$  and  $\|z\| \geq 0$ , (21) implies:

$$\begin{aligned} V(z, m) &\leq \alpha_2(\|z\| + c_1) + \|z\| + c_1 \\ &= \bar{\alpha}_2(\|z\| + c_1) \\ \Rightarrow \bar{\alpha}_2^{-1}(V(z, m)) &\leq \|z\| + c_1. \end{aligned} \quad (26)$$

Let  $\xi(s)$  be any  $\mathcal{K}_{\infty}$ -function, for example  $\xi(s) = s$ .

- If  $c_1 \leq \|z\|$ :

$$\begin{aligned} c_1 \leq \|z\| &\Leftrightarrow \frac{\|z\| + c_1}{2} \leq \|z\| \\ \Rightarrow \alpha_3\left(\frac{\|z\| + c_1}{2}\right) &\leq \alpha_3(\|z\|) \leq \alpha_3(\|z\|) + \xi(c_1). \end{aligned} \quad (27)$$

- If  $c_1 > \|z\|$ :

$$\begin{aligned} \|z\| < c_1 &\Leftrightarrow \frac{\|z\| + c_1}{2} < c_1 \\ \Rightarrow \xi\left(\frac{\|z\| + c_1}{2}\right) &\leq \xi(c_1) \leq \alpha_3(\|z\|) + \xi(c_1). \end{aligned} \quad (28)$$

Let’s define  $\underline{\alpha}_3(s) \triangleq \min\{\xi(\frac{s}{2}), \alpha_3(\frac{s}{2})\}$ . Eq.s (27),(28) yield:

$$\underline{\alpha}_3(\|z\| + c_1) \leq \alpha_3(\|z\|) + \xi(c_1) \quad (29)$$

We notice that  $\underline{\alpha}_3 \in \mathcal{K}_{\infty}$ , in particular  $\underline{\alpha}_3$  is strictly increasing, which implies (with (26),(29)):

$$\underline{\alpha}_3(\bar{\alpha}_2^{-1}(V(z, m))) \leq \alpha_3(\|z\| + c_1) \leq \alpha_3(\|z\|) + \xi(c_1).$$

Let’s define  $\alpha_4 \triangleq \underline{\alpha}_3 \circ \bar{\alpha}_2^{-1}$ , then:

$$\begin{aligned} \alpha_4(V(z, m)) &\leq \alpha_3(\|z\|) + \xi(c_1) \\ (22) \Rightarrow \Delta V(z, m) &\leq -\alpha_3(\|z\|) + c_2 - \xi(c_1) + \xi(c_1) \\ &\leq -\alpha_4(V(z, m)) + c_2 + \xi(c_1). \end{aligned} \quad (30)$$

Let  $\rho$  be a  $\mathcal{K}_{\infty}$ -function such that  $(id - \rho)$  is also a  $\mathcal{K}_{\infty}$ -function.  $\rho(s) = \frac{s}{2}$  is an example. We define  $\Omega \subset \mathbb{R}^{n_z} \times \mathbb{M}$ :

$$\Omega = \{(z, m) \in \mathbb{R}^{n_z} \times \mathbb{M} : V(z, m) \leq \omega(c_3)\}, \quad (31)$$

where  $\omega \triangleq \alpha_4^{-1} \circ \rho^{-1}$  and  $c_3 \triangleq c_2 + \xi(c_1)$ .

We assume that  $(id - \alpha_4)$  is a  $\mathcal{K}_\infty$ -function. Lemma B.1 in [15] proves that if  $(id - \alpha_4)$  is not a  $\mathcal{K}_\infty$ -function, there exists a  $\mathcal{K}_\infty$ -function  $\hat{\alpha}_4$  such that  $\hat{\alpha}_4(s) \leq \alpha_4(s)$  and  $(id - \hat{\alpha}_4)$  is a  $\mathcal{K}_\infty$ -function that can be used hereafter to lead to the same result.

Let's now assume that  $(z, m) \in \Omega$ :

$$\begin{aligned} (30) \Rightarrow V(f_{v^*}(z, u^*), v^*) - V(z, m) &\leq -\alpha_4(V(z, m)) + c_3 \\ \Rightarrow V(f_{v^*}(z, u^*), v^*) &\leq (id - \alpha_4)(V(z, m)) + c_3 \\ &\leq (id - \alpha_4)(\omega(c_3)) + c_3 \\ &= \omega(c_3) - \alpha_4(\omega(c_3)) + c_3 \\ &= \omega(c_3) - \alpha_4(\omega(c_3)) + \rho \circ \alpha_4(\omega(c_3)) \\ &= \omega(c_3) - (id - \rho)(\alpha_4(\omega(c_3))), \end{aligned} \quad (32)$$

where we have used the fact that  $\rho \circ \alpha_4(\omega(s)) = s$ . Since  $(id - \rho)(s) \geq 0$  (being a  $\mathcal{K}_\infty$ -function), (32) yields:

$$V(f_{v^*}(z, u^*), v^*) \leq \omega(c_3),$$

thus proving that  $\Omega$  is an invariant set for the closed loop system (19).

*Step 2:* Let's now prove that the invariant set  $\Omega$  is an attractive set, *i.e.* that for any  $(z_0, m_0) \notin \Omega$ , there exists a finite  $\bar{k}$  such that  $(z_{\bar{k}}, m_{\bar{k}}) \in \Omega$ . Let  $\bar{k}$  be the first time index where the system enters  $\Omega$ , for the initial condition  $(z_0, m_0)$ :

$$\bar{k} \triangleq \min \{k \in \mathbb{Z}_{\geq 0} : (z_k, m_k) \in \Omega\} \leq \infty, \quad (33)$$

where  $\bar{k}$  is infinite when the trajectories never enter  $\Omega$ . To prove that  $\Omega$  is attractive, we need to prove that  $\bar{k}$  is finite. We start by noticing that if  $(z, m) \notin \Omega$ , then:

$$V(z, m) > \omega(c_3) = \alpha_4^{-1} \circ \rho^{-1}(c_3) \quad (34)$$

$$\Rightarrow \rho \circ \alpha_4(V(z, m)) > c_3$$

$$\Leftrightarrow \rho \circ \alpha_4(V(z, m)) - c_3 > 0. \quad (35)$$

Moreover:

$$\begin{aligned} (30) \Rightarrow \Delta V(z, m) &\leq -\alpha_4(V(z, m)) + c_3 \\ &= -(id - \rho) \circ \alpha_4(V(z, m)) - \rho \circ \alpha_4(V(z, m)) + c_3 \end{aligned}$$

$$(35) \Rightarrow \Delta V(z, m) \leq -(id - \rho) \circ \alpha_4(V(z, m)). \quad (36)$$

Hence, for all  $k < \bar{k}$ ,  $\Delta V(z_k, m_k) \leq -\alpha_5(V(z_k, m_k))$ , where  $\alpha_5(s) \triangleq (id - \rho) \circ \alpha_4(s)$  is a  $\mathcal{K}_\infty$ -function, and thus is in particular a  $\mathcal{K}$ -function. According to [24, Lemma 4.3], this implies that there exists a  $\mathcal{KL}$ -function  $\hat{\gamma}(s, k)$  such that:

$$V(z_k, m_k) \leq \hat{\gamma}(V(z_0, m_0), k), \quad \forall k < \bar{k}. \quad (37)$$

The function  $\hat{\gamma}(s, k)$  is decreasing in  $k$  and goes to 0 as  $k \rightarrow \infty$ , then there exists a finite  $\tilde{k}$  such that:

$$\hat{\gamma}(V(z_0, m_0), \tilde{k}) < \omega(c_3) \quad (38)$$

This implies that  $\tilde{k} \geq \bar{k}$ . Indeed, if  $\tilde{k}$  was  $\tilde{k} < \bar{k}$ , then Eq.s (34),(37) would hold, but Eq.s (37),(38) would imply that  $V(z_k, m_k) < \omega(c_3)$ , in contradiction with (34).

This ends the proof that  $\Omega$  is attractive since  $\bar{k} \leq \tilde{k} < \infty$ .

*Step 3:* Finally, we want to prove that Eq. (20) holds. We collect the results from the previous steps,  $\forall (z_0, m_0) \in \mathbb{R}^{n_z} \times \mathbb{M}$ ,  $\forall k \in \mathbb{Z}_{\geq 0}$ :

- if  $(z_k, m_k) \in \Omega$ , then  $V(z_k, m_k) \leq \omega(c_3)$ ,
- if  $(z_k, m_k) \notin \Omega$ , then  $V(z_k, m_k) \leq \hat{\gamma}(V(z_0, m_0), k)$ .

Eq. (21) implies that  $\|z_k\| \leq \alpha^{-1}(V(z_k, m_k))$ , we thus obtain:

- if  $(z_k, m_k) \in \Omega$ , then  $\|z_k\| \leq \alpha^{-1}(\omega(c_3))$ ,
- if  $(z_k, m_k) \notin \Omega$ , then  $\|z_k\| \leq \alpha^{-1}(\hat{\gamma}(V(z_0, m_0), k))$ .

In any case, we have:

$$\|z_k\| \leq \alpha^{-1}(\hat{\gamma}(V(z_0, m_0), k)) + \alpha^{-1}(\omega(c_3)).$$

Eq. (21) implies that  $V(z_0, m_0) \leq \alpha_2(\|z_0\|) + c_1$ , which implies:

$$\|z_k\| \leq \alpha^{-1}(\hat{\gamma}(\alpha_2(\|z_0\|) + c_1, k)) + \alpha^{-1}(\omega(c_3)). \quad (39)$$

Then, we notice that, for any function  $\alpha(s)$  of class  $\mathcal{K}_\infty$ ,  $\forall (s_1, s_2) \in \mathbb{R}_{\geq 0}$ , the following holds:

$$\begin{aligned} \alpha(s_1 + s_2) &\leq \begin{cases} \alpha(2s_1), & \text{if } s_1 \geq s_2 \\ \alpha(2s_2), & \text{if } s_1 \leq s_2 \end{cases} \\ \Rightarrow \alpha(s_1 + s_2) &\leq \alpha(2s_1) + \alpha(2s_2). \end{aligned}$$

Since, for a given  $k$ ,  $\alpha^{-1}(\hat{\gamma}(s, k))$  is a function of class  $\mathcal{K}_\infty$  w.r.t.  $s$ , we have:

$$\begin{aligned} \alpha^{-1}(\hat{\gamma}(\alpha_2(\|z_0\|) + c_1, k)) &\leq \\ \alpha^{-1}(\hat{\gamma}(2\alpha_2(\|z_0\|), k)) &+ \alpha^{-1}(\hat{\gamma}(2c_1, k)). \end{aligned} \quad (40)$$

As the function  $k \alpha^{-1}(\hat{\gamma}(2c_1, k))$  is decreasing w.r.t.  $k$ , it attains its maximum for  $k = 0$ :

$$\alpha^{-1}(\hat{\gamma}(2c_1, k)) \leq \alpha^{-1}(\hat{\gamma}(2c_1, 0)), \quad \forall k \in \mathbb{Z}_{\geq 0}. \quad (41)$$

Notice that  $\alpha^{-1}(\hat{\gamma}(2\alpha_2(s), k))$  is a  $\mathcal{KL}$ -function. Eq.s (39)-(41) imply:

$$\begin{aligned} \|z(z_0, m_0, k)\| &\leq \gamma(\|z_0\|, k) + c \\ \text{with } \gamma(s, k) &= \alpha^{-1}(\hat{\gamma}(2\alpha_2(s), k)) \\ c &= \alpha^{-1}(\omega(c_3)) + \alpha^{-1}(\hat{\gamma}(2c_1, 0)). \end{aligned}$$

**Remark 3** *The choice of the  $\mathcal{K}_\infty$ -functions  $\xi(s)$ ,  $\rho(s)$  influence how  $\gamma(s, k)$  give a more or less conservative bound.*

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