Smart Energy-Aware Sensors for Event-Based Control (with appendix)

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Complementary notes on Smart Energy-Aware Sensors for Event-Based Control

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***Abstract*— This document completes the paper "Smart Energy-Aware Sensors for Event-Based Control" submitted to the 51st IEEE Conference on Decision and Control by the same authors. It is not intended to be self contained; it only gives the proof of Lemma 2.**

I. APPENDIX

We recall from [25] the following elements.

The closed loop system (the system (5) with the policy (18) and the initial conditions (z_0, m_0)), that we note $z_k(z_0, m_0)$, evolves as follows:

$$\begin{cases} z_{k+1}(z_0, m_0) = f_{v_k^*}(z_k(z_0, m_0), u_k^*) \\ m_{k+1} = v_k^* = \eta(z_k, m_k) \\ u_k^* = \mu(z_k, m_k). \end{cases} \tag{19}$$

Definition 1 *The closed loop system* (19) *is said to be Input-to-State practically Stable (ISpS) if there exist a* $\mathcal{KL}$*-function* γ*, and a constant* $c \geq 0$*, such that, for all* $z_0 \in \mathbb{R}^{n_z}$ *and for all* $m_0 \in \mathbb{M}$*:*

$$\|z_k(z_0, m_0)\| \leq \gamma(\|z_0\|, k) + c, \quad k \in \mathbb{Z}_{\geq 0}. \tag{20}$$

Definition 2 $V : \mathbb{R}^{n_z} \times \mathbb{M} \to \mathbb{R}_{\geq 0}$ *is called a ISpS-Lyapunov function for the closed loop system* (19) *if:*

- *there exist a pair of* $\mathcal{K}_\infty$*-functions* α_1*,* α_2*, and a constant* $c_1 \geq 0$ *such that, for all* $z \in \mathbb{R}^{n_z}$ *and for all* $m \in \mathbb{M}$*:*

$$\alpha_1(\|z\|) \leq V(z, m) \leq \alpha_2(\|z\|) + c_1, \tag{21}$$

- *there exist a suitable* $\mathcal{K}_\infty$*-function* α_3 *and a constant* $c_2 \geq 0$ *such that, for all* $z \in \mathbb{R}^{n_z}$ *and for all* $m \in \mathbb{M}$*:*

$$\begin{aligned} \Delta V(z, m) &\triangleq V(f_{v*}(z, u^*), v^*) - V(z, m)) \\ &\leq -\alpha_3(\|z\|) + c_2. \end{aligned} \tag{22}$$

Lemma 2 *If the closed loop system* (19) *admits an ISpS-Lyapunov function, then it is ISpS.*

Proof: This proof is based on the proofs of ISS and ISpS from [15], [22].

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We assume that Eq.s (21)-(22) hold, *i.e.* that the closed loop system (19) admits an ISpS-Lyapunov function, denoted $V(z, m)$ hereafter. Let's prove that the closed loop system is ISpS, *i.e.* that Eq. (20) holds.

Step 1: First, we prove that the closed loop system (19) admits an invariant set $\Omega \subset \mathbb{R}^{n_z} \times \mathbb{M}$, *i.e.* , for all $(z, m) \in \Omega$, $f_{v^*}(z, u^*) \in \Omega$.
We define $\overline{\alpha}_2(s) \triangleq \alpha_2(s) + s$, then, noting that $c_1 \geq 0$ and $\|z\| \geq 0$, (21) implies:

$$\begin{aligned} V(z, m) &\leq \alpha_2(\|z\| + c_1) + \|z\| + c_1 \\ &= \overline{\alpha}_2(\|z\| + c_1) \\ \Rightarrow \overline{\alpha}_2^{-1}(V(z, m)) &\leq \|z\| + c_1. \end{aligned} \tag{26}$$

Let $\xi(s)$ be any $\mathcal{K}_\infty$-function, for example $\xi(s) = s$.

- If $c_1 \leq \|z\|$:

$$\begin{aligned} c_1 \leq \|z\| &\Leftrightarrow \frac{\|z\| + c_1}{2} \leq \|z\| \\ \Rightarrow \alpha_3\left(\frac{\|z\|+c_1}{2}\right) &\leq \alpha_3(\|z\|) \leq \alpha_3(\|z\|) + \xi(c_1). \end{aligned} \tag{27}$$

- If $c_1 > \|z\|$:

$$\begin{aligned} \|z\| < c_1 &\Leftrightarrow \frac{\|z\| + c_1}{2} < c_1 \\ \Rightarrow \xi\left(\frac{\|z\| + c_1}{2}\right) &\leq \xi(c_1) \leq \alpha_3(\|z\|) + \xi(c_1). \end{aligned} \tag{28}$$

Let's define $\underline{\alpha_3}(s) \triangleq \min\left\{\xi\left(\frac{s}{2}\right), \alpha_3\left(\frac{s}{2}\right)\right\}$. Eq.s (27),(28) yield:

$$\underline{\alpha_3}(\|z\| + c_1) \leq \alpha_3(\|z\|) + \xi(c_1) \tag{29}$$

We notice that $\underline{\alpha_3} \in \mathcal{K}_\infty$, in particular $\underline{\alpha_3}$ is strictly increasing, which implies (with (26),(29)):

$$\underline{\alpha}_3\left(\overline{\alpha}_2^{-1}(V(z, m))\right) \leq \alpha_3(\|z\| + c_1) \leq \alpha_3(\|z\|) + \xi(c_1).$$

Let's define $\alpha_4 \triangleq \underline{\alpha}_3 \circ \overline{\alpha}_2^{-1}$, then:

$$\begin{aligned} \alpha_4\left(V(z, m)\right) &\leq \alpha_3(\|z\|) + \xi(c_1) \\ (22) \Rightarrow \Delta V(z, m) &\leq -\alpha_3(\|z\|) + c_2 - \xi(c_1) + \xi(c_1) \\ &\leq -\alpha_4\left(V(z, m)\right) + c_2 + \xi(c_1). \end{aligned} \tag{30}$$

Let ρ be a $\mathcal{K}_\infty$-function such that $(id - \rho)$ is also a $\mathcal{K}_\infty$-function. $\rho(s) = \frac{s}{2}$ is an example. We define $\Omega \subset \mathbb{R}^{n_z} \times \mathbb{M}$:

$$\Omega = \left\{(z, m) \in \mathbb{R}^{n_z} \times \mathbb{M} : V(z, m) \leq \omega(c_3)\right\}, \tag{31}$$

where $\omega \triangleq \alpha_4^{-1} \circ \rho^{-1}$ and $c_3 \triangleq c_2 + \xi(c_1)$.

We assume that $(id-\alpha_4)$ is a $\mathcal{K}_\infty$-function. Lemma B.1 in [15] proves that if $(id-\alpha_4)$ is not a $\mathcal{K}_\infty$-function, there exists a $\mathcal{K}_\infty$-function $\hat{\alpha}_4$ such that $\hat{\alpha}_4(s) \leq \alpha_4(s)$ and $(id-\hat{\alpha}_4)$ is a $\mathcal{K}_\infty$-function that can be used hereafter to lead to the same result.

Let's now assume that $(z,m) \in \Omega$:

$$\begin{aligned}
(30) \Rightarrow\ & V(f_{v^*}(z,u^*),v^*) - V(z,m) \leq -\alpha_4\left(V(z,m)\right) + c_3 \\
\Rightarrow\ & V(f_{v^*}(z,u^*),v^*) \leq (id-\alpha_4)\left(V(z,m)\right) + c_3 \\
& \leq (id-\alpha_4)\left(\omega(c_3)\right) + c_3 \\
& = \omega(c_3) - \alpha_4\left(\omega(c_3)\right) + c_3 \\
& = \omega(c_3) - \alpha_4\left(\omega(c_3)\right) + \rho \circ \alpha_4(\omega(c_3)) \\
& = \omega(c_3) - (id-\rho)(\alpha_4\left(\omega(c_3)\right)), \qquad (32)
\end{aligned}$$

where we have used the fact that $\rho \circ \alpha_4(\omega(s)) = s$. Since $(id-\rho)(s) \geq 0$ (being a $\mathcal{K}_\infty$-function), (32) yields:

$$V(f_{v^*}(z,u^*),v^*) \leq \omega(c_3),$$

thus proving that Ω is an invariant set for the closed loop system (19).

Step 2: Let's now prove that the invariant set Ω is an attractive set, *i.e.* that for any $(z_0,m_0) \notin \Omega$, there exists a finite $\bar{k}$ such that $(z_{\bar{k}},m_{\bar{k}}) \in \Omega$. Let $\bar{k}$ be the first time index where the system enters Ω, for the initial condition (z_0,m_0):

$$\bar{k} \triangleq \min\left\{k \in \mathbb{Z}_{\geq 0} : (z_k,m_k) \in \Omega\right\} \leq \infty, \qquad (33)$$

where $\bar{k}$ is infinite when the trajectories never enter Ω. To prove that Ω is attractive, we need to prove that $\bar{k}$ is finite. We start by noticing that if $(z,m) \notin \Omega$, then:

$$\begin{aligned}
& V(z,m) > \omega(c_3) = \alpha_4^{-1} \circ \rho^{-1}(c_3) \qquad (34) \\
\Rightarrow\ & \rho \circ \alpha_4(V(z,m)) > c_3 \\
\Leftrightarrow\ & \rho \circ \alpha_4(V(z,m)) - c_3 > 0. \qquad (35)
\end{aligned}$$

Moreover:

$$\begin{aligned}
(30) \Rightarrow \Delta V(z,m) & \leq -\alpha_4\left(V(z,m)\right) + c_3 \\
& = -(id-\rho) \circ \alpha_4\left(V(z,m)\right) - \rho \circ \alpha_4\left(V(z,m)\right) + c_3 \\
(35) \Rightarrow \Delta V(z,m) & \leq -(id-\rho) \circ \alpha_4\left(V(z,m)\right). \qquad (36)
\end{aligned}$$

Hence, for all $k < \bar{k}$, $\Delta V(z_k,m_k) \leq -\alpha_5(V(z_k,m_k))$, where $\alpha_5(s) \triangleq (id-\rho) \circ \alpha_4(s)$ is a $\mathcal{K}_\infty$-function, and thus is in particular a $\mathcal{K}$-function. According to [24, Lemma 4.3], this implies that there exists a $\mathcal{KL}$-function $\hat{\gamma}(s,k)$ such that:

$$V(z_k,m_k) \leq \hat{\gamma}(V(z_0,m_0),k), \quad \forall k < \bar{k}. \qquad (37)$$

The function $\hat{\gamma}(s,k)$ is decreasing in k and goes to 0 as $k \to \infty$, then there exists a finite $\tilde{k}$ such that:

$$\hat{\gamma}(V(z_0,m_0),\tilde{k}) < \omega(c_3) \qquad (38)$$

This implies that $\tilde{k} \geq \bar{k}$. Indeed, if $\tilde{k}$ was $\tilde{k} < \bar{k}$, then Eq.s (34),(37) would hold, but Eq.s (37),(38) would imply that $V(z_k,m_k) < \omega(c_3)$, in contradiction with (34).
This ends the proof that Ω is attractive since $\bar{k} \leq \tilde{k} < \infty$.

Step 3: Finally, we want to prove that Eq. (20) holds. We collect the results from the previous steps, $\forall (z_0,m_0) \in \mathbb{R}^{n_z} \times \mathbb{M}$, $\forall k \in \mathbb{Z}_{\geq 0}$:

- if $(z_k,m_k) \in \Omega$, then $V(z_k,m_k) \leq \omega(c_3)$,
- if $(z_k,m_k) \notin \Omega$, then $V(z_k,m_k) \leq \hat{\gamma}\left(V(z_0,m_0),k\right)$.

Eq. (21) implies that $\|z_k\| \leq \alpha^{-1}(V(z_k,m_k))$, we thus obtain:

- if $(z_k,m_k) \in \Omega$, then $\|z_k\| \leq \alpha^{-1}\left(\omega(c_3)\right)$,
- if $(z_k,m_k) \notin \Omega$, then $\|z_k\| \leq \alpha^{-1}\left(\hat{\gamma}\left(V(z_0,m_0),k\right)\right)$.

In any case, we have:

$$\|z_k\| \leq \alpha^{-1}\left(\hat{\gamma}\left(V(z_0,m_0),k\right)\right) + \alpha^{-1}\left(\omega(c_3)\right).$$

Eq. (21) implies that $V(z_0,m_0) \leq \alpha_2(\|z_0\|) + c_1$, which implies:

$$\|z_k\| \leq \alpha^{-1}(\hat{\gamma}\left(\alpha_2(\|z_0\|) + c_1, k\right)) + \alpha^{-1}(\omega(c_3)). \qquad (39)$$

Then, we notice that, for any function $\alpha(s)$ of class $\mathcal{K}_\infty$, $\forall (s_1,s_2) \in \mathbb{R}_{\geq 0}$, the following holds:

$$\begin{aligned}
\alpha(s_1+s_2) & \leq \begin{cases} \alpha(2s_1), & \text{if } s_1 \geq s_2 \\ \alpha(2s_2), & \text{if } s_1 \leq s_2 \end{cases} \\
\Rightarrow \alpha(s_1+s_2) & \leq \alpha(2s_1) + \alpha(2s_2).
\end{aligned}$$

Since, for a given k, $\alpha^{-1}\left(\hat{\gamma}\left(s,k\right)\right)$ is a function of class $\mathcal{K}_\infty$ w.r.t. s, we have:

$$\begin{aligned}
& \alpha^{-1}\left(\hat{\gamma}\left(\alpha_2(\|z_0\|) + c_1, k\right)\right) \leq \\
& \quad \alpha^{-1}\left(\hat{\gamma}\left(2\alpha_2(\|z_0\|), k\right)\right) + \alpha^{-1}\left(\hat{\gamma}\left(2c_1, k\right)\right). \qquad (40)
\end{aligned}$$

As the function $k\ \alpha^{-1}\left(\hat{\gamma}\left(2c_1,k\right)\right)$ is decreasing w.r.t. k, it attains its maximum for $k=0$:

$$\alpha^{-1}\left(\hat{\gamma}\left(2c_1,k\right)\right) \leq \alpha^{-1}\left(\hat{\gamma}\left(2c_1,0\right)\right), \ \forall\ k \in \mathbb{Z}_{\geq 0}. \qquad (41)$$

Notice that $\alpha^{-1}\left(\hat{\gamma}\left(2\alpha_2(s),k\right)\right)$ is a $\mathcal{KL}$-function. Eq.s (39)-(41) imply:

$$\begin{aligned}
& \|z(z_0,m_0,k)\| \leq \gamma(\|z_0\|,k) + c \\
& \text{with } \gamma(s,k) = \alpha^{-1}\left(\hat{\gamma}\left(2\alpha_2(s),k\right)\right) \\
& \qquad c = \alpha^{-1}\left(\omega(c_3)\right) + \alpha^{-1}\left(\hat{\gamma}\left(2c_1,0\right)\right).
\end{aligned}$$

■

Remark 3 *The choice of the $\mathcal{K}_\infty$-functions $\xi(s)$, $\rho(s)$ influence how $\gamma(s,k)$ give a more or less conservative bound.*

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