Deformation of Kähler manifolds

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ABSTRACT. It has been shown by Claire Voisin in 2003 that one cannot always deform a compact Kähler manifold into a projective algebraic manifold, thereby answering negatively a question raised by Kodaira. In this article, we prove that under an additional semipositivity or seminegativity condition on the canonical bundle, the answer becomes positive, namely such a compact Kähler manifold can be approximated by deformations of projective manifolds.

CONTENTS

1. INTRODUCTION

It is well known that the curvature of the canonical bundle controls the structure of projective varieties. C.Voisin has given a counterexample to the Kodaira conjecture which states that one cannot always deform a compact Kähler manifold to a projective manifold. In her counterexample one can see that the canonical bundle is neither nef nor anti-nef. Therefore it is interesting to ask whether for a Kähler manifold with a nef or anti-nef canonical bundle, one can deform it to a projective variety. In this article, we discuss the deformation properties of Kähler manifolds in the following three cases:

(1) Compact Kähler manifolds with hermitian semipositive anticanonical bundles.

(2) Compact Kähler manifolds with real analytic metrics and nonpositive bisectional curvatures.

(3) Compact Kähler manifolds with nef tangent bundles.

We first recall some definitions about numerical effective (nef) bundles (cf. [DPS] for details).

Definition 1.1. *A vector bundle E is said to be numerically effective (nef) if the canonical bundle $\mathcal{O}_E(1)$ is nef on $\mathbb{P}(E)$, the projective bundle of hyperplanes in the fibres of E. For a nef line bundle L on a compact Kähler manifold, the numerical dimension* $\mathrm{nd}(L)$ *is defined to be the largest number v, such that $\mathrm{c}_1(L)^v \neq 0$. A holomorphic vector bundle E over X is said to be numerically flat if both E and E^* are nef (or equivalently if E and $(\det E)^{-1}$ are nef).*

Definition 1.2. *Let X be a compact Kähler manifold. We say that X can be approximated by projective varieties, if there exists a deformation of X: $\mathcal{X} \to \Delta$ such that the central fiber X_0 is X, and there exists a sequence $t_i \to 0$ in Δ such that all the fibers X_{t_i} are projective.*

The main result of this article is

Main Theorem. *If X is a compact Kähler manifold in one of the above three cases, then X can be approximated by projective varieties.*

The proof for these three types of manifolds relies on their respective structure theorems. We first sketch the strategy of the proof when X is a compact Kähler manifold with hermitian semipositive anticanonical bundle. We first recall that a compact Kähler manifold X is said to be deformation unobstructed, if there exists a smooth deformation of X, $\pi : \mathcal{X} \to \Delta$, such that the Kodaira-Spencer map $T_\Delta \to H^1(X, T_X)$ is surjective. For this type of manifolds, we have the following proposition:

Proposition 3.3 in [Voi1]. *Assume that a deformation unobstructed compact Kähler manifold X has a Kähler class ω satisfying the following condition: the interior product*

$$\omega\wedge : H^1(X, T_X) \to H^2(X, \mathcal{O}_X)$$

is surjective. Then X can be approximated by projective varieties.

In [DPS 96], it is proved that after a finite cover, a compact Kähler manifold with hermitian semipositive anticanonical bundle has a smooth fibration to a compact Kähler manifold with trivial canonical bundle and the fibers Y_t satisfy the vanishing property:

$$H^q(Y_t, \mathcal{O}) = 0 \qquad \text{for } q \geq 1.$$

Therefore the Dolbeault cohomology of X is easy to calculate. One can thus construct explicitly a deformation of X satisfying the surjectivity in Proposition 3.3 in [Voi1]. Therefore this type of manifolds can be approximated by projective varieties.

When X is a compact Kähler manifold with nef tangent bundle, the proof is more difficult. It is based on the structure theorem in [DPS] which can be stated as follows.

Theorem 1.1. *Let X be a compact Kähler manifold with nef tangent bundle T_X. Let $\widetilde{X}$ be a finite étale cover of X of maximum irregularity $q = h^1(\widetilde{X}, \mathcal{O}_{\widetilde{X}})$. Then the Albanese map $\pi : \widetilde{X} \to T$ is a smooth fibration over a q-dimensional torus, and $-K_{\widetilde{X}}$ is relatively ample.*

Remark. *We will prove that after passing to some finite Galois cover $\widetilde{X} \to X$ with group G, there exists a commutative diagram*

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & X \\ \big\downarrow{\scriptstyle \widetilde{\pi}} & & \big\downarrow{\scriptstyle \pi} \\ T & \longrightarrow & T/G \end{array}$$

and T/G is smooth.

In [DPS], when X is a projective variety with nef tangent bundle, it is proved that $\pi_*(-mK_X)$ is numerically flat for all $m \geq 1$. One of the main ingredient of this article is to prove that this is also true when X is a compact Kähler manifold.

Theorem 1.2. *Let X be a compact Kähler manifold of dimension n with nef tangent bundle such that the Albanese map $\pi : X \to T$ is a smooth fibration onto a torus T of dimension r, and $-K_X$ is relatively ample. Then $\mathrm{nd}(-K_X) = n - r$, and $\pi_*(-mK_X)$ is numerically flat for all $m \geq 1$.*

We combine this with a result in [Sim] which states that any numerically flat bundle over a compact Kähler manifold is in fact a local system [1] :

Theorem 1.3. *Let E be a numerically flat holomorphic vector bundle on a Galois quotient of a torus T, then the transformation matrices can be choosen to be constant matrices.*

Using Theorem 1.2 and 1.3, we will see that one can approximate Kähler manifolds with nef tangent bundles by projective varieties.

We now sketch the proof of Theorem 1.2. Thanks to a formula in [Ber], $\pi_*(-mK_X)$ is nef. Then using the argument in [DPS], the only difficult part is to prove $\mathrm{nd}(-K_X) = n - r$. If X is projective, the equality $\mathrm{nd}(-K_X) = n - r$ comes directly from the Kawamata-Viehweg vanishing theorem. Since X is just a compact Kähler manifold in our case, the proof is more difficult. We get it by contradiction. Let $\pi : X \to T$ be the fibration in Theorem 1.2. If $\mathrm{nd}(-K_X) \geq n - r + 1$, there are two cases:
(*i*) The (1,1)-class $\pi_*((-K_X)^{n-r+1})$ is trivial on T.

[1] If the base manifold is a torus, an explicite construction of the local system would be found in the author's forthcoming Phd thesis.

(*ii*) The (1,1)-class $\pi_*((-K_X)^{n-r+1})$ is effective (non trivial) on T.

In the case (*i*), thanks to Corollary 2.6 in [DPS], we can prove that $\pi_*(-mK_X)$ is numerically flat. By Theorem 1.3, we can thus deform X to a projective manifold by preserving $\mathrm{nd}(-K_X)$. Using the Kawamata-Viehweg vanishing theorem in the projective case, we can therefore prove that $\mathrm{nd}(-K_X) = n - r$. Thus we get a contradiction.

In the case (*ii*), the argument is more complicated. By solving a Monge-Ampère equation, we can prove that $-K_X - c\pi^*(\pi_*((-K_X)^{n-r+1}))$ is pseudo-effective for some $c > 0$. Therefore we can construct a singular metric h on $-K_X$ with a good control on its eigenvalues and with $\mathcal{I}(h) = \mathcal{O}_X$, where $\mathcal{I}(h)$ is the multiplier ideal sheaf associated to the singular metric h (cf. [Dem2] for the definition of multiplier ideal sheaf). Thanks to the construction of the metric h, we can prove that

$$H^r(X, (K_X - K_X) \otimes \mathcal{I}(h)) = 0,$$

where $r = \dim T$. Therefore $H^r(X, \mathcal{O}_X) = 0$, which implies that $H^r(T, \mathcal{O}_T) = 0$ by the observation that $-K_X$ is relatively ample. Since the torus T is of dimension r, we get a contradiction.

The organization of the article is as follows. Let $\pi : \widetilde{X} \to T$ be the smooth fibration of Theorem 1.2. In Section 2, we gather some useful propositions. In particular, we prove a nefness result by using the formula (4.8) of [Ber]. In Section 3, we prove our main theorem when X is in the case (1) or (2). As an interesting application, the dual cone conjecture in [BDPP] is proved for the case (1). In the following sections, we concentrate on the proof of our main theorem when X is a compact Kähler manifold with nef tangent bundle. In Section 4, we prove a deformation lemma which allows us to deform a Kähler manifold to a projective variety under certain conditions and discuss how one can deform X to a projective variety by keeping the numerical dimension. In Section 5, we prove a very special Kawamata-Viehweg vanishing theorem which will play a central role in the proof of Theorem 1.2. Using the results in Section 4 and 5, we finally complete the proof of our main theorem in Section 6.

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2. Preliminaries

We first prove some preparatory propositions which are useful in the proof of our main theorem.

Proposition 2.1. *Let X be a compact Kähler manifold possessing a smooth submersion $\pi : X \to T$ to a compact Kähler manifold T. If $-K_X$ is nef on*

X and is relatively ample for π, then the direct image $E = \pi_*(-mK_X)$ is a nef vector bundle for all $m \in \mathbb{N}$.

Proof. Let us first show that the direct image E is locally free. Let X_t be the fiber of π over $t \in T$. Thanks to the Kodaira vanishing theorem, we have

$$H^q(X_t, -mK_{X_t}) = 0 \qquad \text{for } q \geq 1.$$

By the Riemann-Roch theorem,

$$\sum_q (-1)^q h^q(X_t, -mK_{X_t})$$

is a constant independent of t. Therefore $h^0(X_t, -mK_{X_t})$ is also a constant and by a standard result of H.Grauert, the direct image $E = \pi_*(-mK_X)$ is locally free.

Since $-K_X$ is nef, for any $\epsilon > 0$ fixed, there exists a smooth metric φ on $-mK_X$ such that

$$i\Theta_\varphi(-mK_X) \geq -\epsilon\omega_T.$$

Since E is known to be locally free, we can use formula (4.8) in [Ber]. In particular, the Bergman Kernel on E gives a metric on E and we write its curvature as

$$\Theta^E = \sum_{j,k} \Theta^E_{jk,\varphi} dt_j \wedge d\bar{t}_k$$

where $\{t_i\}$ are the coordinates of T. Using the terminology in [Ber], we assume that $\{u_i\}$ is a base of local holomorphic sections of E such that $D^{1,0}u_i = 0$ at a given point. We now calculate the curvature at this point. Let

$$T_u = \sum_{j,k} (u_j, u_k) \widehat{dt_j \wedge d\bar{t}_k}.$$

Then

$$i\partial\overline{\partial} T_u = -\sum_{j,k} (\Theta^E_{jk,\varphi} u_j, u_k) dV_t.$$

By the formula (4.8) in [Ber], we obtain[2]

$$-i\partial\overline{\partial} T_u \geq c\pi_*(\widehat{u} \wedge \overline{\widehat{u}} \wedge i\partial\overline{\partial}\varphi e^{-\varphi})$$

where the constant c is independent of φ. Since $i\partial\overline{\partial}\varphi \geq -\epsilon\omega_T$ by the choice of φ, we have

$$-i\partial\overline{\partial} T_u \geq -c\epsilon\pi_*(\widehat{u} \wedge \overline{\widehat{u}} \wedge \omega_T e^{-\varphi})$$
$$= -c\epsilon(\int_{X_t} \sum_j (u_j, u_j) e^{-\varphi}) dV_t$$
$$= -c\epsilon \|u\|^2 dV_t.$$

[2]The $i\partial\overline{\partial}\varphi$ below is just $i\Theta_\varphi(-mK_X)$.

In other words, we have

$$\sum_{j,k}(\Theta^E_{jk,\varphi}u_j, u_k) \geq -c\epsilon\|u\|^2.$$

Therefore we get the proposition.

□

Proposition 2.2. *Let $T = \mathbb{C}^n/\Gamma$ be a complex torus of dimension n, and $\alpha \in H^{1,1}(T, \mathbb{Z})$ an effective non trivial element. Then T possess a submersion*

$$\pi : T \to S$$

to an abelian variety S. Moreover $\alpha = \pi^* c_1(A)$ for some ample line bundle A on S.

Proof. Since T is a torus, we can suppose that α is a constant semipositive $(1,1)$-form. As α is an integral class, it defines a bilinear form

$$G_{\mathbb{Q}} : (\Gamma \otimes \mathbb{Q}) \times (\Gamma \otimes \mathbb{Q}) \to \mathbb{Q}.$$

We denote its extension to $\Gamma \otimes \mathbb{R}$ by $G_{\mathbb{R}}$. Let V be the maxium subspace of $\Gamma \otimes \mathbb{Q}$, on which $G_{\mathbb{Q}}$ is zero. Therefore $V_{\mathbb{R}} = V \otimes \mathbb{R}$ is also the kernel of $G_{\mathbb{R}}$, and $(\Gamma \cap V_{\mathbb{R}}) \otimes \mathbb{R} = V_{\mathbb{R}}$. Moreover, since α is an $(1,1)$-form, $V_{\mathbb{R}}$ is a complex subspace of $\mathbb{C}^n$. Hence $V_{\mathbb{R}}/(\Gamma \cap V_{\mathbb{R}})$ is a complex torus. We denote it T_1. Observing that T/T_1 is also a complex torus, we have thus a natural holomorphic submersion $T \to T/T_1$. We denote the complex torus T/T_1 by S. Since $V_{\mathbb{R}}$ is the kernel of $G_{\mathbb{R}}$, α is well defined on S and is moreover strictly positive on it. The proposition is proved. □

Proposition 2.3. *Let E be a numerically flat bundle on a compact Kähler manifold. Then E is a local system.*

Proof. Thanks to Thoerem 1.18 in [DPS], all numerically flat vector bundles are successive extensions of hermitian flat bundles. By the section 3 of [Sim], all such types of bundles are local systems. The proposition is proved. □

Remark. *This simple proof is due to C.Simpson. When X is just a finite étale quotient of a torus, one can give a more elementary proof. Since that proof is a bit long and technical, we omit the proof here and refer instead to our forthcoming PhD thesis.*

We need a partial vanishing theorem with multiplier ideal sheaf (cf.[Dem2] for the definition of multiplier ideal sheaves and analytic singularities).

Proposition 2.4. *Let L be a line bundle on a compact Kähler manifold (X, ω) of dimension n and let φ be a metric on L with analytic singularities. Let $\lambda_1(z) \leq \lambda_2(z) \leq \cdots \leq \lambda_n(z)$ be the eigenvalues of $\frac{i}{2\pi}\Theta_\varphi(L)$ with respect to ω. If*

$$\sum_{i=1}^{p} \lambda_i(z) \geq c \tag{4.1}$$

for some constant $c > 0$ independent of $z \in X$, then

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0 \qquad \text{for } q \geq p.$$

Proof. Since φ has analytic singularities, there exists an analytic subvariety Y such that φ is smooth on $X \setminus Y$. Moreover it is known that there exists a quasi-psh function ψ on X, smooth on $X \setminus Y$ such that (cf. [Dem1])

$$\mathcal{I}(\varphi) = \mathcal{I}(\varphi + \psi)$$

and $\widetilde{\omega} = c_1\omega + i\partial\overline{\partial}\psi$ is a complete metric on $X \setminus Y$ for some fixed constant c_1 with $0 < c_1 \ll c$. To prove the proposition, it is therefore equivalent to prove that

$$H^q(X, K_X \otimes L \otimes \mathcal{I}(\varphi + \psi)) = 0 \qquad \text{for } q \geq p. \tag{4.2}$$

We consider the new metric $\phi = \varphi + \psi$ on L (i.e. the new metric is $\|\cdot\|_\varphi e^{-\psi}$). Then

$$\frac{i}{2\pi}\Theta_\phi(L) = \frac{i}{2\pi}\Theta_\varphi(L) + dd^c\psi = (\frac{i}{2\pi}\Theta_\varphi(L) - c_1\omega) + \widetilde{\omega}. \tag{4.3}$$

We claim that the sum of p-smallest eigenvalues of $\frac{i}{2\pi}\Theta_\phi(L)$ with respect to $\omega_\tau = \omega + \tau\widetilde{\omega}$ is larger than $\frac{c}{2}$ when τ is small enough with respect to c_1.

Proof of the claim: By the minimax principle, it is sufficient to prove that for any p-dimensional subspace V of $(T_X)_x$, we have

$$\sum_{i=1}^{p} \langle \frac{i}{2\pi}\Theta_\phi(L)e_i, e_i\rangle \geq \frac{c}{2} \tag{4.4}$$

where $\{e_i\}$ is an orthonormal basis of V with respect to ω_τ.

We first consider the case when V contains an element e such that

$$\widetilde{\omega}(e,e) \geq \frac{c_1}{\tau} \qquad \text{and} \qquad |e|_\omega = 1.$$

Since φ is a quasi-psh function, there exists a constant M independent of τ such that

$$(\frac{i}{2\pi}\Theta_\varphi(L) - c_1\omega) \geq -M\omega.$$

Thanks to (4.3), we have

$$\langle \frac{i}{2\pi}\Theta_\phi(L)e, e\rangle \geq -M + \widetilde{\omega}(e,e) \geq \frac{\widetilde{\omega}(e,e)}{2},$$

where the last inequality comes from the facts that τ is small enough with respect to c_1 and $\widetilde{\omega}(e,e) \geq \frac{c_1}{\tau}$. Observing moreover that the construction of ω_τ implies

$$\langle e, e\rangle_{\omega_\tau} \leq 1 + \tau \cdot \widetilde{\omega}(e,e),$$

then

$$\frac{\langle \frac{i}{2\pi}\Theta_\phi(L)e, e\rangle}{\langle e, e\rangle_{\omega_\tau}} \geq \frac{\widetilde{\omega}(e,e)}{2 + 2\tau\widetilde{\omega}(e,e)}. \tag{4.5}$$

Since τ is small enough with respect to c_1, the inequality (4.5) implies that $\frac{\langle \frac{i}{2\pi}\Theta_\phi(L)e,e\rangle}{\langle e,e\rangle_{\omega_\tau}}$ is large enough with respect to M and c. Noting that (4.3) implies

$$\langle \frac{i}{2\pi}\Theta_\phi(L)e',e'\rangle \geq -M\omega(e',e') \geq -M\omega_\tau(e',e')$$

for any $e' \in V$, the inequality (4.5) implies thus the inequality (4.4).

In the case when

$$\tau \cdot \widetilde{\omega}(e,e) \leq c_1 \qquad \text{for any } e \in V \text{ with } |e|_\omega = 1,$$

we have

$$|\omega_\tau - \omega|_\omega \leq c_1 \qquad \text{on } V, \tag{4.6}$$

i.e. for considering just the restriction on V, the difference between ω_τ and ω is controled by $c_1\omega$. On the other hand, using again the minimax principle, (4.1) implies that

$$\sum_{i=1}^{p}\langle \frac{i}{2\pi}\Theta_\varphi(L)\widetilde{e}_i,\widetilde{e}_i\rangle \geq c$$

for any orthonormal basis $\{\widetilde{e}_i\}$ of V with respect to ω. Combining with (4.3) and the smallness assumption on c_1, we have

$$\sum_{i=1}^{p}\langle \frac{i}{2\pi}\Theta_\phi(L)\widetilde{e}_i,\widetilde{e}_i\rangle \geq \frac{3c}{4}. \tag{4.7}$$

Since c_1 is a fixed constant small enough with respect to c, (4.6) and (4.7) imply the inequality (4.4). The claim is proved.

Let f be a L-valued closed (n,q)-form such that

$$\int_X |f|^2 e^{-2\varphi-2\psi}\omega^n < +\infty.$$

To prove the proposition, it is equivalent to find a L-valued $(n,q-1)$-form g such that

$$f = \overline{\partial} g \qquad \text{and} \qquad \int_X |g|^2 e^{-2\varphi-2\psi}\omega^n < +\infty.$$

Thanks to our claim, we can use the standard L^2 estimate on

$$(X \setminus Y, \omega_\tau, L, e^{-\varphi-\psi}).$$

In particular, we can find a g_τ such that $f = \overline{\partial} g_\tau$ and [3]

$$\int_{X\setminus Y} |g|^2 e^{-2\phi}\omega_\tau^n \leq C\int_{X\setminus Y} |f|^2 e^{-2\phi}\omega_\tau^n < +\infty.$$

[3]It is known that $\int_{X\setminus Y} |f|^2 e^{-2\phi}\omega_\tau^n \leq \int_{X\setminus Y} |f|^2 e^{-2\phi}\omega^n < +\infty$.

for a constant C depending only on c (i.e. C is independent of τ). Letting $g = \lim_{\tau \to 0} g_\tau$, we get $f = \overline{\partial} g$ on $X \setminus Y$ and

$$\int_{X \setminus Y} |g|^2 e^{-2\phi} \omega^n < +\infty.$$

Lemma (11.10) in [Dem2] implies that such g can be extended to the whole space X, and $f = \overline{\partial} g$ on X. Therefore (4.2) is proved. □

As a corollary of the main theorem in [DPS], we prove that every compact Kähler manifold with nef tangent bundle admits a smooth fibration to an étale Galois quotient of a torus.

Lemma 2.5. *Let X be a compact Kähler manifold with nef tangent bundle and let $\widetilde{X} \to X$ be an étale Galois cover with group G such that $\widetilde{X}$ satisfies Theorem 1.1 (i.e. Main theorem in* [DPS]*). Then G induces a free automorphism group on $T = \mathrm{Alb}(\widetilde{X})$ and we have the following commutative diagram*

$$\begin{array}{ccc} \widetilde{X} & \longrightarrow & X \\ \big\downarrow{\scriptstyle \widetilde{\pi}} & & \big\downarrow{\scriptstyle \pi} \\ T & \longrightarrow & T/G \end{array}$$

where $\widetilde{\pi} : \widetilde{X} \to T$ is the Albanese map in Theorem 1.1, and T/G is an étale Galois quotient of the torus T.

Proof. By the universal property of Albanese map, for any $g \in G$, g induces an automorphism on T, and the action of g on $\widetilde{X}$ maps fibers to fibers. We need hence only to prove that G acts on T freely. Suppose by contradiction that $g(t_0) = t_0$ for some $t_0 \in T$ and $g \in G$. Since g acts on $\widetilde{X}$ without fixed point, g induces an automorphism on $\widetilde{X}_{t_0}$ without fixed points, where $\widetilde{X}_{t_0}$ is the fiber of $\widetilde{\pi}$ over t_0. Combining with the fact that $\widetilde{X}_{t_0}$ is a Fano manifold, the quotient $\widetilde{X}_{t_0}/g$ is hence also a Fano manifold. Thus the Nadel vanishing theorem implies that

$$\chi(\widetilde{X}_{t_0}, \mathcal{O}_{\widetilde{X}_{t_0}}) = \chi(\widetilde{X}_{t_0}/g, \mathcal{O}_{\widetilde{X}_{t_0}/g}) = 1. \tag{5.1}$$

(5.1) contradicts with the fact that the double cover $\widetilde{X}_{t_0} \to \widetilde{X}_{t_0}/g$ implies

$$\chi(\widetilde{X}_{t_0}, \mathcal{O}_{\widetilde{X}_{t_0}}) = 2\chi(\widetilde{X}_{t_0}/g, \mathcal{O}_{\widetilde{X}_{t_0}/g}).$$

Then G factorizes to an étale Galois action on T, and the lemma is proved. □

3. Deformation of compact Kähler manifolds with hermitian semipositive anticanonical bundles or nonpositive bisectional curvatures

We first treat a special case, i.e., how to appproximate compact manifolds with numerically trivial canonical bundles by projective varieties. To prove the statement, we need the following two propositions.

Proposition 3.3 in [Voi1]. *Assume that a deformation unobstructed compact Kähler manifold X has a Kähler class ω satisfying the following condition: the interior product*

$$\omega\wedge : H^1(X, T_X) \to H^2(X, \mathcal{O}_X)$$

is surjective. Then X can be approximated by projective varieties.

Remark. *The proof of this proposition is based on a density criterion (cf. Proposition 5.20 in* [Voi2]*) which will also be used in Proposition 3.3 and Proposition 3.5. We need moreover a slightly generalized version of Proposition 3.3 in* [Voi1]*. In fact, we can suppose ω to be a nef class in X, since the surjectivity is preserved under small perturbation. Moreover, if X is not necessarily unobstructed, we just need a deformation unobstructed subspace V of $H^1(X, T_X)$ such that*

$$\omega \wedge V \to H^2(X, \mathcal{O}_X)$$

is surjective. In summary, we have the following variant of the above proposition.

Version B of Proposition 3.3 in [Voi1]. *Let $\mathcal{X} \to \Delta$ be a deformation of a compact Kähler manifold X and V be the image of Kodaira-Spencer map of this deformation. If there exists a nef class ω in $H^{1,1}(X)$ such that*

$$\omega \wedge V \to H^2(X, \mathcal{O}_X)$$

is surjective, then there exists a sequence $t_i \to 0$ in Δ such that all the fibers X_{t_i} are projective.

In general, it is difficult to check the surjectivity in the above proposition. By a well-known observation communicated to us by J-P. Demailly, one can prove that the above morphism is surjective when $-K_X$ is hermitian semipositive by using the following Hard Lefschetz theorem.

Hard Lefschetz theorem. *(cf. Corollary 15.2 in* [Dem2]*) Let (L, h) be a semi-positive line bundle on a compact Kähler manifold (X, ω) of dimension n i.e., h is a smooth metric on L and $i\Theta_h(L) \geq 0$. Then the wedge multiplication operator $\omega^q\wedge$ induces a surjective morphism*

$$\omega^q\wedge : H^0(X, \Omega_X^{n-q} \otimes L) \to H^q(X, \Omega_X^n \otimes L).$$

Using the above two propositions, we can reprove the following well-known fact.

Proposition 3.1. *Let X be a compact Kähler manifold with $c_1(X)_{\mathbb{R}} = 0$. Then it can be approximated by projective varieties.*

Proof. By a theorem due to Beauville, there exists a finite Galois cover $\widetilde{X} \to X$ such that $K_{\widetilde{X}}$ is trivial. Then K_X is a torsion line bundle. Using the Tian-Todorov theorem (cf. the torsion version in [Ran]), X is unobstructed. To use Proposition 3.3 in [Voi1], we just need to check that

$$\omega\wedge : H^1(X, T_X) \to H^2(X, \mathcal{O}_X) \tag{3.1}$$

is surjective for some Kähler class ω.

In fact, since $c_1(K_X)_{\mathbb{R}} = 0$, there exists a smooth metric h on $-K_X$ such that $i\Theta_h(-K_X) = 0$. Thus $(-K_X, h)$ is semipositive. Then the Hard Lefschetz theorem above told us that for any Kähler metric ω, the morphism

$$\omega^2\wedge : H^0(X, \Omega^{n-2} \otimes (-K_X)) \to H^2(X, K_X \otimes (-K_X))$$

is surjective. Observing moreover that

$$\omega^2 \wedge H^0(X, \Omega^{n-2} \otimes (-K_X))$$

is contained in the image of

$$\omega \wedge H^1(X, \Omega^{n-1} \otimes (-K_X)) = \omega \wedge H^1(X, T_X),$$

(3.1) is thus surjective. Using Proposition 3.3 in [Voi1], the proposition is proved. □

We now begin to prove the main proposition in this section, i.e., one can approximate compact Kähler manifolds with hermitian semipositive anticanonical bundles by projective varieties. The main tool is the structure theorem of [DPS 96]:

Structure Theorem. *Let X be a compact Kähler manifold with $-K_X$ hermitian semipositive. Then*

(i) The universal cover $\widetilde{X}$ admits a holomorphic and isometric splitting

$$\widetilde{X} = \mathbb{C}^q \times Y_1 \times Y_2$$

with Y_1 being the product of either Calabi-Yau manifolds or symplectic manifolds, and Y_2 being projective. Moreover $H^0(Y_2, \Omega_{Y_2}^{\otimes q}) = 0$ for $q \geq 1$.

(ii) There is a normal subgroup $\Gamma_1 \subset \pi_1(X)$ of finite index such that $\widetilde{X}/\Gamma_1$ has a smooth fibration to a compact manifold: $Z = (\mathbb{C}^q \times Y_1)/\Gamma_1$, and the fibers are Y_2.

Remark. *Since $\Omega_{Y_2}^q \subset \Omega_{Y_2}^{\otimes q}$, the above structure theorem implies that*

$$H^0(Y_2, \Omega_{Y_2}^q) = 0.$$

Therefore $H^q(Y_2, \mathcal{O}) = 0$ by duality.

We need also the following lemma.

Lemma 3.2. *Let X be a compact Kähler manifold with $K_X = \mathcal{O}_X$, and G a finite subgroup of the biholomorphic group* $\mathrm{Aut}(X)$*. Then there exists a local deformation of X: $\mathcal{X} \to \Delta$ such that the image of the Kodaira-Spencer map of this deformation is equal to $H^1(X, T_X)^{G-\mathrm{inv}}$ and $\mathcal{X}$ admits a holomorphic G-action fiberwise, where $H^1(X, T_X)^{G-\mathrm{inv}}$ is the G-invariant subspace of $H^1(X, T_X)$.*

Proof. By the Kuranishi deformation theory, it is sufficient to construct a vector valued $(0,1)$-form

$$\varphi(t) = \sum_{k_i \geq 0} \varphi_{k_1 \cdots k_m} t_1^{k_1} \cdots t_m^{k_m}$$

such that

$$(*) \qquad \varphi(0) = 0 \qquad \text{and} \qquad \overline{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]$$

where $\varphi_{k_1 \cdots k_m}$ are G-invariant vector valued $(0,1)$-forms, $\{\varphi_{k_1 \cdots k_m}\}_{\sum k_i = 1}$ gives a basis of $H^1(X, T_X)^{G-\mathrm{inv}}$ and $t_1, ..., t_m$ are parameters of Δ. By [MK], it is equivalent to find G-invariant vector valued $(0,1)$-forms φ_μ such that

$$(**) \qquad \overline{\partial}\varphi_\mu = \frac{1}{2} \sum_{|\lambda| < |\mu|} [\varphi_\lambda, \varphi_{\mu - \lambda}]$$

for any μ.

Suppose that we have already found φ_μ for $|\mu| \leq N$ such that $(**)$ is satisfied for all $|\mu| \leq N$. If $|\mu| = N + 1$, thanks to [Tian], there exists a vector valued $(0,1)$-form s_μ satisfying

$$\overline{\partial} s_\mu = \frac{1}{2} \sum_{|\lambda| \leq N} [\varphi_\lambda, \varphi_{\mu - \lambda}].$$

Recall that if Y_1, Y_2 are two G-invariant vector valued $(0,1)$-forms, then $[Y_1, Y_2]$ is also a G-invariant vector valued $(0,2)$-form [4]. Therefore $\overline{\partial} s_\mu$ is a G-invariant vector valued $(0,2)$-form. The finiteness of G and the above G-invariance of $\overline{\partial} s_\mu$ imply hence that $\frac{1}{|G|} \sum\limits_{g \in G} g^* s_\mu$ is a G-invariant vector valued $(0,1)$-form satisfying $(**)$. The lemma is proved. □

The following proposition tells us that for a compact Kähler manifold with numerically trivial canonical bundle, if it admits "more automorphisms", then it is "more algebraic". More precisely, we have

Proposition 3.3. *Let $\pi : \mathcal{X} \to \Delta$ be the deformation constructed in Lemma 3.2. Then there exists a sequence $t_i \to 0 \in \Delta$ such that X_{t_i} are projective varieties.*

[4] Let $\alpha \in G$, $f \in C^\infty(X)$ and $x \in X$. Using the G-invariance of Y_1 and Y_2, we have $\alpha^*(Y_1 Y_2)(f)(x) = Y_1 Y_2(f \circ \alpha)(\alpha^{-1}(x)) = Y_1(Y_2(f) \circ \alpha)(\alpha^{-1}(x)) = Y_1(Y_2(f))(x)$. Thus $[Y_1, Y_2]$ is also G-invariant.

Proof. We first prove that $H^2(X,\mathbb{Q})^{G-\mathrm{inv}}$ admits a sub-Hodge structure of $H^2(X,\mathbb{Q})$. In fact, we have the equality $H^2(X,\mathbb{Q})^{G-\mathrm{inv}}\otimes\mathbb{R} = H^2(X,\mathbb{R})^{G-\mathrm{inv}}$ by observing that G acts continuous on $H^2(X,\mathbb{R})$. Combining with the obvious Hodge decomposition

$$H^2(X,\mathbb{C})^{G-\mathrm{inv}} = \oplus_{p+q=2} H^{p,q}(X,\mathbb{C})^{G-\mathrm{inv}},$$

$H^2(X,\mathbb{Q})^{G-\mathrm{inv}}$ is thus a sub-Hodge structure of $H^2(X,\mathbb{Q})$. Then π induces a VHS of $H^2(X,\mathbb{Q})^{G-\mathrm{inv}}$.

Let ω_X be a G-invariant Kähler metric on X. (3.1) of Proposition 3.1 implies that

$$\omega_X \wedge H^1(X,T_X) \to H^2(X,\mathcal{O})$$

is surjective. Thanks to the G-invariance of ω_X,

$$\omega_X \wedge H^1(X,T_X)^{G-\mathrm{inv}} \to H^2(X,\mathcal{O})^{G-\mathrm{inv}}$$

is also surjective. Using the density criterion Proposition 5.20 in [Voi2] and the same argument of Proposition 3.3 in [Voi1], the proposition is proved. □

We now prove the main result in this section.

Proposition 3.4. *Let X be a compact Kähler manifold with $-K_X$ hermitian semipositive. Then it can be approximated by projective varieties.*

Proof. We prove it in three steps.

Step 1: We use the terminology of the Structure Theorem in this section. Let $G=\pi_1(X)/\Gamma_1$ and $X_1=\widetilde{X}/\Gamma_1$. Then G acts on X_1 and $X=X_1/G$. Thanks to *(ii)* of the Structure Theorem, we have the smooth fibration

$$\pi : X_1 \to Z \tag{4.1}$$

with the fibers Y_2. We prove in this step that

$$H^q(X_1,\mathcal{O}) = H^q(Z,\mathcal{O}) \qquad \text{and} \qquad H^q(X_1,\mathcal{O})^{G-\mathrm{inv}} = \pi^*(H^q(Z,\mathcal{O})^{G-\mathrm{inv}}).$$

Thanks to the smooth fibration (4.1), we can calculate $H^q(X_1,\mathcal{O})$ by the Leray spectral sequence. Then the first equality comes directly from the fact that

$$H^q(Y_2,\mathcal{O}) = 0 \qquad \text{for } q \geq 1$$

(cf. the remark of the Structure Theorem in this section). As for the second equality, we just need to check that the image of the injective map

$$\pi^* : H^q(Z,\mathcal{O})^{G-\mathrm{inv}} \to H^q(X_1,\mathcal{O}) \tag{4.2}$$

is $H^q(X_1,\mathcal{O})^{G-\mathrm{inv}}$. Let $\gamma\in G$ and α a smooth differential form on Z. Since $\pi_1(X)$ acts on $\mathbb{C}^q\times Y_1$ and Y_2 separately, we have the diagram

$$\begin{tikzcd} X_1 \arrow[r, "\gamma"] \arrow[d, "\pi"] & X_1 \arrow[d, "\pi"] \\ Z \arrow[r, "\gamma"] & Z \end{tikzcd} .$$

Then the equality

$$\gamma^*(\pi^*\alpha) = \pi^*(\gamma^*\alpha)$$

implies that the image of the morphism (4.2) is contained in $H^q(X_1, \mathcal{O})^{G-\mathrm{inv}}$. As for the surjectivity, we suppose that $\beta \in H^q(X_1, \mathcal{O})^{G-\mathrm{inv}}$. Thanks to the proved equality

$$H^q(X_1, \mathcal{O}) = H^q(Z, \mathcal{O}),$$

we can find an element $\mu \in H^q(Z, \mathcal{O})$ such that $\pi^*\mu = \beta$ as an element in $H^q(X_1, \mathcal{O})$. Since

$$\pi^*(\gamma^*\mu) = \gamma^*(\pi^*\mu) = \gamma^*(\beta) = \beta = \pi^*(\mu)$$

in $H^q(X_1, \mathcal{O})$, the injectivity of (4.2) implies that $\gamma^*(\mu) = \mu$ in $H^q(Z, \mathcal{O})$. Then μ is G-invariant and (4.2) gives an isomorphism from $H^q(Z, \mathcal{O})^{G-\mathrm{inv}}$ to $H^q(X_1, \mathcal{O})^{G-\mathrm{inv}}$.

Step 2: Let $\omega_Z^{G-\mathrm{inv}}$ be a G-invariant Kähler metric on Z. We construct in this step a deformation of Z: $\mathcal{Z} \to \Delta$ such that

$$\omega_Z^{G-\mathrm{inv}} \wedge V \to H^2(Z, \mathcal{O})^{G-\mathrm{inv}}$$

is surjective, where V is the image of the Kodaira-Spencer map of this deformation. Moreover, $\mathcal{Z}$ should admit a holomorphic G-action fiberwise.

Since $c_1(Z)_{\mathbb{R}} = 0$ by construction, Proposition 3.1 implies that

$$\omega_Z^{G-\mathrm{inv}} \wedge H^1(Z, T_Z) \to H^2(Z, \mathcal{O})$$

is surjective. Then

$$\omega_Z^{G-\mathrm{inv}} \wedge H^1(Z, T_Z)^{G-\mathrm{inv}} \to H^2(Z, \mathcal{O})^{G-\mathrm{inv}} \tag{4.3}$$

is also surjective. Thanks to Lemma 3.2, there exists a deformation of Z satisfying the requirements of deformation in this step, especially it admits a holomorphic G-action fiberwise.

Step 3: Final conclusion.

Since X_1 is the quotient of $\Gamma_1 \curvearrowright \mathbb{C}^q \times Y_1 \times Y_2$ and Γ_1 acts on $\mathbb{C}^q \times Y_1$ and Y_2 separately, the deformation of $(\mathbb{C}^q \times Y_1)/\Gamma_1$ in Step 2 induces a deformation of X_1:

$$\mathcal{X}_1 \to \Delta$$

by preserving the complex structure of Y_2. By construction, we have a natural fibration

$$\widetilde{\pi} : \mathcal{X}_1 \to \mathcal{Z}.$$

Moreover, since G is holomorphic for every Z_t, the quotient $\mathcal{X} = \mathcal{X}_1/G$ is a smooth deformation of X. In summary, we have the following diagrams:

$$\begin{array}{ccc} X_1 & \xrightarrow{G} & X \\ \downarrow{\scriptstyle \pi} & & \\ Z & & \end{array} \quad \text{and} \quad \begin{array}{ccc} \mathcal{X}_1 & \xrightarrow{G} & \mathcal{X} \\ \downarrow{\scriptstyle \widetilde{\pi}} & & \\ \mathcal{Z} & & \end{array}.$$

Thanks to Proposition 3.3, there exists a sequence $t_i \to 0 \in \Delta$ such that Z_{t_i} are projective. Since the fibers of $\widetilde{\pi}$ are also projective by the Structure Theorem, X_{t_i} are projective. The proposition is proved. $\square$

Remark. *For the further application, we need study the deformation $\mathcal{X}$ in detail. Since $\pi^*\omega_Z^{G-\mathrm{inv}}$ is a G-invariant semipositive form on X_1, it comes from a nef class on X. We denote it also $\pi^*\omega_Z^{G-\mathrm{inv}}$ for simplicity. Let V be the image of Kodaira-Spencer map of the deformation $\mathcal{X} \to \Delta$. We now prove that*

$$\pi^*\omega_Z^{G-\mathrm{inv}} \wedge V \to H^2(X, \mathcal{O}) \tag{4.4}$$

is surjective. Thanks to the construction of $\mathcal{X}_1$ and the surjectivity of (4.3)*, the morphism*

$$\pi^*\omega_Z^{G-\mathrm{inv}} \wedge W \to \pi^*(H^2(Z, \mathcal{O})^{G-\mathrm{inv}}) \tag{4.5}$$

is surjective on X_1, where W is the image of Kodaira-Spencer map of the deformation $\mathcal{X}_1 \to \Delta$. Since

$$\pi^*(H^2(Z, \mathcal{O})^{G-\mathrm{inv}}) = H^q(X_1, \mathcal{O})^{G-\mathrm{inv}}$$

which is proved in Step 1, (4.5) *implies that*

$$\pi^*\omega_Z^{G-\mathrm{inv}} \wedge W \to H^q(X_1, \mathcal{O})^{G-\mathrm{inv}}$$

is surjective. Thus (4.4) *is surjective.*

As an application, we prove the Conjecture 2.3 and 10.1 in [BDPP] for compact Kähler manifolds with hermitian semipositive anticanonical bundles.

Proposition 3.5. *If X is a compact Kähler manifold with $-K_X$ hermitian semipositive, then the Conjecutre 2.3 and 10.1 in* [BDPP] *are all true.*

Proof. By the remark of Proposition 3.4, there exists a local deformation of X

$$\pi : \mathcal{X} \to \Delta,$$

such that

$$\alpha \wedge V \to H^2(X, \mathcal{O}) \tag{5.1}$$

is surjective for some nef class $\alpha \in H^{1,1}(X, \mathbb{R})$, where V is the image of the Kodaira-Spencer map of π.

Let β be any smooth closed $(1,1)$-form on X. Thanks to the surjectivity of (5.1),

$$(\beta + s\alpha) \wedge V \to H^2(X, \mathcal{O})$$

is also surjective for any $s \neq 0$ small enough. By the proof of Proposition 5.20 in [Voi2], we can hence find a sequence of smooth closed 2-forms $\{\beta_t\}$ on X, such that

$$\lim_{t \to 0} \beta_t = \beta + s\alpha$$

in C^∞-topology and $\beta_t \in H^{1,1}(X_t, \mathbb{Q})$. By the same argument as in Theorem 10.12 of [BDPP], the proposition is proved. $\square$

We now study the case when X has a real analytic metric and nonpositive bisectional curvatures. Recall first the structure theorem E in [WZ]

Proposition 3.6. *Let X be a compact Kähler manifold of dimension n with real analytic metric and nonpositive bisectional curvature, and let $\widetilde{X}$ be its universal cover. Then*

(i) There exists a holomorphically isometric decomposition $\widetilde{X} = \mathbb{C}^{n-r} \times Y^r$, where Y^r is a complete manifold with nonpositve bisectional curvature and the Ricci tensor of Y^r is strictly negative somewhere.

(ii) (cf. Claim 2 of Theorem E in [WZ]*) There exists a finite index subnormal group Γ' of $\Gamma = \pi_1(X)$ such that Y^r/Γ' is a compact manifold and $\widetilde{X}/\Gamma'$ possess the smooth fibrations to Y^r/Γ' and $\mathbb{C}^{n-r}/\Gamma'$.*

Remark. *By Claim 2 of Theorem E in* [WZ]*, $\mathbb{C}^{n-r}/\Gamma'$ is a torus. We should notice that in contrast to the case when $-K_X$ is semipositive, Y^r is not necessary compact in this proposition. The universal covers of curves of genus $g \geq 2$ are typical exemples. The good news here is that Y^r/Γ' is a projective variety of general type thanks to (i).*

Proposition 3.7. *Let X be a compact Kähler manifold of dimension n with real analytic metric and nonpositive bisectional curvature. Then it can be approximated by projective varieties.*

Proof. Keeping the notation in Proposition 3.6, $T = \mathbb{C}^{n-r}/\Gamma'$ is a torus with the finite group action $G = \Gamma/\Gamma'$. By Lemma 3.2, there exists a deformation of T

$$\pi : \mathcal{T} \to \Delta$$

such that G acts holomorphically fiberwise. Therefore this deformation induces the deformations of $\widetilde{X}/\Gamma'$ and X by preserving the complex structure on Y^r. We denote

$$\widetilde{\mathcal{X}}/\Gamma' \to \Delta \qquad \text{and} \qquad \mathcal{X} \to \Delta.$$

Thanks to the construction, X_t is the G-quotient of $\widetilde{X}_t/\Gamma'$, where X_t and $\widetilde{X}_t/\Gamma'$ are the fibers over $t \in \Delta$ of the above deformations.

Let $t_i \to 0$ be the sequence in Proposition 3.3 such that T_{t_i} are projective. By Proposition 3.6, we have two fibrations:

$$\widetilde{X}_{t_i}/\Gamma' \to T_{t_i} \qquad \text{and} \qquad \widetilde{X}_{t_i}/\Gamma' \to Y^r/\Gamma'.$$

Thanks to the projectivity of T_{t_i} and the remark of Proposition 3.6, $\widetilde{X}_{t_i}/\Gamma'$ is thus projective. Therefore all X_{t_i} are projective and the proposition is proved. □

4. A deformation proposition

The following sections are devoted to the deformation problem of compact Kähler manifolds with nef tangent bundles. We discuss in this section how to deform varieties that are defined by certain numerically flat fibrations. We first prove a preparatory lemma.

Lemma 4.1. *Let T be a torus and let E be a numerical flat bundle on the torus possessing a filtration*

$$0 = E_0 \subset E_1 \subset \cdots \subset E_m = E \tag{1.1}$$

such that the quotients E_i/E_{i-1} are irreducible hermitian flat vector bundles. Then all elements in $H^0(T, E)$ are parallel with respect to the natural local system induced by the filtration (1.1)*.*

In particular, if there are two such filtrations, the transformation matrices between these two filtrations should be locally constant.

Proof. Thanks to [Sim], the filtration (1.1) induces a natural local system on E. If E_i/E_{i-1} is a non trivial irreducible hermitian flat bundle. then $H^0(T, E_i/E_{i-1}) = 0$. Using the recurrence process, to prove that all elements in $H^0(T, E)$ are parallel with respect to the local system, it is sufficient to prove that if F is a non trivial extension

$$0 \longrightarrow \mathcal{O}_T \xrightarrow{\ i\ } F \longrightarrow \mathcal{O}_T \longrightarrow 0,$$

then $H^0(T, F) = i(H^0(T, \mathcal{O}_T))$. In fact, we have the exact sequence

$$0 \longrightarrow H^0(T, \mathcal{O}_T) \xrightarrow{\ i\ } H^0(T, F) \longrightarrow H^0(T, \mathcal{O}_T) \xrightarrow{\ \delta\ } H^1(T, \mathcal{O}_T).$$

Since $h^0(T, \mathcal{O}_T) = 1$ and F is a non trivial extension, δ is injective. Therefore $i(H^0(T, \mathcal{O}_T)) = H^0(T, F)$. Then all elements in $H^0(T, F)$ should be parallel with respect to the natural local system induced by (1.1).

If there is another filtration

$$0 = E'_0 \subset E'_1 \subset \cdots \subset E'_n = E,$$

then it induces a filtration on E^*. Using this filtration on E^* and the filtration (1.1) on E, we get a natural filtration on $\mathrm{Hom}(E, E) = E^* \otimes E$. Applying this lemma, the natural identity element $\mathrm{id} \in H^0(T, \mathrm{Hom}(E, E))$ should be parallel with respect to the filtration. In other words, the transformation matrices between these two filtrations should be locally constant. □

Proposition 4.2. *Let X be a Kähler manifold possessing a submersion $\pi : X \to T$, where T is a finite étale quotient of a torus. Assume that $-K_X$ is nef and relatively ample. If moreover $E_m = \pi_*(-mK_X)$ is numerically flat for any $m > 0$, then there is a smooth deformation of the fibration which can be realized as:*

$$\mathcal{X} \xrightarrow{\ \pi\ } \mathcal{T} \xrightarrow{\ \pi_1\ } \Delta$$

such that $\pi_1 : \mathcal{T} \to \Delta$ is the local universal deformation of T and the central fiber is $X \to T$.

Moreover, let T_s be the fiber of π_1 over $s \in \Delta$, and let X_s be the fiber of $\pi \circ \pi_1$ over $s \in \Delta$. Then the anticanonical bundle of X_s is also nef and relatively ample with respect to the fibration $X_s \to T_s$ for any $s \in \Delta$.

Proof. Thanks to Theorem 3.20 of [DPS], we have the embeddings $X \hookrightarrow \mathbb{P}(E_m)$ and $V_{m,p} = \pi_*(\mathcal{I}_X \otimes \mathcal{O}_{\mathbb{P}(E_m)}(p)) \subset S^p E_m$ for m, p large enough. More importantly, $V_{m,p}$ and $S^p E_m$ are numerically flat vector bundles. By

Proposition 2.3, S^pE_m is in fact a local system on T which be represented by locally constant transformation matrices and its subbundle $V_{m,p}$ can be represented by the upper blocks of the transformation matrices.

Thanks to Proposition 2.3 of [Ran], the deformation of T is unobstructed. Let $\pi_1 : \mathcal{T} \to \Delta$ be the local universal deformation of T. Then the transformation matrices of $S^pE_m, V_{m,p}$ are always holomorphic under the deformation of the complex structure on T. Therefore we get the holomorphic deformations of these vector bundles by changing the complex strucutre on T:

$$\begin{tikzcd} \mathcal{V}_{m,p} \arrow[r, hook] \arrow[dr] & S^p\mathcal{E}_m \arrow[d] \\ & \mathcal{T} \arrow[d] \\ & \Delta \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} \mathcal{V}_{m,p} \times \mathbb{P}(\mathcal{E}_m) \arrow[d] \\ \mathcal{T} \arrow[d] \\ \Delta \end{tikzcd} .$$

Let $U \subset T$ be any small open neighborhood. By the discussion after Proposition 3.19 in [DPS], a local basis of $V_{m,p}$ gives the determinant polynomials of X in $\mathbb{P}(E_m)$ over U. The proof of Proposition 2.3 also implies that $V_{m,p}$ is a sub-local system of S^pE_m. Now we have two filtrations on $S^p(E_m)$, one is induced by the inclusion $V_{m,p} \subset S^pE_m$ and the another is induced by a filtration on E_m. Thanks to Lemma 4.1, on the small open set U, we can choose a local basis of $V_{m,p}$ with constant coefficients with respect to E_m, i.e the coefficients of the defining polynomials of X in $\mathbb{P}(E_m)$ over U can be choosen as constants. Then the defining equations of $(V_{m,p})$ over $U \times s$ are the same as $(V_{m,p})$ over $U \times \{0\}$ for $s \in \Delta$. Therefore $\mathcal{V}_{m,p}$ defines a smooth deformation of X, we denote it

$$\mathcal{X} \xrightarrow{\ \pi\ } \mathcal{T} \xrightarrow{\ \pi_1\ } \Delta\cdot$$

As for the second part of the proposition, we first prove that $-K_{X_t}$ is ample on X_t where X_t is the fiber of $\mathcal{X} \to \mathcal{T}$ over $t \in \mathcal{T}$ and t is in a neighborhood of T in $\mathcal{T}$. Since $-K_{X_{t_0}}$ is ample for for $t_0 \in T$, by [Yau] there exists a Kähler metric ω_{t_0} on X_{t_0} such that $i\Theta_{\omega_{t_0}}(-K_{X_{t_0}}) > 0$. By a standard continuity argument (cf. Theorem 3.1 of [Sch] for exemple), we can construct Kähler metrics ω_t on X_t for t in a neighborhood of t_0 in $\mathcal{T}$ and by continuity the curvatures $i\Theta_{\omega_t}(-K_{X_t})$ are all positive for t in a neighborhood of t_0 in $\mathcal{T}$. Therefore $-K_{X_t}$ is ample on X_t for t near t_0 in $\mathcal{T}$. Letting t_0 run over T, then $-K_{X_t}$ is ample for all t in a neighborhood of T in $\mathcal{T}$.

We need also prove that $-K_{X_s}$ is nef on X_s, where X_s is the fiber of $\pi \circ \pi_1$ over $s \in \Delta$. Let $(E_m)_s$ be the fiber of $\mathcal{E}_m \to \Delta$ over s. By construction, $(E_m)_s$ is numerically flat on T_s, where T_s is fiber of π_1 over s. Then $\mathcal{O}_{\mathbb{P}(\mathcal{E}_m)}(1)$ is nef on $\mathbb{P}(E_m)_s$. Since X_s is embedded in $\mathbb{P}(E_m)_s$, $\mathcal{O}_{\mathbb{P}(\mathcal{E}_m)}(1)|_{X_s}$ is also nef for any $s \in \Delta$. If $s = 0$, we have

$$\mathcal{O}_{\mathbb{P}(\mathcal{E}_m)}(1)|_{X_s} = -mK_X.$$

Therefore

$$c_1(\mathcal{O}_{\mathbb{P}(\mathcal{E}_m)}(1)|_{X_s}) = c_1(-mK_{X_s})$$

for $s \in \Delta$ by the rigidity of integral classes. Then the nefness of $\mathcal{O}_{\mathbb{P}(\mathcal{E}_m)}(1)|_{X_s}$ implies that $-mK_{X_s}$ is nef for all $s \in \Delta$.

The proposition is proved.

□

Remark. *In general, nefness is not an open condition in families (cf. Theorem 1.2.17* [Laz]*). Thanks to the above construction, nefness is preserved under deformation in our special case.*

Thanks to Proposition 4.2, we immediately get the following corollary.

Corollary 4.3. *Let X be a compact Kähler manifold satisfying the condition in Proposition 4.2. Then X can be approximated by projective varieties. Moreover,* $\mathrm{nd}(-K_X) = n - \dim T$.

Proof. By Proposition 4.2, there exists a deformation of $X \to T$:

$$\mathcal{X} \xrightarrow{\pi} \mathcal{T} \xrightarrow{\pi_1} \Delta$$

such that $\mathcal{T} \to \Delta$ is the local universal deformation of T and $X \to T$ is the central fiber of this deformation. By Proposition 3.1, there exists a sequence $s_i \to 0$ in Δ such that all T_{s_i} are projective. Using Proposition 4.2, we know that the fibers of

$$X_{s_i} \to T_{s_i}$$

are Fano manifolds. Then all X_{s_i} are projective and X can be approximated by projective manifolds.

As for the second part of the corollary, by observing that $-K_X$ is relatively ample, we have $\mathrm{nd}(-K_X) \geq n-r$. If $\mathrm{nd}(-K_X) \geq n-r+1$, by the definition of numerical dimension we have

$$\int_X (-K_X)^{n-r+1} \wedge \omega_X^{r-1} > 0.$$

By continuity,

$$\int_{X_{s_i}} (-K_{X_{s_i}})^{n-r+1} \wedge \omega_{X_{s_i}}^{r-1} > 0 \tag{3.1}$$

for $|s_i| \ll 1$. Thanks to Proposition 4.2, $-K_{X_{s_i}}$ are nef. Then (3.1) implies the existence of a projective variety X_{s_i} such that $-K_{X_{s_i}}$ is nef and $\mathrm{nd}(-K_{X_{s_i}}) \geq n - r + 1$, which contradicts with the Kawamata-Viehweg vanishing theorem for projective varieties. We get a contradiction and the corollary is proved. □

5. A Kawamata-Viehweg vanishing theorem

As pointed out in the introduction, when X is a projective variety of dimension n and L is a nef line bundle on X with $\mathrm{nd}(L) = k$, we have the Kawamata-Viehweg vanishing theorem:

$$H^r(X, K_X + L) = 0 \qquad \text{for } r > n - k.$$

But it is probably a difficult problem to prove this vanishing theorem for a non projective compact Kähler manifold. We will prove in this section a Kawamata-Viehweg vanishing theorem for certain Kähler manifolds. More precisely, we say that a compact Kähler manifold X of dimension n and a nef line bundle L satisfy conditions $(*)$, if

Conditions $(*)$: Let L be a nef line bundle on a compact Kähler manifold X of dimension n . We say that (X, L) satisfies conditions $(*)$, if

(i) There exists a smooth two steps tower fibration

$$X \xrightarrow{\pi} T \xrightarrow{\pi_1} S$$

where π is a submersion to a smooth variety T of dimension r, and π_1 is a submersion to a smooth curve S.

(ii) The nef line bundle L is relatively ample with respect to π and

$$\pi_*(L^{n-r+1}) = \pi_1^*(\mathcal{O}_S(1))$$

(this implies that $\mathrm{nd}(L) > n - r$) for some ample line bundle $\mathcal{O}_S(1)$ on S.

(iii) Moreover, we suppose that [5]

$$L^{n-r+t} \wedge \pi^*\pi_1^*(\mathcal{O}_S(1)) = 0 \qquad \text{where } n - r + t = \mathrm{nd}(L).$$

We will prove in this section that if (X, L) satisfies the conditions $(*)$, then

$$H^p(X, K_X + L) = 0 \qquad \text{for } p \geq r.$$

Before the proof of this vanishing theorem, we first prove a useful lemma.

Lemma 5.1. *Assume that (X, L) satisfies the conditions $(*)$. Then $L - c\pi^*\pi_1^*(\mathcal{O}_S(1))$ is pseudo-effective for some constant $c > 0$.*

Proof. We first explain the idea of the proof. By using a Monge-Ampère equation, we can construct a sequence of metrics $\{\varphi_\epsilon\}$ on L, such that

$$\frac{i}{2\pi}\Theta_{\varphi_\epsilon}(L) \geq c\pi^*\pi_1^*(\mathcal{O}_S(1)) \qquad \text{for all small } \epsilon.$$

Then $\frac{i}{2\pi}\Theta_\varphi(L) \geq c\pi^*\pi_1^*\mathcal{O}_S(1)$, where φ is a limit of some subsequence of $\{\varphi_\epsilon\}$. In this way, the lemma would therefore be proved. This idea comes from [DP], but the proof here is in some sense much simpler because we do not need their diagonal trick in our case.

[5] It is redundant when X is projective. See the remark after Lemma 5.1 for the explanation of this assumption.

Observing first that (ii) of the condition $(*)$ implies that $\mathrm{nd}(L) > n-r$, we can thus suppose that $\mathrm{nd}(L) = n-r+t$, for some $t \geq 1$. For simplicity, we denote $\pi^*\pi_1^*\mathcal{O}_S(1)$ by A. Let $s \in S$, and X_s the fiber of $\pi \circ \pi_1$ over s. We first fix a smooth metric h_0 on $\mathcal{O}_S(1)$. Thanks to the semi-positivity of A, we can choose a sequence of smooth functions ψ_ϵ on X such that for the metrics $h_0 e^{-\psi_\epsilon}$ on A, the curvature forms $\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A)$ are semi-positive [6], and

$$\int_{V_\epsilon} \frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) \wedge \omega^{n-1} \geq C_1 \qquad \text{for } \epsilon \to 0 \tag{1.1}$$

where V_ϵ is an ϵ open neighborhood of X_s, and $C_1 > 0$ is a uniform constant. (All the constants C_i below will be uniformly strictly positive. When the uniform boundedness comes from obvious reasons, we will not make it explicit.)

Let τ_1, τ_2 two constants such that $1 \gg \tau_1 \gg \tau_2 > 0$ which will be made precise later. Let h be a fixed smooth metric on L. Thanks to the nefness of L, we can solve a Monge-Ampère equation:

$$\left(\frac{i}{2\pi}\Theta_h(L) + \tau_1\omega + dd^c\varphi_\epsilon\right)^n = C_{2,\epsilon}\frac{\tau_1^{r-t}}{\tau_2^{n-1}}\left(\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) + \tau_2\omega\right)^n, \tag{1.2}$$

where

$$C_{2,\epsilon} = \frac{(\frac{i}{2\pi}\Theta_h(L) + \tau_1\omega)^n \tau_2^{n-1}}{\tau_1^{r-t}(\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) + \tau_2\omega)^n}.$$

Since $\mathrm{nd}(L) = n-r+t$, we have $\inf_\epsilon C_{2,\epsilon} > 0$.

Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be the eigenvalues of $\frac{i}{2\pi}\Theta_h(L) + \tau_1\omega + dd^c\varphi_\epsilon$ with respect to $\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) + \tau_2\omega$. Then the Monge-Ampère equation (1.2) implies that

$$\prod_{i=1}^n \lambda_i(z) = C_{2,\epsilon}\frac{\tau_1^{r-t}}{\tau_2^{n-1}} \qquad \text{for any } z \in X. \tag{1.3}$$

We claim that there exists a constant $\delta > 0$ independent of ϵ, τ_1, τ_2, such that

$$\int_{V_\epsilon \setminus E_{\delta,\epsilon}} \frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) \wedge \omega^{n-1} \geq \frac{C_1}{2} \qquad \text{for any } \epsilon, \tag{1.4}$$

where

$$E_{\delta,\epsilon} = \{z \in V_\epsilon | \prod_{i=2}^n \lambda_i(z) \geq C_{2,\epsilon}\frac{\tau_1^{r-t}}{\delta\tau_2^{n-1}}\}.$$

We postphone the proof of the claim in Lemma 5.2 and finish the proof of this lemma. Since

$$\lambda_1(z) \geq \frac{C_2\frac{\tau_1^{r-t}}{\tau_2^{n-1}}}{C_2\frac{\tau_1^{r-t}}{\delta\tau_2^{n-1}}} = \delta \qquad \text{for } z \in V_\epsilon \setminus E_{\delta,\epsilon}$$

---

[6]Note that here ψ_ϵ are functions, but the φ's in Proposition 2.1 are metrics! Therefore in this lemma, $\frac{i}{2\pi}\Theta_{\psi_\epsilon}(\mathcal{O}_S(1)) = \frac{i}{2\pi}\Theta_{h_0}(\mathcal{O}_S(1)) + dd^c\psi_\epsilon$.

by the definition of $E_{\delta,\epsilon}$ and (1.3), (1.4) implies hence that

$$\int_{V_\epsilon} (\frac{i}{2\pi}\Theta_h(L) + \tau_1\omega + dd^c\varphi_\epsilon) \wedge \omega^{n-1} \geq C_8 \int_{V_\epsilon} \lambda_1(z)\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) \wedge \omega^{n-1}$$

$$\geq \delta C_8 \int_{V_\epsilon \setminus E_{\delta,\epsilon}} \frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) \wedge \omega^{n-1} \geq \delta \cdot C_8 \cdot \frac{C_1}{2}. \tag{1.5}$$

Letting $\epsilon \to 0$, the choice of V_ϵ and (1.5) imply that the weak limit of

$$\frac{i}{2\pi}\Theta_h(L) + \tau_1\omega + dd^c\varphi_\epsilon$$

is more positive than $C_9[X_s]$. Thus $L+\tau_1\omega - C_9[X_s]$ is pseudo-effective. Since C_9 is independent of τ_1, when $\tau_1 \to 0$, we obtain that $L - C_9\pi^*\pi_1^*(\mathcal{O}_S(1))$ is pseudo-effective. The lemma is proved.

□

Remark. *If one could generalize the Hovanskii-Teissier type inequality (5.4) in* [Dem93] *to the Kähler case, the hypothesis*

$$L^{n-r+t} \wedge \pi^*\pi_1^*(\mathcal{O}_S(1)) = 0$$

made here would become redundant. When we prove the main theorem (Theorem 6.2), we could easily exploit this hypothesis by induction on dimension.

Lemma 5.2. *We now prove the claim in Lemma 5.1*

Proof. By construction,

$$\int_X (\prod_{i=2}^n \lambda_i(z))(\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) + \tau_2\omega)^n \tag{2.1}$$

$$\leq C_3 \int_X (c_1(L) + \tau_1\omega + dd^c\varphi_\epsilon)^{n-1} \wedge (\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) + \tau_2\omega)$$

$$= C_3 \int_X (c_1(L) + \tau_1\omega)^{n-1} \wedge (c_1(A) + \tau_2\omega).$$

On the other hand, using (iii) of the conditions $(*)$, we have

$$\int_X (c_1(L) + \tau_1\omega)^{n-1} \wedge (c_1(A) + \tau_2\omega) \tag{2.2}$$

$$= C_4\tau_1^{r-t}c_1(L)^{n-r+t-1} \wedge c_1(A) + O(\tau_2) \leq C_5\tau_1^{r-t}.$$

where the last inequality comes from the fact that $\tau_2 \ll \tau_1$. Combining (2.1) and (2.2), we have

$$\int_X (\prod_{i=2}^n \lambda_i(z))(\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) + \tau_2\omega)^n \leq C_6\tau_1^{r-t}. \tag{2.3}$$

For any δ fixed, (2.3) and the definition of $E_{\delta,\epsilon}$ imply that

$$\int_{E_\delta} C_{2,\epsilon}\frac{\tau_1^{r-t}}{\delta\tau_2^{n-1}}(\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) + \tau_2\omega)^n \leq C_6\tau_1^{r-t}.$$

Combining with the fact that $\inf_{\epsilon} C_{2,\epsilon} > 0$, we get

$$(2.4) \qquad \int_{E_{\delta,\epsilon}} (\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) + \tau_2\omega)^n \leq C_7\delta\tau_2^{n-1}.$$

Since $\frac{i}{2\pi}\Theta_{\psi_\epsilon}(A)$ is semi-positive, (2.4) implies that

$$(2.5) \qquad \int_{E_{\delta,\epsilon}} \frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) \wedge \omega^{n-1} \leq C_7\delta.$$

By taking $\delta = \frac{C_1}{2C_7}$, (1.1) of Lemma 5.1 and (2.5) imply that

$$\int_{V_\epsilon \setminus E_{\delta,\epsilon}} \frac{i}{2\pi}\Theta_{\psi_\epsilon}(A) \wedge \omega^{n-1} \geq \frac{C_1}{2}.$$

The lemma is proved. □

Using Lemma 5.1, we would like to prove a Kawamata-Viehweg vanishing theorem. Recall that T.Ohsawa proved in [Ohs] that if $X \to T$ is a smooth fibration and (E,h) is a hermitian line bundle on X with $\frac{i}{2\pi}\Theta_h(E) \geq \pi^*\omega_T$. Then

$$H^q(T, R^0\pi_*(K_X \otimes E)) = 0$$

for $q \geq 1$. In his proof, he uses the metrics $\pi^*\omega_T + \tau\omega_X$ on X and lets $\tau \to 0$ to preserve the information on T. The idea of our proof comes from this technique.

Proposition 5.3. *Assume that (X,L) satisfies the conditions $(*)$. Then*

$$H^p(X, K_X + L) = 0 \qquad \textit{for } p \geq r.$$

Proof. Thanks to the conditions $(*)$, we have a smooth fibration

$$X \xrightarrow{\pi} T \xrightarrow{\pi_1} S\cdot$$

Using the fixed metric ω_X, we have a C^∞-decomposition of the tangent bundle of X:

$$T_X = T_{X/T} \oplus E_1 \oplus E_2,$$

where $T_{X/T}$ is the relative tangent bundle of $\pi : X \to T$, E_1 is the relative tangent bundle of $\pi_1 : T \to S$ and E_2 is the tangent bundle of S.

We first construct a metric h with analytic singularities on L such that

(i). $i\Theta_h(L)$ is strictly positive on $T_{X/T}$.

(ii). The restrictions of $i\Theta_h(L)$ on E_1 maybe negative, but the positivity of the restrictions on E_2 is large enough with respect to the negativity on E_1.

(iii). $\mathcal{I}(h) = \mathcal{O}_X$.

By Lemma 5.1 and Demailly's regularization theorem, for any $\epsilon > 0$, there is a singular metric h_1 with analytic singularities such that

$$(3.1) \qquad i\Theta_{h_1}(L) \geq c\pi^*(\omega_S) - \epsilon\omega_X.$$

We will make the choice of ϵ more explicit later on. Moreover, since L is relatively ample, there is a smooth metric h_2 on L, such that the restriction of $i\Theta_{h_2}(L)$ on $T_{X/T}$ is strictly positive. Thanks to the nefness of L, we can also choose a smooth metric h_ϵ on L such that

$$i\Theta_{h_\epsilon}(L) \geq -\epsilon\omega_X. \tag{3.2}$$

We now define a new metric h on L by combining the above three metrics:

$$h = \epsilon_1 h_1 + \epsilon_2 h_2 + (1 - \epsilon_1 - \epsilon_2) h_\epsilon$$

for some $1 \gg \epsilon_1 \gg \epsilon_2 \gg \epsilon > 0$.

We now check that the new metric h satisfies the above three conditions. Since $1 \gg \epsilon_1 \gg \epsilon_2$ and h_ϵ is smooth, Condition (*iii*) follows. To check the first two properties, we first observe that (3.1) and (3.2) imply that

$$i\Theta_h(L) \geq c\epsilon_1 \pi^*(\omega_S) + \epsilon_2 i\Theta_{h_2}(L) - \epsilon\omega_X.$$

Therefore it is enough to check the two conditions for $c\epsilon_1\pi^*(\omega_S) + \epsilon_2 i\Theta_{h_2}(L) - \epsilon\omega_X$. Condition (*i*) follows by the observation that $i\Theta_{h_2}$ is strictly positive on $T_{X/T}$ and $\epsilon_2 \gg \epsilon$. Since $\epsilon_1 \gg \epsilon_2$, the positivity of $i\Theta_h(L)$ on the direction of E_2 is large enough with respect to the negativity on the directions of E_1, which comes from $\epsilon_2 i\Theta_{h_2}(L)$. Condition (*ii*) follows.

Let ω_T be a Kähler metric on T and let $\omega_\tau = \tau\omega_X + \pi^*(\omega_T)$ for $\tau > 0$. When $\tau \to 0$, Condition (*i*) and Condition (*ii*) of h imply that the pair (X, ω_τ, L, h) satisfies the conditions in Proposition 2.4. Thus

$$H^p(X, K_X + L \otimes \mathcal{I}(h)) = 0 \qquad \text{for } p \geq r.$$

Since $\mathcal{I}(h) = \mathcal{O}_X$ by our construction, we get

$$H^p(X, K_X + L) = 0 \qquad \text{for } p \geq r.$$

□

6. Deformation of compact Kähler manifolds with nef tangent bundles

Before giving the proof of our main theorem, we need a technical lemma.

Lemma 6.1. *Assume that X has a two step tower smooth fibration:*

$$X \xrightarrow{\pi} T \xrightarrow{\pi_1} S,$$

where T is a torus of dimension r, and S is an abelian variety of dimension s. We suppose also that the fibers of π are Fano manifolds. Let S_p be a complete intersection of zero divisors of p general global sections of a very ample line bundle $\mathcal{O}_S(1)$ on S. Let T_p and X_p be the inverse images of S_p in T and X. Then

$$H^{r-p}(X_p, K_{X_p} - K_X) \neq 0 \qquad \textit{for } 0 \leq p \leq s-1.$$

Proof. By the adjunction formula $-K_{X_p} + p\pi^*\pi_1^*\mathcal{O}_S(1) = -K_X|_{X_p}$, we have

$$H^{r-p}(X_p, K_{X_p} - K_X) = H^{r-p}(X_p, p\pi^*\pi_1^*\mathcal{O}_S(1)). \tag{1.1}$$

On the other hand, the assumption that the fibers of π are Fano manifolds implies that

$$H^{r-p}(X_p, p\pi^*\pi_1^*\mathcal{O}_S(1)) = H^{r-p}(T_p, p\pi_1^*\mathcal{O}_S(1)) \tag{1.2}$$

by using the Leray spectral sequence. Observing moreover that $K_{T_p} = p\pi_1^*\mathcal{O}_S(1)$, equalities (1.1) and (1.2) imply that

$$H^{r-p}(X_p, K_{X_p} - K_X) = H^{r-p}(T_p, K_{T_p}) \qquad \text{for } 0 \leq p \leq s-1. \tag{1.3}$$

To prove the lemma, it is therefore enough to check the non vanishing of $H^{r-p}(T_p, K_{T_p})$.

Since $\dim T_p - \dim S_p = r - s$ for any p, by a standard vanishing theorem (cf. Theorem 4.1 in Chapter VII of [Dem1]) we have

$$H^{r-s+i}(T_p, K_{T_p} + \pi_1^*\mathcal{O}_S(1)) = 0 \tag{1.4}$$

for $i \geq 1$ and $p = 0, 1, ..., s-1$. Thanks to the exact sequence

$$0 \to \mathcal{O}_{T_{p-1}}(K_{T_{p-1}}) \to \mathcal{O}_{T_{p-1}}(K_{T_{p-1}} + \pi_1^*\mathcal{O}_S(1)) \to \mathcal{O}_{T_p}(K_{T_p}) \to 0,$$

(1.4) implies that

$$H^{r-s+i}(T_p, K_{T_p}) = H^{r-s+i+1}(T_{p-1}, K_{T_{p-1}}) \qquad \text{for } i \geq 1. \tag{1.5}$$

In particular, if we take $i = s - p$ in (1.5), then

$$H^{r-p}(T_p, K_{T_p}) = H^{r-p+1}(T_{p-1}, K_{T_{p-1}}). \tag{1.6}$$

Since $T_0 = T$ is a torus, we have $H^r(T_0, K_{T_0}) \neq 0$. Then (1.6) implies by induction that

$$H^{r-p}(T_p, K_{T_p}) \neq 0 \qquad \text{for } 0 \leq p \leq s-1.$$

Using (1.3), the lemma is proved. □

Theorem 6.2. *Let X be a compact Kähler manifold of dimension n with nef tangent bundle, and $\pi : X \to T$ a smooth fibration onto a torus T of dimension r such that $-K_X$ is nef and relatively ample. Then* $\mathrm{nd}(-K_X) = n - r$.

Proof. We prove the theorem by contradiction. Observing that the relative ampleness of $-K_X$ already implies that $\mathrm{nd}(-K_X) \geq n - r$, we can thus assume by contradiction that $\mathrm{nd}(-K_X) \geq n - r + 1$. There are two cases.

Case 1: $\pi_*((-K_X)^{n-r+1})$ is trivial on T.

Then

$$\int_X c_1(-K_X)^{n-r+1} \wedge (\pi^*\omega_T)^{r-1} = 0, \tag{2.1}$$

where ω_T is a Kähler form on T. Since T_X is nef, by Corollary 2.6 in [DPS], (2.1) implies that all degree $n - r + 1$ Chern polynomals P of T_X satisfy

$$\int_X P(T_X) \wedge (\pi^*\omega_T)^{r-1} = 0. \tag{2.2}$$

Let $E_m = \pi_*(-mK_X)$. By the Riemann-Roch-Grothendick theorem, we have

$$\mathrm{Ch}(E_m) = \pi_*(c_1(-K_X) \cdot \mathrm{Todd}(T_X)). \tag{2.3}$$

Then (2.2) and (2.3) imply that $c_1(E_m) \wedge (\omega_T)^{r-1} = 0$. But by Proposition 3.1, E_m is nef on T. Therefore E_m is numerically flat. By Corollary 4.3, $\mathrm{nd}(-K_X) = n - r$. We get a contradiction.

Case 2: $\pi_*((-K_X)^{n-r+1})$ is a non trivial effective class on T.

By Proposition 2.2, we have the following smooth fibration

$$X \xrightarrow{\pi} T \xrightarrow{\pi_1} S,$$

where S is an abelian variety of dimension m, and

$$\pi_*((-K_X)^{n-r+1}) = c \cdot \pi_1^* \mathcal{O}_S(1) \tag{2.4}$$

for some $c > 0$ and a very ample line bundle $\mathcal{O}_S(1)$ on S. Let S_{m-1} be a complete intersection of $m-1$ general global sections of $H^0(S, \mathcal{O}_S(1))$. Let X_{m-1} and T_{m-1} be the inverse images of S_{m-1} in X and T. Then we have a smooth fibration

$$X_{m-1} \xrightarrow{\widetilde{\pi}} T_{m-1} \xrightarrow{\widetilde{\pi}_1} S_{m-1}. \tag{2.5}$$

By (2.4), $\mathrm{nd}(-K_X|_{X_{m-1}}) > n - r$. We can thus suppose that

$$\mathrm{nd}(-K_X|_{X_{m-1}}) = n - r + t$$

for some $t \geq 1$. We claim that $(X_{m-1}, -K_X|_{X_{m-1}})$ satisfies the conditions $(*)$ in Section 5 for the fibration (2.5).

Proof of the claim: All claims are obvious except for the fact that

$$(-K_X)^{n-r+t} \wedge \widetilde{\pi}^* \widetilde{\pi}_1^* (\mathcal{O}_{S_{m-1}}(1)) = 0.$$

Let $s_0 \in S_{m-1}$. We denote its inverse image in T and X by T_{s_0} and X_{s_0}. Then we have a fibration

$$\pi_2 : X_{s_0} \to T_{s_0}.$$

Since s_0 is a point on the torus S, T_{s_0} is also a torus and X_{s_0} has nef tangent bundle. Moreover, since $-K_X|_{X_{s_0}} = -K_{X_{s_0}}$, $-K_{X_{s_0}}$ is also nef and relatively ample for π_2. Then X_{s_0} satisfies the conditions in this theorem. By induction on dimension in this theorem, we get

$$(-K_{X_{s_0}})^{n-r+t} = 0.$$

Then

$$(-K_X|_{X_{s_0}})^{n-r+t} = (-K_{X_{s_0}})^{n-r+t} = 0. \tag{2.6}$$

Since $[X_{s_0}] = c \cdot c_1(\mathcal{O}_{S_{m-1}}(1))$ in X_{m-1}, (2.6) implies that

$$(-K_X|_{X_{m-1}})^{n-r+t} \wedge \widetilde{\pi}^* \widetilde{\pi}_1^* \mathcal{O}_{S_{m-1}}(1) = 0.$$

The claim is proved.

Since $\dim T_{m-1} = r-(m-1)$, the claim and Proposition 5.2 imply therefore that

$$H^{r-(m-1)}(X_{m-1}, K_{X_{m-1}} - K_X) = 0.$$

On the other hand, Lemma 6.1 implies that

$$H^{r-(m-1)}(X_{m-1}, K_{X_{m-1}} - K_X) \neq 0.$$

We obtain again a contradiction.

Since Case 1 and Case 2 are both impossible, we infer that $\mathrm{nd}(-K_X) = n - r$.

□

Now we can prove our main result:

Theorem 6.3. *Let X be a compact Kähler manifold of dimension n with nef tangent bundle. Then X can be approximated by projective varieties.*

Proof. By Lemma 2.5, there exists a finite étale Galois cover $\widetilde{X} \to X$ with group G such that one has a commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r] \arrow[d, "\widetilde{\pi}"] & X \arrow[d, "\pi"] \\ T \arrow[r] & T/G \end{tikzcd}$$

where the fibers of π are Fano manifolds. We suppose that $\dim T = r$. Thanks to Theorem 6.2, we have $\mathrm{nd}(-K_{\widetilde{X}}) = n - r$, which is equivalent to say that $\mathrm{nd}(-K_X) = n - d$.

Let $E_m = \pi_*(-mK_X)$, for $m \geq 1$. By Proposition 2.1, E_m is a nef vector bundle. By the Riemann-Roch-Grothendick theorem, we have

$$\mathrm{Ch}(E_m) = \pi_*(\mathrm{Ch}(-K_X)\,\mathrm{Todd}(T_X)).$$

Since $\mathrm{nd}(-K_X) = n-r$, the above equality implies that $c_1(E_m) = 0$ by using Corollary 2.6 of [DPS]. Combining the fact that E_m is nef by Proposition 2.1, E_m is thus numerically flat by definition. Using Corollary 4.3, we conclude that X can be approximated by projective varieties.

□

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