Stochastic integration with respect to multifractional Brownian motion via tangent fractional Brownian motions

Erick Herbin, Joachim Lebovits, Jacques Lévy Véhel

Stochastic integration with respect to multifractional Brownian motion *via* tangent fractional Brownian motions

Erick HERBIN * Joachim LEBOVITS *§ Jacques LÉVY VÉHEL *

November 18, 2012

Abstract

Stochastic integration w.r.t. fractional Brownian motion (fBm) has raised strong interest in recent years, motivated in particular by applications in finance and Internet traffic modelling. Since fBm is not a semi-martingale, stochastic integration requires specific developments. Multifractional Brownian motion (mBm) generalizes fBm by letting the local Hölder exponent vary in time. This is useful in various areas, including financial modelling and biomedicine. The aim of this work is twofold: first, we prove that an mBm may be approximated in law by a sequence of "tangent" fBms. Second, using this approximation, we show how to construct stochastic integrals w.r.t. mBm by "transporting" corresponding integrals w.r.t. fBm. We illustrate our method on examples such as the Wick-Itô, Skorohod and pathwise integrals.

Keywords: Fractional and multifractional Brownian motions, Gaussian processes, convergence in law, white noise theory, Wick-Itô integral, Skorohod integral, pathwise integral.

1 Motivation and Background

Fractional Brownian motion (fBm) is a centred Gaussian process with features that make it a useful model in various applications such as financial and Internet traffic modeling, image analysis and synthesis, physics, geophysics and more. These features include self-similarity, long range dependence and the ability to match any prescribed constant local regularity. Its covariance function R_H reads:

$$R_H(t,s) := \frac{\gamma_H}{2}(|t|^{2H} + |s|^{2H} - |t-s|^{2H}),$$

where γ_H is a positive constant and H, which is usually called the Hurst exponent, belongs to $(0,1)$. When $H = \frac{1}{2}$, fBm reduces to standard Brownian motion. Various integral representations of fBm are known, including the harmonizable and moving average ones [25], as well as representations by integrals over a finite domain [2, 9].

The fact that most of the properties of fBm are governed by the single real H restricts its application in some situations. In particular, its Hölder exponent remains the same all along its trajectory. This does not seem to be adapted to describe adequately natural terrains, for instance. In addition, long range dependence requires $H > 1/2$, and thus imposes paths smoother than the ones of Brownian motion. Multifractional Brownian motion was introduced to overcome these limitations. The basic idea is to replace the real H by a function $t \mapsto h(t)$ ranging in $(0,1)$.

Several definitions of multifractional Brownian motion exist. The first ones were proposed in [21] and in [4]. A more general approach was introduced in [26]. In this work, we shall use a new definition that includes all previously known ones and which is, in our opinion, both more flexible and retains the essence of this class of Gaussian processes. We first need to define a fractional Brownian field:

*Regularity team, INRIA Saclay and MAS Laboratory, Ecole Centrale Paris, Grande Voie des Vignes, 92295 Chatenay-Malabry Cedex, France. erick.herbin@ecp.fr, joachim.lebovits@ecp.fr and jacques.levy-vehel@inria.fr.

§Laboratoire de Probabilités et Modèles Aléatoires, C.N.R.S (UMR 7599), Université Pierre et Marie Curie (Paris VI), case 188, 4, pl. Jussieu, F-75252 Paris Cedex 5, France.

Definition 1.1 (Fractional Brownian field)**.** *Let $(\Omega, \mathcal{F}, P)$ be a probability space. A fractional Brownian field on $\mathbf{R} \times (0,1)$ is a Gaussian field, noted $(\mathbf{B}(t,H))_{(t,H)\in\mathbf{R}\times(0,1)}$, such that, for every H in $(0,1)$, the process $(B_t^H)_{t\in\mathbf{R}}$ defined by $B_t^H := \mathbf{B}(t,H)$ is a fractional Brownian motion with Hurst parameter H*[1]*.*

A multifractional Brownian motion is simply a "path" traced on a fractional Brownian field. More precisely, it is defined as follows:

Definition 1.2 (Multifractional Brownian motion)**.** *Let $h : \mathbf{R} \to (0,1)$ be a deterministic continuous function. A multifractional Brownian motion (mBm) with functional parameter h is the Gaussian process $B^h := (B_t^h)_{t\in\mathbf{R}}$ defined by $B_t^h := \mathbf{B}(t, h(t))$ for all t in $\mathbf{R}$.*

A word on notation: $B^H_{\cdot}$ or $B^{h(t)}_{\cdot}$ will always denote an fBm with Hurst index H or $h(t)$, while $B^h_{\cdot}$ will stand for an mBm. Note that $B_t^h := \mathbf{B}(t,h(t)) = B_t^{h(t)}$, for every real t. The function h is called the *regularity function* of mBm.
It is straightforward to check that any multifractional Brownian motion in the sense of [26, Def.1.1] is also an mBm with our definition. Fractional fields $(\mathbf{B}(t,H))_{(t,H)\in\mathbf{R}\times(0,1)}$ leading to previously considered mBms include:

$$\mathbf{B}_1(t,H) := \frac{1}{c_H}\int_{\mathbf{R}} \frac{e^{itu}-1}{|u|^{H+1/2}}\widetilde{\mathbb{W}_1}(du)\ , \qquad \mathbf{B}_2(t,H) := \int_{\mathbf{R}} \left(|t-u|^{H-1/2} - |u|^{H-1/2}\right)\ \mathbb{W}_2(du),$$

$$\mathbf{B}_3(t,H) := \int_{\mathbf{R}} \left((t-u)_+^{H-1/2} - (-u)_+^{H-1/2}\right)\ \mathbb{W}_3(du)\ , \qquad \mathbf{B}_4(t,H) := \int_0^T \mathbb{1}_{\{0\le u<t\le T\}}(t,u)\ K_H(t,u)\ \mathbb{W}_4(du),$$

where $c_H := \left(\frac{2\cos(\pi H)\Gamma(2-2H)}{H(1-2H)}\right)^{1/2}$, $d_H := \left(\frac{2H\Gamma(3/2-H)}{\Gamma(1/2+H)\Gamma(2-2H)}\right)^{1/2}$ and

$$K_H(t,s) := d_H\ (t-s)^{H-1/2} + c_H(1/2-H)\int_s^t (u-s)^{H-3/2}\big(1-\left(\tfrac{s}{u}\right)^{1/2-H}\big)du,$$

and where, for $i \in \{1;2;3;4\}$, $\mathbb{W}_i$ denotes an independently scattered standard Gaussian measure on $\mathbf{R}$, and $\widetilde{\mathbb{W}_1}$ denotes the complex-valued Gaussian measure which can be associated in a unique way to $\mathbb{W}_1$ (see [26, p.203-204] and [25, p.325-326] for more details). Replacing H with $h(t)$ in $\mathbf{B}_1(t,H)$ and $\mathbf{B}_2(t,H)$ leads to the so-called harmonisable mBm, first considered in [4]. The same operation on $\mathbf{B}_3(t,H)$ yields the moving average mBm defined in [21]. Both are particular cases of mBms in the sense of [26]. Finally, $\mathbf{B}_4(t,h(t))$ corresponds to the Volterra multifractional Gaussian process studied in [9]. This last process is an mBm in our sense.
The definition of a fractional Brownian field does not specify its "inter-line" behaviour, *i.e.* the relations between $(B_t^H)_{t\in\mathbf{R}}$ and $(B_t^{H'})_{t\in\mathbf{R}}$ for $H \neq H'$. In order to obtain a useful theory, we need to control these relations to some extent. It turns out that the following condition is sufficient to prove all the results we will need in this paper. We shall denote $\mathbf{E}[Y]$ the expectation of a random variable Y in $L^1(\Omega,\mathcal{F},P)$.

$$(\mathcal{H}_1) : \forall [a,b] \subset \mathbf{R}, \forall [c,d] \subset (0,1), \exists (\Lambda,\delta) \in (\mathbf{R}_+^*)^2, \text{ such that } \mathbf{E}[(\mathbf{B}(t,H) - \mathbf{B}(t,H'))^2] \le \Lambda\ |H-H'|^{\delta},$$
$$\text{for all } (t,H,H') \text{ in } [a,b]\times[c,d]^2.$$

Using the equality $\mathbf{E}[(\mathbf{B}(t,H) - \mathbf{B}(s,H))^2] = |t-s|^{2H}$ and the triangular inequality for the L^2-norm, Assumption $(\mathcal{H}_1)$ is seen to be equivalent to the following one:

$$(\mathcal{H}) : \forall [a,b]\times[c,d] \subset \mathbf{R}\times(0,1), \exists (\Lambda,\delta) \in (\mathbf{R}_+^*)^2, s.t.\ \mathbf{E}[(\mathbf{B}(t,H) - \mathbf{B}(s,H'))^2] \le \Lambda\ \left(|t-s|^{2c} + |H-H'|^{\delta}\right),$$
$$\text{for all } (t,s,H,H') \in [a,b]^2\times[c,d]^2.$$

Thus, we will refer either to assumption $(\mathcal{H}_1)$ or $(\mathcal{H})$ in the sequel.

[1]Alternatively, one might start from a family of fBms $(B^H)_{H\in(0,1)}$ (*i.e.* $B^H := (B_t^H)_{t\in\mathbf{R}}$ is an fBm for every H in $(0,1)$) and define from it the field $(\mathbf{B}(t,H))_{(t,H)\in\mathbf{R}\times(0,1)}$ by $\mathbf{B}(t,H) := B_t^H$. However it is not true, in general, that the field $(\mathbf{B}(t,H))_{(t,H)\in\mathbf{R}\times(0,1)}$ obtained in this way is Gaussian.

Remark 1. *(i) Assumption* $(\mathcal{H})$ *entails that the map* $(t, s, H, H') \mapsto \mathbf{E}[\mathbf{B}(t, H)\ \mathbf{B}(s, H')]$ *is continuous on* $\mathbf{R}^2 \times (0,1)^2$.
(ii) It is well-known that, since $\mathbf{B}$ *is Gaussian, Assumption* $(\mathcal{H})$ *and Kolmogorov's criterion entail that the field* $\mathbf{B}$ *has a d-Hölder continuous version for any* d *in* $(0, \frac{\delta}{2} \wedge c)$*. In the sequel we will always work with such a version.*

In many cases, $\mathbf{B}$ will be specified through an integral representation. It is thus relevant to recast assumption $(\mathcal{H})$ in terms of the kernel used in these representations. We distinguish between two situations: the case where the integral is over a compact interval, and where it is over $\mathbf{R}$.

Integral on a compact set $[0, T]$

In this situation (see, *e.g* [2]), the fractional field $(\mathbf{B}(t, H))_{(t,H)\in[0,T]\times(0,1)}$ is defined by

$$\mathbf{B}(t, H) := \int_0^T K(t, u, H)\ \mathbb{W}(du),$$

where $\mathbb{W}$ is a Gaussian measure, K is defined on $[0, T]^2 \times (0,1)$ and is such that $u \mapsto K(t, u, H)$ belongs to $L^2([0, T], du)$, for all (t, H) in $[0, T] \times (0,1)$. This is for instance the case of $\mathbf{B}_4$. As one can easily see, the following condition (C_K) entails $(\mathcal{H})$:

$$(C_K) : \forall(c, d) \text{ with } 0 < c < d < 1, H \mapsto K(t, u, H) \text{ is Hölder continuous on } [c, d], \text{uniformly in } (t, u) \text{ in } [0, T]^2,$$
$$i.e\ \exists(M, \delta) \in (\mathbf{R}_+^*)^2,\ \forall(t, u) \text{ in } [0, T]^2,\ |K(t, u, H) - K(t, u, H')| \leq M\ |H - H'|^\delta.$$

Condition (C_K) is fulfilled by the kernel K defining $\mathbf{B}_4$ (see [9, Proposition 3, (5)]).

Integral over R

A representation with an integral over $\mathbf{R}$ is used for instance in [7, 12]. The fractional field $(\mathbf{B}(t, H))_{(t,H)\in\mathbf{R}\times(0,1)}$ is then defined by $\mathbf{B}(t, H) := \int_{\mathbf{R}} M(t, u, H)\ \widetilde{\mathbb{W}}(du)$ where M is defined on $\mathbf{R}^2 \times (0,1)$, and is such that $u \mapsto M(t, u, H)$ belongs to $L^2(\mathbf{R}, du)$, for every (t, H) in $\mathbf{R} \times (0,1)$. This is the case for the fields $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$. Condition (C_M) entails $(\mathcal{H})$:

$$(C_M) : \forall[a, b] \subset \mathbf{R}, \forall[c, d] \subset (0,1), \exists\delta \in \mathbf{R}_+^*, \forall t \in [a, b], \exists\Phi_t \in L^2(\mathbf{R}, du), \text{ verifying } \sup_{t\in[a,b]} \int_{\mathbf{R}} |\Phi_t(u)|^2\ du < +\infty,$$
$$s.t.\ \forall(u, H, H') \in \mathbf{R} \times [c, d]^2,\ |M(t, u, H) - M(t, u, H')| \leq \Phi_t(u)\ |H - H'|^\delta.$$

Condition (C_M) is fulfilled by the kernel M defining $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$. See Appendix A for a proof.

Outline of the paper

The remaining of this paper is organized as follows. Our main result in Section 2 (Theorem 2.1 Point 2 (**i**)) is that an mBm may be approximated in law (as well as in the L^2 and almost sure senses) by a sequence of "tangent" fBms. In Section 3 we show how to define a stochastic integral w.r.t. mBm as a limit of integrals w.r.t. approximating fBms. The main result is Theorem 3.3 that provides a condition on the stochastic integral w.r.t. fBm that guarantees convergence of the sequence of approximations. In other words, as soon as a method of integration w.r.t. fBm verifies this condition, then our method allows to "transport" it into an integral w.r.t. mBm. We apply this construction to the cases of the Wick-Itô, Skorohod and pathwise integrals respectively in Sections 4, 5 and 6.

2 Approximation of multifractional Brownian motion

Since an mBm is just a continuous path traced on a fractional Brownian field, a natural question is to enquire whether it may be approximated by patching adequately chosen fBms, and in which sense.

Heuristically, for $a < b$, we divide $[a, b)$ into "small" intervals $[t_i, t_{i+1})$, and replace on each of these B^h by the fBm B^{H_i} where $H_i = h(t_i)$. It seems reasonable to expect that the resulting process $\sum_i B_t^{H_i} \mathbb{1}_{[t_i,t_{i+1})}(t)$ will converge, in a sense to be made precise, to B^h when the sizes of the intervals $[t_i, t_{i+1})$ go to 0.
Our aim in this section is to make this line of thought rigorous.

2.1 Approximation of mBm by piecewise fBms

In the sequel, we fix a fractional Brownian field $\mathbf{B}$ and a continuous function h, thus an mBm, noted B^h. We aim to prove that this mBm can be approximated on every compact interval $[a, b]$ by patching together fractional Brownian motions defined on a sequence of partitions of $[a, b]$. In that view, we choose an increasing sequence $(q_n)_{n\in\mathbb{N}}$ of integers such that $q_0 := 1$ and $2^n \leq q_n \leq 2^{2^n}$ for all n in $\mathbb{N}^*$. For a compact interval $[a, b]$ of $\mathbf{R}$ and n in $\mathbb{N}$, let $x^{(n)} := \{x_k^{(n)}; k \in [\![0, q_n]\!]\}$ where $x_k^{(n)} := a + k\frac{(b-a)}{q_n}$ for k in $[\![0, q_n]\!]$ (for integers p and q with $p < q$, $[\![p, q]\!]$ denotes the set $\{p; p + 1; \cdots ; q\}$). Define, for n in $\mathbb{N}$, the partition $\mathcal{A}_n := \{[x_k^{(n)}, x_{k+1}^{(n)}); k \in [\![0, q_n - 1]\!]\} \cup \{x_{q_n}^{(n)}\}$. It is clear that $\mathcal{A} := (\mathcal{A}_n)_{n\in\mathbb{N}}$ is a decreasing nested sequence of subdivisions of $[a, b]$ (*i.e.* $\mathcal{A}_{n+1} \subset \mathcal{A}_n$, for every n in $\mathbb{N}$).
For t in $[a, b]$ and n in $\mathbb{N}$ there exists a unique integer p in $[\![0, q_n - 1]\!]$ such that $x_p^{(n)} \leq t < x_{p+1}^{(n)}$. We will note $x_t^{(n)}$ the real $x_p^{(n)}$ in the sequel. The sequence $(x_t^{(n)})_{n\in\mathbb{N}}$ is increasing and converges to t as n tends to $+\infty$. Besides, define for n in $\mathbb{N}$, the function $h_n : [a, b] \to (0, 1)$ by setting $h_n(b) = h(b)$ and, for any t in $[a, b)$, $h_n(t) := h(x_t^{(n)})$. The sequence of step functions $(h_n)_{n\in\mathbb{N}}$ converges pointwise to h on $[a, b]$.
Define, for t in $[a, b]$ and n in $\mathbb{N}$, the process

$$B_t^{h_n} := \mathbf{B}(t, h_n(t)) = \sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t)\ \mathbf{B}(t, h(x_k^{(n)})) + \mathbb{1}_{\{b\}}(t)\ \mathbf{B}(b, h(b)). \tag{2.1}$$

Note that, despite the notation, the process B^{h_n} is not an mBm, as h_n is not continuous. We believe however there is no risk of confusion in using this notation. B^{h_n} is almost surely càdlàg and discontinuous at times $x_k^{(n)}$, k in $[\![0, q_n]\!]$.

The following theorem shows that mBm appears naturally as a limit object of sums of fBms:

Theorem 2.1 (Approximation theorem)**.** *Let* $\mathbf{B}$ *be a fractional Brownian field,* $h : \mathbf{R} \to (0, 1)$ *be a continuous deterministic function and* B^h *be the associated mBm. Let* $[a, b]$ *be a compact interval of* $\mathbf{R}$*,* $\mathcal{A}$ *be a sequence of partitions as defined above, and consider the sequence of processes defined in* (2.1)*. Then:*

1. *If* $\mathbf{B}$ *is such that the map* $C : (t, s, H, H') \mapsto \mathbf{E}[\mathbf{B}(t, H)\ \mathbf{B}(s, H')]$ *is continuous on* $[a, b]^2 \times h([a, b])^2$ *then the sequence of processes* $(B^{h_n})_{n\in\mathbb{N}}$ *converges in* $L^2(\Omega)$ *to* B^h*,i.e.*

$$\forall t \in [a, b], \lim_{n\to+\infty} \mathbf{E}\left[\left(B_t^{h_n} - B_t^h\right)^2\right] = 0.$$

2. *If* $\mathbf{B}$ *satisfies assumption* $(\mathcal{H})$ *and if* h *is* β*-Hölder continuous for some positive real* β*, then the sequence of processes* $(B^{h_n})_{n\in\mathbb{N}}$ *converges*

   (**i**) *in law to* B^h*, i.e.* $\{B_t^{h_n};\ t \in [a, b]\} \xrightarrow[n\to+\infty]{law} \{B_t^h;\ t \in [a, b]\}$.

   (**ii**) *almost surely to* B^h*, i.e.* $P\left(\{\forall t \in [a, b], \lim_{n\to+\infty} B_t^{h_n} = B_t^h\}\right) = 1.$

Before we proceed to the proof, we note that Point 2 (**i**) above is a much stronger statement than the well-known localisability of mBm, *i.e.* the fact that the moving average (see [21]), harmonisable (see [4]) and Volterra mBms (see [9]) are all "tangents" to fBms in the following sense: for every real u,

$$\left\{\frac{B_{u+rt}^h - B_u^h}{r^{h(u)}};\ t \in [a, b]\right\} \xrightarrow[r\to0^+]{\text{law}} \{B_t^{h(u)};\ t \in [a, b]\}.$$

Proof:
The outline of the proof is as follows: the only delicate point is 2 (**i**). The difficulty lies in the fact that the sequence of processes $(B^{h_n})_{n\in\mathbf{N}}$ is not continuous but merely càdlàg. As the limit process is continuous, this situation may be dealt with the theorem on page 92 in [22]. In order to show tightness, we shall make use of [1, (2.6) p.43]. This requires finding an upper bound for the quantity $\mathbf{E}\big[\sup_{H\in\mathcal{K}^{(i)}} \mathbf{B}(t_i,H)-\mathbf{B}(t_i,h(t_i))\big]$, where $\mathcal{K}^{(i)}$ denotes a compact set of $[a,b]$. This is done in Lemma 2.2, using chaining arguments from [27].
Let us now present the detailed proof.

1. Let $t\in[a,b]$. For any n in $\mathbb{N}$, one computes

$$\mathbf{E}\left[\left(B_t^{h_n}-B_t^{h}\right)^2\right]=C(t,t,h(x_t^{(n)}),h(x_t^{(n)}))-2\,C(t,t,h(x_t^{(n)}),h(t))+C(t,t,h(t),h(t)).$$

The continuity of the maps h, $(t,H,H^{'})\mapsto C(t,t,H,H^{'})$ and the fact that $\lim_{n\to+\infty} x_t^{(n)}=t$ entail that $\lim_{n\to\infty}\mathbf{E}\left[(B_t^{h_n}-B_t^{h})^2\right]=0$.
2. By assumption, there exists (η,β) in $\mathbf{R}_+^*\times\mathbf{R}_+^*$ such that for all (s,t) in $[a,b]$,

$$|h(s)-h(t)|\le\eta\;|s-t|^{\beta}. \tag{2.2}$$

(**i**) We proceed as usual in two steps (see for examples [8, 23]), **a)**: finite-dimensional convergence and **b)**: tightness of the sequence of probability measures $(P\circ B^{h_n})_{n\in\mathbf{N}}$.

a) Finite dimensional convergence
Since the processes B^h and B^{h_n} defined by (2.1) are centred and Gaussian, it is sufficient to prove that $\lim_{n\to\infty}\mathbf{E}\left[B_t^{h_n}B_s^{h_n}\right]=\mathbf{E}\left[B_t^{h}B_s^{h}\right]$ for every (s,t) in $[a,b]^2$.
The cases where $t=b$ or $s=t$ are consequences of point 1. above. We now assume that $a\le s<t<b$. One computes

$$\mathbf{E}\left[B_t^{h_n}B_s^{h_n}\right]=\sum_{(k,j)\in[\![0,q_n-1]\!]^2}\mathbb{1}_{[x_k^{(n)},x_{k+1}^{(n)})}(t)\;\mathbb{1}_{[x_j^{(n)},x_{j+1}^{(n)})}(s)\;\mathbf{E}\left[\mathbf{B}(t,h_n(t))\mathbf{B}(s,h_n(s))\right].$$

Hence, $\mathbf{E}\left[B_t^{h_n}B_s^{h_n}\right]=\mathbf{E}\left[\mathbf{B}(t,h(x_t^{(n)}))\mathbf{B}(s,h(x_s^{(n)}))\right]$ for all large enough integers n (*i.e.* such that $x_s^{(n)}\le s<x_t^{(n)}\le t$). The continuity of h, (i) of Remark 1, and the fact that $\lim_{n\to\infty}(x_t^{(n)},x_s^{(n)})=(t,s)$, entail that $\lim_{n\to\infty}\mathbf{E}\left[B_t^{h_n}B_s^{h_n}\right]=\mathbf{E}\left[B_t^{h}B_s^{h}\right]$.

b) Tightness of the sequence of probability measures $(P\circ B^{h_n})_{n\in\mathbb{N}}$.
We are in the particular case where a sequence of càdlàg processes converges to a continuous one. The theorem on page 92 of [22] applies to this situation: it is sufficient to show that, for every positive reals ε and τ, there exist an integer m and a grid $\{t_i\}_{i\in[\![0,m]\!]}$ such that $a=t_0<t_1<\cdots<t_m=b$ that verify

$$\limsup_{n\to+\infty}\;P\left(\left\{\max_{0\le i\le m}\;\sup_{t\in[t_i,t_{i+1})}|B_t^{h_n}-B_{t_i}^{h_n}|>\tau\right\}\right)<\varepsilon. \tag{2.3}$$

Let us then fix (ε,τ) in $(\mathbf{R}_+^*)^2$. Define $[H_1,H_2]:=[\inf_{u\in[a,b]}h(u),\;\sup_{u\in[a,b]}h(u)]$ and set, until the end of this proof, $q_n:=2^{2^n}, n\in\mathbb{N}$ and $F:=[a,b]\times[H_1,H_2]$. The process $(\mathbf{B}(t,H))_{(t,H)\in F}$ is Gaussian and the space of continuous real-valued functions defined on F endowed with the sup-norm is a separable Banach space. Fernique's theorem (see [11, Theorem 2.6 p.37]) applies to the effect that there exists a positive real α such that $A_\alpha:=\mathbf{E}\Big[\exp\{\alpha\sup_{(t,H)\in F}\mathbf{B}(t,H)^2\}\Big]<+\infty$. Set $G:=L\,\sum_{p=0}^{+\infty}\frac{2^{p/2}}{{q_p}^{\delta/2}}$, where L is the universal constant given in [27, Theorem 2.1.1 p. 33] . Define also $D:=\Lambda\;(\eta\;(b-a)^{\beta})^{\delta}$ where Λ and δ are the constants appearing in $(\mathcal{H})$ and η in (2.2). Let N be the smallest integer n such that

$$\max\left\{A_\alpha\;(1+q_n)\;\exp\left\{\tfrac{-\alpha\;\tau^2}{4|b-a|^{2H_2}}\;q_n^{2H_2}\right\};4\;(1+q_n)\;\exp\left\{\tfrac{-\;\tau^2}{2^7\;D}\;q_n^{\delta\beta}\right\};\tfrac{b-a}{q_n};\tfrac{G\;D^{1/2}}{q_n^{\delta\beta/2}}\right\}<1\wedge\tfrac{\tau}{8}\wedge\tfrac{\varepsilon}{2}. \tag{2.4}$$

Set $m := m(\tau, \varepsilon) = q_N$ and $t_i := x_i^{(N)}$ for i in $[\![0, m]\!]$. Note that (2.4) entails that, for all n larger than N, $h_n(t_i) = h(t_i)$. Besides, also as soon as $n \geq N$, $h_n(t)$ belongs to the set $h([t_i, t_{i+1}])$ when $t \in [t_i, t_{i+1}]$. Let $J_n^{\tau,m} := P\big(\big\{\max\limits_{0\leq i\leq m} \sup\limits_{t\in[t_i,t_{i+1})} |B_t^{h_n} - B_{t_i}^{h_n}| > \tau\big\}\big)$. Then

$$J_n^{\tau,m} \leq (1+m) \max_{0\leq i\leq m} P\big(\big\{\sup_{t\in[t_i,t_{i+1})} |\mathbf{B}(t, h_n(t)) - \mathbf{B}(t_i, h_n(t_i))| > \tau\big\}\big) \leq (1+m) \big(\max_{0\leq i\leq m} L_n^{\tau,i} + \max_{0\leq i\leq m} Q_n^{\tau,i}\big), \tag{2.5}$$

where

$$L_n^{\tau,i} := P\big(\big\{\sup_{t\in[t_i,t_{i+1})} |\mathbf{B}(t, h_n(t)) - \mathbf{B}(t_i, h_n(t))| > \tau/2\big\}\big)$$

and

$$Q_n^{\tau,i} := P\big(\big\{\sup_{t\in[t_i,t_{i+1})} |\mathbf{B}(t_i, h_n(t)) - \mathbf{B}(t_i, h_n(t_i))| > \tau/2\big\}\big).$$

- Upper bound for $(1+m) \max\limits_{0\leq i\leq m} L_n^{\tau,i}$:

The couple (i, n) being fixed in $[\![0, m]\!] \times \mathbb{N}$, the process $(\mathbf{B}(s, h_n(t)))_{s\in[t_i,t_{i+1}]}$ is a fractional Brownian motion of Hurst index $h_n(t)$. Using increment-stationarity and self-similarity of fractional Brownian motion yields:

$$\begin{aligned} L_n^{\tau,i} &= P\big(\sup_{t\in[t_i,t_{i+1})} |\mathbf{B}(t - t_i, h_n(t))| > \tau/2\big) = P\big(\sup_{u\in[0,t_{i+1}-t_i)} |t_{i+1} - t_i|^{h_n(u+t_i)} \ \big|\mathbf{B}\big(\tfrac{u}{t_{i+1}-t_i}, h_n(u + t_i)\big)\big| > \tau/2\big) \\ &\leq P\Big(\sup_{v\in[0,1]} |\mathbf{B}(v, h_n(t_i + v(t_{i+1} - t_i)))| > \tfrac{\tau}{2|t_{i+1}-t_i|^{H_2}}\Big). \end{aligned} \tag{2.6}$$

Using Markov identity and Fernique's theorem we get,

$$\begin{aligned} L_n^{\tau,i} &\leq P\big(\sup_{(v,H)\in F} |\mathbf{B}(v, H)| > \tfrac{\tau}{2|t_{i+1}-t_i|^{H_2}}\big) = P\Big(\exp\big\{\alpha \sup_{(v,H)\in F} |\mathbf{B}(v, H)|^2\big\} > \exp\big\{\tfrac{\alpha\,\tau^2}{4|t_{i+1}-t_i|^{2H_2}}\big\}\Big) \\ &\leq E\big[\exp\big\{\alpha \sup_{(t,H)\in F} \mathbf{B}(t, H)^2\big\}\big] \ \exp\big\{\tfrac{-\alpha\tau^2}{4|t_{i+1}-t_i|^{2H_2}}\big\} = A_\alpha \ \exp\big\{\tfrac{-\alpha\tau^2 \ q_N^{2H_2}}{4|b-a|^{2H_2}}\big\} < \tfrac{\varepsilon}{2(1+q_N)} = \tfrac{\varepsilon}{2(1+m)}. \end{aligned}$$

We have shown that

$$\forall i \in [\![0, q_N]\!],\ \forall n \geq N, \qquad (1+m) \max_{0\leq i\leq m} L_n^{\tau,i} < \frac{\varepsilon}{2}. \tag{2.7}$$

- Upper bound for $(1+m) \max\limits_{0\leq i\leq m} Q_n^{\tau,i}$:

Fix a couple (i, n) in $[\![0, m]\!] \times \mathbb{N}$. Recall that $h_n(t)$ belongs to $h([t_i, t_{i+1}]) =: \mathcal{K}^{(i)}$ for every t in $[t_i, t_{i+1})$. We hence have,

$$Q_n^{\tau,i} \leq P\big(\big\{\sup_{H\in\mathcal{K}^{(i)}} |\mathbf{B}(t_i, H) - \mathbf{B}(t_i, h(t_i))| > \tau/2\big\}\big) \leq 2\ P\big(\big\{\sup_{H\in\mathcal{K}^{(i)}} \overbrace{\mathbf{B}(t_i, H) - \mathbf{B}(t_i, h(t_i))}^{=:X_i(H)} > \tau/4\big\}\big). \tag{2.8}$$

Observe that the right hand side of the previous inequality does not depend on n any more.
Our aim is to apply [1, (2.6) p.43]. In that view, we first prove the following estimate:

Lemma 2.2. *For all i in $[\![0, q_N]\!]$,* $\qquad \mu_i := \mathbf{E}\Big[\sup\limits_{H\in\mathcal{K}^{(i)}} X_i(H)\Big] < \frac{G\ D^{1/2}}{q_N^{\delta\beta/2}} < \frac{\tau}{8}$.

Proof of Lemma 2.2: Fix i in $[\![0, q_N]\!]$. We recall some notions of chaining from [27]. A sequence $C^{(i)} := (C_n^{(i)})_{n\in\mathbb{N}}$ of partitions of $\mathcal{K}^{(i)}$ is called admissible if it is increasing and such that $\mathrm{card}(C_n^{(i)}) \leq 2^{2^n}$, for every n in $\mathbb{N}$. Let d_i denote the pseudo-distance associated to the Gaussian process $(X_i(H))_{H\in\mathcal{K}^{(i)}}$, (*i.e.* $d_i(H, H') := (\mathbf{E}[(X_i(H) - X_i(H'))^2])^{1/2}$, for (H, H') in $\mathcal{K}^{(i)} \times \mathcal{K}^{(i)}$).

For (H,p) in $\mathcal{K}^{(i)}\times\mathbb{N}$, let $C_p^{(i)}(H)$ be the unique element of the partition $C_p^{(i)}$ which contains H. The diameter of $C_p^{(i)}(H)$ is by definition $\Delta_i(C_p^{(i)}(H)) := \sup_{(H,H')\in C_p(H)^2} d_i(H,H')$. [27, Theorem 2.1.1] entails that:

$$\mu_i \le L\ \gamma_2(\mathcal{K}^{(i)}, d_i), \tag{2.9}$$

where $\gamma_2(\mathcal{K}^{(i)}, d_i) := \inf \sup_{H\in\mathcal{K}^{(i)}} \sum_{p\ge 0} 2^{p/2}\ \Delta_i(C_p^{(i)}(H))$ and where the infimum is taken over all admissible sequences of partitions of $\mathcal{K}^{(i)}$. Let $H_1^{(i)}$ and $H_2^{(i)}$ be such that $\mathcal{K}^{(i)} =: [H_1^{(i)}, H_2^{(i)}]$. Consider the sequence of partitions $C^{(i)} := (C_n^{(i)})_{n\in\mathbb{N}}$ of $\mathcal{K}^{(i)}$ defined, for every integer n, by setting $C_n^{(i)} := \{[y_k^{(n)}, y_{k+1}^{(n)}); k \in [\![0, q_n-1]\!]\} \cup \{y_{q_n}^{(n)}\}$, where $y_0^{(n)} = H_1^{(i)}$ and $y_{k+1}^{(n)} - y_k^{(n)} = \frac{H_2^{(i)}-H_1^{(i)}}{q_n}$, for (n,k) in $\mathbb{N}\times[\![0, q_n-1]\!]$.
It is clear that $C^{(i)}$ is a decreasing nested sequence of partitions of $\mathcal{K}^{(i)}$ and is hence admissible. For (H_0,p) in $\mathcal{K}^{(i)}\times\mathbb{N}$, denote $[y_{k_0}^{(p)}, y_{k_0+1}^{(p)})$ the unique element of $C_p^{(i)}$ which contains H_0. Then, using $(\mathcal{H})$,

$$\begin{aligned}
\mu_i &\le L \sum_{p\ge 0} 2^{p/2} \sup_{H_0\in\mathcal{K}^{(i)}} \sup_{(H,H')\in[y_{k_0}^{(p)}, y_{k_0+1}^{(p)})^2} \left(\mathbf{E}[(X_i(H)-X_i(H'))^2]\right)^{1/2} \\
&\le L \sum_{p\ge 0} 2^{p/2} \sup_{k\in[\![0,q_p-1]\!]} \sup_{(H,H')\in[y_{k}^{(p)}, y_{k+1}^{(p)})^2} \left(\mathbf{E}[(X_i(H)-X_i(H'))^2]\right)^{1/2} \\
&\le L\ \Lambda^{1/2} \sum_{p\ge 0} 2^{p/2} \sup_{k\in[\![0,q_p-1]\!]} |y_{k+1}^{(p)} - y_k^{(p)}|^{\delta/2} = G\ \Lambda^{1/2}\ \left(H_2^{(i)} - H_1^{(i)}\right)^{\delta/2}.
\end{aligned}$$

By Hölder continuity of h, $H_2^{(i)} - H_1^{(i)} \le \eta\, |t_{i+1}-t_i|^\beta = \eta\ \frac{|b-a|^\beta}{q_N^\beta}$. Using (2.4), we finally get $\mu_i < \frac{GD^{1/2}}{q_N^{\delta\beta/2}} < \tau/8$. □

Note that Lemma 2.2 implies that $(\tau/8)^2 < (\tau/4-\mu_i)^2$. Let us now go back to the proof of tightness. [1, (2.6) p.43] yields, for all $n\ge N$,

$$Q_n^{\tau,i} \le 4\exp\left\{\frac{-(\tau/4-\mu_i)^2}{2\sigma_i^2}\right\},$$

where $\sigma_i^2 := \sup_{H\in\mathcal{K}^{(i)}} \mathbf{E}\big[X_i(H)^2\big]$. By definition of $(X_i(H))_{H\in\mathcal{K}^{(i)}}$ and assumption $(\mathcal{H})$:

$$\sigma_i^2 \le \Lambda \sup_{H\in\mathcal{K}^{(i)}} |H-h(t_i)|^\delta \le \Lambda\ \eta^\delta\ |t_{i+1}-t_i|^{\beta\delta} = \Lambda\ \eta^\delta\ \frac{|b-a|^{\beta\delta}}{q_N^{\beta\delta}} = \frac{D}{q_N^{\beta\delta}}.$$

This yields that $Q_n^{\tau,i} \le\ 4\ \exp\left\{\frac{-\tau^2\ q_N^{\delta\beta}}{2^7\ D}\right\}$ and finally:

$$\forall i\in[\![0,q_N]\!],\ \forall n\ge N, \qquad (1+m)\ \max_{0\le i\le m} Q_n^{\tau,i} < \frac{\varepsilon}{2}.$$

Using (2.7) and the previous inequality, Inequality (2.5) then becomes $J_n^{\tau,m} < \varepsilon$, for every $n\ge N$, which ends **(i)**.

(ii) Almost sure convergence

Denote $\widetilde{\Omega}$ the measurable subset of Ω, verifying $P(\widetilde{\Omega}) = 1$, such that for every ω in $\widetilde{\Omega}$, $(t,H)\mapsto B(t,H)(\omega)$ is continuous on $[a,b]\times[H_1,H_2]$. Then, for every ω in Ω, we get:

$$|B_t^{h_n}(\omega') - B_t^h(\omega')| = |B(t,h_n(t))(\omega') - B(t,h(t))(\omega')| = \big|B(t,h\big(x_t^{(n)}\big))(\omega') - B(t,h(t))(\omega')\big| \underset{n\to+\infty}{\longrightarrow} 0.$$

This ends the proof. □

Remark 2. *With some additional work, one may establish the almost sure convergence of* $(B^{h_n})_{n\in\mathbb{N}}$ *under the sole condition of continuity of* h.

3 Stochastic integrals w.r.t. mBm as limits of integrals w.r.t. fBm

The results of the previous section, especially 2 (**i**) of Theorem 2.1, suggest that one may define stochastic integrals with respect to mBm as limits of integrals with respect to approximating fBms. We formalize this intuition in the present section.
We consider as above a fractional field $(\mathbf{B}(t,H))_{(t,H)\in\mathbf{R}\times(0,1)}$, but assume in addition that the field is C^1 in H on $(0,1)$ in the $L^2(\Omega)$ sense, *i.e.* we assume that the map $H\mapsto\mathbf{B}(t,H)$, from $(0,1)$ to $L^2(\Omega)$, is C^1 for every real t. We will denote $\frac{\partial\mathbf{B}}{\partial H}(t,H')$ the $L^2(\Omega)$-derivative at point H' of the map $H\mapsto\mathbf{B}(t,H)$. The field $(\frac{\partial\mathbf{B}(t,H)}{\partial H})_{(t,H)\in\mathbf{R}\times(0,1)}$ is of course Gaussian. We will need that the derivative field satisfies the same assumption $(\mathcal{H}_1)$ as $\mathbf{B}(t,H)$. More precisely, from now on, we assume that $\mathbf{B}(t,H)$ satisfies $(\mathcal{H}_2)$:

$(\mathcal{H}_2)$: For all $[a,b]\times[c,d]\subset\mathbf{R}\times(0,1), H\mapsto\mathbf{B}(t,H)$ is C^1 in the $L^2(\Omega)$ sense from $(0,1)$ to $L^2(\Omega)$ for every t in $[a,b]$, and there exists $(\Delta,\alpha,\lambda)\in(\mathbf{R}_+^*)^3$ such that, for all (t,s,H,H') in $[a,b]^2\times[c,d]^2$,

$$\mathbf{E}\left[\left(\tfrac{\partial\mathbf{B}}{\partial H}(t,H)-\tfrac{\partial\mathbf{B}}{\partial H}(s,H')\right)^2\right]\le\Delta\ \left(|t-s|^\alpha+|H-H'|^\lambda\right).$$

Proposition 3.1. *The fractional Brownian fields* $\mathbf{B}_i:=(\mathbf{B}_i(t,H))_{(t,H)\in\mathbf{R}\times(0,1)}$, $i\in[\![1,4]\!]$, *verify Assumption* $(\mathcal{H}_2)$.

Proof: The proof of this proposition in the case of $\mathbf{B}_1$ and $\mathbf{B}_2$ may be found in Appendix B. The ones for $\mathbf{B}_3$ and $\mathbf{B}_4$ are easily obtained using results from [21] and [9] and are left to the reader. □

In the remaining of this paper, we consider a C^1 deterministic function $h:\mathbf{R}\to(0,1)$, a fractional field $\mathbf{B}$ which fulfills assumptions $(\mathcal{H}_1)$ and $(\mathcal{H}_2)$, and the associated mBm $B_t^h:=\mathbf{B}(t,h(t))$.

We now explain in a heuristic way how to define an integral with respect to mBm using approximating fBms. Write the "differential" of $\mathbf{B}(t,H)$:

$$d\mathbf{B}(t,H)=\frac{\partial\mathbf{B}}{\partial t}(t,H)\ dt+\frac{\partial\mathbf{B}}{\partial H}(t,H)\ dH.$$

Of course, this is only formal as $t\mapsto\mathbf{B}(t,H)$ is not differentiable in the L^2-sense nor almost surely with respect to t. It is, however, in the sense of Hida distributions, but we are not interested in this fact at this stage. With a differentiable function h in place of H, this (again formally) yields

$$d\mathbf{B}(t,h(t))=\frac{\partial\mathbf{B}}{\partial t}(t,h(t))\ dt+h'(t)\ \frac{\partial\mathbf{B}}{\partial H}(t,h(t))\ dt. \tag{3.1}$$

The second term on the right-hand side of (3.1) is defined for almost every ω and every real t by assumption. Moreover, it is almost surely continuous as a function of t and thus Riemann integrable on compact intervals.
On the other hand, the first term of (3.1) has no meaning *a priori* since mBm is not differentiable with respect to t. However, since stochastic integrals with respect to fBm do exist, we are able to give a sense to $t\mapsto\frac{\partial\mathbf{B}}{\partial t}(t,H)$ for every fixed H in $(0,1)$. Continuing with our heuristic reasoning, we then approximate $\frac{\partial\mathbf{B}}{\partial t}(t,h(t))$ by $\lim\limits_{n\to+\infty}\sum_{k=0}^{q_n-1}\mathbb{1}_{[x_k^{(n)},x_{k+1}^{(n)})}(t)\ \frac{\partial\mathbf{B}}{\partial t}(t,h_n(t))$. This formally yields:

$$d\mathbf{B}(t,h(t))\approx\lim_{n\to+\infty}\sum_{k=0}^{q_n-1}\mathbb{1}_{[x_k^{(n)},x_{k+1}^{(n)})}(t)\ \frac{\partial\mathbf{B}}{\partial t}(t,h_n(t))\ dt+h'(t)\ \frac{\partial\mathbf{B}}{\partial H}(t,h(t))\ dt. \tag{3.2}$$

Assuming we may exchange integrals and limits, we would thus like to define, for suitable processes Y,

$$\int_0^1 Y_t\ d\mathbf{B}(t,h(t))=\lim_{n\to+\infty}\sum_{k=0}^{q_n-1}\int_{x_k^{(n)}}^{x_{k+1}^{(n)}}Y_t\ dB_t^{h(x_k^{(n)})}+\int_0^1 Y_t\ h'(t)\ \frac{\partial\mathbf{B}}{\partial H}(t,h(t))\ dt, \tag{3.3}$$

where the first term of the right-hand side of (3.3) is a limit, in a sense to be made precise depending on the method of integration, of a sum of integrals with respect to fBms and the second term is a Riemann integral or an integral in a weaker sense (see Section 4).
In order to make the above ideas more precise, let us fix some notations. $(\mathcal{M})$ will denote a given method of integration with respect to fBm (*e.g* Skorohod, white noise, pathwise, $\cdots$). For the sake of notational simplicity, we will consider integrals over the interval $[0,1]$. For H in $(0,1)$, denote $\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^H$ the integral of $Y := (Y_t)_{t\in[0,1]}$ on $[0,1]$ with respect to the fBm B^H, in the sense of method $(\mathcal{M})$, assuming it exists. The following notation will be useful:
Notation (integral with respect to lumped fBms) *Let $Y := (Y_t)_{t\in[0,1]}$ be a real-valued process on $[0,1]$ which is integrable with respect to all fBms of index H in $h([0,1])$ in the sense of method $(\mathcal{M})$. We denote the integral with respect to lumped fBms in the sense of method $(\mathcal{M})$ by:*

$$\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^{h_n} := \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) \; Y_t \, d^{(\mathcal{M})} B_t^{h(x_k^{(n)})}, \quad n \in \mathbb{N} \tag{3.4}$$

(we use the same notations as in subsection 2.1: $(q_n)_{n\in\mathbb{N}}$ is a sequence of integers and the family $x^{(n)} := \{x_k^{(n)}; k \in [\![0, q_n]\!]\}$ is defined by $x_k^{(n)} := \frac{k}{q_n}$ for k in $[\![0, q_n]\!]$).
With this notation, our tentative definition of an integral w.r.t. to mBm (3.3) reads:

$$\int_0^1 Y_t \, d\mathbf{B}(t, h(t)) = \lim_{n\to+\infty} \int_0^1 Y_t \, d^{(\mathcal{M})} B_t^{h_n} + \int_0^1 Y_t \, h'(t) \, \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \, dt, \tag{3.5}$$

The interest of (3.3) is that it allows to use any of the numerous definitions of stochastic integrals with respect to fBm, and automatically obtain a corresponding integral with respect to mBm. It is worthwhile to note that, with this approach, an integral with respect to mBm is a sum of two terms: the first one seems to depend only on the chosen method for integrating with respect to fBm (for instance, a white noise or pathwise Riemann integral), while the second is an integral which appears to depend only on the field used to define the chosen mBm, *i.e.* essentially on its correlation structure. This second term will imply that the integral with respect to the moving average mBm, for instance, is different from the one with respect to the harmonisable mBm. As the example of simple processes in the next subsection will show, the second term does however also depend on the integration method with respect to fBm.
Note that the nature of $\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^{h_n}$ depends on $(\mathcal{M})$. For example, $\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^H$ and hence $\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^{h_n}$ will belong to $L^2(\Omega)$ if $(\mathcal{M})$ denotes the Skorohod integral, whereas $\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^H$ and hence $\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^{h_n}$ belong to the space $(S)^*$ of stochastic distributions when $(\mathcal{M})$ denotes the integral in the sense of white noise theory.

We will write $\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^h$ for the integral of Y on $[0,1]$ with respect to mBm in the sense of $(\mathcal{M})$ (which is yet to be defined). When we do not want to specify a particular method but instead wish to refer to all methods at the same time, we will write $\int_0^1 Y_t \, dB_t^{h_n}$ and $\int_0^1 Y_t \, dB_t^h$ instead of $\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^{h_n}$ and $\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^h$.
In order to gain a better understanding of our approach, we explore in the following subsection the particular cases of simple deterministic and then random integrands.

3.1 Example: simple integrands

3.1.1 Deterministic simple integrands

Any reasonable definition of an integral must be linear. Thus, to determine the integral of deterministic simple functions w.r.t. mBm, it suffices to consider the case of $Y = 1$. Obviously, we should find that $\int_0^1 1 \, dB_t^h = B_1^h$. In order to verify this fact, let us compute the limit of the sequence $(\int_0^1 1 \, dB_t^{h_n})_{n\in\mathbb{N}}$.

Proposition 3.2. *Assume that, almost surely, for all real t, the real-valued map $H \mapsto \mathbf{B}(t, H)(\omega)$ belongs to $C^2((0,1))$. The sequence $(\int_0^1 1 \, dB_t^{h_n})_{n\in\mathbb{N}}$ then converges almost surely and in $L^2(\Omega)$ to $\mathbf{B}(1, h(1)) - \int_0^1 h'(t) \, \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \, dt$, where the second term is a pathwise integral.*

Proposition 3.2 implies that, for regular enough fields $\mathbf{B}$ and h functions, Formula (3.3) does indeed yield $\int_0^1 1\, dB_t^h = B_1^h$.

Proof: By definition of $\int_0^1 1\, dB_t^{h_n}$, for almost all ω,

$$I_n := \int_0^1 1\, dB_t^{h_n}(\omega) = B_1^{h(x_{q_n-1}^{(n)})}(\omega) - \sum_{k=1}^{q_n-1} \big(B_{x_k^{(n)}}^{h(x_k^{(n)})}(\omega) - B_{x_k^{(n)}}^{h(x_{k-1}^{(n)})}(\omega)\big) - B_0^{h(0)}(\omega). \tag{3.6}$$

Denote $K_n(\omega) := \sum_{k=1}^{q_n-1} \big(B_{x_k^{(n)}}^{h(x_k^{(n)})}(\omega) - B_{x_k^{(n)}}^{h(x_{k-1}^{(n)})}(\omega)\big)$ Since both $H \mapsto \mathbf{B}(t,H)(\omega)$ and h are smooth, the finite increments theorem applied twice allows one to write, for some θ_k and φ_k in $(x_{k-1}^{(n)}, x_k^{(n)})$,

$$K_n(\omega) = \sum_{k=1}^{q_n-1} q_n^{-1}\, h'(\varphi_k)\, \tfrac{\partial \mathbf{B}}{\partial H}(x_k^{(n)}, h(\theta_k))(\omega).$$

Denote $F := [0,1] \times h([0,1])$. Using again the finite increments theorem, the fact that the processes $(\frac{\partial \mathbf{B}}{\partial H}(t,H))_{(t,H)\in F}$ and $(\frac{\partial^2 \mathbf{B}}{\partial H^2}(t,H))_{(t,H)\in F}$ are Gaussian, and the uniform continuity of h' we get $\lim_{n\to\infty} K_n(\omega) = \int_0^1 h'(t)\, \frac{\partial \mathbf{B}}{\partial H}(t,h(t))\, dt(\omega)$. Equality (3.6) entails that the sequence $(\int_0^1 1\, dB_t^{h_n})_{n\in\mathbb{N}}$ converges almost surely to $\mathbf{B}(1,h(1)) - \int_0^1 h'(t)\, \frac{\partial \mathbf{B}}{\partial H}(t,h(t))\, dt$.
It remains to prove that the convergence also holds in $L^2(\Omega)$. First, we remark that $\int_0^1 1\, dB_t^{h_n}$ is in $L^2(\Omega)$ since $\mathbf{B}$ is a Gaussian process. Let us now show that $\int_0^1 h'(t)\, \frac{\partial \mathbf{B}}{\partial H}(t,h(t))\, dt$ belongs to $L^2(\Omega)$. Cauchy-Schwarz inequality and Assumption $(\mathcal{H}_2)$ entail:

$$\mathbf{E}\big[\big(\int_0^1 h'(t) \tfrac{\partial \mathbf{B}}{\partial H}(t,h(t))\, dt\big)^2\big] \le \Delta\, \|h'\|^2_{L^2(\mathbf{R})} \big(\int_0^1 (|t|^\alpha + |h(t)|^\lambda)\, dt\big) < +\infty.$$

Now, almost sure convergence of the sequence $(\int_0^1 1\, dB_t^{h_n})_{n\in\mathbb{N}}$ implies convergence in $L^2(\Omega)$ provided it is bounded by a random variable $X \in L^2(\Omega)$. Assumption $(\mathcal{H})$ entails that the sequence $(B_1^{h(x_{q_n-1}^{(n)})})_{n\in\mathbb{N}}$ converges to $B_1^{h(1)}$ in $L^2(\Omega)$. It thus remains to study the random variable K_n defined above. By almost sure continuity (which follows from Assumption $(\mathcal{H}_2)$), the centred Gaussian process $\big(\frac{\partial \mathbf{B}}{\partial H}(t,H)\big)_{(t,H)\in F}$ has bounded sample paths with probability one. Moreover it is well-known (see [1, (2.4) p.43] for example) that this entails that $\sup_{(t,H)\in F} \big|\frac{\partial \mathbf{B}}{\partial H}(t,H)\big|$ belongs to $L^2(\Omega)$. $L^2(\Omega)$ convergence follows. □

Remark 3. *Proposition 3.2 applies to the four fields considered in the introduction.*

3.1.2 Simple processes

We now consider a particular case of a simple process that will show that Formula (3.3) does not always yield the expected result, and must be modified in certain situations. We take $Y_t = Y_0$ with Y_0 a centred Gaussian random variable which is $\mathcal{F}(B^h)$-measurable where $\mathcal{F}(B^h)$ denotes the σ-field generated by B^h.

Case of the integral in the sense of white noise theory
The reader who is not familiar with the integral with respect to fBm in the sense of white noise theory (also called fractional Wick-Itô integral) may refer to Subsection 4.1 or [7, 12]. We denote this integral $\int Y_i\ d^\diamond B_t^H$, with a similar notation for the integral w.r.t. mBm. Denote $(W_t^H)_{t\in[0,T]}$ the fractional white noise process (see Section 4 and references therein for more details). Set, for $n \in \mathbb{N}$ $S_n := \int_0^1 Y_t\ d^\diamond B_t^{h_n}$. One computes:

$$\begin{aligned} S_n &= \sum_{k=0}^{q_n-1} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} Y_0\, d^\diamond B_t^{h(x_k^{(n)})} = \sum_{k=0}^{q_n-1} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} Y_0 \diamond W_t^{h(x_k^{(n)})}\, dt = \sum_{k=0}^{q_n-1} Y_0 \diamond \big(\int_{x_k^{(n)}}^{x_{k+1}^{(n)}} W_t^{h(x_k^{(n)})}\, dt\big) \\ &= Y_0 \diamond \big(\sum_{k=0}^{q_n-1} \big(B_{x_{k+1}^{(n)}}^{h(x_k^{(n)})} - B_{x_k^{(n)}}^{h(x_k^{(n)})}\big)\big) = Y_t \diamond \big(\sum_{k=0}^{q_n-1} \big(B_{x_{k+1}^{(n)}}^{h(x_k^{(n)})} - B_{x_k^{(n)}}^{h(x_k^{(n)})}\big)\big), \end{aligned} \tag{3.7}$$

where $\diamond$ denotes the Wick product.

In the proof of Proposition 3.2 we have shown that the sequence $\left(\sum_{k=0}^{q_n-1}\left(B^{h(x_k^{(n)})}_{x_{k+1}^{(n)}} - B^{h(x_k^{(n)})}_{x_k^{(n)}}\right)\right)_{n\in\mathbb{N}}$ converges, in $L^2(\Omega)$, to $B_1^h - \int_0^1 h'(t)\ \frac{\partial \mathbf{B}}{\partial H}(t,h(t))\ dt$. By continuity of the Wick product, we get:

$$\lim_{n\to\infty} S_n = Y_t \ \diamond\ B_1^h - Y_0 \diamond \int_0^1 h'(t)\ \frac{\partial \mathbf{B}}{\partial H}(t,h(t))\ dt = Y_t \ \diamond\ B_1^h - \int_0^1 h'(t)\ Y_0 \ \diamond\ \frac{\partial \mathbf{B}}{\partial H}(t,h(t))\ dt, \quad (3.8)$$

where the limit holds in $L^2(\Omega)$. Now, for the Wick-Itô integral w.r.t. mBm defined in [18], one has:

$$\int_0^1 Y_0\ \ d^\diamond B_t^h = Y_0 \ \diamond\ B_1^h = Y_t \ \diamond\ B_1^h.$$

Formula (3.8) then reads $Y_t \ \diamond\ B_1^h = \lim_{n\to\infty}\int_0^1 Y_t\ \ d^\diamond B_t^{h_n} + \int_0^1 h'(t)\ Y_t \ \diamond\ \frac{\partial \mathbf{B}}{\partial H}(t,h(t))\ dt$. We see that, for this integration method, Formula (3.3) should be modified into

$$\int_0^1 Y_t\ \ d^\diamond B_t^h := \lim_{n\to\infty}\int_0^1 Y_t\ \ d^\diamond B_t^{h_n} + \int_0^1 h'(t)\ Y_t \ \diamond\ \frac{\partial \mathbf{B}}{\partial H}(t,h(t))\ dt. \quad (3.9)$$

The natural spaces of white noise theory are the spaces $(\mathcal{S}_{-p})$, p being a positive integer. Equality (3.9) will be used in Section 4 to define the integral of an $(\mathcal{S}_{-p})$-valued process $Y := (Y_t)_{t\in[0,1]}$ with respect to mBm, where the limit and the last integral (which is in the sense of Bochner) will hold in $(\mathcal{S}_{-q})$ for some integer q, assuming they both exist. Note that the previous equality will also hold in $(\mathcal{S}_{-q})$.

Case of the integral in the sense of Skorohod
We denote this integral with the symbol δ . Thus, for instance, $\int_0^1 X\ \delta B_t^H$ denotes the Skorohod integral of the process X w.r.t. B^H.
Continuing with our example $Y_t = Y_0$, where Y_0 is a centred Gaussian random variable, we use Theorem 7.40 in [16, section 7], that yields the general form of a Skorohod integral with respect to a Gaussian process. In our very simple case, this reads:

$$\int_0^1 Y_0\ \delta B_t^h = Y_0 \ \diamond\ \int_0^1 1\ \delta B_t^h = Y_0 \ \diamond\ B_1^h.$$

Besides, [19, Proposition 8] and [6, Theorem 6.2] yield that $\int_0^1 Y_t\ \ \delta B_t^H = \int_0^1 Y_t\ \ d^\diamond B_t^H$, as soon as $\int_0^1 Y_t\ \ \delta B_t^H$ is defined. Thus, writing $T_n := \int_0^1 Y_t\ \ \delta B_t^{h_n}$, we have $T_n = S_n$ for all n (recall (3.7)).
This prompts us to defining the Skorohod integral w.r.t. to mBm again with a Wick product, *i.e.*, using a formula analogous to (3.9):

$$\int_0^1 Y_t\ \ \delta B_t^h := \lim_{n\to\infty}\int_0^1 Y_t\ \ \delta B_t^{h_n} + \int_0^1 h'(t)\ Y_t \ \diamond\ \frac{\partial \mathbf{B}}{\partial H}(t,h(t))\ dt, \quad (3.10)$$

but where the equality and limit would now hold in $L^2(\Omega)$.

An advantage of these definitions is that they will ensure, by construction, the equality $\int_0^1 Y_t\ \ d^\diamond B_t^h = \int_0^1 Y_t\ \ \delta B_t^h$ as soon as Y is integrable w.r.t. mBm in the sense of Skorohod.

Case of pathwise integrals
In the case of the pathwise fractional integral in the sense of [28], denoted $\int X_t\ dB_t^h$, the use of Formula (22) in [28] with g an mBm and $f = Y_0$ yields

$$\int_0^1 Y_0\ dB_t^h = Y_0\ \ B_1^h.$$

Thus the correct way to define our integral w.r.t. mBm in this case is to use a standard product, *i.e.* to set:

$$\int_0^1 Y_t\ \ dB_t^h := \lim_{n\to\infty}\int_0^1 Y_t\ \ dB_t^{h_n} + \int_0^1 h'(t)\ Y_t\ \frac{\partial \mathbf{B}}{\partial H}(t,h(t))\ dt. \quad (3.11)$$

3.2 Integral with respect to mBm through approximating fBms

We now define in a precise way our integral with respect to mBm. Let $(E, \|\|_E)$ and $(F, \|\|_F)$ be two normed linear spaces, endowed with their Borel σ-field $\mathcal{B}(E)$ and $\mathcal{B}(F)$. Let $Y := (Y_t)_{t\in[0,1]}$ be an E-valued process (i.e Y_t belongs to E for every real t in $[0,1]$ and $t \mapsto Y_t$ is measurable from $(0,1)$ to $(E, \mathcal{B}(E))$). Fix an integration method $(\mathcal{M})$. As explained in the previous subsection, we wish to define the integral w.r.t. an mBm B^h in the sense of $(\mathcal{M})$ by a formula of the kind:

$$\int_0^1 Y_t \;\; d^{(\mathcal{M})} B_t^h := \lim_{n\to\infty} \int_0^1 Y_t \;\; d^{(\mathcal{M})} B_t^{h_n} + \int_0^1 h'(t)\, Y_t * \frac{\partial \mathbf{B}}{\partial H}(t, h(t))\, dt, \tag{3.12}$$

where the meaning of the limit depends on $(\mathcal{M})$ and where $*$ denotes the ordinary product (in the case of pathwise integrals) or Wick product (in other cases) depending on $(\mathcal{M})$. For this formula to make sense, it is certainly necessary that Y be $(\mathcal{M})-$integrable w.r.t. fBm of all exponents α in $h([0,1])$.
We thus define, for $\alpha \in (0,1)$,

$$\mathcal{H}_E^\alpha := \big\{ Y \in E^{[0,1]} : \textstyle\int_{[0,1]} Y_t \; d^{(\mathcal{M})} B_t^\alpha \text{ exists and belongs to } F \big\},$$

and

$$\mathcal{H}_E = \bigcap_{\alpha\in h([0,1])} \mathcal{H}_E^\alpha.$$

We will always assume that there exists a subset Λ_E of $\mathcal{H}_E$ (maybe equal to $\mathcal{H}_E$) which may be endowed with a norm $\| \, \|_{\Lambda_E}$ such that $(\Lambda_E, \| \, \|_{\Lambda_E})$ is complete and which satisfies the following property: there exists $M > 0$ and a real χ such that for all partitions of $[0,1]$ in intervals $A_1, \dots, A_n$ of equal size $\frac{1}{n}$,

$$\|Y.\mathbb{1}_{A_1}\|_{\Lambda_E} + \cdots + \|Y.\mathbb{1}_{A_n}\|_{\Lambda_E} \le M\; n^\chi \|Y\|_{\Lambda_E}. \tag{3.13}$$

When Y belongs to Λ_E, Definition (3.12) will be a valid one as soon as the limit and the last term on the right hand side exist. It turns out that a simple sufficient condition guarantees the existence of the limit of the integral w.r.t. lumped fBms. Define, for $n \in \mathbb{N}$, the map

$$\begin{aligned} L_n : \Lambda_E &\to F \\ Y &\mapsto \int_{[0,1]} Y_t \; d^{(\mathcal{M})} B_t^{h_n}. \end{aligned} \tag{3.14}$$

The following theorem provides a sufficient condition under which $(L_n(Y))_{n\in\mathbb{N}}$ converges in F.

Theorem 3.3. *Let $(a_n)_{n\in\mathbb{N}}$ be an increasing sequence of positive integers such that $2^n \le \prod_{0\le k\le n-1} a_k \le 2^{2^n}$ for every n in $\mathbb{N}$ and such that $\lim_{n\to+\infty} (n(a_n-1)) \, \big(\prod_{0\le k\le n-1} a_k\big)^{-1} = 0$. Choose the sequence $(q_n)_{n\in\mathbb{N}}$ used in (3.4) such that $q_0 = 1$ and $q_{n+1} = a_n\, q_n$ for all n in $\mathbb{N}$. Assume that the function $\mathcal{I} : \Lambda_E \times (0,1) \to F$ defined by*

$$\forall\, Y \in \Lambda_E,\ \forall \alpha \in (0,1), \quad \mathcal{I}(Y,\alpha) := \int_{[0,1]} Y_t \; d^{(\mathcal{M})} B_t^\alpha,$$

is θ-Hölder continuous with respect to α uniformly in Y for a real number $\theta > \chi$, i.e. there exists $K > 0$ such that

$$\forall\, Y \in \Lambda_E,\ \forall (\alpha,\alpha') \in (0,1)^2, \quad \|\mathcal{I}(Y,\alpha) - \mathcal{I}(Y,\alpha')\|_F \le K\, |\alpha-\alpha'|^\theta\, \|Y\|_{\Lambda_E}. \tag{3.15}$$

Then the sequence of functions $(L_n)_{n\in\mathbf{N}}$ defined in (3.14) converges pointwise to a function $L : \Lambda_E \to F$.

Proof: For the sake of simplicity, we will establish the result in the case where $a_n \equiv 2$ which obviously fulfils the required conditions. The general case is similar. For $n \in \mathbf{N}$ and $Y \in \Lambda_E$, $L_n(Y)$ may be decomposed as $L_n(Y) = \sum_{k=0}^{2^n-1} \big(\int_{[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}})} Y_t \; d^{(\mathcal{M})} B_t^{h(\frac{k}{2^n})} + \int_{[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}})} Y_t \; d^{(\mathcal{M})} B_t^{h(\frac{k}{2^n})} \big)$. Now

$$L_{n+1}(Y) = \sum_{k=0}^{2^n-1} \big(\int_{[\frac{2k}{2^{n+1}}, \frac{2k+1}{2^{n+1}})} Y_t \, d^{(\mathcal{M})} B_t^{h(\frac{2k}{2^{n+1}})} + \int_{[\frac{2k+1}{2^{n+1}}, \frac{2k+2}{2^{n+1}})} Y_t \, d^{(\mathcal{M})} B_t^{h(\frac{2k+1}{2^{n+1}})} \big).$$

Using assumptions (3.13) and (3.15), one obtains

$$\begin{aligned}
\|L_n(Y) - L_{n+1}(Y)\|_F &= \big\|\sum_{k=0}^{2^n-1} \big(\mathcal{I}\big(Y.\mathbb{1}_{\big[\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}\big)}, h(\tfrac{k}{2^n})\big) - \mathcal{I}\big(Y.\mathbb{1}_{\big[\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}\big)}, h(\tfrac{2k+1}{2^{n+1}})\big)\big)\big\|_F \\
&\leq \sum_{k=0}^{2^n-1} \big\|\big(\mathcal{I}\big(Y.\mathbb{1}_{\big[\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}\big)}, h(\tfrac{k}{2^n})\big) - \mathcal{I}\big(Y.\mathbb{1}_{\big[\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}\big)}, h(\tfrac{2k+1}{2^{n+1}})\big)\big)\big\|_F \\
&\leq K \ \sum_{k=0}^{2^n-1} \big\|Y.\mathbb{1}_{[(2k+1).2^{-(n+1)},(k+1).2^{-n})}\big\|_{\Lambda_E} \ \big|h((2k+1).2^{-(n+1)}) - h(k.2^{-n})\big|^\theta \\
&\leq K \ 2^{-\theta(n+1)} \ \sup_{t\in[0,1]} |h'(t)|^\theta \ \sum_{k=0}^{2^n-1} \big\|Y.\mathbb{1}_{[(2k+1).2^{-(n+1)},(k+1).2^{-n})}\big\|_{\Lambda_E} \\
&\leq K \ M \ 2^{-\theta(n+1)} \ \sup_{t\in[0,1]} |h'(t)|^\theta 2^{n\chi} \ \|Y\|_{\Lambda_E}. \qquad (3.16)
\end{aligned}$$

It follows that the series $\sum_{n\in\mathbf{N}} (L_{n+1}(Y) - L_n(Y))$ converges absolutely for any fixed $Y \in \Lambda_E$, and consequently $(L_n(Y))_{n\in\mathbb{N}}$ converges to a limit $L(Y)$ as n goes to infinity. □

For a process Y in Λ_E, we will say that $t \mapsto h'(t) \ Y_t * \frac{\partial \mathbf{B}}{\partial H}(t, h(t))$ is integrable on $[0,1]$ if

- $h'(t)Y(t) \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t))$ is almost surely Riemann integrable and $\int_0^1 h'(t)Y(t) \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \, dt$ belongs to $L^2(\Omega)$ in the case of the Skorohod integral,
- $\int_0^1 h'(t) \ Y_t \diamond \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \, dt$ exists in the sense of Bochner and belongs to F in the case of the Wick-Itô integral (in this situation, $L^2(\Omega) \subset F$). The reader who is not familiar with the Bochner integral may refer to Section 4 below and references therein,
- $\int_{[0,1]} h'(t) \ Y_t \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \, dt$ exists for almost every ω in the case of a pathwise integral.

We are finally able to define our integral:

Definition 3.1 (Integral with respect to mBm in the sense of $(\mathcal{M})$)**.** *Assume that Method $(\mathcal{M})$ fulfils condition* (3.15) *and let $Y := (Y_t)_{t\in[0,1]}$ be an element of Λ_E such that the map $t \mapsto h'(t) \ Y_t * \frac{\partial \mathbf{B}}{\partial H}(t, h(t)$ is integrable. The integral of Y with respect to B^h in the sense of $(\mathcal{M})$ is defined as:*

$$\int_0^1 Y_t \, d^{(\mathcal{M})} B_t^h := \lim_{n\to\infty} \int_0^1 Y_t \, d^{(\mathcal{M})} B_t^{h_n} + \int_0^1 h'(t) \ Y_t * \frac{\partial \mathbf{B}}{\partial H}(t, h(t)) \, dt, \qquad (3.17)$$

where the limit and equality both hold in F.

Remark 4. (i) *Contrary to what one might expect in view of Proposition 3.2, the second term on the right-hand side of* (3.17) *does not only depend on the choice of the fractional field* $\mathbf{B}$ *but also on the method $(\mathcal{M})$. Of course, the same is true of the first term on the right-hand side of* (3.17).
(ii) *The main advantage of the above definition is that any known stochastic integral with respect to fBm (e.g. pathwise integrals, Skorohod or Wick-Itô integral) gives rise to a corresponding stochastic integral with respect to mBm.*
(iii) *Once again, note that E is not necessary a space of random variables (e.g $E := (\mathcal{S}_{-p})$ for some positive integer p; see Section 4 below) and that E may be different from F (this will be the case in section 4).*

Sections 4, 5 and 6 provide three examples of application of Theorem 3.3.

4 Wick-Itô integral with respect to mBm through approximating fBms

Our aim in this section is to construct a Wick-Itô integral w.r.t. mBm using approximating fBms. A direct approach to Wick-Itô integration w.r.t. mBm is presented in [18], where Itô and Tanaka formulas are also obtained. An application of this integral in mathematical finance may be found in [10]. We shall compare the integral obtained through approximating fBms with the direct approach of [18] in Subsection 4.4.
For definiteness, we will use the field $\mathbf{B}_1$ (as in [18]), but any other field would lead to similar developments.
We first briefly recall some basic facts about white noise theory and the Bochner integral, as well as on the construction of the integral w.r.t. fBm in the spirit of [5, 6, 7, 12].

4.1 Recalls on white noise theory and the Bochner integral

4.1.1 White noise Theory

Define the measurable space $(\Omega, \mathcal{F})$ by setting $\Omega := \mathscr{S}'(\mathbf{R})$ and $\mathcal{F} := \mathcal{B}(\mathscr{S}'(\mathbf{R}))$, where $\mathcal{B}$ denotes the σ-algebra of Borel sets. There exists a unique probability measure μ on $(\Omega, \mathcal{F})$ such that, for every f in $L^2(\mathbf{R})$, the map $< .,f >: \Omega \to \mathbf{R}$ defined by $< .,f > (\omega) =< \omega, f >$ (where $< \omega, f >$ is by definition $\omega(f)$, *i.e.* the action of the distribution ω on the function f) is a centred Gaussian random variable with variance equal to $\|f\|^2_{L^2(\mathbf{R})}$ under μ. For every n in $\mathbb{N}$, define $e_n(x) := (-1)^n\, \pi^{-1/4}(2^n n!)^{-1/2} e^{x^2/2} \frac{d^n}{dx^n}(e^{-x^2})$ the $n-$th Hermite function. Let $(|\ \ |_p)_{p\in\mathbf{Z}}$ be the family norms defined by $|f|^2_p := \sum_{k=0}^{+\infty} (2k+2)^{2p} < f, e_k >^2_{L^2(\mathbf{R})}$, for all (p, f) in $\mathbf{Z} \times L^2(\mathbf{R})$. The operator A defined on $\mathscr{S}(\mathbf{R})$ by $A := -\frac{d^2}{dx^2} + x^2 + 1$ admits the sequence $(e_n)_{n\in\mathbb{N}}$ as eigenfunctions and the sequence $(2n+2)_{n\in\mathbb{N}}$ as eigenvalues.
We denote (L^2) the space $L^2(\Omega, \mathcal{G}, \mu)$ where $\mathcal{G}$ is the σ-field generated by $(< ., f >)_{f\in L^2(\mathbf{R})}$. For every random variable Φ of (L^2) there exists, according to the Wiener-Itô theorem, a unique sequence $(f_n)_{n\in\mathbb{N}}$ of functions f_n in $\widehat{L}^2(\mathbf{R}^n)$ such that Φ can be decomposed as $\Phi = \sum_{n=0}^{+\infty} I_n(f_n)$, where $\widehat{L}^2(\mathbf{R}^n)$ denotes the set of all symmetric functions f in $L^2(\mathbf{R}^n)$ and $I_n(f)$ denotes the $n-$th multiple Wiener-Itô integral of f with the convention that $I_0(f_0) = f_0$ for constants f_0. Moreover the equality $\mathbf{E}[\Phi^2] = \sum_{n=0}^{+\infty} n!\, \|f_n\|^2_{L^2(\mathbf{R}^n)}$ holds, where $\mathbf{E}$ denotes the expectation with respect to μ. For any $\Phi := \sum_{n=0}^{+\infty} I_n(f_n)$ satisfying the condition $\sum_{n=0}^{+\infty} n!\, |A^{\otimes n} f_n|^2_0 < +\infty$, define the element $\Gamma(A)(\Phi)$ of (L^2) by $\Gamma(A)(\Phi) := \sum_{n=0}^{+\infty} I_n(A^{\otimes n} f_n)$, where $A^{\otimes n}$ denotes the $n-$th tensor power of the operator A (see [16, Appendix E] for more details about tensor products of operators). The operator $\Gamma(A)$ is densely defined on (L^2). It is invertible and its inverse $\Gamma(A)^{-1}$ is bounded. Let, for φ in (L^2), $\|\varphi\|^2_0 := \|\varphi\|^2_{(L^2)}$ and, for n in $\mathbb{N}$, let $\mathbb{D}\mathrm{om}(\Gamma(A)^n)$ be the domain of the $n-$th iteration of $\Gamma(A)$. Define the family of norms $(\|\ \ \|_p)_{p\in\mathbf{Z}}$ by:

$$\|\Phi\|_p := \|\Gamma(A)^p\Phi\|_0 = \|\Gamma(A)^p\Phi\|_{(L^2)}, \qquad \forall p \in \mathbf{Z}, \quad \forall \Phi \in (L^2) \cap \mathbb{D}\mathrm{om}(\Gamma(A)^p).$$

For p in $\mathbb{N}$, define $(\mathcal{S}_p) := \{\Phi \in (L^2) : \Gamma(A)^p\Phi \text{ exists and belongs to } (L^2)\}$ and define $(\mathcal{S}_{-p})$ as the completion of the space (L^2) with respect to the norm $\|\ \ \|_{-p}$. As in [17], we let $(\mathcal{S})$ denote the projective limit of the sequence $((\mathcal{S}_p))_{p\in\mathbb{N}}$ and $(\mathcal{S})^*$ the inductive limit of the sequence $((\mathcal{S}_{-p}))_{p\in\mathbb{N}}$. This means that we have the equalities $(\mathcal{S}) = \underset{p\in\mathbb{N}}{\cap}(\mathcal{S}_p)$ (resp. $(\mathcal{S})^* = \underset{p\in\mathbb{N}}{\cup}(\mathcal{S}_{-p})$) and that convergence in $(\mathcal{S})$ (resp. in $(\mathcal{S})^*$) means convergence in $(\mathcal{S}_p)$ for every p in $\mathbb{N}$ (resp. convergence in $(\mathcal{S}_{-p})$ for some p in $\mathbb{N}$).
The space $(\mathcal{S})$ is called the space of stochastic test functions and $(\mathcal{S})^*$ the space of Hida distributions. One can show that, for any p in $\mathbb{N}$, the dual space $(\mathcal{S}_p)^*$ of $\mathcal{S}_p$ is $(\mathcal{S}_{-p})$. Thus we will write $(\mathcal{S}_{-p})$, in the sequel, to denote the space $(\mathcal{S}_p)^*$. Note also that $(\mathcal{S})^*$ is the dual space of $(\mathcal{S})$. We will note $\ll , \gg$ the duality bracket between $(\mathcal{S})^*$ and $(\mathcal{S})$. If ϕ, Φ belong to (L^2) then we have the equality $\ll \Phi, \varphi \gg\ = < \Phi, \varphi >_{(L^2)} = \mathbf{E}[\Phi\, \varphi]$. A function $\Phi : \mathbf{R} \to (\mathcal{S})^*$ is called a stochastic distribution process, or an $(\mathcal{S})^*-$process, or a Hida process. A Hida process Φ is said to be differentiable at $t_0 \in \mathbf{R}$ if $\lim_{r\to 0} r^{-1}(\Phi(t_0 + r) - \Phi(t_0))$ exists in $(\mathcal{S})^*$.

For f in $L^2(\mathbf{R})$, we define the *Wick exponential* of $< .,f >$, noted $: e^{<.,f>} :$, as the (L^2) random variable equal to $e^{<.,f>-\frac{1}{2}|f|_0^2}$. The S-transform of an element Φ of $(\mathcal{S}^*)$, noted $S(\Phi)$, is defined as the function from $\mathscr{S}(\mathbf{R})$ to $\mathbf{R}$ given by $S(\Phi)(\eta) := \ll \Phi, : e^{<.,\eta>} : \gg$ for every η in $\mathscr{S}(\mathbf{R})$. Finally for every $(\Phi, \Psi) \in (\mathcal{S})^* \times (\mathcal{S})^*$, there exists a unique element of $(\mathcal{S})^*$, called the Wick product of Φ and Ψ and noted $\Phi \diamond \Psi$, such that $S(\Phi \diamond \Psi)(\eta) = S(\Phi)(\eta)\ S(\Psi)(\eta)$ for every η in $\mathscr{S}(\mathbf{R})$.

4.1.2 Fractional and multifractional White noise

We introduce two operators, denoted M_H and $\frac{\partial M_H}{\partial H}$, that will prove useful for the definition of the integral with respect to fBm and mBm.

Operators M_H and $\frac{\partial M_H}{\partial H}$

Let H be a fixed real in $(0,1)$. Following [12] and references therein, define the operator M_H, specified in the Fourier domain, by $\widehat{M_H(u)}(y) := \frac{\sqrt{2\pi}}{c_H}\ |y|^{1/2-H}\widehat{u}(y)$ for every y in $\mathbf{R}^*$. This operator is well defined on the homogeneous Sobolev space of order $1/2-H$, denoted $L^2_H(\mathbf{R})$ and defined by $L^2_H(\mathbf{R}) := \{u \in \mathscr{S}'(\mathbf{R})\ :\ \widehat{u} = T_f;\ f \in L^1_{loc}(\mathbf{R})\ \text{ and }\ \|u\|_H < +\infty\}$, where the norm $\|\ \|_H$ derives from the inner product $<,>_H$ defined on $L^2_H(\mathbf{R})$ by $< u,v >_H\ := \frac{1}{c_H^2}\int_{\mathbf{R}} |\xi|^{1-2H}\widehat{u}(\xi)\overline{\widehat{v}(\xi)}d\xi$ and where c_H was given in the definition of the fractional field $\mathbf{B}_1$ in Section 1.
The definition of the operator $\frac{\partial M_H}{\partial H}$ is quite similar. More precisely, define for every H in $(0,1)$, the space $\Gamma_H(\mathbf{R}) := \{u \in \mathscr{S}'(\mathbf{R})\ :\ \widehat{u} = T_f;\ \ f \in L^1_{loc}(\mathbf{R})\ \text{ and }\ \|u\|_{\delta_H(\mathbf{R})} < +\infty\}$, where the norm $\|\ \|_{\delta_H(\mathbf{R})}$ derives from the inner product on $\Gamma_H(\mathbf{R})$ defined by $< u,v >_{\delta_H}\ := \frac{1}{c_H^2}\int_{\mathbf{R}} (\beta_H + \ln|\xi|)^2\ |\xi|^{1-2H}\ \widehat{u}(\xi)\ \overline{\widehat{v}(\xi)}\ d\xi$. Following [18], define the operator $\frac{\partial M_H}{\partial H}$ from $(\Gamma_H(\mathbf{R}), <,>_{\delta_H(\mathbf{R})})$ to $(L^2(\mathbf{R}), <,>_{L^2(\mathbf{R})})$, in the Fourier domain, by: $\widehat{\frac{\partial M_H}{\partial H}(u)}(y) := -(\beta_H + \ln|y|)\ \frac{\sqrt{2\pi}}{c_H}\ |y|^{1/2-H}\ \widehat{u}(y)$, for every y in $\mathbf{R}^*$. The reader interested in the properties of M_H and $\frac{\partial M_H}{\partial H}$ may refer to [18, Sections 2.2 and 4.2].

Fractional and multifractional White noise

Recall the following result ([18, (5.10)]): Almost surely, for every t,

$$B^h_t = \mathbf{B}_1(t,h(t)) = < ., M_{h(t)}(\mathbb{1}_{[0,t]}) > = \sum_{k=0}^{+\infty} \left(\int_0^t M_{h(t)}(e_k)(s)\ ds\right) < ., e_k > . \tag{4.1}$$

We now define the derivative in the sense of $(\mathcal{S})^*$ of mBm. Define the $(\mathcal{S}^*)$-valued process $W^h := (W^h_t)_{t\in[0,1]}$ by

$$W^h_t := \sum_{k=0}^{+\infty} \left[\tfrac{d}{dt}\left(\int_0^t M_{h(t)}(e_k)(s)\ ds\right)\right]\ < ., e_k > . \tag{4.2}$$

Theorem-Definition 4.1. *[18, Theorem-definition 5.1] The process W^h defined by* (4.2) *is an $(\mathcal{S})^*$-process which verifies, in $(\mathcal{S})^*$, the following equality:*

$$W^h_t = \sum_{k=0}^{+\infty} M_{h(t)}(e_k)(t) < ., e_k > + h'(t)\ \sum_{k=0}^{+\infty} \left(\int_0^t \tfrac{\partial M_H}{\partial H}(e_k)(s)\big|_{H=h(t)} ds\right) < ., e_k > . \tag{4.3}$$

Moreover the process B^h is $(\mathcal{S})^*$-differentiable on $[0,1]$ *and verifies $\frac{dB^h}{dt}(t) = W^h_t$ in $(\mathcal{S})^*$.*

When the function h_0 is constant and identically equal to H, we will write $W^H := (W^H_t)_{t\in[0,1]}$ and call the $(\mathcal{S})^*$-process W^H a fractional white noise. Note that (4.3) may be written as

$$W^h_t = W^{h(t)}_t + h'(t)\ \tfrac{\partial \mathbf{B}_1}{\partial H}(t,h(t)), \tag{4.4}$$

where $W^{h(t)}_t$ is nothing but $W^H_t|_{H=h(t)}$ and where the equality holds in $(\mathcal{S})^*$.

4.1.3 Bochner integral

Since the objects we are dealing with are no longer random variables in general, the Riemann or Lebesgue integrals are not relevant here. However, taking advantage of the fact that we are working with vector linear spaces, we may use Pettis or Bochner integrals. In the frame of the Wick-Itô integral, the space E, defined at the beginning of Section 3.2 will be a space $(\mathcal{S}_{-p})$ for some integer p. The fact that we need a norm on $\mathcal{H}_E$ suggests the use of Bochner integral. A nice survey of this topic may be found in [17, p.247]. We only recall here the definition and two basic results.

Definition 4.1 (Bochner integral [17], p.247)**.** *Let I be a subset of $[0,1]$ endowed with the Lebesgue measure. One says that $\Phi : I \to (\mathcal{S})^*$ is Bochner integrable of index p on I if it satisfies the two following conditions:*

1. *Φ is weakly measurable on I, i.e. $t \mapsto \ll \Phi_t, \varphi \gg$ is measurable on I for every φ in $(\mathcal{S})$.*
2. *There exists p in $\mathbb{N}$ such that $\Phi_t \in (\mathcal{S}_{-p})$ for almost every t in I and such that $t \mapsto \|\Phi_t\|_{-p}$ belongs to $L^1(I, dt)$.*

The Bochner-integral of Φ on I is denoted $\int_I \Phi_t \, dt$.

Proposition 4.1. *If Φ: $I \to (\mathcal{S})^*$ is Bochner-integrable on I with index p then $\| \int_I \Phi_t \, dt \|_{-p} \leq \int_I \|\Phi_t\|_{-p} \, dt$.*

Theorem 4.2 ([17], Theorem 13.5)**.** *Let $\Phi := (\Phi_t)_{t\in[0,1]}$ be an $(\mathcal{S})^*$-valued process such that:*

(i) $t \mapsto S(\Phi_t)(\eta)$ is measurable for every η in $\mathscr{S}(\mathbf{R})$.

(ii) There exist p in $\mathbb{N}$, b in $\mathbf{R}^+$ and a function L in $L^1([0,1], dt)$ such that, for a.e. t in $[0,1]$, $|S(\Phi_t)(\eta)| \leq L(t) \, e^{b|\eta|_p^2}$, for every η in $\mathscr{S}(\mathbf{R})$.

Then Φ is Bochner integrable on $[0,1]$ and $\int_0^1 \Phi(s) \, ds \in (\mathcal{S}_{-q})$ for every $q > p$ such that $2be^2 D(q-p) < 1$ where e denotes the base of the natural logarithm and where $D(r) := \frac{1}{2^{2r}} \sum_{n=1}^{+\infty} \frac{1}{n^{2r}}$ for r in $(1/2, +\infty)$.

4.2 Wick-Itô integral with respect to fBm

The fractional Wick-Itô integral with respect to fBm (or integral w.r.t. fBm in the white noise) was introduced in [12] and extended in [5] using the Pettis integral. However, in order to use Theorem 3.3, we need that Y_s belongs to $(\mathcal{S}_{-p})$ for almost every real s. It then seems reasonable to assume that $(Y_s)_{s\in[0,1]}$ is Bochner integrable on $[0,1]$. For this reason, we now particularize the fractional Wick-Itô integral with respect to fBm of [12] and [5] in the framework of the Bochner integral.

Definition 4.2 (Wick-Itô integral w.r.t fBm in the Bochner sense)**.** *Let $H \in (0,1)$, I be a Borel subset of $[0,1]$, $B^H := (B_t^H)_{t\in I}$ be a fractional Brownian motion of Hurst index H, and $Y := (Y_t)_{t\in I}$ be an $(\mathcal{S})^*$-valued process verifying:*

(i) there exists $p \in \mathbb{N}$ such that $Y_t \in (\mathcal{S}_{-p})$ for almost every $t \in I$,

(ii) the process $t \mapsto Y_t \diamond W_t^H$ is Bochner integrable on I.

Then, Y is said to be Bochner-integrable with respect to fBm on I and its integral is defined by:

$$\int_I Y_s \, d^\diamond B_s^H := \int_I Y_s \diamond W_s^H ds. \tag{4.5}$$

Lemma 4.3. *Let $Y := (Y_t)_{t\in[0,1]}$ be an $(\mathcal{S})^*$-valued process, Bochner integrable of index $p_0 \in \mathbb{N}$. Then Y is integrable on $[0,1]$, with respect to fBm of any Hurst index H, in the Bochner sense. Moreover, for any H in $(0,1)$, $\int_{[0,1]} Y_s \, d^\diamond B_s^H$ belongs to $(\mathcal{S}_{-r_0})$ for every $r_0 \geq p_0 + 1$ if $p_0 \geq 2$ and for every $r_0 \geq p_0 + 2$ if $p_0 \in \{0,1\}$.*

Proof: Fix $H \in (0,1)$, $p_0 \geq 2$ and $r_0 \geq p_0 + 1$. The map $t \mapsto Y_t \diamond W_t^H$ is weakly measurable since $t \mapsto S(Y_t \diamond W_t^H)(\eta)$ is measurable for all $\eta \in \mathscr{S}(\mathbf{R})$. Using [17, Remark 2 p.92], one obtains that, for almost all t in $[0,1]$, $\|Y_t \diamond W_t^H\|_{-r_0} \leq \|Y_t\|_{-p_0} \, \|W_t^H\|_{-p_0} < +\infty$. Hence $Y_t \diamond W_t^H$ belongs to $(\mathcal{S}_{-r_0})$. Since the map $t \mapsto \|W_t^H\|_{-r}$ is continuous for all integer $r \geq 2$ (see [18, Proposition 5.9]), one also gets:

$$\int_0^1 \|Y_t \diamond W_t^H\|_{-r_0} \, dt \leq \big(\sup_{t \in [0,1]} \|W_t^H\|_{-p_0} \big) \int_0^1 \|Y_t\|_{-p_0} \, dt < +\infty.$$

This shows that $t \mapsto Y_t \diamond W_t^H$ is Bochner-integrable of index r_0.
Let us now assume that $p_0 \in \{0,1\}$. It is sufficient to check that Theorem 4.2 applies. Condition (i) is obviously fulfilled. Moreover, using [17, p.79], we obtain that, for every (t,η) in $[0,1] \times \mathscr{S}(\mathbf{R})$,

$$|S(Y_t \diamond W_t^H)(\eta)| \leq \|Y_t\|_{-p_0} \, e^{\frac{1}{2}|\eta|_2^2} \sup_{t \in [0,1]} \|W_t^H\|_{-2} =: L(t) \, e^{\frac{1}{2}|\eta|_2^2}.$$

Since Y is Bochner integrable of index p_0, it is clear that L belongs to $L^1([0,1], dt)$. Moreover, $e^2 D(r_0 - p_0) < 1$, for every $r_0 \geq p_0 + 2$. Theorem 4.2 then allows to conclude that $t \mapsto Y_t \diamond W_t^H$ is Bochner integrable of index r_0. □

The following lemma, the proof of which is obvious in view of hypothesis $(\mathcal{H}_2)$, will be useful in the proof of Proposition 4.7 below.

Lemma 4.4. *For every p in $\mathbb{N}$, the map $(t,H) \mapsto \frac{\partial \mathbf{B}_1}{\partial H}(t,H)$ is continuous from $[0,1]$ into $((\mathcal{S}_{-p}), \|\|\|_p)$. In particular, for every subset $[a,b]$ of $(0,1)$, there exists a positive real κ such that:*

$$\forall p \in \mathbb{N}, \qquad \sup_{(s,H) \in [0,t] \times [a,b]} \big\| \tfrac{\partial \mathbf{B}_1}{\partial H}(s,H) \big\|_{-p} \leq \kappa. \tag{4.6}$$

4.3 Stochastic integral with respect to mBm with approximating fBms

We construct in this section the Wick-Itô integral w.r.t. mBm using approximating fBms. In that view, we shall apply Theorem 3.3. Fix (p_0, s_0) in $\mathbb{N}^2$ such that $s_0 \geq \max\{p_0+1, 3\}$. Set $E := (\mathcal{S}_{-p_0})$ and $F := (\mathcal{S}_{-s_0})$. By definition we have $\mathcal{H}_E = \big\{ (Y_t)_{t \in [0,1]} \in (\mathcal{S}_{-p_0})^{\mathbf{R}} : \int_{[0,1]} Y_t \, d^\diamond B_t^\alpha \in (\mathcal{S}_{-s_0}), \ \forall \alpha \in h([0,1]) \big\}$.
Define also the set

$$\Lambda_E := \Big\{ (Y_t)_{t \in [0,1]} \in (\mathcal{S}_{-p_0})^{\mathbf{R}} : Y \text{ is Bochner integrable of index } p_0 \text{ on } [0,1] \Big\}$$

equipped with the norm $\|\Phi\|_{\Lambda_E} := \int_0^1 \, \|\Phi_t\|_{-p_0} \, dt$. The inclusion $\Lambda_E \subset \mathcal{H}_E$ results from Lemma 4.3 whereas the fact that $(\Lambda_E, \| \, \|_{\Lambda_E})$ is complete is a straightforward consequence of [14, Theorem 3.7.7 p.82]. Moreover, $\| \, \|_{\Lambda_E}$ fulfils condition (3.13) with $\chi = 0$.

Lemma 4.5. *The Wick-Itô integral w.r.t. fBm verifies condition* (3.15) *with* $\theta = 1$.

Proof: Since $(\mathcal{S}_{-p_0}) \subset (\mathcal{S}_{-2})$ if p_0 belongs to $\{0;1\}$, we assume from now that $p_0 \geq 2$ and that $s_0 \geq p_0 + 1$. Let $Y \in \Lambda_E$. Lemma 4.3 entails that Y is integrable with respect to fBm in the Bochner sense, for all α in $(0,1)$ and that $\mathcal{I}(Y,\alpha) = \int_{[0,1]} Y_t \, d^\diamond B_t^\alpha$ belongs to $(\mathcal{S}_{-s_0})$. Now for all (α, α') in $(0,1)^2$, using the same arguments we used in the proof of Lemma 4.3,

$$\begin{aligned} \|\mathcal{I}(Y,\alpha) - \mathcal{I}(Y,\alpha')\|_{-s_0} = \big\| \int_{[0,1]} Y_t \diamond (W_t^\alpha - W_t^{\alpha'}) \, dt \big\|_{-s_0} &\leq \int_0^1 \|Y_t\|_{-p_0} \, \|W_t^\alpha - W_t^{\alpha'}\|_{-p_0} \, dt \\ &\leq \big(\sup_{t \in [0,1]} \|W_t^\alpha - W_t^{\alpha'}\|_{-p_0} \big) \|Y\|_{\Lambda_E}, \end{aligned}$$

and thus

$$\sup_{\|Y\|_{\Lambda_E} \leq 1} \|\mathcal{I}(Y,\alpha) - \mathcal{I}(Y,\alpha')\|_{-s_0} \leq \sup_{t \in [0,1]} \|W_t^\alpha - W_t^{\alpha'}\|_{-p_0}. \tag{4.7}$$

By definition, $\|W_t^\alpha - W_t^{\alpha'}\|_{-p_0}^2 = \sum_{k=0}^{+\infty} \frac{(M_\alpha(e_k)(t) - M_{\alpha'}(e_k)(t))^2}{(2k+2)^{2p_0}}$.
For all (t,k) in $[0,1] \times \mathbb{N}$, the function $\alpha \mapsto M_\alpha(e_k)(t)$ is differentiable on $(0,1)$ (this is Lemma 5.5 in [18]). Using point 1 of [18, lemma 5.6] and the mean value theorem, one obtains the following fact: for all $[a,b] \subset (0,1)$, there exists a positive real ρ such that for all $(t,\alpha,\alpha',k) \in [0,1] \times [a,b]^2 \times \mathbb{N}$:

$$|M_\alpha(e_k)(t) - M_{\alpha'}(e_k)(t)| \leq \rho \, (k+1)^{2/3} \, \ln(k+1) \, |\alpha - \alpha'|.$$

As a consequence, we get $\|W_t^{\alpha} - W_t^{\alpha'}\|_{-p_0}^2 \leq \rho^2 \, |\alpha - \alpha'|^2 \sum_{k=0}^{+\infty} \frac{(k+1)^{4/3} \, \ln^2(k+1)}{2^{2p_0}(k+1)^{2p_0}}$. With (4.7), one obtains

$$\sup_{\|Y\|_{\Lambda_E} \leq 1} \|\mathcal{I}(Y, \alpha) - \mathcal{I}(Y, \alpha')\|_{-s_0} \leq |\alpha - \alpha'| \, \gamma_{p_0}, \tag{4.8}$$

where $\gamma_{p_0} := \rho \left(\sum_{k=1}^{+\infty} \frac{\ln^2 k}{k^{2(p_0 - 2/3)}} \right)^{1/2}$ is finite since $p_0 \geq 2$. □

A consequence of the previous lemma is that Theorem 3.3 applies, that is, $\lim_{n \to +\infty} \int_{[0,1]} Y_t \, d^{\diamond} B_t^{h_n}$ exists as an element of $(\mathcal{S}_{-s_0})$.

Lemma 4.6. *For every process $Y := (Y_t)_{t \in [0,1]}$ Bochner-integrable on $[0,1]$, the map $t \mapsto h'(t) \, Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t))$ is integrable.*

Proof: Fix $p_0 \geq 2$ and $s_0 \geq p_0 + 1$. Using the same arguments as in the proof of Lemma 4.3 one easily prove that $t \mapsto h'(t) \, Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t))$ on $[0,1]$ is weakly measurable. Lemma 4.4 entails that, for every p_0,

$$\sup_{(s,H) \in [0,1] \times h([0,1])} \|\tfrac{\partial \mathbf{B}_1}{\partial H}(s, H)\|_{-p_0} \leq \kappa,$$

We hence get $\|h'(t) \, Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t))\|_{-s_0} \leq \|Y_s\|_{-p_0} \big(\sup_{s \in [0,1]} |h'(s)| \big) \sup_{s \in [0,1]} \big| \frac{\partial \mathbf{B}_1}{\partial H}(s, h(s)) \big|_{-p_0} < +\infty$.

Thus there exists $\delta \in \mathbf{R}_+^*$, such that $\int_0^1 \|h'(s) \, Y_s \diamond \frac{\partial \mathbf{B}_1}{\partial H}(s, h(s))\|_{-s_0} \, ds \leq \delta \int_0^1 ||Y_s||_{-p_0} \, ds < +\infty$. As a consequence, $\int_0^1 \, h'(t) \, Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t)) dt$ is well defined in the sense of Bochner. □

As a consequence of Lemmas 4.5 and 4.6, the integral w.r.t. mBm exists as a limit of integrals w.r.t. fBms:

Corollary 4.7. *Let $Y \in \Lambda_E$. Then*

$$\int_0^1 Y_t \, d^{\diamond} B_t^h := \lim_{n \to \infty} \int_0^1 Y_t \, d^{\diamond} B_t^{h_n} + \int_0^1 h'(t) \, Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t)) \, dt, \tag{4.9}$$

where the limit and the equality hold in $(\mathcal{S}_{-s_0})$, is well-defined and belongs to $(\mathcal{S}_{-s_0})$.

4.4 A comparison between multifractional Wick-Itô integral and limiting fractional Wick-Itô integral

A multifractional Wick-Itô integral with respect to mBm was defined in [18]. It is interesting to check whether it coincides with the one defined in Corollary 4.7. In that view, we need to adapt the definition of [18], which used Pettis integrals, to deal with Bochner integrals.

Definition 4.3 (Multifractional Wick-Itô integral in Bochner sense)**.** *Let I be a Borelian connected subset of $[0,1]$, $B^h := (B_t^h)_{t \in I}$ be a multifractional Brownian motion and $Y := (Y_t)_{t \in I}$ be a $(\mathcal{S})^*$-valued process such that:*

- *(i) There exists $p \in \mathbb{N}$ such that $Y_t \in (\mathcal{S}_{-p})$ for almost every $t \in I$,*
- *(ii) the process $t \mapsto Y_t \diamond W_t^h$ is Bochner integrable on I.*

Y is then said to be integrable on I with respect to mBm in the Bochner sense or to admit a multifractional Wick-Itô integral. This integral is defined to be $\int_I Y_s \diamond W_s^h \, ds$.

Remark 5. *From the definition of $\left(W_t^h\right)_{t \in [0,1]}$ [18, proposition 5.9], and the proof of Lemma 4.3, it is clear that every $(\mathcal{S})^*$-valued process $Y := (Y_t)_{t \in I}$ which is Bochner integrable on I of index p_0, is integrable on I with respect to mBm, in the Bochner sense. Moreover $\int_{[0,1]} Y_t \, dB_t^{\diamond h}$ belongs to $(\mathcal{S}_{-r_0})$, where r_0 was defined in Lemma 4.3.*

In order to compare our two integrals with respect to mBm when they both exist, it seems natural to assume that $Y = (Y_t)_{t \in [0,1]}$ is a Bochner integrable process of index $p_0 \in \mathbb{N}$. The space E and the norm $\| \, \|_{\Lambda_E}$ are defined as in the previous subsection.

Theorem 4.8. *Let $Y = (Y_t)_{t\in[0,1]}$ be a Bochner integrable process of index $p_0 \in \mathbb{N}$. Then Y is integrable with respect to mBm in both senses of Corollary 4.7 and Definition 4.3. Moreover $\int_{[0,1]} Y_t \, d^{\diamond} B^h_t$ and $\int_I Y_s \diamond W^h_s \, ds$ are equal in $(\mathcal{S}^*)$.*

Proof: Since Y is a Bochner integrable process of index $p_0 \in \mathbb{N}$, Proposition 4.7 and Remark 5 entail that $\int_{[0,1]} Y_s \, d^{\diamond} B^h_s$ exists in $(\mathcal{S}_{-s_0})$ and that $\int_I Y_s \diamond W^h_s ds$ exist in $(\mathcal{S}_{-r_0})$, where s_0 has been defined just above Proposition 4.7 and r_0 has been defined in Lemma 4.3. Moreover, thanks to (4.9) and using (3.4) and (4.5), we may write, in $(\mathcal{S}_{-s_0})$,

$$\begin{aligned}\int_{[0,1]} Y_s \, d^{\diamond} B^h_s &= \lim_{n\to\infty} \int_{[0,1]} Y_t \, d^{\diamond} B^{h_n}_t + \int_{[0,1]} h'(t) \, Y_t \diamond \tfrac{\partial \mathbf{B}_1}{\partial H}(t, h(t)) \, dt \\ &= \lim_{n\to\infty} \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) \;\; Y_t \, d^{\diamond} B_t^{h(x_k^{(n)})} + \int_0^1 \; h'(t) \, Y_t \diamond \tfrac{\partial \mathbf{B}_1}{\partial H}(t, h(t)) \, dt \\ &= \lim_{n\to\infty} \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) \;\; Y_t \; \diamond W_t^{h(x_k^{(n)})} \; dt + \int_0^1 \; h'(t) \, Y_t \diamond \tfrac{\partial \mathbf{B}_1}{\partial H}(t, h(t)) \, dt. \end{aligned} \tag{4.10}$$

Besides, Definition 4.3 and (4.4) entail that, in $(\mathcal{S}_{-r_0})$,

$$\int_0^1 \; Y_t \diamond W^h_t \, dt = \int_0^1 \; Y_t \diamond W^{h(t)}_t \, dt + \int_0^1 \; h'(t) \, Y_t \diamond \frac{\partial \mathbf{B}_1}{\partial H}(t, h(t)) \, dt. \tag{4.11}$$

Since $s_0 \geq r_0$ we have $(\mathcal{S}_{-r_0}) \subset (\mathcal{S}_{-s_0})$. Thus it remains to show that, in $(\mathcal{S}_{-s_0})$,

$$L(Y) := \lim_{n\to\infty} \sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) \;\; Y_t \; \diamond W_t^{h(x_k^{(n)})} \; dt \text{ is equal to } \; M(Y) := \int_0^1 \; Y_t \diamond W^{h(t)}_t \; dt.$$

Since $L(Y)$ and $M(Y)$ both belong to $(\mathcal{S}_{-s_0})$, it is sufficient to show that they have the same S-transform. Using [17, Theorem 8.6], one has, for η in $\mathscr{S}(\mathbf{R})$,

$$S(L(Y))(\eta) = \lim_{n\to\infty} S\big(\sum_{k=0}^{q_n-1} \int_0^1 \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) \;\; Y_t \; \diamond W_t^{h(x_k^{(n)})} \; dt\big)(\eta).$$

Using now (ii) of [18, Theorem 5.12], one gets

$$\begin{aligned} S(L(Y))(\eta) &= \lim_{n\to+\infty} \sum_{k=0}^{q_n-1} \int_{x_k^{(n)}}^{x_{k+1}^{(n)}} \; S(Y_t)(\eta) \; S(W_t^{h(x_k^{(n)})})(\eta) \; dt \\ &= \lim_{n\to+\infty} \int_{[0,1]} S(Y_t)(\eta) \; \Big(\sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) \; M_{h\left(x_k^{(n)}\right)}(\eta)(t)\Big) \; dt. \end{aligned} \tag{4.12}$$

The map $(t, H) \mapsto M_H(\eta)(t)$ is continuous (see [18, Lemma 5.5]). Define $K_\eta := \sup_{(t,H)\in[0,1]\times h([0,1])} |M_H(\eta)(t)|$. For all n in $\mathbb{N}$ and t in $[0,1]$, one has

$$\big|S(Y_t)(\eta) \sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) \; M_{h(x_k^{(n)})}(\eta)(t)\big| \leq K_\eta \; e^{\frac{1}{2}|\eta|^2_{p_0}} \; \|Y_t\|_{-p_0}. \tag{4.13}$$

The map $t \mapsto K_\eta \; e^{|\eta|^2_{p_0}} \; \|Y_t\|_{-p_0}$ belongs to $L^1(\mathbf{R}, dt)$. In addition, for almost every t in $[0,1]$,

$$\lim_{n\to+\infty} \sum_{k=0}^{q_n-1} \mathbb{1}_{[x_k^{(n)}, x_{k+1}^{(n)})}(t) \; M_{h\left(x_k^{(n)}\right)}(\eta)(t) = M_{h(t)}(\eta)(t),$$

Thus, by dominated convergence and using (4.13), one gets,

$$S(L(Y))(\eta) = \int_{[0,1]} S(Y_t)(\eta) \; M_{h(t)}(\eta)(t) \; dt = \int_{[0,1]} S(Y_t)(\eta) \; S(W_t^{h(t)})(\eta) \; dt = S\big(\int_{[0,1]} Y_t \diamond W_t^{h(t)} \; dt\big)(\eta),$$

Since the map $S : \Phi \mapsto S(\Phi)$ from $(\mathcal{S})^*$ into itself is injective, one deduces that $L(Y) = \int_{[0,1]} Y_t \diamond W^{h(t)}_t \; dt$, and the proof is complete. □

5 Skorohod integral with respect to mBm through approximating fBms

In this section, we apply Theorem 3.3 to define a Skorohod-type integral with respect to mBm. The reference method of integration with respect to fBm here is the one based on Malliavin calculus, as exposed in [2]. We assume throughout this section that $H > 1/2$ and that h ranges in $(1/2, 1)$. We also set $\mathbf{B} = \mathbf{B}_4$ in this section.
Our notations are as follows (for a presentation of Malliavin calculus, see *e.g.* [3, 20]). Let:

$$\mathcal{S} = \left\{ R := f\left(W(h_1), W(h_2), \ldots, W(h_n)\right), f \in C_b^\infty(\mathbf{R}^n), h_i \in L^2([0,T]), i = 1, \ldots, n \right\}$$

where $W(h_i) := \int_{[0,T]} h_i(s)\, dW_s$ with $W := (W_s)_{s\in[0,T]}$ a Brownian motion, $C_b^\infty(\mathbf{R}^n)$ is the set of functions which are bounded as well as all their derivatives. For an element of $\mathcal{S}$, one defines the derivative operator D as:

$$DR = \sum_{i=1}^{n} \partial_i f\left(W(h_1), W(h_2), \ldots, W(h_n)\right) h_i.$$

D extends to the domain $\mathbf{D}$ which is the completion of $\mathcal{S}$ with respect to the norm:

$$\|R\|_{1,2} = \left(\mathbf{E}(R^2) + \mathbf{E}(||DR||^2_{L^2([0,T])})\right)^{\frac{1}{2}}.$$

We denote by δ the adjoint of D, and by $\mathrm{Dom}(\delta)$ its domain. More precisely, $\mathrm{Dom}(\delta)$ is the set of $u \in L^2(\Omega, [0,T])$ such that:

$$|\mathbf{E}(< DR, u >)| \le c_u\, \mathbf{E}(R^2)$$

for all $R \in \mathcal{S}$ (we use $< .,. >$ to denote the scalar product on $L^2([0,T])$, and δ is defined on $\mathrm{Dom}(\delta)$ by the relation:

$$\mathbf{E}(R\delta(u)) = \mathbf{E}(< DR, u >).$$

The operator δ is a closed linear operator on $\mathrm{Dom}(\delta)$. It coincides with the Skorohod integral.
Let us now recall briefly the approach of [2] for the construction of a stochastic integral w.r.t. a class of Gaussian processes. Assume the continuous Gaussian process X may be written:

$$X_t = \int_0^t K(t,s)\, dW_s, \tag{5.1}$$

where the kernel $K(t,s)$ is defined for $0 < s < t < T$ and verifies

$$\sup_{t\in[0,T]} \int_0^t K(t,s)^2 ds < \infty. \tag{5.2}$$

Define the operator K^* on the set of step functions on $[0,T]$:

$$(K^*\varphi)(s) := \varphi(s)K(s^+, s) + \int_s^T \varphi(t)K(dt, s) \tag{5.3}$$

where $K(s^+, s) = K(T,s) - K\left((s,T], s\right)$. Then the stochastic integral w.r.t. X is defined for processes in $\mathrm{Dom}(\delta_X)$ ([2], formula (12)):

$$\mathrm{Dom}(\delta_X) := (K^*)^{-1}(\mathrm{Dom}(\delta)).$$

For a process v in $\mathrm{Dom}(\delta_X)$, one sets:

$$\delta_X(v) := \int_0^T v(s)\delta X(s) := \int_0^T (K^*v)(s)\delta W(s).$$

In the case of fBm, one has:

$$B_t^H = \int_0^t K_H(t,s)\, dW_s,$$

where

$$K_H(t,s) = d_H\big(t-s\big)^{H-1/2} + c_H\big(\tfrac{1}{2}-H\big)\int_s^t (u-s)^{H-3/2}\big(1-\big(\tfrac{s}{u}\big)^{1/2-H}\big)du \tag{5.4}$$

and

$$d_H = \left(\frac{2H\Gamma(\frac{3}{2}-H)}{\Gamma(\frac{1}{2}+H)\Gamma(2-2H)}\right)^{\frac{1}{2}}.$$

We will index our operators and sets with H, *i.e.* we will write:

$$\mathrm{Dom}(\delta_H) = (K_H^*)^{-1}(\mathrm{Dom}(\delta))$$

for the domain of the Skorohod integral with respect to B^H and

$$\delta_H(v) := \int_0^T v(s)\delta B^H(s) := \int_0^T (K_H^* v)(s)\delta W(s)$$

for the integral. In other words, $\delta_H(v) = \delta(K_H^* v)$.

As the methodology of [2] works in a general framework, one may wonder whether it is possible to apply it directly to mBm. The prerequisite is to write an mBm in the form of (5.1). This is exactly how $\mathbf{B}_4$ is defined, and the work [9] develops a Skohorod integral w.r.t. mBm using this approach. In general, [13, Theorem 4.1] ensures that any Gaussian process may be written as $\sum_{i=1}^N \int_0^t K_i(t,u)\, dW_i(u)$, where N is possibly infinite. However it is not an easy task to obtain such a decomposition for a given process. For instance, although a kernel is known for fBm, this is not the case of bifractional motion [15]. Likewise, writing the moving average and harmonizable mBm in this form remains an open problem[2].

We now seek to apply Theorem 3.3 in order to define a Skorohod integral w.r.t. mBm through approximating Skorohod integrals w.r.t. fBms. In that view, we set $F = L^2(\Omega)$ and $\mathcal{I}(Y,\alpha) = \delta_\alpha(Y)$.

It is straightforward to check that, for every (t,s) in $[0,T]^2$, the function $H \mapsto K_H(t,s)$ is C^1. We denote its derivative by G_H, *i.e.* $G_{H_1}(t,s)$ is the derivative of the function $H \mapsto K_H(t,s)$ evaluated at H_1. We associate to G_H an operator G_H^* in a way similar to (5.3). Note that G_H^* is the derivative of the function $H \mapsto K_H^*$. One easily verifies that G_H fulfils (5.2), so that one may define as above $\delta_G(.) := \delta(G_H^* .)$ for a suitable class of processes. Let:

$$\mathcal{D} := \bigcap_{H\in h([0,1])} \mathrm{Dom}(\delta_H),$$

and

$$\mathcal{F} := \bigcap_{H\in h([0,1])} \mathcal{F}_H,$$

where

$$\mathcal{F}_H := (G_H^*)^{-1}(\mathrm{Dom}(\delta)).$$

Set

$$\Lambda := \mathcal{D} \cap \mathcal{F},$$

equipped with the norm $\|v\|_\Lambda = \sup_{H\in h([0,1])} \mathbf{E}\left(\int_0^T (K_H^* v)(s)^2 ds\right) + \sup_{H\in h([0,1])} \mathbf{E}\left(\int_0^T (G_H^* v)(s)^2 ds\right)$, which satisfies condition (3.13) with $\chi = 0$. By definition, $\delta_H(v)$ and $\int_0^T (G_H^* v)(s)\delta W(s)$ both exist for all H in $h([0,1])$ and all v in Λ. Fix (v,s,H,H') in $\Lambda \times [0,T] \times h([0,1])^2$ with $H < H'$. Consider the function $\varphi : \Omega \times [H,H'] \to \mathbf{R}$ defined by:

$$\varphi(\omega, H_1) := (K_{H_1}^* v(\omega))(s) - (K_H^* v(\omega))(s) - (H_1 - H)\frac{(K_{H'}^* v(\omega))(s) - (K_H^* v(\omega))(s)}{H' - H}.$$

For every ω in Ω, $\varphi(\omega,.)$ is C^1 and $\varphi(\omega,H) = \varphi(\omega,H') = 0$. As a consequence, there exists H''_ω in $[H,H']$ such that $\frac{\partial\varphi}{\partial H}(\omega, H''_\omega) = 0$. Thus, the set $A_\omega := \{H_1 \in [H,H'] : \frac{\partial\varphi}{\partial H}(\omega,H_1) = 0\}$ is a non-empty closed subset of $[H,H']$. It has a minimum, that we denote $H_0(\omega)$. The map $\omega \mapsto H_0(\omega)$ is measurable (*i.e.* H_0 is a random variable), and so is the map $(v,s,H,H',\omega) \mapsto H_0(v,s,H,H',\omega)$.

[2] We conjecture that $N > 1$ for the harmonizable mBm, based on the following fact: for all t_1, t_2 in $\mathbf{R}$ and H_1, H_2 in $(0,1)$,

$$\mathbf{E}\left[\mathbf{B}_1(t_1,H_1)\mathbf{B}_1(t_2,H_2)\right] = \mathbf{E}\left[\mathbf{B}_1(t_1,H_2)\mathbf{B}_1(t_2,H_1)\right].$$

This will investigated in a forthcoming work.

We wish to estimate $\|\mathcal{I}(v,H)-\mathcal{I}(v,H')\|_F$ for v in Λ. As we have just seen, there exists a measurable function $H_0 = H_0(H,H',v,s,\omega)$ such that:

$$\begin{aligned} u(s) &:= (K^*_{H'}v)(s) - (K^*_H v)(s) \\ &= (H'-H)(G^*_{H_0}v)(s). \end{aligned}$$

Thus:

$$\mathcal{I}(v,H') - \mathcal{I}(v,H) = \delta(u) = (H'-H)\int_0^T (G^*_{H_0}v)(s)\delta W(s),$$

and

$$\|\mathcal{I}(v,H') - \mathcal{I}(v,H)\|_{L^2(\Omega)} \leq |H-H'|\,\|v\|_\Lambda$$

i.e. (3.15) holds with $\theta = 1$, $E = F := L^2(\Omega)$ and $\Lambda_E := \Lambda$.
In order to define our integral with (3.17), we need to check that $h'(t)Y(t) \diamond \frac{\partial \mathbf{B}}{\partial H}(t,h(t))$ is integrable. Define

$$Z_t := \int_0^t \tfrac{\partial \mathbf{B}}{\partial H}(s,h(s))\,ds = \int_0^t L(t,u)\,dB_u$$

with $L(t,u) := \int_u^t G_{h(s)}(s,u)\,ds$. Thus Z is a Volterra process. It follows from [19, Proposition 7] [3] that any process in $L^2(\Omega\times[0,1])$ is Wick integrable w.r.t. Z. This implies in particular that $\int_0^1 h'(t)Y(t) \diamond \frac{\partial \mathbf{B}}{\partial H}(t,h(t))\,dt$ exists for Y in Λ. In addition, adapting the arguments in Proposition 8 of [19], one may show that $\int_0^1 h'(t)Y(t) \diamond \frac{\partial \mathbf{B}}{\partial H}(t,h(t))\,dt = \int_0^1 h'(t)Y(t)\delta Z_t$ is a Skorohod integral and thus belongs to $L^2(\Omega)$. We are then able to set the following definition and theorem:

Theorem-Definition 5.1. *Let $Y \in \Lambda$. Then the Skorohod integral of Y with respect to mBm is well-defined and given by:*

$$\int_0^1 Y_t\,\delta B^h_t := \lim_{n\to\infty}\int_0^1 Y_t\,\delta B^{h_n}_t + \int_0^1 h'(t)\,Y(t)\diamond\frac{\partial \mathbf{B}}{\partial H}(t,h(t))\,dt. \tag{5.5}$$

where the equality holds in $L^2(\Omega)$.

Remark 6. *One may verify that the integral defined above coincides with the one studied in [9] when they are both defined. Comparing their domains would be an interesting task.*

6 Pathwise integral with respect to mbm through approximating fBms

We discuss in this section the application of our approach for defining an integral w.r.t. mBm using integrals w.r.t. approximating fBms in the sense of [28]. Note first that the method of [28] allows one to define an integral w.r.t. mBm in a direct way. Indeed, the only conditions on both the integrand and integrator are regularity conditions : as an mBm with function h has the same regularity as an fBm with exponent $\min_t(h(t))$, the results of [28] clearly hold without further work. We briefly show in this section that our approximation method applies in this pathwise setting.
Let us recall some notations from [28]: for $0<\alpha<1$, $I^\alpha_{0+}(L^1[0,1])$ is the space of functions that may be represented as the I^α_{0+} integral of an L^1 function φ. Recall that, by definition,

$$I^\alpha_{0+}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}f(y)\,dy.$$

The function φ is uniquely determined and it coincides with the Riemann-Liouville derivative of f or order α:

$$\varphi(t) = D^\alpha_{0+}f(t) := \mathbb{1}_{(0,1)}(t)\tfrac{1}{\Gamma(1-\alpha)}\tfrac{d}{dt}\int_0^t \tfrac{f(u)}{(t-u)^\alpha}du.$$

The fractional integral of f w.r.t. B^H is defined by ([28, Formula (22)]):

$$\int_0^1 f(t)\,dB^H_t = (-1)^\alpha\int_0^1 D^\alpha_{0+}f_{0+}(t)\,D^{1-\alpha}_{1-}B^H_{1-}(t)\,dt + f(0+)B^H_1,$$

[3] it is a straightforward computation to check that the conditions of this proposition are verified by L

provided that $f_{0^+} \in I^{\alpha}_{0^+}(L^1[0,1])$ with $\alpha > 1-H$, where $(-1)^\alpha := e^{i\pi\alpha}$ and

$$B^H_{1^-}(t) := \mathbb{1}_{(0,1)}(t)\left(B^H_t - B^H_1\right), \qquad f_{0^+}(t) := \mathbb{1}_{(0,1)}(t)\left(f(t) - \lim_{\delta \searrow 0} f(\delta)\right),$$

assuming the limit exists (the fractional integral of such f is defined in [28] with respect to much more general integrators. In particular, $\int_0^1 f(t)\, dg(t)$ exists as soon as g is $\beta-$Hölder with $\alpha+\beta > 1$).
Since the integral is computed here ω by ω for almost all ω, our function spaces will be defined in the same way. In other words, E, F and $\mathcal{H}_E$ are all indexed by ω. Fix then $\alpha \in (1-\min(h([0,1])), 1)$ and an ω such that the field $\mathbf{B}$ is Hölder continuous as a function of t and C^1 as a function of H. Set $E = E(\omega) = \mathbf{R}$, $F = F(\omega) = \mathbf{R}$ and

$$\Lambda_E = \mathcal{H}_E = \mathcal{H}_E(\omega) := \{Y(.,\omega) \in E^{[0,1]} : Y(0+,\omega) = 0 \text{ and } Y(.,\omega) \in I^{\alpha}_{0^+}(L^1[0,1])\}.$$

$\mathcal{H}_E$ is endowed with the norm $\|Y(.,\omega)\|_{\Lambda_E} = \|\varphi\|_{L^1}$, where $\varphi(t) = D^{\alpha}_{0^+}Y(.,\omega)(t)$ (see [24, (6.17)]). Let us check that this norm satisfies (3.13). Let Y belong to Λ_E and $\varphi = D^{\alpha}_{0^+}Y$. Since φ is in $L^1[0,1]$, there exists a sequence $(\varphi_n)_n$ of step functions in $L^1[0,1]$ that converges to φ in $L^1[0,1]$. Let $Y_n = I^{\alpha}_{0^+}\varphi_n$. By definition, the sequence $(Y_n)_n$ converges to Y in Λ_E. As a consequence, it is sufficient to verify (3.13) for fractional integrals of step functions. By linearity of $D^{\alpha}_{0^+}$, we may restrict to the case where $Y = I^{\alpha}_{0^+}\mathbb{1}_{(a,b)}$ with $0 \le a < b \le 1$. Straightforward (but lengthy) computations then show that, for a partition of $[0,1]$ in intervals $A_1, \dots, A_n$ of size $\frac{1}{n}$,

$$\|Y.\mathbb{1}_{A_1}\|_{\Lambda_E} + \cdots + \|Y.\mathbb{1}_{A_n}\|_{\Lambda_E} \le M\, n^{\alpha'} \|Y\|_{\Lambda_E}, \tag{6.1}$$

for all $\alpha' > \alpha$ and where M is a constant depending only on α. Thus, (3.13) is fulfilled with any $\chi \in (\alpha, 1)$.
For a measurable process Y with almost all paths in $\mathcal{H}_E$ and for almost every ω,

$$\begin{aligned}
\mathcal{I}(Y,H)(\omega) - \mathcal{I}(Y,H')(\omega) &= \int_0^1 Y_t\, dB^H_t(\omega) - \int_0^1 Y_t\, dB^{H'}_t(\omega) \\
&= (-1)^\alpha \int_0^1 D^{\alpha}_{0^+}Y_{0^+}(t)\, D^{1-\alpha}_{1^-}\left(B^H_{1^-}(\omega) - B^{H'}_{1^-}(\omega)\right)(t)\, dt.
\end{aligned}$$

For better readability, we will omit ω from now on and will sometimes write $B_{1^-}(t,H)$ instead of $B^H_{1^-}(t)$. The map $H \mapsto B_{1^-}(t,H)$ is C^1. Thus there exists $H''(t)$ such that $B^H_{1^-}(t) - B^{H'}_{1^-}(t) = (H - H')\frac{\partial B_{1^-}}{\partial H}(t, H''(t))$. Moreover, the function $t \mapsto \frac{\partial B_{1^-}}{\partial H}(t, H''(t))$ is Hölder continuous of all orders $\beta < \min(h([0,1]))$. Indeed, this is the case of the function $t \mapsto B^H_{1^-}(t)$ and thus

$$\begin{aligned}
\left|\tfrac{\partial B_{1^-}}{\partial H}(t, H''(t)) - \tfrac{\partial B_{1^-}}{\partial H}(t', H''(t'))\right| &= \tfrac{1}{|H-H'|}\left|B^H_{1^-}(t) - B^{H'}_{1^-}(t) - B^H_{1^-}(t') + B^{H'}_{1^-}(t')\right| \\
&= \tfrac{1}{|H-H'|}|B^H_{1^-}(t) - B^H_{1^-}(t') + B^{H'}_{1^-}(t') - B^{H'}_{1^-}(t)| \\
&\le \tfrac{C}{|H-H'|}|t-t'|^\beta,
\end{aligned}$$

for a constant $C > 0$. As a consequence, Y is fractionally integrable w.r.t. $\frac{\partial B_{1^-}}{\partial H}(t, H''(t))$ and:

$$\begin{aligned}
\left|\mathcal{I}(Y,H) - \mathcal{I}(Y,H')\right| &= \left|H - H'\right|\left|\int_0^1 Y_t\, d\tfrac{\partial B_{1^-}}{\partial H}(t, H''(t))\right| \\
&\le C(\mathbf{B})\,|H-H'| \int_0^1 |D^{\alpha}_{0^+}Y_{0^+}(t)|\, dt \\
&= C(\mathbf{B})\,|H-H'|\, \|Y\|_{\Lambda_E},
\end{aligned}$$

where $C(\mathbf{B}) := \sup_{t\in[0,1]}\left|D^{1-\alpha}_{1^-}\frac{\partial B_{1^-}}{\partial H}(t, H''(t))\right| < \infty$ almost surely since $t \mapsto D^{1-\alpha}_{1^-}\frac{\partial B_{1^-}}{\partial H}(t, H''(t))$ is continuous. As a consequence, condition (3.15) of Theorem 3.3 is verified with $\theta = 1 > \chi$. Thus, for a process Y with almost all paths in $\mathcal{H}_E$, the pathwise integral $\int_0^1 Y(t) dB^h_t$ may be defined with (3.11), *i.e.*:

$$\int_0^1 Y_t \; dB_t^h := \lim_{n\to\infty} \int_0^1 Y_t \; dB_t^{h_n} + \int_0^1 h'(t)\, Y_t\, \frac{\partial \mathbf{B}}{\partial H}(t, h(t))\, dt, \tag{6.2}$$

since the second integral on the right-hand side of the above equality exists. We leave it to the reader to verify that the integral w.r.t. mBm defined here through approximating fBms coincides with the one that is obtained with a direct application of the approach in [28].

Appendix

A The fields $\mathbf{B}_1$, $\mathbf{B}_2$ and $\mathbf{B}_3$ fulfil Condition (C_M).

We shall use the notations of [26, (1.5)]. [26, Theorem 3.1] implies that, for $1 \le i \le 3$, the field $\mathbf{B}_i$ can be written almost surely as

$$\widetilde{Y}_{(a^+,a^-)}(t, H) := \alpha_H \int_{\mathbf{R}} \frac{(e^{it\xi} - 1)}{|u|^{H+1/2}}\, U_{(a^+,a^-)}(\xi, H)\, \widetilde{\mathbb{W}_i}(du),$$

for a certain complex measure $\widetilde{\mathbb{W}_i}$, (a^+, a^-) in $\mathbf{R}^2$, where $\alpha_H := \frac{\Gamma(H+1/2)}{\sqrt{2\pi}}$ and where $U_{(a^+,a^-)}(\xi, H)$ is defined by [26, (3.7)]. Let $[a,b] \times [c,d] \subset \mathbf{R} \times (0,1)$. For all (t, H, H') in $[a,b] \times [c,d]^2$,

$$\begin{aligned} I_t^{H,H'} &:= \big|\alpha_H \tfrac{(e^{it\xi}-1)}{|\xi|^{H+1/2}} U_{(a^+,a^-)}(\xi, H) - \alpha_{H'} \tfrac{(e^{it\xi}-1)}{|\xi|^{H'+1/2}} U_{(a^+,a^-)}(\xi, H')\big|^2 \\ &\le \;\; 2\, \big|\tfrac{e^{it\xi}-1}{\xi}\big|^2\, \big|(U_{(a^+,a^-)}(\xi, H) - U_{(a^+,a^-)}(\xi, H'))\alpha_H\, |\xi|^{1/2-H}\big|^2 \\ &\quad + 2\, \big|\tfrac{e^{it\xi}-1}{\xi}\big|^2\, |U_{(a^+,a^-)}(\xi, H')|^2 \big|\alpha_H\, |\xi|^{1/2-H} - \alpha_{H'}\, |\xi|^{1/2-H'}\big|^2. \end{aligned} \tag{A.1}$$

Since the map $H \mapsto \alpha_H$ is C^∞ on $(0,1)$, for ξ in $\mathbf{R}^*$, the map $f_\xi : [c,d] \to \mathbf{R}_+$, defined by $f_\xi(H) := \alpha_H\, |\xi|^{1/2-H}$ is C^1 on $(0,1)$. Moreover [26, (3.0)] allows us to write $|U_{(a^+,a^-)}(\xi, H)|^2 \le 2({a^+}^2 + {a^-}^2) := Q^2$, for every real ξ. Thus there exists a positive real $D \ge 4Q^2$ such that

$$\forall (\xi, H) \in \mathbf{R}^* \times [c,d],\; |f_\xi'(H)| \le D\, |\xi|^{1/2-H}\, (1 + |\ln(|\xi|)|) \le D\, \left(|\xi|^{1/2-c} + |\xi|^{1/2-d}\right)\, (1 + |\ln(|\xi|)|).$$

The mean-value theorem applied to both members of the right hand side of (A.1) yields:

$$\begin{aligned} I_t^{H,H'} &\le\; 2D^2\, |H - H'|^2\, \left[\mathbb{1}_{\mathbf{R}^*}(\xi) \tfrac{|e^{it\xi}-1|^2}{|\xi|^2} \left(|\xi|^{1/2-c} + |\xi|^{1/2-d}\right)^2\, (1 + |\ln(|\xi|)|)^2\right] \\ &\le 2D^2\, |H - H'|^2\, \left(2^3\, \mathbb{1}_{|\xi|>1}\, \tfrac{(1+\ln|\xi|)^2}{|\xi|^{1+2c}}\, d\xi + (2t)^2\, \mathbb{1}_{|\xi|\le 1}\, |\xi|^{1-2d}\, (1 + |\ln(|\xi|)|)^2\, d\xi\right) \\ &\le 2(2^3 + T^2)\, D^2\, \left(\mathbb{1}_{|\xi|>1}\, \tfrac{(1+\ln|\xi|)^2}{|\xi|^{1+2c}}\, d\xi + \mathbb{1}_{|\xi|\le 1}\, |\xi|^{1-2d}\, (1 + |\ln(|\xi|)|)^2\, d\xi\right) |H - H'|^2, \end{aligned}$$

which also reads $I_t^{H,H'} \le \Phi_t^2(\xi)\, |H - H'|^2$, with $\sup_{t\in[a,b]} \int_{\mathbf{R}} |\Phi_t(\xi)|^2\, d\xi < +\infty$. □

B Proof of Proposition 3.1 in the case of $\mathbf{B}_1$ and $\mathbf{B}_2$

It is sufficient to establish the proof for $\mathbf{B}_1$. Almost surely, $\mathbf{B}_1(t, H) :=< ., M_H(\mathbb{1}_{[0,t]}) >$ for every (t, H) in $\mathbf{R} \times (0,1)$ (see [12, section 2] for example). The following result is established in [18]: the map $H \mapsto M_H(\mathbb{1}_{[0,t]})$ is C^1 from $(0,1)$ to $L^2(\mathbf{R})$ for every t. The map is $H \mapsto \mathbf{B}_1(t, H)$ is C^1 for every real t and its derivative is such that, almost surely, $\frac{\partial \mathbf{B}_1}{\partial H}(t, H) =< ., \frac{\partial M_H}{\partial H}(\mathbb{1}_{[0,t]}) >$ for every (t, H) in $(0,1)$. Note moreover that the previous equality also holds in $L^2(\Omega)$ and that the process $(\frac{\partial \mathbf{B}_1}{\partial H}(t, H))_{(t,H)\in \mathbf{R}\times(0,1)}$ is Gaussian and centred. Now, with the notations of [18], $\mathbf{E}[\frac{\partial \mathbf{B}_1}{\partial H}(t, H) \frac{\partial \mathbf{B}_1}{\partial H}(s, H')] = < \frac{\partial M_H}{\partial H}(\mathbb{1}_{[0,t]}), \frac{\partial M_H}{\partial H}(\mathbb{1}_{[0,s]}) >_{L^2(\mathbf{R})}$, for every $(t, H) \in \mathbf{R} \times (0,1)$. Hence, setting $J := \mathbf{E}\big[\big(\frac{\partial \mathbf{B}_1}{\partial H}(t, H) - \frac{\partial \mathbf{B}_1}{\partial H}(s, H)\big)^2\big]$, we get:

$$J := \left\| \frac{\partial M_H}{\partial H}(\mathbb{1}_{[0,t]} - \mathbb{1}_{[0,s]}) \right\|^2_{L^2(\mathbf{R})} = \frac{1}{c_H^2} \int_{\mathbf{R}} (\beta_H + \ln|y|)^2 \, |y|^{1-2H} \, \left| \frac{1-e^{iy(t-s)}}{y^2} \right|^2 dy,$$

where $\beta_H := \frac{c_H'}{c_H}$. Fix τ in $(0, c)$ and let $M := e^{\frac{\ln 2}{\tau}}$. Note that $M > 1$ and that $|y|^\tau \geq 2$ for every y such that $|y| \geq M$. One computes:

$$\begin{aligned} J &= \frac{1}{c_H^2} \int_{|y|>M} (\beta_H + \ln|y|)^2 \, |y|^{1-2H} \, \left| \frac{1-e^{iy(t-s)}}{y^2} \right|^2 dy + \frac{1}{c_H^2} \int_{|y|\leq M} (\beta_H + \ln|y|)^2 \, |y|^{1-2H} \, \left| \frac{1-e^{iy(t-s)}}{y^2} \right|^2 dy \\ &\leq \frac{1}{c_H^2} \int_{|y|>M} (\beta_H + \ln|y|)^2 \, |y|^{1-2H} \, |t-s|^2 \frac{|y|^{2\tau}}{|y|^2} \, dy + |t-s|^2 \, \frac{2^2}{c_H^2} \int_{|y|\leq M} \frac{(\beta_H+\ln|y|)^2}{|y|^{2H-1}} \, dy \\ &\leq |t-s|^2 \, \frac{4}{c_H^2} \left(\int_{|y|>M} \frac{(\beta_H+\ln|y|)^2}{|y|^{1+2(H-\tau)}} \, dy + \int_{|y|\leq M} \frac{(\beta_H+\ln|y|)^2}{|y|^{2H-1}} \, dy \right) =: |t-s|^2 \, Q(H). \end{aligned}$$

Since $\Delta_1 := \sup_{H\in[c,d]} Q(H) < +\infty$, we get:

$$\mathbf{E}\Big[\Big(\frac{\partial \mathbf{B}_1}{\partial H}(t,H) - \frac{\partial \mathbf{B}_1}{\partial H}(s,H)\Big)^2\Big] \leq \Delta_1 \, |t-s|^2. \tag{B.1}$$

Besides, $\mathbf{E}\big[\big(\frac{\partial \mathbf{B}_1}{\partial H}(t,H) - \frac{\partial \mathbf{B}_1}{\partial H}(t,H')\big)^2\big] = \int_{\mathbf{R}} |1-e^{ity}|^2 \, (g_y(H)-g_y(H'))^2 \, dy$, where the map $g_y : (0,1) \to \mathbf{R}$ is defined by $g_y(H) := \frac{(\beta_H+\ln|y|)}{c_H} \, |y|^{1/2-H}$ for every y in $\mathbf{R}^*$. It is easily seen that g_y is C^1 on $[c,d]$ for every y in $\mathbf{R}^*$. The mean value theorem applies and there exists a positive constant K, which only depends on $[c,d]$, such that

$$\mathbf{E}\left[\Big(\frac{\partial \mathbf{B}_1}{\partial H}(t,H) - \frac{\partial \mathbf{B}_1}{\partial H}(t,H')\Big)^2\right] \leq |H-H'|^2 \, K \int_{\mathbf{R}} |1-e^{ity}|^2 \, |\Phi(y)|^2 \, dy,$$

where $\Phi(y) := 1 + \big(|y|^{1/2-c} + |y|^{1/2-d}\big)\big(1 + (1+\ln|y|)\ln|y|\big)$. Since $\Delta_2 := K \int_{\mathbf{R}} |1-e^{ity}|^2 \, |\Phi(y)|^2 \, dy < +\infty$, we have proven that

$$\mathbf{E}\left[\Big(\frac{\partial \mathbf{B}_1}{\partial H}(t,H) - \frac{\partial \mathbf{B}_1}{\partial H}(t,H')\Big)^2\right] \leq \Delta_2 \, |H-H'|^2. \tag{B.2}$$

Using (B.1) and (B.2), one obtains: $\mathbf{E}\big[\big(\frac{\partial \mathbf{B}_1}{\partial H}(t,H) - \frac{\partial \mathbf{B}_1}{\partial H}(s,H')\big)^2\big] \leq \Delta \, \big(|t-s|^2 + |H-H'|^2\big)$, where we set $\Delta := 2(\Delta_1 + \Delta_2)$.

References

[1] R. Adler. *An Introduction to Continuity, Extrema, and Related Topics for General Gaussian Processes processes.* Lecture Notes- Monographs Series. Institute of Mathematica Statistics, 1990.

[2] E. Alòs, O. Mazet, and D. Nualart. Stochastic calculus with respect to Gaussian processes. *Ann. Probab.*, 29(2):766–801, 2001.

[3] V. Bally. An elementary introduction to Malliavin calculus. *Rapport de recherche no 4718, INRIA*, 2003.

[4] A. Benassi, S. Jaffard, and D. Roux. Elliptic Gaussian random processes. *Rev. Mat. Iberoamericana*, 13(1):19–90, 1997.

[5] C. Bender. An Itô formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter. *Stochastic Processes and their Applications*, 104:81–106, 2003.

[6] C. Bender. An S-transform approach to integration with respect to a fractional Brownian motion. *Bernoulli*, 9(6):955–983, 2003.

[7] F. Biagini, A. Sulem, B. Øksendal, and N. Wallner. An introduction to white-noise theory and Malliavin calculus for fractional Brownian motion. *Proc. Royal Society, special issue on stochastic analysis and applications*, pages 347–372, 2004.

[8] P. Billingsley. *Convergence of probability measures.* Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1999. A Wiley-Interscience Publication.

[9] B. Boufoussi, M. Dozzi, and R. Marty. Local time and Tanaka formula for a Volterra-type multifractional Gaussian process. *Bernoulli*, 16(4):1294–1311, 2010.

[10] S. Corlay, J. Lebovits, and J. Lévy Véhel. Multifractional stochastic volatility models. *To appear in Mathematical Finance*, 2012.

[11] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications.* Cambridge University Press, Cambridge, 1992.

[12] R. J. Elliott and J. van der Hoek. A general fractional white noise theory and applications to finance. *Mathematical Finance*, 13(2):301–330, 2003.

[13] T. Hida and M. Hitsuda. *Gaussian processes*, volume 120 of *Translations of Mathematical Monographs*. American Mathematical Society, Providence, RI, 1993. Translated from the 1976 Japanese original by the authors.

[14] E. Hille and R. S. Phillips. *Functional Analysis and Semi-Groups*, volume 31. American Mathematical Society, 1957.

[15] C. Houdré and J. Villa. An example of infinite dimensional quasi-helix. In *Stochastic models (Mexico City, 2002)*, volume 336 of *Contemp. Math.*, pages 195–201. Amer. Math. Soc., Providence, RI, 2003.

[16] S. Janson. *Gaussian Hilbert Spaces*, volume 129 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 1997.

[17] H. Kuo. *White Noise Distribution Theory*. CRC-Press, 1996.

[18] J. Lebovits and J. Lévy Véhel. White Noise-based stochastic calculus with respect to multifractional Brownian motion. *Preprint*, 2011. Available at `http://hal.inria.fr/inria-00580196/en/`.

[19] D. Nualart. A white noise approach to fractional Brownian motion. In *Stochastic analysis: classical and quantum*, pages 112–126. World Sci. Publ., Hackensack, NJ, 2005.

[20] D. Nualart. *The Malliavin Calculus and Related Topics*. Springer, 2006.

[21] R. Peltier and J. Lévy Véhel. Multifractional Brownian motion: definition and preliminary results, 1995. rapport de recherche de l'INRIA, n^0 2645.

[22] D. Pollard. *Convergence of Stochastic Processes*. Springer Series in Statistics. Springer-Verlag, New York, 1984.

[23] D. Revuz and M. Yor. *Continuous martingales and Brownian motion*, volume 293 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, third edition, 1999.

[24] S. G. Samko, A. A. Kilbas, and O. I. Marichev. *Fractional integrals and derivatives*. Gordon and Breach Science Publishers, Yverdon, 1993.

[25] G. Samorodnitsky and M. Taqqu. *Stable Non-Gaussian Random Processes, Stochastic Models with Infinite Variance*. Chapmann and Hall/C.R.C, 1994.

[26] S. Stoev and M. Taqqu. How rich is the class of multifractional Brownian motions? *Stochastic Processes and their Applications*, 116:200–221, 2006.

[27] M. Talagrand. *The Generic Chaining*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005. Upper and lower bounds of stochastic processes.

[28] M. Zähle. Integration with respect to fractal functions and stochastic calculus. I. *Probab. Theory Related Fields*, 111(3):333–374, 1998.