On some historical aspects of Riemann zeta function, 1

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Abstract. Within the variegated framework of Riemann zeta function and related conjecture (Riemann Hypothesis), we would like to start with a study of some quite disregarded or not much in-deep studied historical aspects concerning Entire Function Theory aspects of Riemann zeta function. This first paper essentially would be the *manifesto* of such a historical research program whose main points will be in deep studied and developed later with forthcoming works.

1. Introduction

In reviewing the main moments of the history of Riemann zeta function (RZF) and the related still unsolved conjecture known as *Riemann Hypothesis* (RH), as for instance masterfully exposed in (Bombieri 2006) as well as in the various treatises, textbooks and survey papers on this argument (see Whittaker & Watson 1927, Chandrasekharan 1958, Ivić 1985, Titchmarsh 1986, Patterson 1988, Karatsuba & Voronin 1992, Karatsuba 1994, Edwards 2001, Chen 2003, Conrey 2003, Gonek 2004), one realises that a crucial point which would deserve major historical attention is that concerning the *Hadamard factorization theorem,* which is the central point around which revolve our attention and that has cast a precious bridge with Entire Function Theory. Hadamard (1893) formulated this theorem as a continuation and completion of a previous 1883 theorem stated by Poincaré as regards the order of an entire function factorized according to the Weierstrass factorization theorem, applying the results so obtained to the Riemann ξ function. This celebrated Hadamard result will be the pivotal point through which the Entire Function Theory enters into the realm of Riemann zeta function. After Hadamard, it was Pólya (1923) to achieve some further remarkable outcomes along this research path emphasizing the Entire Function Theory perspective of Riemann zeta function but not following the Hadamard work, until recent results of which we will refer later and that nevertheless have above all emphasized the Pólya way and neglected the Hadamard one. In this first paper, we would like at first to outline those main points of a historical research program just lying on these aspects of Riemann zeta function theory, which goes from entire function theory aspects of RZF to statistical mechanics, as outlined in the next section.

2. A research program in history of mathematics concerning RH

Following (Bombieri 2006), one of the main tool to study the mathematical properties of *Riemann zeta function* $\zeta(s)$ (hereafter RZF), defined by

$$\zeta(s) \doteq \sum_{n\in\mathbb{N}} \frac{1}{n^s} \qquad s \in \mathbb{C}, \qquad \Re(s) > 1,$$

is the related *Riemann functional equation*, which was established in (Riemann 1858) and is defined as follows (see also (Titchmarsh 1986, Sections 2.4 and 2.6), (Katz & Sarnak 1999, Section 1))

$$\pi^{-\frac{s}{2}}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \pi^{-\frac{1-s}{2}}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s).$$

From its *symmetric form* (see (Ivić 1986, Section 1.2), it is possible, in turn, to define the so-called *Riemann ξ function* (Riemann 1858) as follows (see (Whittaker & Watson 1927, Section 13.4))

$$\xi(t) = (1/2\, s(s-1)\, \pi^{-s/2}\, \Gamma(s/2)\zeta(s)\,)_{s=\frac{1}{2}+it}$$

which is an even entire function of order one with simple poles in *s* = 0, 1, and whose zeros verify $|\Im(t)| \leq 1/2$ (Riemann 1858). This last estimate was then improved in $|\Im(t)| < 1/2$ both by Hadamard (1896) and by de la Vallée-Poussin (1896), but independently of each other. The well-known Riemann Hypothesis (RH) asserts that $\Im(t) = 0$, that is to say $t \in \mathbb{R}$. Following (Ivić 1989) and (Gonek 2004), it is plausible to conjecture that all the zeros of RZF, along the critical line, are simple, this assertion being supported by all the existing numerical evidences (see for example (van de Lune et al. 1986)) .

Subsequently, Hadamard (1893) gave a fundamental Weierstrass infinite product expansion of Riemann zeta function, of the following type

$$\xi(t) = ae^{bt} \prod_{\rho \in Z(\zeta)} (1 - \frac{t}{\rho}) e^{\frac{t}{\rho}} \quad \text{(\textit{Hadamard product formula})}$$

where a, b are constants and $Z(\zeta)$ is the set of all the complex non-trivial zeros of Riemann zeta function $\zeta(s)$, so that $Z(\zeta) \subseteq \{t;\ t \in \mathbb{C},\ 0 < \Re(t) < 1\}$, with $card\, Z(\zeta) = \infty$ (G.H. Hardy). This Hadamard paper was considered by H.C.F. von Mangoldt (1854-1925) ‘’the first real progress in the field in 34 years’’ since the only number theory Riemann 1859 paper (see (Edwards 2001, Section 2.1)), having provided the first basic link between RZF and Entire Function Theory. Nevertheless, in relation to the Riemann zeta function, the Hadamard work didn’t have that right historical attention which it would have deserved.

From Hadamard product formula (hereafter HPF), it follows an infinite product expansion of Riemann zeta function of the following type (see (Erdélyi 1981, Section 17.7) and (Titchmarsh 1986, Section 2.12))

$$\zeta(s) = \frac{e^{(\ln \pi - 1 - \frac{\gamma}{2})s}}{2(s-1)\Gamma(1+\frac{s}{2})} \prod_{\rho \in Z(\zeta)} (1 - \frac{s}{\rho}) e^{\frac{s}{\rho}} = \Theta(s) \prod_{\rho \in Z(\zeta)} (1 - \frac{s}{\rho}) e^{\frac{s}{\rho}}$$

where γ is the Euler-Mac Laurin constant. The function $\Theta(s)$ is non-zero into the critical strip $0 < \Re(s) < 1$, so that it is quickly realised as any question about zeros of $\zeta(s)$ might be addressed to the above infinite product factor, which is an entire function; likewise, as regards the above Hadamard product formula for ξ. Therefore, it seems quite obvious to account for the possible relationships existing with Entire Function Theory, following this Hadamard perspective.

Amongst the best treatises on Entire Function Theory, there are those of Boris Ya. Levin (1980, 1996). In particular, the treatise (Levin 1980) is the most complete one on the distributions of zeros of entire functions, which deserves a considerable attention. As regards the above Hadamard product formula, in reviewing the main textbooks on RZF, amongst which (Chandrasekharan 1958, Lectures 4, 5 and 6), (Titchmarsh 1986, Chapter II), (Ivić 1986, Section 1.3), (Patterson 1988, Chapter 3), (Karatsuba & Voronin 1992, Sections 5 and 6), (Edwards 2001, Chapter 2) and (Chen 2003, Chapter 6), it turns out that such a fundamental factorization, like HPF, has been used to study some properties of RFZ, for instance in relation to its Euler infinite product expansion or in relation to its growth order questions. But, in such treatises, it isn’t exposed those results properly related to the possible links between RZF and Entire Function Theory, from Hadamard and Pólya work on.

There have been various studies which have dealt with Entire Function Theory aspects of Riemann ξ function *d'aprés* Pólya work, and, in this regards, we report some valuable considerations very kindly communicated to me by Professor Jeffrey C. Lagarias (see also (Lagarias & Montague 2011)). He first says that, although there are strong circumstantial evidences for RH, no one knows how to prove it and no promising mechanism for a proof is currently known. In particular, there are many approaches to it, and it is not clear whether the complex variables approaches based on *Laguerre-Polya* (LP) and *Hermite-Biehler* (HB) connections with RZF theory via Riemann ξ function (see (Levin 1980, Chapters VII and VIII)) are going to get anywhere. Maybe, Pólya might have been the first to have established the LP connection on the basis of the previous work made by J.L.W. Jensen. The truth of RH requires that ξ(s) falls into the HB class under suitable change of variable (see (Lagarias 2005)), even if Lagarias stresses on the fact that this was already known for a long time, for which reason it requires further historical examination. Also Louis De Branges has made some interesting work in this direction, no matter by his attempts to prove RH which yet deserve as well attention because they follow a historical method, as kindly De Branges himself said to me. Nevertheless, Lagarias himself refers that who has been the first to state this connection to HB class is historically yet not clear. Further studies even along this direction have been then made by G. Csordas, R.S. Varga, W. Smith, A.M. Odlyzko, J.C. Lagarias, D. Montague, D.A. Hejhal, D.A. Cardon, S.R. Adams and some other. Finally, Lagarias concludes stating that the big problem is to find a mechanism that would explain why the Riemann ξ function would fall into this HB class of functions.

On the other hand, the above mentioned Pólya work has also found profitably applications in the mathematical formalism of statistical mechanics (see (Ruelle 1969 Chapter 5)). Indeed, T.D. Lee and C.N. Yang, in two celebrated 1952 works (see (Lee & Yang 1952) and (Yang & Lee 1952)), made use of Pólya work on Riemann ξ function just to prove an important theorem (that will be called *Lee-Yang theorem*) about the zeros of the partition function to forecast possible phase transitions of certain dynamical systems. In this regards, in (Yang 2005, Commentary to $[\![52b,c]\!]$, pp. 14-16), it is referred as this Pólya work was of a crucial importance in proving a conjecture, met along the proof of their theorem, which they were unable to solve. Because of this, they consulted Mark Kac for a help, who just in that period was adjoined editor of Pólya's Collected works, suggesting them how to solve such a conjecture on the basis of (Pólya 1923) which will turn out just to be an essential tool in proving this their theorem, which will begin rightly famous. Indeed, this theorem has opened a new interesting and fruitful chapter in rigorous statistical mechanics as well as in mathematics, until nowadays: see, for instance, the recent papers (Fröhlich & Rodriguez 2012), (Borcea & Brändén 2008, 2009a,b), (Ruelle 2002, 2010) and (Brändén 2011), would deserve major historiographical attention.

What follows, is the content of a valuable email with which Professor Enrico Bombieri has very kindly replied to my request to have some his comments and hints about some historical aspects of Entire Function Theory of Riemann zeta function. He kindly refers that, very likely, the 1893 Hadamard work was motivated by the possible applications to RZF. On the other hand, the general theory of complex and special functions had a great growth impulse just after the middle of 18th Century above all thanks to the pioneering works of Weierstrass, Schwarz, Nevanlinna and others. But Hadamard was the first to found a general theory which will receive its highest appreciation with the next works of Nevanlinna brothers. Thereafter, the attempts to isolate entire function classes comprising RZF (properly modified to avoid its single poles in $s = 0, 1$) have been quite numerous (amongst which those by De Branges), with interesting results but unfruitful as regards the possible applications to RH. Nowadays, few mathematicians go on along this path, amongst whom G. Csordas with interesting works. In a recent conference, in which Bombieri was attended, Csordas proposed to consider the class of Mellin transformation $\mathcal{M}f(x)$ of fast decreasing functions f as $x \to \infty$ such that each $\left(x\frac{d}{dx}\right)^n f(x)$ has exactly n zeros for each $n \in \mathbb{N}_0$. Now, it would seem that the RZF may be related with this class of functions, but, at this moment, there is no exact proof of this idea to which Bombieri himself was attained through other ways. Many mathematicians have besides worked on Lee-Yang theorems hoping to meet the RZF along their paths, but after an initial enthusiasm, every further attempt didn't have any sequel. As regards, then, the complex function theory, this reached its apex around 1960s, above all with the works achieved by the English school of W. Hayman and by the Russian school of B.Ya. Levin. Recently, the statistical mechanics approach seems promising as regards zero distribution of RZF which is quite anomalous and seems to follow a Gaussian Circular Unitary Ensemble (GCUE) law (see (Katz & Sarnak 1999) related to Random Matrix Theory. Bombieri finishes mentioning some very interesting results achieved, amongst others, by Beurling, Nyman and Baez-Duarte, hence saying that, nowadays, there still exists a little but serious group of researchers working on the relationships between Riemann ξ function, its Fourier transform and Entire Function Theorem, *d'aprés* Pólya work.

The next papers would have the aim to develop and deepen the above delineated points related to the possible links between Entire Function Theory and RZF, retracing the historical paths, making the related comparisons and identifying the possible relationships. The first step, for instance, will be oriented to put major historical attention to 1893 Hadamard work hence necessarily starting from the history of entire function factorization theorem which, amongst others, sees involved the figures of Gauss, Weierstrass, Mittag-Leffler, Betti, Dini.

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