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Abstract

*By Boolean category we mean something which is to a Boolean algebra what a category is to a poset. We propose an axiomatic system for Boolean categories, which is different in several respects from the ones proposed recently. In particular everything is done from the start in a *-autonomous category and not in a weakly distributive one, which simplifies issues like the Mix rule. An important axiom, which is introduced later, is a "graphical" condition, which is closely related to denotational semantics and the Geometry of Interaction. Then we show that a previously constructed category of proof nets is the free "graphical" Boolean category in our sense. This validates our categorical axiomatization with respect to a real-life example. Another important aspect of this work is that we do not assume a-priori the existence of units in the *-autonomous categories we use. This has some retroactive interest for the semantics of linear logic, and is motivated by the properties of our example with respect to units.*

1. Introduction

Unlike other mathematicians, proof theorists have access to very few canonical objects. All mathematicians have the integers, the reals, the rationals. Geometers have projective planes and spheres, algebraists have polynomial rings and permutation groups. Indeed, algebraists have access to the *concept* of a group and of a ring, which have been stable for more than a hundred years. In contrast, a proof theorist is always ready to tweak a definition like that of the sequent calculus, to suit his needs. We say *the* sequent calculus but there is no such thing.

Logicians have Boolean and Heyting algebras, but they are of limited interest to proof theorists since they collapse too many things: In a Boolean or Heyting algebra two formulas, a seemigly complex one and a seemingly trivial one, can turn out to have identical denotations—and things are the same, if not worse, for proofs.

We know that much information about a proof is kept if we replace posets by categories. A celebrated example of this is Freyd's proof [13] that higher order intuitionistic logic has the existence and disjunction properties (as a constructive logic should) purely by observing the free elementary topos, and using this very property of freeness. The free topos is a canonical object if there ever was one.

The free elementary topos is one of the many, many examples of a "Heyting category", which is to categories what a Heyting algebra is to posets: a bicartesian closed category. Until very recently it was absolutely mysterious how one could define "Boolean categories" in the same manner. For a long time the only known natural definition of a Boolean category collapsed to a poset. This was first corrected by following closely the approach to term systems for classical logic: in order to prevent collapse, introduce asymmetries, which is what is done for example in Selinger's control categories [17] (which correspond to the $\lambda\mu-$calculus [16]) or the models of Girard's LC and the closely related work of Streicher and Reus on continuations [19], which introduce restrictions by the means of polarities.

But then there appeared several approaches [6, 5, 12, 4] to Boolean categories that do keep the symmetry we associate with Booleanness: all these categories are self-dual, and except for the last one they all are *-autonomous. The present paper is concerned with the category of $\mathbb{B}$-nets of [12], which is a remarkably simple object, a candidate for canonicity: a "beefed up" Boolean algebra. It is surprising that it was not discovered before.

In this paper we present a series of axioms for Boolean categories, in order of increasing strenght. We then show that the category of $\mathbb{B}$-nets of [12] is the free Boolean category for the strongest axioms with the atomic formulas as generators. The axiom of "graphicality" gives it a marked semantical character and relates it to coherences spaces and the Geometry of Interaction.

Our axiomatic approach differs from that of Führmann and Pym [5, 6] in several respects. It is completely 1-categorical and does not use something like an order enrichment. Also, we start with a *-autonomous category and show how to extract (several) weakly distributive categories it contains, while they start with a weakly distributive category and then complete it to a *-autonomous one by adding structure.

Finally, we give a novel answer to the question of defining a *-autonomous category that does not have units, which we need to interpret logics without constants. This retroactively applies to multiplicative [1] and multiplicative-additive [9] proof nets.

2. The axioms

It is very well known how to model a multiple-premise, single-conclusion linear calculus in a symmetric monoidal category that has the $\multimap$ adjoint operator. It is also well-known how to have multiple premises, and/or a negation. If we want zero premise, it is natural to think of the tensor unit as source as representing an empty family of premises: an empty context. But if we have the unit in the category, shouldn't we also have it in the logic? The standard approach to this question is found in [1], where the existence of a unit $\mathbf{I}$ is assumed in the category that is used for the semantics, but its use is very restricted: it can only appear as the source of a semantical map. There is a problem: for example, the category of ordinary multiplicative proof nets without units cannot be used to interpret itself as a theory! We propose a solution to this problem: replace the unit with a functor to $\mathbf{Set}$, which would be the covariant functor represented by the unit, if only there was a unit. This seems to be a very trivial change, but it has interesting consequences. (An alternative approach to our proposal has been very recently presented in [8].)

2.1. *-autonomous categories without units

We will define autonomous (SMC) and *-autonomous categories not to have units by default. This spares us from having *-autonomous categories without units with units.

From now on $\mathscr{C}$ denotes a (small) category. We denote the composition of two maps f, g by either gf or $g \circ f$, depending on readability; the order is the standard (functional, as opposed to diagrammatic) order. Given $X \in \mathscr{C}$, we will write either X or 1_X to represent the identity map on it, according to readability. We use the standard notation for the covariant representable functor associated with X, i.e., $h^X = \mathrm{Hom}_{\mathscr{C}}(X, -)$, and h_X for the contravariant representable $\mathrm{Hom}_{\mathscr{C}}(-, X)$.

The arguments in the following section need familiarity with Yoneda's Lemma: given a functor $F\colon \mathscr{C} \to \mathbf{Set}$ there is a natural bijective correspondence between $F(X)$ and the set of natural transformations $h^X \to F$.

2.1.1 Definition A category $\mathscr{C}$ *has tensors* if it is equipped with a bifunctor $- \otimes -$ with the usual associativity and symmetry isomorphisms

$$\begin{aligned} \mathsf{assoc}_{A,B,C}&\colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C \\ \mathsf{twist}_{A,B}&\colon A \otimes B \to B \otimes A \end{aligned}$$

that obey the usual "pentagon" and "hexagon" (see [15]).

Note that we do not ask for a unit in that definition. Nonetheless the "coherence" theorem for symmetric monoidal categories [14] does also hold in our case, or more precisely everything in it that does not deal with units. In particular, we can simply write $A \otimes B \otimes C \otimes D$ for $((A \otimes B) \otimes C) \otimes D$ or $A \otimes ((B \otimes C) \otimes D)$, or even $(B \otimes D) \otimes (A \otimes C)$, because there is a uniquely defined coherence isomorphism between any two of them.

If it exists, we denote the usual right adjoint to tensoring as $(-) \multimap (-)$ and it defines the usual bivariant bifunctor. We will denote the "internal representable functor" defined by X as $H^X = X \multimap (-)\colon \mathscr{C} \to \mathscr{C}$. The following two natural isomorphisms are trivial but important, and they are natural in both X and Y:

$$H^X H^Y \cong H^{X \otimes Y} \quad \text{and} \quad h^X H^Y \cong h^{X \otimes Y} . \tag{1}$$

It is very well known that a functor $\mathscr{C} \to \mathbf{Set}$ can be profitably seen as a "generalized object" of $\mathscr{C}$; we call such a thing a *virtual object* of $\mathscr{C}$ and we emphasize this fact by writing it as $h^{\mathbb{A}}$, which is a functor, and would be the representable functor associated to the object $\mathbb{A}$ if the latter only existed. Given $X \in \mathscr{C}$, maps $\mathbb{A} \to X$ should morally be in bijective correspondence with natural transformations $h^X \to h^{\mathbb{A}}$, and the latter are truly in bijective correspondence with elements of $h^{\mathbb{A}}(X)$ and this allows us to write an $s \in h^{\mathbb{A}}(X)$ as $\mathbb{A} \overset{s}{\dashrightarrow} X$. In general a dotted arrow will mean that at least one of the source or target is virtual, and it is to be interpreted as a reverse-direction natural transformation between the corresponding functors. For example, given $f\colon X \to Y$ and $t = (h^{\mathbb{A}} f)(s)$, we can write this as a commutative diagram

$$\begin{tikzcd} & \mathbb{A} \arrow[dl, dashed, "s"'] \arrow[dr, dashed, "t"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd} ,$$

which justifies the notation $t = f \circ s$, or simply $t = fs$. But we have to be very careful on how to extend the $\otimes, \multimap$ structure to virtuals. At least one thing works: given a virtual object $\mathbb{A}$ and a real one X we can define a virtual object $\mathbb{A} \otimes X$, by composing their "representables" (the reader should check that this makes perfect sense, by plugging an object of $\mathscr{C}$ in the functors): $h^{\mathbb{A} \otimes X} = h^{\mathbb{A}} H^X$.

So we can only *left*- tensor a virtual object, and only to get a virtual one.[1] This construction in natural in both variables: given $s\colon \mathbb{A} \dashrightarrow \mathbb{B}$ and $f\colon X \to Y$ then there is an

[1]More precisely: everything is a composition of functors, and there can be as many "internal representables" $\mathscr{C} \to \mathscr{C}$ as we want but exactly one $\mathscr{C} \to \mathbf{Set}$, which has to appear at leftmost end. But since we have a sym-

obvious $s \otimes f\colon \mathbb{A} \otimes X \dashrightarrow \mathbb{B} \otimes Y$. Suppose we have a "virtual left unit" $\mathbb{I}$; if it were real we would have a natural isomorphism $\lambda\colon \mathbb{I} \otimes (-) \cong (-)$; this translates, given a real $f\colon X \to X'$, as a commuting square

$$\begin{tikzcd} h^{X'} \arrow[r, "h^f"] \arrow[d, "h^{\lambda_{X'}}"', "\cong"] & h^X \arrow[d, "\cong"', "h^{\lambda_X}"] \\ h^{\mathbb{I}}H^{X'} \arrow[r, "h^{\mathbb{I}}H^f"'] & h^{\mathbb{I}}H^X \end{tikzcd}$$

Since this is a diagram of functors we can plug any map $Y \to Y'$ in there; it is then easy to see that having a "virtual left unit law" isomorphism is equivalent to having an isomorphism $\mathrm{Hom}_{\mathscr{C}}(X, Y) \cong h^{\mathbb{I}}(X \multimap Y)$, natural in both X and Y. This is the point of the whole exercise: a "proof" of an object X can be seen as an element of $h^{\mathbb{I}}(X)$ because a proof of $X \multimap Y$ will just be a map $X \to Y$.

The unit isomorphism in a monoidal category has to interact well with the associativity iso [15, p.159]; for this to happen the following is needed

$$\begin{tikzcd} h^{X\otimes Y} \arrow[r, "\cong"] \arrow[d, "h^{\lambda_{X\otimes Y}}"', "\cong"] & h^X H^Y \arrow[d, "\cong"', "h^{\lambda_X} H^Y"] \\ h^{\mathbb{I}}H^{X\otimes Y} \arrow[r, "\cong"'] & h^{\mathbb{I}}H^X H^Y \end{tikzcd} \tag{2}$$

along with one last axiom: we have to express that the unit laws hold *with the unit itself*: the two ways of going from $\mathbb{I}\otimes\mathbb{I}$ to $\mathbb{I}$ have to coincide. We cannot construct this directly; the equivalent condition for us is to require that for every $s\colon \mathbb{I} \dashrightarrow X$ and $t\colon \mathbb{I} \dashrightarrow Y$, the following diagram of (mostly) virtual maps commutes:

$$\begin{tikzcd} & X \arrow[r, dashed, "\cong"] & \mathbb{I} \otimes X \arrow[r, dashed, "t\otimes X"] & Y \otimes X \arrow[d, "\cong"] \\ \mathbb{I} \arrow[ur, dashed, "s"] \arrow[dr, dashed, "t"'] & & & \\ & Y \arrow[r, dashed, "\cong"'] & \mathbb{I} \otimes Y \arrow[r, dashed, "s\otimes Y"'] & X \otimes Y \end{tikzcd} \tag{3}$$

When all the above hold we have a uniquely defined $s \otimes t\colon \mathbb{I} \dashrightarrow X \otimes Y$. One can then show that the operation $s, t \mapsto s \otimes t\colon h^{\mathbb{I}}(X) \times h^{\mathbb{I}}(Y) \longrightarrow h^{\mathbb{I}}(X \otimes Y)$ agrees well with associativity and twist; in other words, given X, Y, Z with $s \in h^{\mathbb{I}}(X)$, $t \in h^{\mathbb{I}}(Y)$ and $r \in h^{\mathbb{I}}(Z)$ that $t \otimes s = \mathsf{twist}_{X,Y} \circ (s \otimes t)$ and $(s \otimes t) \otimes r = \mathsf{assoc}_{X,Y,Z} \circ (s \otimes (t \otimes r))$. This allows us to simply write $s \otimes t \otimes r\colon \mathbb{I} \dashrightarrow X \otimes Y \otimes Z$. In technical parlance $h^{\mathbb{I}}$ would be a monoidal functor $(\mathscr{C}, \otimes) \to (\mathbf{Set}, \times)$ if $\mathscr{C}$ had a unit (when $\mathscr{C}$ does have a unit $\mathbf{I}$ then $h^{\mathbf{I}}$ is always monoidal).

Notice that it is perfectly natural to write $s \otimes Y$ or $s \otimes 1_Y$ for the (real) horizontal map $Y \to X \otimes Y$ at the bottom of diagram (3). In the same way $X \otimes t$ or $1_X \otimes t$ can stand for the map $X \to X \otimes Y$ which is the top horizontal map followed by the twist.

metry we can play notational tricks; if the logic were non-commutative, we would have access to two implications, which would allow us to attain similar effects.

2.1.2 Definition A category $\mathscr{C}$ with tensors is an *autonomous category* if is has the structure in the previous paragraphs: the adjoint $\multimap$ and the functor $h^{\mathbb{I}}$ along with the natural iso $h^{\mathbb{I}}(X \multimap Y) \cong \mathrm{Hom}_{\mathscr{C}}(X, Y)$, which obeys Equations (2) and (3). The $\mathscr{C}$ is a **-autonomous category* if in addition it has a functor $(-)^{\perp}\colon \mathscr{C}^{op} \to \mathscr{C}$ which is an involution (for simplicity we will later assume that $X^{\perp\perp} = X$, but it could also be a natural isomorphism), and which obeys $X \multimap Y \cong (Y^{\perp} \otimes X)^{\perp}$.

2.1.3 Proposition *Assume that $\mathscr{C}$ is autonomous in the sense above. Then $\mathscr{C}$ is autonomous (SMC) in the usual sense (with the usual units) if and only if $h^{\mathbb{I}}$ is representable.*

In a *-autonomous category, we can define another bifunctor $- \invamp -$ (called *cotensor* or *par*) to be the de Morgan dual of $- \otimes -$, i.e., $X \invamp Y = (Y^{\perp} \otimes X^{\perp})^{\perp}$.[2] Then we have $X \multimap Y \cong X^{\perp} \invamp Y$.

If $\mathscr{C}$ is *-autonomous we also have a "virtual bottom", that we write $h_{\perp\!\!\!\perp}$, given by $h_{\perp\!\!\!\perp}(X) = h^{\mathbb{I}}(X^{\perp})$, and as for $h^{\mathbb{I}}$, thinking of it as an object $\perp\!\!\!\perp$ of $\mathscr{C}$ allows us to write

$$X \overset{s}{\dashrightarrow} \perp\!\!\!\perp$$

for an element $s \in h_{\perp\!\!\!\perp}(X)$. As before, we also get $u \invamp v \invamp w\colon X \invamp Y \invamp T \dashrightarrow \perp\!\!\!\perp$ for $u \in h_{\perp\!\!\!\perp}(X)$ and $v \in h_{\perp\!\!\!\perp}(Y)$ and $w \in h_{\perp\!\!\!\perp}(Z)$.[3]

Given maps $f\colon A \to B \invamp C$ and $g\colon A \otimes B^{\perp} \to C$ where g is the curryfication of f, we say that f and g are *transposes* of each other. More generally, for any objects $A_1, \ldots, A_n$, a map $f\colon A_1^{\perp} \otimes \cdots \otimes A_k^{\perp} \to A_{k+1} \invamp \cdots \invamp A_n$ uniquely determines a map $g\colon A_{p(1)}^{\perp} \otimes \cdots \otimes A_{p(l)}^{\perp} \to A_{p(l+1)} \invamp \cdots \invamp A_{p(n)}$, where $1 \le k, l < n$ and $p\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ is an arbitrary permutation. Obviously f determines in this way a whole family of maps, and we will call such a family an *equivariant family over* $A_1, \ldots, A_n$ [18, 11]. A member of such a family is called a *representative* and it determines the whole family. Given $A_1, \ldots, A_n$ and f as above we write $[\![f]\!]$ to denote the equivariant family determined by f. If we let $l = 0$ in the situation above, we get $\hat{f}\colon \mathbb{I} \dashrightarrow A_1 \invamp \cdots \invamp A_n$, that we call the *name of the equivariant family*. For $l = n$, we get its *coname* $\check{f}\colon A_1^{\perp} \otimes \cdots \otimes A_n^{\perp} \dashrightarrow \perp\!\!\!\perp$. Important examples are the name and the coname of the identity:

$$\mathbb{I} \overset{\hat{1}_A}{\dashrightarrow} A^{\perp} \invamp A \qquad \text{and} \qquad A \otimes A^{\perp} \overset{\check{1}_A}{\dashrightarrow} \perp\!\!\!\perp$$

If we transpose the identity $1_{B \invamp C}\colon B \invamp C \to B \invamp C$, we get the evaluation map $\mathsf{eval}\colon (B \invamp C) \otimes C^{\perp} \to B$. Taking the tensor of this with $1_A\colon A \to A$ and transposing back gives us a map $\mathsf{switch}\colon A \otimes (B \invamp C) \to (A \otimes B) \invamp C$, that

[2] Most of the times we will invert the order when taking the negation, but not always.

[3] Strictly speaking we should use different arrows shape to denote these virtual maps, because they deal with contravariant functors to $\mathbf{Set}$ and not covariant ones, and the two kinds cannot be mixed at all. But there is no risk of such a thing happening here, given the quite conservative use we make of this notation.

is natural in all three arguments, and that we call the *switch map* (like [7, 2] and unlike [4, 3]). For the sake of simplicity (and since we are working in the symmetric world), we will also use switches that are obtained by composing with the twistmap (for $\otimes$ as well as for $\bindnasrepma$). In a similar way we obtain the maps $\mathsf{tens}\colon (A \bindnasrepma B) \otimes (C \bindnasrepma D) \to A \bindnasrepma (B \otimes C) \bindnasrepma D$ and $\mathsf{cotens}\colon A \otimes (B \bindnasrepma C) \otimes D \to (A \otimes B) \bindnasrepma (C \otimes D)$. Note that they are dual to each other and that they both can be obtained by composing two switches. Switch is self-dual.

2.2. Weak units

2.2.1 Definition Let $\mathscr{C}$ be autonomous in the sense above. A *weak unit* in $\mathscr{C}$ is a pair $(\mathbf{I}, e)$ where $e\colon \mathbf{I} \to \mathbf{I}$ is an itempotent map such that splitting h^e in $\mathbf{Set}^{\mathscr{C}}$ gives $h^{\mathbb{I}}$:

$$h^{\mathbf{I}} \longrightarrow h^{\mathbb{I}} \longrightarrow h^{\mathbf{I}} \tag{4}$$

It is well-known that composing with an idempotent is a process of normalization. Let X, Y and $s\colon \mathbf{I} \to X \multimap Y$ be given. We can always normalize s by taking se, and we can say that s is in normal form if $s = se$. The definition above says that there is a natural bijective correspondence between the maps $X \to Y$ and the maps $\mathbf{I} \to X \multimap Y$ that are in normal form. For any X we can transform the virtual maps into real ones, in the following way:

$$\begin{tikzcd}
\mathbf{I} \otimes X \arrow[r, dotted] \arrow[dr, "\ell_X"'] & \mathbb{I} \otimes X \arrow[r, dotted] \arrow[d, dotted, "\lambda_X"] & \mathbf{I} \otimes X \\
& X \arrow[ur, "\ell^*_X"'] &
\end{tikzcd}$$

thus getting two maps ℓ_X, ℓ^*_X with $\ell_X \ell^*_X = 1_X$ and $\ell^*_X \ell_X = e \otimes X$. These are obviously natural in X. The virtual map $\hat{\mathbf{I}}\colon \mathbb{I} \dashrightarrow \mathbf{I}$ induced by (4) is called the *canonical proof of the* $\mathbf{I}$.

Weak units can be used to give "elementary" axiomatization of the ideas of the previous section; we can even define the concept of a "weakly monoidal category", where the unit isomorphism would be replaced by an embedding-retraction pair; it is easy to tweak the standard axioms for that purpose. But they are highly non-canonical: as soon as we have a weak unit we can construct many other weak units from it. Also, having weak units is the same as saying that splitting the idempotents in $\mathscr{C}$ [13] would give us an ordinary symmetrical monoidal closed category.

Notice that an autonomous category can have weak units as well as real ones at the same time. What matters is which one is denoted by $\mathbf{I}$.

A functor between autonomous categories should preserve everything on the nose; this cannot entirely achieved here because of the $h^{\mathbb{I}}$ functor. So given $\mathscr{C}$ and $\mathscr{D}$ autonomous categories we define an autonomous functor $\mathscr{C} \to \mathscr{D}$ to be a pair (F, α) where $F\colon \mathscr{C} \to \mathscr{D}$ is a functor that preserves $\otimes, \multimap$ on the nose and α is a natural isomorphism $h^{\mathbb{I}}_{\mathscr{D}} \circ F \to h^{\mathbb{I}}_{\mathscr{C}}$. If a (weak) unit $(\mathbf{I}, e)$ is defined, we ask F to preserve both the object and the idempotent (if $\mathbf{I}$ is a real unit, e is just the $1_{\mathbf{I}}$).

2.3. Going Classical

Let now $\mathscr{C}$ be *-autonomous. We will change the notation, and use $-\wedge-$ for the tensor and $-\vee-$ for the cotensor. The virtual unit and virtual bottom will be denoted by $\mathbb{t}$ and $\mathbb{f}$, called *virtual truth* and *virtual falsehood*, respectively. In case there are actual objects in the category playing the roles of the units (or weak units), they are denoted by $\mathbf{t}$ and $\mathbf{f}$, respectively. Notice that both, $- \wedge -$ and $- \vee -$, come with their own associativity and twist isos (see Definition 2.1.1); but we will in both cases simply write assoc and twist. The dual of an object A will be denoted $\bar{A}$.

Unsurprisingly, $\wedge$-comonoids and $\vee$-monoids are going to be important. But since we do not have real units for $\wedge, \vee$, we need to adapt the standard definitions of (co)monoid. In order to define the counit to a $\wedge$-comonoid X, which should be a map $X \to \mathbf{t}$ we (unsurprisingly) replace it by a natural transformation $\Pi^X\colon h^{\mathbb{t}} \to h^X$, which we call an X*-pre-projection*. Suppose $A \in \mathscr{C}$. We can construct

$$h^A \xrightarrow{\ \cong\ } h^{\mathbb{t}} H^A \xrightarrow{\ \Pi^X H^A\ } h^X H^A \xrightarrow{\ \cong\ } h^{X \wedge A} \quad ,$$

where the first iso comes from Definition 2.1.2 and the second iso is (1). By Yoneda we get a map $\Pi^X_A\colon X \wedge A \to A$ which is natural in A, i.e., for $f\colon A \to B$,

$$\begin{tikzcd}
X \wedge A \arrow[r, "\Pi^X_A"] \arrow[d, "X \wedge f"'] & A \arrow[d, "f"] \\
X \wedge B \arrow[r, "\Pi^X_B"'] & B
\end{tikzcd} \tag{5}$$

commutes, and thus an X-pre-projection can be seen as natural transformation $\Pi^X\colon X \wedge (-) \to (-)$.

2.3.1 Definition A *cocommutative* $\wedge$*-comonoid* in $\mathscr{C}$ is a triple (X, Δ_X, Π^X) such that $\Delta_X\colon X \to X \wedge X$ is coassociative and cocommutative, i.e.,

$$\begin{aligned} (X \wedge \Delta_X) \circ \Delta_X &= \mathsf{assoc}_{X,X,X} \circ (\Delta_X \wedge X) \circ \Delta_X \\ \Delta_X &= \mathsf{twist}_{X,X} \circ \Delta_X \quad , \end{aligned} \tag{6}$$

and such that $\Pi^X\colon h^{\mathbb{t}} \to h^X$ obeys

$$\Pi^X_X \circ \Delta_X \;=\; 1_X\colon X \to X\ . \tag{7}$$

2.3.2 Definition A *pre-*K*-autonomous category* is a *-autonomous category $\mathscr{K}$, in which every object X is equipped with a cocommutative $\wedge$-comonoid structure (X, Δ_X, Π^X) such that for all A, B, X, Y, we have

$$\begin{tikzcd}
& X \wedge Y \arrow[dl, "\Delta_X \wedge \Delta_Y"'] \arrow[dr, "\Delta_{X \wedge Y}"] & \\
X \wedge X \wedge Y \wedge Y \arrow[rr, "X \wedge \mathsf{twist}_{X,Y} \wedge Y"'] & & X \wedge Y \wedge X \wedge Y
\end{tikzcd} \tag{8}$$

and

$$\Pi^X_A \wedge 1_B \;=\; \Pi^X_{A \wedge B}\colon X \wedge A \wedge B \to A \wedge B\ . \tag{9}$$

and such that all isos preserve this $\wedge$-comonoid structure.

We call Δ_X and Π^X the *diagonal* and *projection* on X. By duality we also have maps $\nabla_X : X \vee X \to X$, called *codiagonal*, and a natural transformation $\amalg^X : (-) \to (-) \vee X$, which we call the *coprojection*, and they give an associative, commutative $\vee$-monoid structure on X, in an obvious sense, slightly different from the standard definition, obeying the dual of equations (8) and (9).

A word on notation: we write $\Pi_A^{\square X}$ for the map $A \wedge X \to A$ obtained by precomposing Π_A^X with the twistmap. In the same line of thought, $\Pi_A^{X\square}$ is just Π_A^X, and more generally, an expression like $\Pi_{A,B}^{X\square Y\square Z}$ is the uniquely defined composite projection $X \wedge A \wedge Y \wedge B \wedge Z \to A \wedge B$. Uniqueness follows from the commutativity of

$$\begin{tikzcd}
A \wedge (X \wedge B) \arrow[rr, "\mathsf{assoc}"] \arrow[dr, "A\wedge\Pi_B^{X\square}"'] & & (A \wedge X) \wedge B \arrow[dl, "\Pi_A^{\square X}\wedge B"] \\
& A \wedge B &
\end{tikzcd} \quad , \tag{10}$$

which is an immediate consequence of (9). By duality, for every A, X there are $\amalg_A^{\square X} : A \to A \vee X$ and $\amalg_A^{X\square} : A \to X \vee A$ which are natural in A. We write $\amalg_A^X$ for $\amalg_A^{\square X}$.

2.3.3 Definition In a pre-K-autonomous category a map $f : X \to Y$ is called *cloneable*, if

$$(f \wedge f) \circ \Delta_X = \Delta_Y \circ f \quad \text{and} \quad f \circ \nabla_X = \nabla_Y \circ (f \vee f) \ .$$

The map f is a *quasientropy* if

$$\begin{tikzcd}
X \wedge A \arrow[rr, "f\wedge 1_A"] \arrow[dr, "\Pi_A^X"'] & & Y \wedge A \arrow[dl, "\Pi_A^Y"] \\
& A &
\end{tikzcd} \quad \text{and} \quad \begin{tikzcd}
& A \arrow[dl, "\amalg_A^X"'] \arrow[dr, "\amalg_A^Y"] & \\
A \vee X \arrow[rr, "1_A\vee f"'] & & A \vee Y
\end{tikzcd}$$

both commute for every A.

2.3.4 Definition

- A $\mathsf{K}^\flat$-*autonomous category* is a pre-K-autonomous category in which Δ, Π, and switch are quasientropies, and quasientropies are closed under $\wedge$ and $\vee$.
- It is a $\mathsf{K}^\natural$-*autonomous category* if the usual units are present and the comonoid structure on $\mathbf{t}$ is the standard degenerate one, obtained from the coherence isos.
- We speak of a $\mathsf{K}^\sharp$-*autonomous category* if the units are weak; we change the preceding condition with the requirement that $\ell_X = \Pi_X^{\mathbf{t}} : \mathbf{t} \wedge X \longrightarrow X$ and $\Delta_{\mathbf{t}} \circ \hat{\mathbf{t}} = \hat{\mathbf{t}} \wedge \hat{\mathbf{t}} : \mathbb{t} \dashrightarrow \mathbf{t} \wedge \mathbf{t}$, where $\hat{\mathbf{t}}$ is the *canonical proof of* $\mathbf{t}$.
- If p is any of $\flat, \natural, \sharp$, we defined a K^p-functor to be an autonomous functor that also preserves negation on the nose, and the obvious monoid and comonoid structures.

We simply say K-autonomous category if the discussion is independent from the units. Thus in a K-autonomous category $\mathscr{K}$, the subcategory $\mathsf{Q}\mathscr{K}$ of quasientropies (with the same objects) inherits the two monoidal structures, switch, and also the involution. It is not *-autonomous in general, but it is weakly distributive [3].

Given two objects A and X, we define $\Lambda_A^X : A \wedge \bar{A} \to X$ by transposing $\amalg_A^{X\square} : A \to X \vee A$, and $\mathrm{V}_A^X : X \to \bar{A} \vee A$ by transposing $\Pi_A^{\square X} : A \wedge X \to A$.

2.3.5 Proposition *For any A, B, X, the map $\mathrm{V}_B^X \circ \Lambda_A^X : A \wedge \bar{A} \to \bar{B} \vee B$ is independent from X.*

Proof: Look at the following:

$$\begin{tikzcd}
& X \arrow[dr, "\mathrm{V}_B^X"] & \\
A \wedge \bar{A} \arrow[ur, "\Lambda_A^X"] \arrow[r, "\Lambda_A^{X\wedge Y}"] \arrow[dr, "\Lambda_A^Y"'] & X \wedge Y \arrow[u, "\Pi_X^{\square Y}"'] \arrow[d, "\Pi_Y^{X\square}"] \arrow[r, "\mathrm{V}_B^{X\wedge Y}"'] & \bar{B} \vee B \\
& Y \arrow[ur, "\mathrm{V}_B^Y"'] &
\end{tikzcd}$$

Taking their transposes, we see that the left triangles commute because projections are quasientropies; and right triangles because projections commute with projections. $\square$

By doing a double transposition on $\mathrm{V}_A \circ \Lambda_B : B \wedge \bar{B} \to \bar{A} \wedge A$ we get the *mix map* $\mathsf{mix}_{A,B} : A \wedge B \to A \vee B$.

2.3.6 Proposition *The following is equal to* $\mathsf{mix}_{A,B}$

$$A \wedge B \xrightarrow{A\wedge\amalg_B^{X\square}} A \wedge (X \vee B) \xrightarrow{\mathsf{switch}} (A \wedge X) \vee B \xrightarrow{\Pi_A^{\square X}\vee B} A \vee B$$

Proof: Transpose $\mathrm{V}_A \circ \Lambda_B$ twice and use the definition of switch. $\square$

From this we get immediately:

2.3.7 Proposition *The map* $\mathsf{mix}_{A,B} : A \wedge B \to A \vee B$ *is natural in A and B.*

It is also very easy to see that mix agrees with the twistmap, i.e.,

$$\begin{tikzcd}
A \wedge B \arrow[r, "\mathsf{mix}_{A,B}"] \arrow[d, "\mathsf{twist}"'] & A \vee B \arrow[d, "\mathsf{twist}"] \\
B \wedge A \arrow[r, "\mathsf{mix}_{B,A}"'] & B \vee A
\end{tikzcd} \tag{11}$$

This gives us a unique map $f \between g : A \wedge B \to C \vee D$, which we call the *disjoint sum* of f and g. This operation is obviously stable under transposes:

2.3.8 Proposition *Le $f : A \wedge B \to C$ and $f' : A' \wedge B' \to C'$ be given, and let $g : B \to \bar{A} \vee C$ and $g' : B' \to \bar{A}' \vee C'$ be their transposes, respectively. Then $g \between g' : B \wedge B' \to A \vee C \vee A' \vee C'$ is a transpose of $f \between f' : A \wedge B \wedge A' \wedge B' \to C \vee C'$.*

We also have the following:

2.3.9 Proposition *In a* K*-autonomous category, the map* $\mathsf{mix}_{A,B}$ *is a quasientropy for every A and B.*

2.3.10 Proposition *Given A, B, and C, then the following commutes:*

$$\begin{tikzcd}
A \wedge (B \wedge C) \arrow[r, "A\wedge\mathsf{mix}_{B,C}"] \arrow[d, "\mathsf{assoc}"'] & A \wedge (B \vee C) \arrow[r, "\mathsf{mix}_{A,B\vee C}"] \arrow[d, "\mathsf{switch}"] & A \vee (B \vee C) \arrow[d, "\mathsf{assoc}"] \\
(A \wedge B) \wedge C \arrow[r, "\mathsf{mix}_{A\wedge B,C}"'] & (A \wedge B) \vee C \arrow[r, "\mathsf{mix}_{A,B\vee C}"'] & (A \vee B) \vee C
\end{tikzcd}$$

For the aficionados, this means that mix would furnish the necessary structure for identity to be a monoidal functor $(\mathscr{K}, \wedge) \to (\mathscr{K}, \vee)$—if we had units, naturally. A consequence of this is that there is a unique way to define a natural n-ary mix map

$$\mathsf{mix}_{A_1,\ldots,A_n} = 1_{A_1} \between \ldots \between 1_{A_n} \colon \\ A_1 \wedge \cdots \wedge A_n \longrightarrow A_1 \vee \cdots \vee A_n \ .$$

Let $f, g \colon A \to B$ be given. We define

$$f + g = \nabla_B \circ (f \between g) \circ \Delta_A \colon A \to B \quad .$$

It is easy to show, using (co)-associativity and (co)-commutativity of Δ and ∇, along with naturality of mix, that the operation $+$ on maps is associative and commutative. Thus every $\mathrm{Hom}_{\mathscr{C}}(A, B)$ has a commutative semigroup structure. In the view of Proposition 2.3.8 this semigroup structure is also present for $h^{\mathbf{t}}(X)$. For $h, k \colon \mathbf{t} \dashrightarrow X$ define $h + k = \nabla_X \circ (h \between k) \colon \mathbf{t} \dashrightarrow X$, where $h \between k = \mathsf{mix}_{X,X} \circ (h \wedge k)$. It immediately follows that $\widehat{f+g} = \hat{f} + \hat{g} \colon \mathbf{t} \dashrightarrow \bar{A} \vee B$, where $f, g \colon A \to B$.

2.3.11 Proposition *Let $f, g \colon A \to B$ and $h, k \colon B \to C$. If h is cloneable, then $h \circ (f + g) = hf + hg$, and if f is cloneable then $(h + k) \circ f = hf + kf$.*

Proof: Immediately from the definitions. □

Note that it does *not* follow that $\mathscr{K}$ is enriched over commutative semigroups.

2.3.12 Proposition *Let $f \colon A \to C$ and $g \colon B \to D$ be given. Then $f \between g = (\amalg_C^{\square D} \circ f \circ \Pi_A^{\square B}) + (\amalg_D^{C\square} \circ g \circ \Pi_B^{A\square})$.*

2.4. Going graphical

Let $\mathscr{K}$ be a K-autonomous category. We define $\mathscr{K}^{\oplus}$ to be the category obtained from $\mathscr{K}$ by formally inverting the mix maps. In other words, for every pair of objects A, B we add a map $\mathsf{mix}^{-1}_{A,B} \colon A \vee B \to A \wedge B$ such that $\mathsf{mix}_{A,B} \circ \mathsf{mix}^{-1}_{A,B} = 1_{A \vee B}$ and $\mathsf{mix}^{-1}_{A,B} \circ \mathsf{mix}_{A,B} = 1_{A \wedge B}$. Looking at the diagram in Proposition 2.3.10 we get a new diagram whose horizontal arrows now go in the reverse direction. This new diagram also commutes for trivial reason; thus it identifies the two associativities and switch. In the same way, the horizontal arrows in (11) can be inverted. The outcome of this is that not only are the bifunctors $\wedge$ and $\vee$ identified in $\mathscr{K}^{\oplus}$, but that this new bifunctor $\oplus$ inherits *a single* symmetric monoidal structure from its two parents: they are identified too.

For trivial reasons the following diagram commutes:

$$\begin{tikzcd} & A \oplus B \arrow[dd, "\mathsf{twist}"] \arrow[dr, "\Pi_B^{A\square}"] & \\ A \arrow[ur, "\amalg_A^{\square B}"] \arrow[dr, "\amalg_A^{B\square}"'] & & B \\ & B \oplus A \arrow[ur, "\Pi_B^{\square A}"'] & \end{tikzcd} \quad . \tag{12}$$

This uniquely determines a map $0_{A,B} \colon A \to B$, that we call the *zero map*. The following is almost trivial.

2.4.1 Proposition *The map $0_{A,B}$ is a quasientropy. For every map $f \colon A \to B$, we have $f + 0_{A,B} = f$. And for every quasientropy $f \colon B \to C$, we have $0_{C,D} \circ f = 0_{B,D}$ and $f \circ 0_{A,B} = 0_{A,C}$.*

2.4.2 Proposition *In $\mathscr{K}^{\oplus}$ the diagram*

$$\begin{tikzcd} A \arrow[r, shift left, "\amalg_A^{\square B}"] & A \oplus B \arrow[l, shift left, "\Pi_A^{\square B}"] \arrow[r, shift right, "\Pi_B^{A\square}"'] & B \arrow[l, shift right, "\amalg_B^{A\square}"'] \end{tikzcd} \tag{13}$$

obeys the standard biproduct equations, i.e.,

$$\begin{aligned} 1_{A \oplus B} &= \amalg_A^{\square B} \Pi_A^{\square B} + \amalg_B^{A\square} \Pi_B^{A\square} \\ 1_A &= \Pi_A^{\square B} \amalg_A^{\square B} \\ 1_B &= \Pi_B^{A\square} \amalg_B^{A\square} \end{aligned}$$

Proof: The first equation is a direct consequence of Proposition 2.3.12. The other two equations are trickier:

$$\begin{aligned} 1_A &= \Pi_A^{\square A} \circ \Delta_A \\ &= \Pi_A^{\square B} \circ (A \oplus 0_{A,B}) \circ \Delta_A \\ &= \Pi_A^{\square B} \circ (A \oplus \Pi_B^{A\square}) \circ (A \oplus \amalg_A^{\square B}) \circ \Delta_A \\ &= \Pi_A^{\square B} \circ (\Pi_A^{\square A} \oplus B) \circ (A \oplus \amalg_A^{\square B}) \circ \Delta_A \\ &= \Pi_A^{\square B} \circ \amalg_A^{\square B} \circ \Pi_A^{\square A} \circ \Delta_A \\ &= \Pi_A^{\square B} \circ \amalg_A^{\square B} \circ 1_A \\ &= \Pi_A^{\square B} \circ \amalg_A^{\square B} \end{aligned}$$

The first equation is (7), the second one uses that $0_{A,B}$ is a quasientropy, the third one is the definition of $0_{A,B}$, the fourth one is (10), the fifth one is naturality of $\amalg^{\square B}$, and the sixth is again (7). □

Notice that this does *not* mean that $\mathscr{K}^{\oplus}$ has biproducts; the semigroup enrichment would be necessary for this.

If we transpose $0_{A,B}$ and compose with the projection, we get a (virtual) map

$$\mathbf{t} \overset{\hat{0}_{A,B}}{\dashrightarrow} \bar{A} \oplus B \xrightarrow{\Pi_B^{\square \bar{A}}} B \quad , \tag{14}$$

that we denote by 0_B. Clearly this is independent from A. By duality we get $B \dashrightarrow \mathbf{f}$, which we also denote by 0_B.

2.4.3 Definition The category $\mathscr{K}^{\oplus}$ is said to be *contractible* if the following commute for all X, Y, and A:

$$\begin{tikzcd} X \arrow[dd, "0_{X,Y}"'] \arrow[dr, "\mathrm{V}_A^X"] & \\ & \bar{A} \oplus A \arrow[dl, "\Lambda_A^Y"] \\ Y & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} A \arrow[r, "A \oplus \hat{1}_{\bar{A}}"] \arrow[dd, "1_A"'] & A \oplus A \oplus \bar{A} \arrow[d, "\nabla_{A \oplus \bar{A}}"] \\ & A \oplus \bar{A} \arrow[d, "\Delta_{A \oplus \bar{A}}"] \\ A & A \oplus A \oplus \bar{A} \arrow[l, "A \oplus \check{1}_A"] \end{tikzcd}$$

2.4.4 Definition Let $\mathscr{K}$ be a K-autonomous category. We say that $\mathscr{K}$ is *graphical* if $\mathscr{K}^{\oplus}$ is contractible and the canonical functor $\mathsf{G}_{\mathscr{K}} \colon \mathscr{K} \to \mathscr{K}^{\oplus}$ is faithful. We say that $\mathscr{K}$ is *purely graphical* if additionally $\mathsf{G}_{\mathscr{K}}$ is full.

Graphicality is quite a powerful property. One can easily show that in a graphical K-autonomous category, the maps

Π, $\amalg$, and switch are cloneable, and that the cloneable maps are closed under $\wedge$ and $\vee$.[4] But note that it does not follow that Δ and ∇ are cloneable.[5] Full graphicality is an even more powerful property, since it enters the realm of degeneracy: it obviously identifies $\wedge, \vee$ in $\mathscr{K}$. But it is useful technically.

2.4.5 Definition A K-autonomous category $\mathscr{K}$ is called Δ-∇-*strong* if Δ and ∇ are cloneable.

2.4.6 Remark In a graphical K-autonomous category which is Δ-∇-strong, the subcategory $\mathsf{CQ}\mathscr{K}$ of cloneable quasientropies behaves quite nicely: not only is it weakly distributive, in addition, since every object is equipped with both a monoid and comonoid structure *which is preserved by every map*, the category has binary products and coproducts, and the semigroup structure on the hom-sets of $\mathsf{CQ}\mathscr{K}$ is an enrichment, in the usual sense. This works in reverse: the properties just stated suffice to show Δ-∇-strength and graphicality [4].

The action of inverting the mix maps introduces some amount of degeneracy, which creates a "meeting ground" for the "higher-order" (*-autonomous) and the "structural" (monoids and comonoids) structures. Given the right additional axioms (like Δ-∇-strength) this meeting ground turns out to be a *familiar place*.

2.4.7 Theorem *In a graphical* K*-autonomous category which is* Δ-∇*-strong, we have that* $1_A + 1_A = 1_A$.

Proof: We show the statement for $\mathscr{K}^{\oplus}$. By graphicality it follows for $\mathscr{K}$.

$$
\begin{aligned}
1_A &= (A \oplus \check{1}_A) \circ (\Delta_A \oplus \bar{A}) \circ (\nabla_A \oplus \bar{A}) \circ (A \oplus \hat{1}_{\bar{A}}) \\
&= (A \oplus \check{1}_A) \circ (\nabla_A \oplus \nabla_A \oplus \bar{A}) \circ (A \oplus \mathsf{twist} \oplus A \oplus \bar{A}) \\
&\quad \circ (\Delta_A \oplus \Delta_A \oplus \bar{A}) \circ (A \oplus \hat{1}_{\bar{A}}) \\
&= (A \oplus \check{1}_A \oplus \check{1}_A) \circ (A \oplus A \oplus \mathsf{twist} \oplus \bar{A}) \\
&\quad \circ (\nabla_A \oplus A \oplus A \oplus \Delta_{\bar{A}}) \circ (A \oplus \mathsf{twist} \oplus A \oplus \bar{A}) \\
&\quad \circ (\Delta_A \oplus A \oplus A \oplus \nabla_{\bar{A}}) \\
&\quad \circ (A \oplus A \oplus \mathsf{twist} \oplus \bar{A}) \circ (A \oplus \hat{1}_{\bar{A}} \oplus \hat{1}_{\bar{A}}) \\
&= (A \oplus \check{1}_A) \circ (\nabla_A \oplus A \oplus \bar{A}) \circ (A \oplus \mathsf{twist} \oplus \bar{A}) \\
&\quad \circ (\Delta_A \oplus A \oplus \bar{A}) \circ (A \oplus \hat{1}_{\bar{A}}) \\
&= 1_A \circ \nabla_A \circ (1_A \oplus 1_A) \circ \Delta_A \circ 1_A \\
&= 1_A + 1_A
\end{aligned}
$$

The first equation is just Definition 2.4.3. The second one is Δ-∇-strength together with (8). The third equation uses that Δ and ∇ are dual. The fourth equation uses again the right diagram in Definition 2.4.3, and the fifth equation is a twisted form of $1_A = (\check{1}_A \vee A) \circ \mathsf{switch} \circ (A \wedge \hat{1}_A)$ which holds in every *-autonomous category. $\square$

[4] In fact, for showing these facts, a much weaker property than graphicality (the presence of a *medial map* [2]) is sufficient. But since graphicality implies medial and is needed anyway, we do not deal with medial in this paper.

[5] We do not need this fact here and a proof of it would go beyond the scope of this paper.

Note that this proof does not make any use of the projections nor the notion of quasientropy, i.e., is independent from the treatment of the units.

2.4.8 Corollary *In a graphical* $\mathsf{K}^\sharp$*-autonomous category which is* Δ-∇*-strong, we have that* $\hat{\mathbf{t}} + \hat{\mathbf{t}} = \hat{\mathbf{t}}$.

2.4.9 Definition A K-autonomous category is *idempotent* if $f + f = f$ for every map f.

In such a category every Hom has a semilattice structure.

Note that Theorem 2.4.7 does *not* imply that a graphical and Δ-∇-strong K-autonomous category is idempotent. However by an inductive argument, which is implicitely contained in the construction of the next section, one can show that the *free* graphical Δ-∇-strong K-autonomous category is idempotent.

3. Proof nets

We will recall the notion of proof nets that has been introduced in [12]. We consider only the case of $\mathbb{B}$-nets.

3.1. Cut-free prenets

For a given set $\mathscr{A} = \{a, b, c, \ldots\}$ of *propositional variables*, the set of $\mathsf{K}^\sharp$*-formulas* over $\mathscr{A}$ is generated from the set $\mathscr{A} \cup \bar{\mathscr{A}} \cup \{\mathbf{t}, \mathbf{f}\}$ via the binary connectives $\wedge$ (*conjunction*) and $\vee$ (*disjunction*). Here $\bar{\mathscr{A}} = \{\bar{a}, \bar{b}, \bar{c}, \ldots\}$ is the set of *negated propositional variables*, and $\mathbf{t}$ and $\mathbf{f}$ are the *constants* representing "true" and "false", respectively. The elements of the set $\mathscr{A} \cup \bar{\mathscr{A}} \cup \{\mathbf{t}, \mathbf{f}\}$ are called *atoms*. The formulas in which the constants do not appear are called $\mathsf{K}^\flat$*-formulas*. A finite list of formulas $\Gamma = A_1, A_2, \ldots, A_n$ is called a *sequent*. We will consider formulas as binary trees (and sequents as forests), whose leaves are decorated by atoms, and whose inner nodes are decorated by the connectives. Given a formula A or a sequent Γ, we write $\mathscr{L}(A)$ or $\mathscr{L}(\Gamma)$, respectively, to denote its set of leaves. For simplicity, we will suppose, that this is actually the set $\{1, \ldots, n\}$ if there are n leaves. We can accomplish this by agreeing that for example if $\mathscr{L}(A) = \{1, \ldots, n\}$ and $\mathscr{L}(B) = \{1, \ldots, m\}$, then $\mathscr{L}(A \wedge B) = \{1, \ldots, n+m\}$ with $\mathscr{L}(A)$ and $\mathscr{L}(B)$ embedded as complementary subsets $\{1, \ldots, n\}$ and $\{n+1, \ldots, n+m\}$. We will write a_u to say that the leaf u is decorated by the atom a. If no ambiguity is possible, we will omit the index or the decoration, i.e., just write a or u for a_u.

We define the negation $\bar{A}$ of a formula A as follows:

$$
\begin{array}{lll}
\bar{\bar{a}} = a & \bar{\mathbf{t}} = \mathbf{f} & \overline{(A \wedge B)} = \bar{B} \vee \bar{A} \\
\bar{a} = \bar{a} & \bar{\mathbf{f}} = \mathbf{t} & \overline{(A \vee B)} = \bar{B} \wedge \bar{A}
\end{array}
\qquad (15)
$$

Here a ranges over the set $\mathscr{A}$, and there is a slight abuse of notation. However, from now on we will use a to denote an arbitrary atom (including constants), and $\bar{a}$ to denote its negation according to (15). Note that (15) implies $\bar{\bar{A}} = A$ for all A.

3.1.1 Definition A *linking* for a sequent Γ is an undirected graph P whose set of vertices is $\mathscr{L}(\Gamma)$ and whose set of edges obeys the following condition: whenever there is an edge between two leaves $u, v \in \mathscr{L}(\Gamma)$, denoted as $u \frown v$, then one of the following two cases holds:

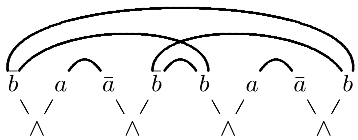
- either, u is decorated by an atom a and v by its dual $\bar{a}$,
- or, $u = v$ and it is decorated by $\mathbf{t}$.

A *prenet*[6] consists of a sequent Γ and a linking P for it. It will be denoted by $P \rhd \Gamma$.

Since no ambiguity is possible, we will identify a linking with its set of edges. Here is an example:

$$\begin{array}{c} \{\, \bar{b}_1 \frown b_5 \,,\, \bar{b}_1 \frown b_8 \,,\, \bar{b}_4 \frown b_5 \,,\, \bar{b}_4 \frown b_8 \,,\, a_2 \frown \bar{a}_3 \,,\, a_6 \frown \bar{a}_7 \,\} \\ \triangledown \\ \bar{b}_1 \wedge a_2, \bar{a}_3 \wedge \bar{b}_4, b_5 \wedge a_6, \bar{a}_7 \wedge b_8 \end{array} \tag{16}$$

One can draw it in the proof net tradition as



, (17)

as it has been done in [12].

On the set of prenets we define the following two operations: Let $P \rhd \Gamma$ and $Q \rhd \Gamma$ and $R \rhd \Theta$ be given. Then $(P + Q) \rhd \Gamma$ is obtained by taking the union of the two graphs P and Q (the set of vertices does not change), and $(P \oplus R) \rhd \Gamma, \Theta$ is obtained by taking the disjoint union of the two graphs (i.e., they are simply put next to each other).

Let $P \rhd \Gamma$ be a prenet and $L \subseteq \mathscr{L}(\Gamma)$ an arbitrary subset of leaves. Then $P|_L$ denotes the subgraph of P induced by L. We also have a subforest $\Gamma' = \Gamma|_L$ of Γ, whose set of leaves is precisely L and such that an inner node s of Γ is in $\Gamma|_L$ if *one or two* of its children is in $\Gamma|_L$. We will say that $P|_L \rhd \Gamma'$ is a *sub-prenet* of $P \rhd \Gamma$. Since this sub-prenet is entirely determined by Γ', we can also write it as $P|_{\Gamma'} \rhd \Gamma'$ without mentioning L any further.

3.2. Cuts and cut elimination

A *cut* is a formula of the shape $A \between \bar{A}$, where $\between$ is called the cut connective. It is allowed only at the root of a formula tree. A *prenet with cuts* is a prenet $P \rhd \Gamma$, where Γ may contain cuts. On these, the cut reduction relation $\to$ is defined by

$$\begin{array}{rcl} P \rhd (A \wedge B) \between (\bar{B} \vee \bar{A}), \Gamma & \to & P \rhd A \between \bar{A}, B \between \bar{B}, \Gamma \\ P \rhd a_u \between \bar{a}_v, \Gamma & \to & (P|_\Gamma + Q) \rhd \Gamma \end{array}$$

where

$$\begin{array}{rcl} Q & = & \{\, i \frown j \mid i, j \in \mathscr{L}(\Gamma) \text{ and } i \frown u \,,\, v \frown j \in P \,\} \cup \\ & & \{\, i \frown i \mid i \in \mathscr{L}(\Gamma) \text{ and } i \frown u \,,\, v \frown v \in P \,\} \cup \\ & & \{\, j \frown j \mid j \in \mathscr{L}(\Gamma) \text{ and } u \frown u \,,\, v \frown j \in P \,\} \end{array}$$

If we think of graphs as matrices, this definition is a version of the execution formula in the Geometry of Interaction.

3.2.1 Theorem *The cut reduction relation on prenets is confluent and terminating.*

Proof: See [12]. □

3.3. Prenet categories

An important consequence of Theorem 3.2.1 is that we can construct a category of prenets: The objects are the formulas and the arrows are the two-conclusion prenets. More precisely, any prenet $P \rhd \bar{A}, B$ is an arrow from A to B. The composition of two arrows $P \rhd \bar{A}, B$ and $Q \rhd \bar{B}, C$ is defined by eliminating the cut from $P \oplus Q \rhd \bar{A}, B \between \bar{B}, C$. Identity maps are given by the obvious prenets.

We denote this category by $\mathbf{Pre}^\flat(\mathscr{A})$, resp. $\mathbf{Pre}^\sharp(\mathscr{A})$, if the objects are the $\mathsf{K}^\flat$-, resp. the $\mathsf{K}^\sharp$-formulas, generated from $\mathscr{A}$. $\mathbf{Pre}^\flat(\mathscr{A})$ is a full subcategory of $\mathbf{Pre}^\sharp(\mathscr{A})$.

3.3.1 Proposition *For every $\mathscr{A}$, the category $\mathbf{Pre}^\flat(\mathscr{A})$ is a $\mathsf{K}^\flat$-autonomous category, and $\mathbf{Pre}^\sharp(\mathscr{A})$ is a $\mathsf{K}^\sharp$-autonomous category.*

Proof: The maps assoc, twist, Δ, ∇, Π, and $\amalg$ are given by the obvious prenets. If we let $h^{\mathtt{tt}}(A)$ to be the set of all prenets $P \rhd A$, we have all necessary structure. Checking that all the needed properties hold (in particular that $\mathbf{t}$ is the weak unit), is a trivial computation on prenets. □

3.3.2 Proposition *$\mathbf{Pre}^\flat(\mathscr{A})$ and $\mathbf{Pre}^\sharp(\mathscr{A})$ are purely graphical and Δ-∇-strong.*

Proof: In both categories $\wedge$ and $\vee$ are isomorphic. Δ-∇-strength and the equations in Definition 2.4.3 can be shown by performing cut elimination on prenets. □

3.4. Prenets and equivariant families

Purely graphical K-autonomous categories are pretty absurd creatures, since they implement the same structure twice under the different names of $\wedge$ and $\vee$. But they are useful for us.

Let $\mathscr{K}$ be a purely graphical and Δ-∇-strong K-autonomous category, and $G^\circ \colon \mathscr{A} \to \mathrm{Obj}(\mathscr{K})$ a map that chooses an object $a^\bullet$ of $\mathscr{K}$ for every atom $a \in \mathscr{A}$. It is obvious how to extend this map to every formula of the logic, since we want things to be preserved on the nose. We can now give a construction that assigns to every prenet $P \rhd \Gamma$ with $\Gamma = A_1, \dots, A_n$ an equivariant family over $A_1, \dots, A_n$ in $\mathscr{K}$, and this in a unique way. We will start with the cut-free case and then extend the construction to the prenets with cuts.

3.4.1 Definition Let $P \rhd \Gamma$ be given and let $u \in \mathscr{L}(\Gamma)$ and $\mathscr{S}(u) = \{v \in \mathscr{L}(\Gamma) \mid u \frown v \in P\}$. We call u *celibate* if $|\mathscr{S}(u)| = 0$, we say u is *monogamous* if $|\mathscr{S}(u)| = 1$, and *polygamous* if $|\mathscr{S}(u)| \geq 2$. The *size* of $P \rhd \Gamma$ is the sum of

- the number of $\wedge$-nodes and $\vee$-nodes in Γ,
- the number of polygamous and celibate leaves in Γ,

[6] What we call *prenet* is sometimes also called a *proof structure.*

- the number of edges in P.

Note that the monogamous leaves are not counted.

3.4.2 Equivariant family construction (cut-free case) The unique family $[\![g]\!]$ that we are going to construct will be denoted by $[\![P \rhd \Gamma]\!]^\bullet$. We proceed by induction on the size of $P \rhd \Gamma$. We have the following cases:

0. If there are no edges in P, then $[\![P \rhd \Gamma]\!]^\bullet$ is the all-zero-maps equivariant family.
1. If $P \rhd \Gamma$ is $\{\, \bar{a} \frown a \,\} \rhd a, \bar{a}$ for some atom a, then $[\![P \rhd \Gamma]\!]^\bullet$ is determined by the identity on a. That this is indeed the unique choice follows from Theorem 2.4.7.
2. If it is $\{\, \mathbf{t} \frown \mathbf{t} \,\} \rhd \mathbf{t}$, then $[\![P \rhd \Gamma]\!]^\bullet$ has only one member: $\hat{\mathbf{t}} \in h^{\mathbf{tt}}(\mathbf{t})$ (see Section 2.2 and Definition 2.3.4). Uniqueness follows from Corollary 2.4.8.
3. If one of the A_j is a $\vee$-formula, say $A_1 = B \vee C$, then by induction hypothesis we have already $\bar{A}_2^\bullet \wedge \ldots \wedge \bar{A}_n^\bullet \to B^\bullet \vee C^\bullet$.
4. If one of the A_j is a $\wedge$-formula, the situation is the same (here we make crucial use of the fact that $\wedge$ and $\vee$ are isomorphic in $\mathscr{K}$).
5. If $P \rhd \Gamma$ falls into two disjoint subnets $P' \rhd \Gamma'$ and $P'' \rhd \Gamma''$, we can apply the induction hypothesis to them and take the disjoint sum $f ⅋ g$, where f and g are representatives of $[\![P' \rhd \Gamma']\!]^\bullet$ and $[\![P'' \rhd \Gamma'']\!]^\bullet$.
6. If there is a formula in Γ, whose leaves are all celibate, say it is A_1, then we apply the induction hypothesis to the prenet with A_1 removed and compose with $\amalg^{\bar{A}_1^\bullet}$.[7]
7. If one of the A_i is a polygamous atom, say $A_1 = a$, then we obtain $P' \rhd \Gamma'$ by replacing a with $k = |\mathscr{S}(a)|$ copies of a and the obvious modification in P. We can apply the induction hypothesis and construct $\bar{A}_2^\bullet \wedge \ldots \wedge \bar{A}_n^\bullet \to a^\bullet \vee \ldots \vee a^\bullet \to a^\bullet$.

3.4.3 Equivariant family construction (with cuts) Consider $P \rhd \Gamma$ with $\Gamma = A_1, \ldots, A_n, B_1 \Diamond \bar{B}_1, \ldots, B_m \Diamond \bar{B}_m$ (for some $n \geq 1, m \geq 0$), where $A_1, \ldots, A_n$ are not cuts. We construct $[\![P \rhd \Gamma]\!]^\bullet$ by first applying our construction to the prenet $P \rhd \Gamma'$ in which all cuts are replaced by $\wedge$-formulas. Then we get

$$\bar{A}_1^\bullet \wedge \ldots \wedge \bar{A}_n^\bullet \xrightarrow{g} (B_1^\bullet \wedge \bar{B}_1^\bullet) \vee \ldots \vee (B_m^\bullet \wedge \bar{B}_m^\bullet) \overset{h}{\rightsquigarrow} \mathsf{ff}$$

which represents $[\![P \rhd \Gamma]\!]^\bullet$. Here, g represents $[\![P \rhd \Gamma']\!]^\bullet$ and h is the $\vee$ of the family $(\check{1}_{B_j^\bullet})$, the conames of the identities for $B_j^\bullet$ in $\mathscr{K}$.

The important fact about this construction is that it is preserved by cut elimination:

3.4.4 Lemma *Let $P \rhd \Gamma$ be a prenet, and $P' \rhd \Gamma'$ be the result of applying the cut elimination procedure to it. Then $[\![P \rhd \Gamma]\!]^\bullet$ and $[\![P' \rhd \Gamma']\!]^\bullet$ are same equivariant family.*

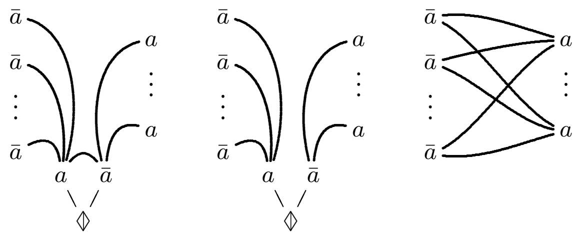
Proof (Sketch): The basic idea is the same as in [18, 11]. There are two cases to consider:

If a compound cut has been reduced, we use the following commuting diagram in $\mathscr{K}$:

$$\begin{tikzcd} & A^\bullet \wedge B^\bullet \wedge (\bar{B}^\bullet \vee \bar{A}^\bullet) \arrow[dd, "\mathsf{cotens}"] \arrow[dr, dotted, "\check{1}_{A^\bullet \wedge B^\bullet}"] & \\ \bigwedge \bar{\Theta}^\bullet \arrow[ur, "g"] \arrow[dr, "g'"'] & & \mathsf{ff} \\ & (A^\bullet \wedge \bar{A}^\bullet) \vee (B^\bullet \wedge \bar{B}^\bullet) \arrow[ur, dotted, "\check{1}_{A^\bullet} \wedge \check{1}_{B^\bullet}"'] & \end{tikzcd} \tag{18}$$

The upper path represents $[\![\pi]\!]^\bullet$ and the lower path $[\![\pi']\!]^\bullet$.

For the reduction of an atomic cut, look at the prenets



All three of them yield the same equivariant family. For the left and the middle ones use the contractible property, and for the middle and the right ones use Δ-∇-strength. If a is a unit, the situation is similar. $\Box$

An immediate consequence of the equivariant-family-construction is

3.4.5 Theorem $\mathbf{Pre}^\flat(\mathscr{A})$*, resp.* $\mathbf{Pre}^\sharp(\mathscr{A})$*, is the free purely graphical and Δ-∇-strong $\mathsf{K}^\flat$-autonomous category, resp. $\mathsf{K}^\sharp$-autonomous category, generated from $\mathscr{A}$.*

3.5. From prenets to proof nets

In this section we will consider those prenets, that come from actual proofs—the proof nets.

3.5.1 Definition A *conjunctive pruning*[8] of a prenet $P \rhd \Gamma$ is a sub-prenet $P|_{\Gamma'} \rhd \Gamma'$ where Γ' has been obtained by deleting one child subformula for every conjunction node and every cut node of Γ (i.e., in $P|_{\Gamma'} \rhd \Gamma'$ every $\wedge$-node and every $\Diamond$-node is unary).

3.5.2 Definition A prenet $P \rhd \Gamma$ is said to be *correct* if for every one of its conjunctive prunings $P|_{\Gamma'} \rhd \Gamma'$ the graph $P|_{\Gamma'}$ has at least one edge. A *proof net* is a correct prenet.

The examples in (16) and in the proof of Lemma 3.4.4 are proof nets.

3.5.3 Theorem *The cut reduction relation $\to$ preserves correctness.*

Observe that the identity nets, as well as the nets defining Δ, Π, assoc, twist, and switch are all correct. The only net that is not correct is the one representing mix^{-1}. Therefore we immediately have that also the two conclusion proof nets form a graphical K-autonomous category, which is Δ-∇-strong. But it is no longer purely graphical. We call this category $\mathbf{Net}^\flat(\mathscr{A})$, resp. $\mathbf{Net}^\sharp(\mathscr{A})$. It is a wide subcategory of $\mathbf{Pre}^\flat(\mathscr{A})$, resp. $\mathbf{Pre}^\sharp(\mathscr{A})$. We now have:

[7] Note that here this case is redundant. But it becomes important when we deal with proof nets instead of prenets (see proof of Theorem 3.5.4).

[8] What is called "pruning" here, has been called "resolution" in [9, 12].

3.5.4 Theorem $\mathbf{Net}^\flat(\mathscr{A})$*, resp.* $\mathbf{Net}^\sharp(\mathscr{A})$*, is the free graphical and* Δ-∇*-strong* $\mathsf{K}^\flat$*-autonomous category, resp.* $\mathsf{K}^\sharp$*-autonomous category, generated from* $\mathscr{A}$*.*

Proof (Sketch): The proof is almost the same as for prenets. The only thing that we have to show is that the construction of the equivariant families can also be done for proof nets. Inspecting the cases in 3.4.2 show that only cases 4 and 5 are problematic. We modify them as follows.

4. If $A_1 = B \wedge C$, let $\Theta = A_2, \ldots, A_n$. We construct

$$\begin{array}{rl} \bigwedge \bar{\Theta}^\bullet & \xrightarrow{(\Delta\wedge 1)\circ\Delta} \bigwedge \bar{\Theta}^\bullet \wedge \bigwedge \bar{\Theta}^\bullet \wedge \bigwedge \bar{\Theta}^\bullet \\ & \xrightarrow{g_1 \wedge g_3 \wedge g_2} B^\bullet \wedge (C^\bullet \vee B^\bullet) \wedge C^\bullet \\ & \xrightarrow{\mathsf{cotens}} (B^\bullet \wedge C^\bullet) \vee (B^\bullet \wedge C^\bullet) \\ & \xrightarrow{\nabla} B^\bullet \wedge C^\bullet \ . \end{array} \tag{19}$$

   where g_1, g_2, and g_3 represent the nets $P|_{B,\Theta} \rhd B, \Theta$ and $P|_{C,\Theta} \rhd C, \Theta$ and $P|_{B,C,\Theta} \rhd B, C, \Theta$, which are all correct and of smaller size.

5. We apply that case only if both subnets are correct.

Note that now we need case 6 because case 0 is no longer available. It follows from graphicality (and Theorem 2.4.7), that this construction yields the same map as the one in 3.4.2. □

4. Conclusions and Future Work

There are not enough examples yet for anybody to be able to give a definitive answer to the question "what is a Boolean category?". The final axiomatization will be the product of a succession of refinements. But we believe we have made a significant progress in that quest: the axioms for a K-autonomous category are general and easy to verify; they should inspire new semantics. The conditions of graphicality and Δ-∇-strength build a bridge for denotational semantics and the Geometry of Interaction; they also show that the world is very big and that our category of proof nets is still at the degenerate end of the spectrum. From Theorem 2.4.7 we learned that things like $\mathbb{B}$-nets have limitations if we want to construct Boolean categories that are not idempotent. In the near future, we intend to work on

- finding Boolean categories that are not idempotent.
- incorporating Hyland's recent work [10] in that framework.
- the study of the Kleisli categories associated with comonoids of the form (X, Δ_X, Π^X). As Lambek has pointed out a long time ago, this corresponds to theories that are no longer pure, but where X has been added as an axiom. We can now try to relate the complexity of X to the structure of that category, and ask questions like "when does such a category of have cut-elimination?".
- extension to first-order logic.

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