# Large time behavior for a simplified $N-$dimensional model of fluid-solid interaction

Alexandre Munnier*

Institut Élie Cartan

Université Henri Poincaré Nancy 1,

B.P. 239, F-54506 Vandœuvre-lès-Nancy Cedex, France

alexandre.munnier@iecn.u-nancy.fr

Enrique Zuazua†

Departamento de Matemáticas, Universidad Autónoma

28049 Madrid, Spain

enrique.zuazua@uam.es

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**Abstract**

In this paper, we study the large time behavior of solutions of a parabolic equation coupled with an ordinary differential equation (ODE). This system can be seen as a simplified $N$-dimensional model for the interactive motion of a rigid body (a ball) immersed in a viscous fluid in which the pressure of the fluid is neglected. Consequently, the motion of the fluid is governed by the heat equation and the standard conservation law of linear momentum determines the dynamics of the rigid body. In addition, the velocity of the fluid and that of the rigid body coincide on its boundary. The time variation of the ball position, and consequently of the domain occupied by the fluid, are not known a priori, so we deal with a free boundary problem. After proving the existence and uniqueness of a strong global in time solution, we get its decay rate in $L^p$ $(1 \leq p \leq \infty)$, assuming the initial data to be integrable. Then, working in suitable weighted Sobolev spaces, and using the so-called similarity variables and scaling arguments, we compute the first term in the asymptotic development of solutions. We

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prove that the asymptotic profile of the fluid is the heat kernel with an appropriate total mass. The $L^\infty$ estimates we get allow us to describe the asymptotic trajectory of the center of mass of the rigid body as well. We compute also the second term in the asymptotic development in $L^2$ under further regularity assumptions on the initial data.

**Keywords and Phrases:** Fluid-solid interaction, heat-ODE coupled system, large time behavior, similarity variables, heat kernel.

**AMS Subject Classification:** 35B40, 35K15, 35R35, 35K05, 34E05.

# 1 Introduction and main results

The aim of this paper is to describe the large time asymptotic behavior for a coupled system of partial and ordinary differential equations. The system under consideration is a simplified $N-$dimensional model for the motion of a rigid body inside a fluid flow.

The governing equation for the fluid is merely the heat equation whereas the motion of the solid is governed by the balance equation for linear momentum. For the sake of simplicity, we assume the solid to be a moving ball of radius 1 occupying the domain $B(t)$ of $\mathbb{R}^N$ whose center of mass lies in the point $\mathbf{h}(t)$. Thus, the system we shall deal with is the following one:

$$\begin{cases} \mathbf{u}_t - \Delta \mathbf{u} = \mathbf{0}, & \mathbf{x} \in \Omega(t), \quad t > 0, \\ \mathbf{u}(\mathbf{x}, t) = \mathbf{h}'(t), & \mathbf{x} \in \partial B(t), \quad t > 0, \\ m\mathbf{h}''(t) = -\displaystyle\int_{\partial\Omega(t)} \mathbf{n} \cdot \nabla \mathbf{u} d\sigma_x, & \qquad\qquad t > 0, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega(0), & \mathbf{h}(0) = \mathbf{h}_0, \quad \mathbf{h}'(0) = \mathbf{h}_1, \end{cases} \tag{1.1}$$

where $\Omega(t) := \mathbb{R}^N \setminus B(t)$ and $m > 0$ stands for the mass of the ball. The vector $\mathbf{n}(\mathbf{x}, t)$ is the unit normal to $\partial\Omega(t)$ at the point $\mathbf{x}$ directed to the interior of $B(t)$. In the above system the unknowns are $\mathbf{u}(\mathbf{x}, t)$ (that can be seen as the Eulerian velocity field of the fluid) and $\mathbf{h}(t)$. The coupling condition (1.1-ii) ensures that the velocity of the body is the same as the one of the fluid on its boundary. The equation (1.1-iii) results from the standard conservation law of linear momentum.

Let us stress the main differences between our model and a full model of fluid-structure interaction, namely:

$$\begin{cases} \mathbf{u}_t - \Delta \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p = \mathbf{0}, & \mathbf{x} \in \Omega(t), \quad t > 0, \\ \text{div } \mathbf{u} = 0, & \mathbf{x} \in \Omega(t), \quad t > 0, \\ \mathbf{u}(\mathbf{x}, t) = \mathbf{h}'(t), & \mathbf{x} \in \partial B(t), \quad t > 0, \\ m\mathbf{h}''(t) = -\displaystyle\int_{\partial\Omega(t)} T\mathbf{n} d\sigma_x, & \qquad\qquad t > 0, \\ \mathbf{u}(\mathbf{x}, 0) = \mathbf{u}_0(\mathbf{x}), \quad \mathbf{x} \in \Omega(0), & \mathbf{h}(0) = \mathbf{h}_0, \quad \mathbf{h}'(0) = \mathbf{h}_1, \end{cases} \tag{1.2}$$

where $T$ is the stress tensor in the fluid whose components are defined by

$$T_{ij}(\mathbf{x},t) := -p(\mathbf{x},t)\delta_{ij} + \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right),$$

and $p$ stands for the pressure. When $N = 2$, on account of (1.2-ii) and (1.2-iii), the relation (1.2-iv) can be rewritten as:

$$m\mathbf{h}''(t) = -\int_{\partial\Omega(t)} [\mathbf{n}\cdot\nabla\mathbf{u} - p\mathbf{n}]\, d\sigma_x. \tag{1.3}$$

Indeed, from (1.2-ii), we deduce that $\mathbf{n}\cdot\nabla\mathbf{u}^T = (\mathbf{n}^\perp\cdot\nabla\mathbf{u})^\perp = 0$, because $\mathbf{u}$ is constant, equal to $\mathbf{h}'$, along $\partial B(t)$. Therefore, we obtain that $T\mathbf{n} = -p\mathbf{n} + \mathbf{n}\cdot\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right)$ and so $T\mathbf{n} = -p\mathbf{n} + \mathbf{n}\cdot\nabla\mathbf{u}$. That yields relation (1.3). Note that the formulation (1.3) is quite close to (1.1-iii). In our model (1.1), the pressure term has been neglected. Note that the convective quadratic non-linearity has also been neglected in (1.1-i). However, this simplification is less relevant since most of our developments can also be carried out in the presence of a non-linear convective term. Thus, the main difference between system (1.1) and the more realistic one (1.2) is that in (1.1), we neglect the pressure term. Extending the results of this paper to the full system (1.2) is an interesting open problem.

The model (1.2) and other more complete and complex ones involving several bodies with rotational motions, were extensively studied during the last years. Concerning the existence and uniqueness of weak solutions, see for example [4], [5] and [7], [8], [6] and [14], [13] and the references therein. Recently and independently of the present work, M. Tucsnak and T. Takahashi in [18] in the whole space and T. Takahashi in [17] for a bounded domain, proved the existence and uniqueness of a strong solution for a more complete version of the model (1.2), adding the rotational motion for the ball. Moreover, it was shown that the solution is global in time provided the ball does not collide with the boundary of the domain. Whether finite time collision occurs is one of the most interesting and challenging problems in this field. Recently, the problem was solved in 1-d by J.L. Vázquez and E. Zuazua [20] showing that finite time collision may not occur.

In a previous paper by J.L. Vázquez and E. Zuazua [19], the large time behavior for a simplified one dimensional model of fluid-structure interaction was analyzed. In this paper, a sharp description of the asymptotic behavior as time goes to infinity of a point particle, which floats in a fluid governed by the viscous Burgers equation was given. More precisely, it was proved that the velocity $u$ of the fluid behaves, for $t$ large, like the unique self-similar solution of the Burger's equation on $\mathbb{R}$ with source type initial data $M\delta_0$. The constant $M$ is defined by $M := \int_{\mathbb{R}} u_0 dx + mh_1$, the functions $u_0$ and $h_1$ being the initial velocities of the fluid and of the particle respectively. The present work is a natural extension of this one to the case of several space dimensions. However, in the present paper, the equation governing $\mathbf{u}$ is assumed to be linear although similar results could be proved for a model including a convective non-linearity in the parabolic equation.

It is also of interest to compare our results with the existing ones on the asymptotic behavior of the Navier-Stokes equations (without rigid-bodies) in [2] and [3] (and references given there). In these papers it is shown that, roughly speaking, the first order approximation is given by the heat kernel with an appropriate total mass. The same result holds for the solution of the Navier-Stokes equations in $\mathbb{R}^2$ and $\mathbb{R}^3$ (see [2] and [3]). One can expect the same result to be true for the Navier-Stokes equations coupled with the motion of a rigid body (as in (1.2)) but this result has not been proved so far.

Let us go back now to system (1.1) we are dealing with. It is a linear, free boundary problem since the position of $B(t)$ is to be determined. Applying the change of variables $\mathbf{v}(\mathbf{x},t) = \mathbf{u}(\mathbf{x}+\mathbf{h}(t),t)$ and $\mathbf{g}(t) = \mathbf{h}'(t)$, we can rewrite system (1.1) using $\mathbf{v}$ and $\mathbf{g}$ as new unknown functions and the system turns out to be non-linear but in a fixed domain, independent of $t$. Indeed, we get

$$\begin{cases} \mathbf{v}_t - \Delta\mathbf{v} - \mathbf{g}\cdot\nabla\mathbf{v} = \mathbf{0}, & \mathbf{x}\in\Omega, \qquad t>0, \\ \mathbf{v}(\mathbf{x},t) = \mathbf{g}(t), & \mathbf{x}\in\partial B, \qquad t>0, \\ m\mathbf{g}'(t) = -\displaystyle\int_{\partial\Omega} \mathbf{n}\cdot\nabla\mathbf{v}\, d\sigma_x, & \qquad\qquad t>0, \\ \mathbf{v}(\mathbf{x},0) = \mathbf{v}_0(\mathbf{x}), \quad \mathbf{x}\in\Omega, & \mathbf{g}(0) = \mathbf{h}_1. \end{cases} \tag{1.4}$$

Here $B$ stands for the fixed ball of center $\mathbf{0}$ and radius 1, $\Omega := \mathbb{R}^N/B$ and $\mathbf{n}(\mathbf{x})$ is the unit normal to $\partial\Omega$ at the point $\mathbf{x}$ directed to the interior of $B$.

In view of (1.4), it is clear that the $N$ components of the fluid field ($\mathbf{u}$ or $\mathbf{v}$) are coupled through the unknowns ($\mathbf{h}$ or $\mathbf{g}$) describing the motion of the solid. Thus, although it might seem not to be the case, all the components of $\mathbf{u}$ are coupled in (1.1).

## 1.1 Notations

Throughout this article, we shall use bold print notations for $N-$dimensional vectors like $\mathbf{x}$, $\mathbf{y}$, $\mathbf{u}$, $\mathbf{v}$, $\boldsymbol{\zeta}$ whereas we keep the usual characters for real valued functions : $u$, $v$, $\zeta$. The generic notation $v$ will be used for any of the components $v_i$ of the vector $\mathbf{v}$.

In the same way, $\mathbf{L}^2(\omega)$, $\mathbf{H}^1(\omega)$ will stand for $L^2(\omega)^N$ and $H^1(\omega)^N$ respectively, $\omega$ being a measurable subset of $\mathbb{R}^N$.

A $N\times N$ matrix is denoted $[\mathbf{M}]$. Its entries are $M_{ij}$, $1\le i,j\le N$ and $\mathbf{M}_i$ stands for the $i-$th row. However, to shorten notation, we sometimes drop the index $i$ and denote generically by $\mathbf{M}$ any row of the matrix $[\mathbf{M}]$. For instance, according to these simplifications, the matrix identity $[\mathbf{M}] = \mathbf{U}\mathbf{V}^T$ leads to the vectors equality (equality of the rows of the matrices) $\mathbf{M} = U\mathbf{V}$ and can also be rewritten as $N$ scalar equalities $M_i = UV_i$. For vectors and matrices, the classical

Euclidean norms are defined $|\mathbf{V}| = \left(\sum_{i=1}^N V_i^2\right)^{1/2}$ and $|[\mathbf{M}]| = \left(\sum_{i,j=1}^N M_{ij}^2\right)^{1/2}$. When $\mathbf{V}(\mathbf{x})$ and $[\mathbf{M}](\mathbf{x})$ are a vector valued function and a matrix valued function respectively, on an open set $\omega \subset \mathbb{R}^N$, we denote: $\|\mathbf{V}\|_p = \left(\sum_{i=1}^N \int_\omega V_i^p d\mathbf{x}\right)^{1/p}$ and $\|[\mathbf{M}]\|_p = \left(\sum_{i,j=1}^N \int_\omega M_{ij}^p d\mathbf{x}\right)^{1/p}$, for all $1 \leq p < \infty$.

Non negative constants shall be denoted by $C$ along the computations. The value of $C$ can change from one line to the other. We sometimes use $C_1$ and $C_2$ when its values need to be followed along the computations. The notation $C_p$ allows to emphasize the dependence with respect to $p$, the exponent of the Sobolev or $L^p$ space we are working in. Finally, in some equalities, $C(t)$ will stand for a real valued function such that $|C(t)| \leq C$ for all $t > 0$.

$L^2(F, \omega)$ and $H^1(F, \omega)$ stand for weighted spaces where $F$ is a positive function (the weight) on the subset $\omega$ of $\mathbb{R}^N$. They are endowed with the scalar products $\int_\omega uvF(\mathbf{x})d\mathbf{x}$ and $\int_\omega \nabla u \cdot \nabla v F(\mathbf{x})d\mathbf{x} + \int_\omega uvF(\mathbf{x})d\mathbf{x}$ respectively. To shorten notations, we will write $L^2(F)$ and $H^1(F)$ instead of $L^2(F, \mathbb{R}^N)$ and $H^1(F, \mathbb{R}^N)$ respectively when $\omega = \mathbb{R}^N$.

Finally, $\dot{H}^1(\omega)$ is the closure of $C_c^1(\omega)$ (the space of $C^1$ functions with compact support in $\omega$) for the norm $\left(\int_\omega |\nabla u|^2 d\mathbf{x}\right)^{\frac{1}{2}}$.

## 1.2 The scalar version of system (1.1)

Any component $(v_i, g_i)$, $i = 1, \ldots, N$ of the solution $(\mathbf{v}, \mathbf{g})$ of system (1.4), that we shall merely denote by $v$ and $g$, is a vector valued function with two scalar components, which solves:

$$\begin{cases} v_t - \Delta v - \mathbf{g} \cdot \nabla v = 0, & \mathbf{x} \in \Omega, \quad t > 0, \\ v(\mathbf{x}, t) = g(t), & \mathbf{x} \in \partial B, \quad t > 0, \\ mg'(t) = -\displaystyle\int_{\partial\Omega} \frac{\partial v}{\partial \mathbf{n}} d\sigma_x, & \qquad\qquad t > 0, \\ v(\mathbf{x}, 0) = v_0(\mathbf{x}), \quad \mathbf{x} \in \Omega, & g(0) = h_1. \end{cases} \tag{1.5}$$

Note in particular that in (1.5), $v$ satisfies a scalar heat equation. However, all the scalar equations satisfied by the components $v_i$, $i = 1, \ldots, N$ are coupled through the convective term and in particular, through the vector field $\mathbf{g}$ describing the motion of the solid.

As far as the first term in the large time asymptotic development is concerned, we shall prove that the term $\mathbf{g} \cdot \nabla v$ can be neglected.

To simplify notations, we will sometimes work with these scalar functions $(v, g)$.

## 1.3 Main results

**Theorem 1.1 (Existence and uniqueness of solutions)** *For any* $(\mathbf{v}_0, \mathbf{g}_0) \in \mathbf{L}^2(\Omega) \times \mathbb{R}^N$*, there exists a unique global strong solution* $(\mathbf{v}, \mathbf{g})$ *of system* (1.4) *such that:*

$$\mathbf{v} \in C\left([0,+\infty), \mathbf{L}^2(\Omega)\right) \cap L^2\left((0,\infty), \dot{\mathbf{H}}^1(\Omega)\right) \quad \textit{and} \quad \mathbf{g} \in C\left([0,+\infty)\right).$$

The proof of this Theorem is quite classical and follows the same ideas as in [19]. A complete proof can be found in the self-contained version of the present article [16].

If we integrate the first equation of system (1.4), use the Stokes formula and the transmission condition (1.4-iii) on the boundary of the ball, we deduce that

$$\mathbf{M}_1 := \int_\Omega \mathbf{v} d\mathbf{x} + m\mathbf{g}, \tag{1.6}$$

is independent of time. This *first momentum* plays a crucial role in the description of the large time behavior of $\mathbf{v}$.This idea will be made more precise in the following Theorem.

Let us introduce the weight function $K(\mathbf{x}) := \exp\left(|x|^2/4\right)$ and the constant $\sigma_N$, the area of the unit sphere, necessary to state the main results of this paper. Note that $\sigma_N/N$ is therefore the volume of the unit sphere.

**Theorem 1.2 (First term in the asymptotic development)** *Assume that* $\mathbf{v}_0 \in \mathbf{L}^2(K,\Omega)$ *and* $\mathbf{g}_0 \in \mathbb{R}^N$*. Then there exist constants* $C_p > 0$ *depending on the dimension* $N$*, on the mass* $m$ *of the solid and on* $p$ *such that the following inequalities hold:*

$$t^{\frac{N}{2}\left(1-\frac{1}{p}\right)} \|\mathbf{v}(t) - \mathbf{M}_1 G(t)\|_{\mathbf{L}^p(\Omega)} \leq C_p R_1(t), \tag{1.7}$$

$$t^{\frac{N}{2}} |\mathbf{g}(t) - \mathbf{M}_1 (4\pi t)^{-\frac{N}{2}}| \leq C_\infty R_2(t), \quad \forall t \geq 1. \tag{1.8}$$

*In these estimates,* $G$ *stands for the heat kernel on* $\mathbb{R}^N$ *defined by* $G(t,\mathbf{x}) := (4\pi t)^{-\frac{N}{2}} \exp\left(-|\mathbf{x}|^2/4t\right)$*, and the first asymptotic momentum* $\mathbf{M}_1$ *is given by* $\mathbf{M}_1 := \int_\Omega \mathbf{v}_0 d\mathbf{x} + m\mathbf{g}_0$*. The error functions* $R_1$ *and* $R_2$ *are given in the following tables:*

| | $m = \sigma_N/N$ | |
|---|---|---|
| | $N=2$ | $N \geq 3$ |
| $R_2(t)$ | $\lvert\log(t)\rvert t^{-\frac{1}{2}}$ | $t^{-\frac{1}{2}}$ |
| | $1 \leq p \leq \infty$ | |
| $R_1(t)$ | $\lvert\log(t)\rvert t^{-\frac{1}{2}}$ | $t^{-\frac{1}{2}}$ |

(1.9)

| | $m \neq \sigma_N/N$ | | | |
|---|---|---|---|---|
| | $N=2$ | | $N \geq 3$ | |
| $R_2(t)$ | $\lvert\log(t)\rvert^{\frac{1}{2}} t^{-\frac{1}{4}}$ | | $t^{-\frac{1}{N+2}}$ | |
| | $1 \leq p \leq 2$ | $2 < p \leq \infty$ | $1 \leq p \leq N$ | $N < p \leq \infty$ |
| $R_1(t)$ | $\lvert\log(t)\rvert t^{-\frac{1}{2}}$ | $\lvert\log(t)\rvert^{\frac{p}{2p-1}} t^{-\frac{1}{2}+\theta(2,p)}$ | $t^{-\frac{1}{2}}$ | $t^{-\frac{1}{2}+\theta(N,p)}$ |

(1.10)

where:

$$\theta(N,p) := \frac{N}{2}\frac{(p-1)(p-N)}{p(2p+N(p-1))}. \tag{1.11}$$

**Remark 1.1** *The embedding* $\mathbf{L}^2(K,\Omega) \subset \mathbf{L}^1(\Omega)$ *ensures that* $\mathbf{M}_1$ *is well defined.*

**Remark 1.2** *Note that the decay rates we obtain for* $\mathbf{g}$ *are the same as those for* $\mathbf{v}$ *in the* $\mathbf{L}^\infty-$*norm. This is perfectly natural in view of the coupling condition in (1.4-ii).*

**Remark 1.3** *Theorem 1.2 provides different decay rates in the case* $m = \sigma_N/N$ *and* $m \neq \sigma_N/N$. *This appears naturally along the proof when analyzing the behavior of* $\bar{\mathbf{v}} := \mathbf{v} - \mathbf{M}_1 G$ *and* $\bar{\mathbf{g}} := \mathbf{g} - \mathbf{M}_1 G|_{\partial\Omega}$. *More precisely the estimates on* $|m\bar{\mathbf{g}}' + \int_{\partial\Omega} \mathbf{n}\cdot\nabla\bar{\mathbf{v}} d\sigma_x|$ *depend on the mass* $m$ *of the solid ball. This term decays like* $t^{-1/2-2}$ *when* $m = \sigma_N/N$ *but only as* $t^{-N/2-1}$ *when* $m \neq \sigma_N/N$.

According to Theorem 1.2, in a first approximation, $\mathbf{v}$ behaves, as $t \to \infty$, as the fundamental solution $G$ of the heat equation. Note that this Gaussian profile is multiplied by $\mathbf{M}_1 := \int_\Omega \mathbf{v}_0 d\mathbf{x} + m\mathbf{g}_0$ which indicates that the fluid component of the system absorbs asymptotically as $t \to \infty$ the initial momentum introduced by the solid mass.

The values of $R_1$ in (1.9) and (1.10) of Theorem 1.2 are sharp for $p = 2$ and all $N \geq 2$. This clearly appears when exhibiting the second term in the asymptotic development of $\mathbf{v}$ in $\mathbf{L}^2(\Omega)$ in Theorem 1.3 below. We do not know yet if the error estimates for the $\mathbf{L}^p-$norms with $p > N$ are sharp or not. At this respect it is important to observe that, despite the fact that the estimates we obtain for $p \leq N$ are similar to those that one obtains for the linear heat equation where one gains an extra $t^{-1/2}$ factor of decay when subtracting the fundamental solution, our estimates deteriorate as $p$ increases beyond the exponent $p = N$ due to the additional factor $t^{\theta(N,p)}$.

**Remark 1.4** *In Theorem 1.2 the dynamics of* $\mathbf{g}$ *is rather simple since, for t large, the action of the fluid on the ball can be neglected. This can be easily predicted by a scaling argument. According to the scaling properties of the heat equation, given* $(\mathbf{v},\mathbf{g})$ *solution of* (1.4), *it is natural to introduce* $\mathbf{v}_\lambda(\mathbf{x},t) := \lambda^N \mathbf{v}(\lambda\mathbf{x}, \lambda^2 t)$ *and* $\mathbf{g}_\lambda(t) := \lambda^N \mathbf{g}(\lambda^2 t)$, *for all* $\lambda > 0$. *Then,* $(\mathbf{v}_\lambda, \mathbf{g}_\lambda)$ *is a solution of the following system:*

$$\begin{cases} \mathbf{v}_{\lambda,t} - \Delta\mathbf{v}_\lambda - \lambda^{-N+1}\mathbf{g}_\lambda\cdot\nabla\mathbf{v}_\lambda = \mathbf{0}, & \mathbf{x}\in\Omega_\lambda, \qquad t>0,\\ \mathbf{v}_\lambda(\mathbf{x},t) = \mathbf{g}_\lambda(t), & \mathbf{x}\in\partial B_\lambda, \qquad t>0,\\ (m/\lambda)\mathbf{g}_\lambda'(t) = -\displaystyle\int_{\partial\Omega_\lambda} \mathbf{n}\cdot\nabla\mathbf{v}_\lambda d\sigma_x, & \qquad\qquad t>0,\\ \mathbf{v}_\lambda(\mathbf{x},0) = \lambda^N\mathbf{v}_0(\lambda\mathbf{x}), \quad \mathbf{x}\in\Omega_\lambda, & \mathbf{g}_\lambda(0) = \lambda^N\mathbf{h}_1, \end{cases} \tag{1.12}$$

*where $B_\lambda$ is the ball centered at the origin and of radius $1/\lambda$ and $\Omega_\lambda = \mathbb{R}^N \setminus B_\lambda$. Formally, as $\lambda \to \infty$ the convective term in the first equation vanishes, and the equation for the acceleration of the ball tends to the trivial identity. Taking this into account, the rescaled solution of the heat equation can be shown to converge to the Gaussian kernel with an appropriate mass. Thus, denoting by $\widetilde{\mathbf{v}}$ and $\widetilde{\mathbf{g}}$ the limits of $\mathbf{v}$ and $\mathbf{g}$ as $\lambda \to \infty$, one expects as well that $\widetilde{\mathbf{v}}(\mathbf{x}, t) = \mathbf{M}_1 G(\mathbf{x}, t)$ and $\widetilde{\mathbf{g}}(t) = \mathbf{M}_1 G(\mathbf{0}, t)$, where $\mathbf{M}_1$ can be identified by the property of conservation of momentum.*

In view of Theorem 1.2, the solution $\mathbf{u}$ of system (1.1) behaves as follows: $\mathbf{u}(\mathbf{x}, t) \to \mathbf{M}_1 G(\mathbf{x} - \mathbf{h}(t), t)$ as $t \to \infty$, in all the $\mathbf{L}^p$ spaces. Moreover, Theorem 1.2 yields precise estimates of the velocity of the ball, $\mathbf{g} := \mathbf{h}'$. Integrating these relations, we deduce that:

- When $N = 2$: $\mathbf{h}(t) = (\mathbf{M}_1/4\pi)\log(t) + O(1)$, and then, the ball goes to infinity as $t \to \infty$.
- When $N \geq 3$: $|\mathbf{h}(t)| \leq C_0$, for all $t \geq 0$, where $C_0$ depends on the initial data. The ball remains in a bounded domain as $t \to \infty$.

**Theorem 1.3 (Second term in the asymptotic development)**
*Let $(\mathbf{v}_0, \mathbf{g}_0) \in \mathbf{H}^2(\Omega, K) \times \mathbb{R}^N$ s. t. $\mathbf{v}_0|_{\partial\Omega} = \mathbf{g}_0$. Then, as far as the $\mathbf{L}^2-$norm is concerned, we can improve Theorem 1.1: there exist two constants $a > 0$ and $b > 0$ such that:*

- *When $N = 2$,*

$$\|\mathbf{v} - \mathbf{M}_1 G - |\log(1+t)|[\mathbf{M}_2^1]\nabla G - [\mathbf{M}_2^2]\nabla G\|_{\mathbf{L}^2(\Omega)} \leq C\,|\log(1+t)|^b t^{-1-a}. \tag{1.13a}$$

- *When $N \geq 3$,*

$$\|\mathbf{v} - \mathbf{M}_1 G - [\mathbf{M}_2]\nabla G\|_{\mathbf{L}^2(\Omega)} \leq C\,|\log(1+t)|^b t^{-\frac{N}{4}-\frac{1}{2}-a}, \tag{1.13b}$$

*where $a = a(N)$, $b = b(N)$ and where the second asymptotic momenta $[\mathbf{M}_2^1]$, $[\mathbf{M}_2^2]$ and $[\mathbf{M}_2]$ are $N \times N$ matrices defined by*

- *When $N = 2$,*

$$[\mathbf{M}_2^1] := \frac{1}{4\pi}\mathbf{M}_1\mathbf{M}_1^T, \tag{1.14a}$$

*and*

$$[\mathbf{M}_2^2] := -\int_\Omega \mathbf{v}_0\mathbf{x}^T d\mathbf{x} - \frac{m}{(4\pi)^2}\mathbf{M}_1\mathbf{M}_1^T - \int_0^\infty\left(\int_{\partial\Omega}(\mathbf{n}\cdot\nabla\mathbf{v})\mathbf{x}^T d\sigma_x\right)dt$$
$$+\int_0^\infty (1+t)^{-\frac{3}{4}}\mathbf{M}_1\boldsymbol{\beta}^T - \frac{m}{4\pi}(1+t)^{-\frac{7}{4}}(\mathbf{M}_1\boldsymbol{\beta}^T + \boldsymbol{\beta}\mathbf{M}_1^T) - m(1+t)^{-2}\boldsymbol{\beta}\boldsymbol{\beta}^T dt. \tag{1.14b}$$

- *When* $N \geq 3$,

$$[\mathbf{M}_2] := -\int_\Omega \mathbf{v}_0 \mathbf{x}^T d\mathbf{x} - \frac{1}{(4\pi)^N}\mathbf{M}_1\mathbf{M}_1^T\left[\frac{m}{N-1} - (4\pi)^{\frac{N}{2}}\frac{2}{N-2}\right]$$
$$-\int_0^\infty\left(\int_{\partial\Omega}(\mathbf{n}\cdot\nabla\mathbf{v})\mathbf{x}^T d\sigma_x\right)dt + \int_0^\infty (1+t)^{-\frac{N-1}{2}-\frac{1}{2+N}}\mathbf{M}_1\boldsymbol{\beta}^T dt$$
$$-\int_0^\infty \frac{m}{(4\pi)^{\frac{N}{2}}}(1+t)^{-\frac{2N-1}{2}-\frac{1}{2+N}}(\mathbf{M}_1\boldsymbol{\beta}^T + \boldsymbol{\beta}\mathbf{M}_1^T) + m(1+t)^{-\frac{2N-1}{2}-\frac{2}{2+N}}\boldsymbol{\beta}\boldsymbol{\beta}^T dt, \tag{1.14c}$$

*where*

$$\boldsymbol{\beta} := (1+t)^{\frac{N}{2}+\frac{1}{2+N}}\left(\mathbf{g} - \mathbf{M}_1(4\pi t)^{-\frac{N}{2}}\right), \tag{1.14d}$$

*is a bounded quantity. Moreover, all the integrals involved in the definition of* $[\mathbf{M}_2^2]$ *for* $N=2$ *and* $[\mathbf{M}_2]$ *for* $N \geq 3$ *are well defined.*

**Remark 1.5** *The second term in the asymptotic expansion of the solution contains some terms that may not be explicitly computed in terms of the initial data. This is the case both in dimension* $N=2$ *and* $N=3$*. When* $N=2$*, the definition of* $[\mathbf{M}_2^2]$ *contains several time integrals that involve the solution* $(\mathbf{v},\mathbf{g})$ *for all time* $t \geq 0$*. The same phenomenon occurs (see Theorem 3, [22]) for scalar convection-diffusion equations on the whole space* $\mathbb{R}^N$*.*

It is convenient to display the results of Theorem 1.3 as an asymptotic development as $t \to \infty$ in $\mathbf{L}^2(\Omega)$:

- When $N=2$:

$$\mathbf{v}(t) = \mathbf{M}_1 G(t) + |\log(1+t)|[\mathbf{M}_2^1]\nabla G(t) + [\mathbf{M}_2^2]\nabla G(t) + \mathcal{O}\left(|\log t|^b t^{-1-a}\right). \tag{1.15a}$$

- When $N \geq 3$:

$$\mathbf{v}(t) = \mathbf{M}_1 G(t) + [\mathbf{M}_2]\nabla G(t) + \mathcal{O}\left(|\log t|^b t^{-\frac{N}{4}-\frac{1}{2}-a}\right). \tag{1.15b}$$

For the solution $\tilde{v}$ of the heat equation on the whole space $\mathbb{R}^N$, with initial data $\tilde{v}_0 \in L^1(\mathbb{R}^N, 1+|\mathbf{x}|^2)$, we have the asymptotic expansion in $L^2(\mathbb{R}^N)$:

$$\tilde{v}(t) = \widetilde{M_1}G(t) + \widetilde{\mathbf{M}}_2\cdot\nabla G(t) + \mathcal{O}\left(t^{-\frac{N}{4}-1}\right), \tag{1.15c}$$

where $\widetilde{M_1} = \int_{\mathbb{R}^N}\tilde{v}_0 d\mathbf{x}$ and $\widetilde{\mathbf{M}}_2 = -\int_{\mathbb{R}^N}\tilde{v}_0\mathbf{x}d\mathbf{x}$. Comparing (1.15a) for $N=2$ and (1.15b) for $N=3$ with the known results for the heat equation (1.15c) we observe some slight differences due to the presence of the solid mass. In dimension $N = 2$ the main difference is due to the presence of a time logarithmic multiplicative factor on the second term of the asymptotic expansion involving $\nabla G$. This was already observed to be the case in [22] for the quadratic convective nonlinearity in dimension $N = 2$. We also see the presence of this time logarithmic factor on the error term. The main difference in the case $N = 3$ comes from the definition of the factor $[\mathbf{M}_2]$ multiplying the second term $\nabla G$, which reflects the coupling between the heat equation and the solid mass.

## 1.4 Sketches of the proofs of Theorems 1.2 and 1.3

The first step to prove Theorem 1.2 consists in establishing the decay rate of the solution $(\mathbf{v}, \mathbf{g})$ of system (1.4) in $\mathbf{L}^p$ $(1 \leq p \leq \infty)$. We get this result componentwise by multiplying the heat equation by non-linear functions of $v$, integrating by parts and using Hölder, Sobolev and interpolation inequalities. The problem is then reduced to solve an ordinary differential inequality and the conclusion arises by exhibiting a suitable super-solution.

In a second step, we introduce $\bar{\mathbf{v}}(\mathbf{x}, t) := \mathbf{v}(\mathbf{x}, t) - \mathbf{M}_1 G(\mathbf{x}, t)$ for all $\mathbf{x} \in \Omega$ and $\bar{\mathbf{g}}(t) := \mathbf{g}(t) - \mathbf{M}_1 J(t)$ where $J(t) := G(t, \mathbf{x}) |_{\mathbf{x} \in \partial B} = (4\pi t)^{-N/2} e^{-1/4t}$. Since $G$ is the fundamental solution of the heat equation, $\bar{\mathbf{v}}$ solves in $\Omega \times (0, \infty)$:

$$\bar{\mathbf{v}}_t - \Delta \bar{\mathbf{v}} - \mathbf{g} \cdot \nabla \bar{\mathbf{v}} = \mathbf{M}_1 \mathbf{g} \cdot \nabla G. \tag{1.16}$$

Simple computations yield $J'(t) = (4\pi)^{-N/2} t^{-N/2-1} e^{-1/4t} \left(-N/2 + 1/4t\right)$, for all $t \geq 0$ and also $\int_{\partial\Omega} \frac{\partial G}{\partial \mathbf{n}} d\sigma_x = 1/2 (4\pi)^{-N/2} \sigma_N t^{-N/2-1} e^{-1/4t}$. Thus, with the correcting term,

$$\varepsilon(t) := \frac{1}{2} m (4\pi)^{-\frac{N}{2}} t^{-\frac{N}{2}-1} e^{-\frac{1}{4t}} \left( \frac{\sigma_N}{m} - N + \frac{1}{2t} \right), \tag{1.17}$$

it follows that $mJ'(t) = -\int_{\partial\Omega} \frac{\partial G}{\partial \mathbf{n}} d\sigma_x + \varepsilon(t)$. Therefore, the ODE governing the evolution of $\bar{\mathbf{g}}$ reads as follows:

$$m\bar{\mathbf{g}}' = -\int_{\partial\Omega} \mathbf{n} \cdot \nabla \bar{\mathbf{v}} d\sigma_x - \mathbf{M}_1 \varepsilon(t). \tag{1.18}$$

In order to prove that $\mathbf{M}_1 G$ is the first term in the asymptotic development of $\mathbf{v}$, we have to prove that $\bar{\mathbf{v}}$ decreases faster than $\mathbf{v}$ and $G$ separately do. The decay rate for $\bar{\mathbf{v}}$ is obtained by using the same arguments employed when analyzing the decay rate of $\mathbf{v}$. However the proof is technically more involved due to the presence of the correcting terms on the right hand side of (1.16) and (1.18).

In a third step, we rewrite equations (1.16) and (1.18), using the so-called similarity variables and rescaled functions. Working in weighted Sobolev spaces, we determine the decay rate of $\bar{\mathbf{v}} = \mathbf{v} - \mathbf{M}_1 G$ in these similarity variables. Expressing this result in the classical variables, we prove, in particular, the decay of the $\mathbf{L}^1$ norm of $\bar{\mathbf{v}}$.

The conclusion of Theorem 1.2 follows by interpolation of the $\mathbf{L}^p$ estimate with the $\mathbf{L}^1$ decay of the solution.

The outline of the proof of Theorem 1.3 is the following: we begin by determining the expressions of $[\mathbf{M}_2]$ distinguishing the dimension $N = 2$ and $N \geq 3$, using scaling arguments and similarity variables. Then, following in similarity variables, we compute the decay rate in $L^2(K)$ of $\mathbf{v} - \mathbf{M}_1 G - |\log(1+t)|[\mathbf{M}_2^1]\nabla G - [\mathbf{M}_2^2]\nabla G$ when $N = 2$ and of $\mathbf{v} - \mathbf{M}_1 G - [\mathbf{M}_2]\nabla G$ when $N \geq 3$. The expressions of these results in classical variables yield the conclusion of Theorem 1.3.

### 1.5 Plan of the paper

This article is organized as follows: at the beginning of the following section, we give some basic estimates like, for example, the energy dissipation law. Then, we study the decay rate in $\mathbf{L}^p$ of a solution of a generalized version of system (1.4). This system is similar to (1.4), but a little more complex because it contains some additional non-linear terms. As an application of these results, we deduce the decay rate of the solution $\mathbf{v}$ of system (1.4), as well as the decay rate of $\bar{\mathbf{v}} = \mathbf{v} - \mathbf{M}_1 G$. The decay of the $\mathbf{L}^1$ norm is proved in section 3 by classical parabolic techniques, using similarity variables and scaling arguments. However, in our case, the presence of the second unknown $\mathbf{g}$ requires special care. These arguments allow us to perform the proof of Theorem 1.2, combining the decay rate of the $\mathbf{L}^1$ norm with the results of section 2.1. Afterwards, in section 4, we identify the second term in the asymptotic development in similarity variables and give the proof of Theorem 1.3.

## 2 Decay rates

From now on, we shall work with the scalar functions $v$ and $g$ introduced in subsection 1.2 to denote any of the components $v_i, g_i$ of the vectors $\mathbf{v}$, $\mathbf{g}$.

### 2.1 Basic a priori estimates

*Energy dissipation:*
Multiplying by $v$ and integrating by parts the first equation of system (1.5), we find:

$$\frac{1}{2}\left[\int_\Omega v^2(t,x)d\mathbf{x} + m|g(t)|^2\right] + \int_0^t \int_\Omega |\nabla v|^2 d\mathbf{x}ds = \frac{1}{2}\left[\int_\Omega v^2(0,\mathbf{x})d\mathbf{x} + m|g(0)|^2\right]. \quad (2.1)$$

*$L^p$ estimates:*
In the same way as above, we multiply the equation by $j'(v)$, with $j$ a real valued

convex function and we integrate with respect to $\mathbf{x}$ to obtain:

$$\frac{d}{dt}\left[\int_\Omega j(v)d\mathbf{x} + mj(g(t))\right] = -\int_\Omega |\nabla v|^2 j''(v)d\mathbf{x}.$$

If we choose for $j(v)$ an approximation of the function $|v|^p$, we deduce that the quantity

$$\int_\Omega |v|^p d\mathbf{x} + m|g|^p, \tag{2.2}$$

decreases in time whenever $v_0 \in L^p(\Omega)$ for all $1 \le p < \infty$. The first step in the analysis of the large time behavior of (1.5) consists in establishing the decay rate of the solution. But, instead of studying directly (1.5), we prefer considering the following more general framework in which the same decay properties hold.

## 2.2 General decay results

We consider, in this subsection, any smooth global in time solution $(\mathbf{v}, \mathbf{g})$:

$$\begin{aligned}
&\mathbf{v} \in C\left([0,\infty), \mathbf{L}^2(\Omega)\right) \cap L^2\left((0,\infty), \dot{\mathbf{H}}^1(\Omega)\right) \cap L^\infty((0,\infty), \mathbf{L}^1(\Omega)),\\
&\mathbf{g} \in C\left([0,\infty), \mathbb{R}^N\right),
\end{aligned}$$

of the following non-linear system:

$$\begin{cases}
\mathbf{v}_t - \Delta\mathbf{v} - [\mathbf{U}]\mathbf{g} - \mathbf{V}(t)\cdot\nabla\mathbf{v} = \boldsymbol{\varepsilon}_1(\mathbf{x},t), & \text{in } \Omega\times(0,\infty),\\
\mathbf{v} = \mathbf{g}, & \text{in } \partial\Omega\times(0,\infty),\\
m\mathbf{g}'(t) = -\displaystyle\int_{\partial\Omega} \mathbf{n}\cdot\nabla\mathbf{v}d\sigma_x + \boldsymbol{\varepsilon}_2(t), & \text{on } (0,\infty),\\
\mathbf{v}(0) = \mathbf{v}_0, \quad \mathbf{g}(0) = \mathbf{g}_0,
\end{cases} \tag{2.3}$$

where $[\mathbf{U}](\mathbf{x},t)$ is a matrix valued function and $\mathbf{V}(t)$, $\boldsymbol{\varepsilon}_1(t)$ and $\boldsymbol{\varepsilon}_2(t)$ three vector valued functions which will be specified later.

In the sequel, we will apply the results obtained for the general system (2.3) in the following particular cases:

**Application 1** *If we specify* $[\mathbf{U}] = [\mathbf{0}]$, $\mathbf{V} = \mathbf{g}$, $\boldsymbol{\varepsilon}_1 = \mathbf{0}$ *and* $\boldsymbol{\varepsilon}_2 = \mathbf{0}$ *we obtain system* (1.4). *This case will be considered in subsection 2.3, Proposition 2.2.*

**Application 2** *In view of equations* (1.16) *and* (1.18), $(\bar{\mathbf{v}}, \bar{\mathbf{g}})$ *solves system* (2.3) *with* $\mathbf{V} = \mathbf{g}$, $[\mathbf{U}] = [\mathbf{0}]$, $\boldsymbol{\varepsilon}_1 := \mathbf{M}_1\mathbf{g}\cdot\nabla G$ *and* $\boldsymbol{\varepsilon}_2(t) := -\mathbf{M}_1\varepsilon(t)$ *where* $\varepsilon(t)$ *is defined by* (1.17). *This case will be treated in section 3, Proposition 3.1.*

In the following Proposition, we describe the decay rate in $\mathbf{L}^p$ of the solution $(\mathbf{v}, \mathbf{g})$ of the general system (2.3).

**Proposition 2.1** *Let us denote:*

$$\delta_1 := 2N \sup_{t\in(0,\infty)} (\|\mathbf{v}\|_1 + m|\mathbf{g}|) \quad \textit{and} \quad \epsilon_p(t) := \max\left(\|\boldsymbol{\varepsilon}_1\|_p, \frac{1}{m}|\boldsymbol{\varepsilon}_2|\right). \tag{2.4}$$

*Fix $1 < p < \infty$ and assume also that there exists $C_p > 0$ and $\alpha_p > 0$ such that the functions:*

$$\vartheta_1(t) := t\|[\mathbf{U}]\|_p \quad \textit{and} \quad \vartheta_2(t) := \epsilon_p(t) t^{\frac{N}{2}\left(1-\frac{1}{p}\right)+1}, \tag{2.5}$$

*fulfil the estimate:*

$$\vartheta_1(t) + \vartheta_2(t) \leq C_p\left(1 + t^{-\alpha_p}\right), \quad \forall t > 0. \tag{2.6}$$

*Then, any smooth solution $(\mathbf{v}, \mathbf{g})$ of system (2.3) satisfies the following decay properties:*

$$\|\mathbf{v}\|_p \leq C(p)\delta_p t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)}, \qquad |\mathbf{g}| \leq C(p)\delta_p t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)}, \quad \forall t \geq 1, \tag{2.7}$$

*where $\delta_p$ is a positive constant defined by:*

$$\delta_p := \delta_1 \max\left\{(1+\alpha_p)^{\frac{N}{2}\left(1-\frac{1}{p}\right)}, \left[\frac{1}{\delta_1}\sup_{t\in(1,\infty)}\vartheta_2(t)\right]^{\frac{N(p-1)}{2p+N(p-1)}}, \sup_{t\in(1,\infty)}\vartheta_1(t)^{\frac{N}{2}\left(1-\frac{1}{p}\right)}\right\}. \tag{2.8}$$

*Moreover, the constant $C(p)$ in estimates (2.7) depend on $p$ and $N$ only. Assume furthermore that:*

$$C_p \textit{ and } \alpha_p \textit{ in (2.6) are uniformly bounded for all } p \textit{ large enough.} \tag{2.9}$$

*In this case, estimates (2.7) remain valid for $p = \infty$ with $\delta_p$ as in (2.8) with $p = \infty$.*

**Remark 2.1** *The following comments are in order:*

- *In view of the definitions (2.5), it is obvious that $\vartheta_1$ and $\vartheta_2$ depend on $p$. Nevertheless, to shorten notations, we have not made this dependence explicit.*
- *We do not make any assumption on the decay properties of the potential $\mathbf{V}$ because the term $\mathbf{V}\cdot\nabla\mathbf{v}$ vanishes in all the estimates, since $\mathbf{V}$ depends only on $t$.*
- *The decay rate (2.7) we obtain for $v$ coincides with the one of the solution of the heat equation on $\mathbb{R}^N$ and with those of the 1-d model for fluid-solid interaction in [19].*

**Proof of Proposition 2.1:** We treat only the case $1 < p < \infty$. The case $p = \infty$ is obtained applying an iterative argument inspired by a work of L. Véron [21] and used in [19] for a fluid-solid interaction model. We refer to [16] for details. We proceed componentwise, using the rules of notation of section 1.1: $v$ and $g$ stand for any component $v_i$ and $g_i$ of $\mathbf{v}$ and $\mathbf{g}$. The corresponding first momentum will be denoted by $M_1$ although it stands for the quantity $M_{1,i}$.

Multiplying the equation (2.3-i) by $v|v|^{p-2}$ and integrating by parts, the term $\int_\Omega \mathbf{V}\cdot\nabla v|v|^{p-2}v d\mathbf{x}$ vanishes according to Green's formula and we get:

$$\frac{1}{p}\frac{d}{dt}[\|v\|_p^p + m|g|^p] = -\frac{4(p-1)}{p^2}\|\nabla|v|^{\frac{p}{2}}\|_2^2 + I(t), \tag{2.10}$$

where $I(t) := \int_\Omega \mathbf{g}\cdot\mathbf{U}v|v|^{p-2}d\mathbf{x} + \int_\Omega \varepsilon_1 v|v|^{p-2}d\mathbf{x} + \varepsilon_2 g|g|^{p-2} = I_1(t)+I_2(t)+I_3(t)$ can be estimated as follows:

**Lemma 2.1** *There exists a constant $C > 0$ depending on $m$ and $N$ only, such that, for all $t \geq 0$:*

$$|I(t)| \leq C\|[\mathbf{U}]\|_p \left[\|v\|_p^p + m\left(\max_i |g_i|\right)^p\right] + C\epsilon_p(t)\left[\|v\|_p^p + m|g|^p\right]^{1-1/p}. \quad (2.11)$$

**Proof of Lemma 2.1:** Concerning $I_1$, we have $|I_1(t)| \leq \int_\Omega |\mathbf{g}\cdot\mathbf{U}||v|^{p-1}d\mathbf{x} \leq \int_\Omega N^{1/2}\left(\max_i |g_i|\right)|\mathbf{U}||\mathbf{v}|^{p-1}d\mathbf{x}$. Applying Hölder's inequality we get $|I_1(t)| \leq N^{1/2}\left(\max_i |g_i|\right)\|[\mathbf{U}]\|_p\|v\|_p^{p-1}$ and since

$$\left(\max_i |g_i|\right)\|v\|_p^{p-1} \leq C\left[\|v\|_p^p + m\left(\max_i |g_i|\right)^p\right],$$

it comes:

$$|I_1(t)| \leq C\|[\mathbf{U}]\|_p\left[\|v\|_p^p + m\left(\max_i |g_i|\right)^p\right], \quad \forall\, t \geq 0. \tag{2.12}$$

For $I_2$, one checks easily, by Hölder's inequality that $|I_2(t)| \leq \|\varepsilon_1\|_p\|v\|_p^{p-1} \leq \|\boldsymbol{\varepsilon}_1\|_p\|v\|_p^{p-1}$ and $I_3$ satisfies $|I_3(t)| \leq |\varepsilon_2||g|^{p-1} \leq |\boldsymbol{\varepsilon}_2||g|^{p-1}$. Using the notation (2.4), these results yield $|I_2|+|I_3| \leq \epsilon_p\left[\|v\|_p^{p-1} + |g|^{p-1}\right]$. The function $x \mapsto x^{1-1/p}$ being concave, we get:

$$|I_2| + |I_3| \leq C\epsilon_p\left[\|v\|_p^p + m|g|^p\right]^{1-1/p}, \quad \forall\, t \geq 0, \tag{2.13}$$

with $C = C(m)$. Putting together (2.12) and (2.13), we obtain (2.11). ■

Going back to equation (2.10), we give now estimates for the term involving the gradient of $|v|^{\frac{p}{2}}$.

**Lemma 2.2** *For any $N \geq 2$ and $p > 1$, we have:*

$$\frac{\|v\|_p^{p\left(1+\frac{2}{N(p-1)}\right)}}{\left[\|v\|_1 + m|g|\right]^{\frac{2p}{N(p-1)}}} \leq C\|\nabla |v|^{\frac{p}{2}}\|_2^2, \tag{2.14}$$

$$\frac{(m|g|^p)^{1+\frac{2}{N(p-1)}}}{\left[\|v\|_1 + m|g|\right]^{\frac{2p}{N(p-1)}}} \leq C\|\nabla |v|^{\frac{p}{2}}\|_2^2, \quad \forall\, t \geq 0, \tag{2.15}$$

*where the constant $C > 0$ depends on $N$ and $m$ only.*

The proof of this Lemma is quite similar to the proof of Lemma 1 in [12]. The complete proof is given in [16].

Observe now that, by a convexity argument:

$$\left[\|v\|_p^p + m|g|^p\right]^{1+\frac{2}{N(p-1)}} \leq 2^{\frac{2}{N(p-1)}} \left[\|v\|_p^{p(1+\frac{2}{N(p-1)})} + (m|g|^p)^{1+\frac{2}{N(p-1)}}\right]. \tag{2.16}$$

Using the inequalities (2.14) and (2.15), one obtains:

$$\left[\|v\|_p^p + m|g|^p\right]^{1+\frac{2}{N(p-1)}} \leq C2^{\frac{2}{N(p-1)}} \left[\|v\|_1 + m|g|\right]^{\frac{2p}{N(p-1)}} \|\nabla |v|^{\frac{p}{2}}\|_2^2, \tag{2.17}$$

with $C$ uniform with respect to $p$. In view of the definition (2.4) of $\delta_1$, we get $\|v\|_1 + m|g| \leq \delta_1/2N$, and then

$$\left[\|v\|_p^p + m|g|^p\right]^{1+\frac{2}{N(p-1)}} \leq C\left(\frac{\delta_1}{N}\right)^{\frac{2p}{N(p-1)}} \|\nabla |v|^{\frac{p}{2}}\|_2^2, \tag{2.18}$$

where $C$ does not depend on $p$. In all the sequel we will be very careful on how the constants in the estimates depend on $\delta_1$ and $p$. Introducing the functions:

$$X_p := \left[\|v\|_p^p + m|g|^p\right]^{1/p} \quad \text{and} \quad Y_p := \left[\|\mathbf{v}\|_p^p + m|\mathbf{g}|^p\right]^{1/p}, \tag{2.19}$$

we can summarize (2.10), (2.11) and (2.18) by:

$$\frac{1}{p}(X_p^p)' + C\frac{(p-1)}{p^2}\left(\frac{\delta_1}{N}\right)^{-\frac{2p}{N(p-1)}} X_p^{p+\frac{2p}{N(p-1)}} - C\|[\mathbf{U}]\|_p \left[\|v\|_p^p + m\left(\max_i |g_i|\right)^p\right] - C\epsilon_p X_p^{p-1} \leq 0. \tag{2.20}$$

This last inequality holds for each component $v$ and $g$ of $\mathbf{v}$ and $\mathbf{g}$. Adding together these $N$ inequalities, we get for all $t \geq 0$:

$$\frac{1}{p}(Y_p^p)' + C\frac{(p-1)}{p^2}\left(\frac{\delta_1}{N}\right)^{-\frac{2p}{N(p-1)}} \sum_{i=1}^N \left[\|v_i\|_p^p + m|g_i|^p\right]^{1+\frac{2}{N(p-1)}}$$
$$- C\|[\mathbf{U}]\|_p \left[\sum_{i=1}^N \|v_i\|_p^p + mN\left(\max_i |g_i|\right)^p\right] - C\epsilon_p \sum_{i=1}^N \left[\|v_i\|_p^p + m|g_i|^p\right]^{1-\frac{1}{p}} \leq 0. \tag{2.21}$$

A convexity argument yields $Y_p^{p+\frac{2p}{N(p-1)}} \leq N^{\frac{2}{N(p-1)}} \sum_{i=1}^N \left[\|v_i\|_p^p + m|g_i|^p\right]^{1+\frac{2}{N(p-1)}}$ and then, since

$$\delta_1^{-\frac{2p}{N(p-1)}} Y_p^{p+\frac{2p}{N(p-1)}} = (\delta_1/N)^{-\frac{2p}{N(p-1)}} N^{-\frac{2p}{N(p-1)}} Y_p^{p+\frac{2p}{N(p-1)}},$$

we have $\delta_1^{-\frac{2p}{N(p-1)}} Y_p^{p+\frac{2p}{N(p-1)}} \leq (\delta_1/N)^{-\frac{2p}{N(p-1)}} \sum_{i=1}^N \left[\|v_i\|_p^p + m|g_i|^p\right]^{1+\frac{2}{N(p-1)}}$. One proves as well, by concavity of the function $x \mapsto x^{1-\frac{1}{p}}$, that:

$$\sum_{i=1}^N \left[\|v_i\|_p^p + m|g_i|^p\right]^{1-\frac{1}{p}} \leq C \left[\sum_{i=1}^N \|v_i\|_p^p + m|g_i|^p\right]^{1-\frac{1}{p}} = CY_p^{p-1}, \tag{2.22a}$$

with $C := N^{1/p} \leq N$. On the other hand, since $\left(\max_i |g_i|\right) \leq \left(\sum_{i=1}^N |g_i|^p\right)^{\frac{1}{p}}$, for all $1 \leq p < \infty$, we deduce that:

$$\sum_{i=1}^N \|v_i\|_p^p + mN\left(\max_i |g_i|\right)^p \leq NY_p^p. \tag{2.22b}$$

Relations (2.22) together with (2.21) yield, for all $t \geq 0$:

$$\frac{1}{p}(Y_p^p)' + C\frac{(p-1)}{p^2}\delta_1^{-\frac{2p}{N(p-1)}} Y_p^{p+\frac{2p}{N(p-1)}} - C\|[\mathbf{U}]\|_p Y_p^p - C\epsilon_p Y_p^{p-1} \leq 0. \tag{2.23}$$

According to notations (2.5) of Proposition 2.1, (2.20) reads:

$$\frac{1}{p}(Y_p^p)' + C\frac{(p-1)}{p^2}\delta_1^{-\frac{2p}{N(p-1)}} Y_p^{p+\frac{2p}{N(p-1)}} - Ct^{-1}\vartheta_1 Y_p^p - Ct^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-1}\vartheta_2 Y_p^{p-1} \leq 0,$$

for all $t \geq 0$. Multiplying both sides by $Y_p^{1-p}$, it comes:

$$Y_p' + C(p)\delta_1^{-\frac{2p}{N(p-1)}} Y_p^{1+\frac{2p}{N(p-1)}} - Ct^{-1}\vartheta_1 Y_p - Ct^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-1}\vartheta_2 \leq 0, \tag{2.24}$$

with $C(p) := C\,(p-1)/p^2$. We introduce then the function $Z_p$ defined on $(0,\infty)$ by $Z_p(t) := \bar{C}_p\delta_p\left(N/2 + \gamma_1 t^{-\alpha_p}\right)^{(N/2)(1-1/p)} t^{-N/2(1-1/p)}$ where $\delta_p$ and $\alpha_p$ are the constants defined by (2.8) and in the hypothesis (2.6) respectively and $\gamma_1$ and $\bar{C}_p$ are as follows:

$$\gamma_1 := \sup_{t\in(0,1)} \vartheta_1(t)t^{\alpha_p} + \delta_1^{-1} \sup_{t\in(0,1)} \vartheta_2(t)t^{\alpha_p}, \tag{2.25a}$$

$$\bar{C}_p := \left[\frac{3}{C(p)} \max\left\{\frac{N}{2}, C(p), C\right\}\right]^{\frac{N}{2}}, \tag{2.25b}$$

$C(p)$ and $C$ being the constants in (2.24). The assumption (2.6) of the Proposition ensures that $\gamma_1$ is well defined. Direct computations yield:

$$Z_p' + C(p)\delta_1^{-\frac{2p}{N(p-1)}} Z_p^{1+\frac{2p}{N(p-1)}} - Ct^{-1}\vartheta_1 Z_p - Ct^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-1}\vartheta_2 = F(t), \qquad (2.26)$$

for all $t \geq 0$, where $F(t)$ is defined on $(0,\infty)$ by:

$$F(t) := \frac{Z_p}{t}\left\{ -\frac{N}{2}\left(1-\frac{1}{p}\right) + \frac{\gamma_1}{t^{\alpha_p}}\left[ C(p)\left(\bar{C}_p\frac{\delta_p}{\delta_1}\right)^{\frac{2p}{N(p-1)}} - \alpha_p \frac{\frac{N}{2}\left(1-\frac{1}{p}\right)}{\left(\frac{N}{2}+\gamma_1/t^{\alpha_p}\right)}\right]\right.$$
$$\left. + C(p)\frac{N}{2}\left(\bar{C}_p\frac{\delta_p}{\delta_1}\right)^{\frac{2p}{N(p-1)}} - C\vartheta_1 - \frac{C(\bar{C}_p\delta_p)^{-1}}{\left(\frac{N}{2}+\gamma_1/t^{\alpha_p}\right)^{\frac{N}{2}\left(1-\frac{1}{p}\right)}}\vartheta_2 \right\}. \qquad (2.27)$$

We are going to prove that any solution $Y_p$ of (2.24) is a sub-solution of (2.26) and hence, by Theorem 1.5.3 of [15], that:

$$Y_p(t) \leq Z_p(t), \quad \forall\, t \geq 0. \qquad (2.28)$$

Observe that $t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)}$ in the definition of $Z_p$ is the term we need in the right hand side of (2.28) in order to conclude the proof of the decay rate (2.7). The term added in the expression of $Z_p$ in which $t^{-\alpha_p}$ appears has no incidence on the asymptotic behavior as $t \to \infty$, but it is required to get (2.28) in the neighborhood of $t = 0$. Thus, it is sufficient to prove that:

$$F(t) \geq 0, \quad \forall\, t \geq 0. \qquad (2.29)$$

Note that the term $t^{-1}Z_p$, in the definition (2.27) of $F(t)$, is positive. Since $N/2\,(1-1/p)/(N/2+\gamma_1 t^{-\alpha_p}) \leq 1$, we obtain also that:

$$C(p)(\delta_1^{-1}\bar{C}_p\delta_p)^{\frac{2p}{N(p-1)}} - \alpha_p \frac{\frac{N}{2}\left(1-\frac{1}{p}\right)}{\left(\frac{N}{2}+\gamma_1 t^{-\alpha_p}\right)} \geq C(p)(\delta_1^{-1}\bar{C}_p\delta_p)^{\frac{2p}{N(p-1)}} - \alpha_p.$$

On the other hand $(N/2+\gamma_1 t^{-\alpha_p})^{-\frac{N}{2}\left(1-\frac{1}{p}\right)} \leq 1$, because $N \geq 2$, and $\bar{C}_p^{-1} \leq 1$ (obvious with the definition (2.25)) and $N/2 \geq 1$. Hence, to get (2.29), it is sufficient to prove that:

$$-\frac{N}{2}+\frac{\gamma_1}{t^{\alpha_p}}\left[C(p)\left(\bar{C}_p\frac{\delta_p}{\delta_1}\right)^{\frac{2p}{N(p-1)}} - \alpha_p\right] + C(p)\left(\bar{C}_p\frac{\delta_p}{\delta_1}\right)^{\frac{2p}{N(p-1)}} - C\vartheta_1 - \frac{C}{\delta_p}\vartheta_2 \geq 0. \quad (2.30)$$

Dividing in (2.30) by $\max\{N/2, C(p), C\} \geq 1$, the problem is reduced to prove that:

$$\gamma_1 t^{-\alpha_p}\left[\tilde{C}(\delta_1^{-1}\bar{C}_p\delta_p)^{\frac{2p}{N(p-1)}} - \alpha_p\right] + \tilde{C}(\delta_1^{-1}\bar{C}_p\delta_p)^{\frac{2p}{N(p-1)}} - 1 - \vartheta_1 - \delta_p^{-1}\vartheta_2 \geq 0, \quad (2.31)$$

for all $t \geq 0$, where $0 < \tilde{C} < 1$ is defined by

$$\tilde{C} := C(p) \max\left\{\frac{N}{2}, C(p), C\right\}^{-1}. \tag{2.32}$$

We proceed in two steps proving first that (2.31) holds on $(1, \infty)$ and then on $(0, 1)$.

**The case $t \in (1, \infty)$:**

According to the definition (2.8), we have $\delta_p \geq \delta_1 \delta_1^{-\frac{N(p-1)}{2p+N(p-1)}} \sup_{t\in(1,\infty)} \vartheta_2(t)^{\frac{N(p-1)}{2p+N(p-1)}}$. We deduce that, for all $t \geq 1$:

$$\tilde{C}(\bar{C}_p \delta_1^{-1} \delta_p)^{\frac{2p}{N(p-1)}} \geq \tilde{C}\bar{C}_p^{\frac{2p}{N(p-1)}} \delta_1^{-\frac{2p}{2p+N(p-1)}} \sup_{t\in(1,\infty)} \vartheta_2(t)^{\frac{2p}{2p+N(p-1)}}, \tag{2.33a}$$

$$\delta_p^{-1}\vartheta_2 \leq \delta_p^{-1} \sup_{t\in(1,\infty)} \vartheta_2(t) \leq \delta_1^{-\frac{2p}{2p+N(p-1)}} \sup_{t\in(1,\infty)} \vartheta_2(t)^{\frac{2p}{2p+N(p-1)}}. \tag{2.33b}$$

Combining (2.33a) and (2.33b), we get, for all $t \geq 1$:

$$\tilde{C}(\bar{C}_p \delta_1^{-1} \delta_p)^{\frac{2p}{N(p-1)}} \geq \tilde{C}\bar{C}_p^{\frac{2p}{N(p-1)}} \delta_p^{-1}\vartheta_2. \tag{2.34}$$

Comparing the definitions (2.25) and (2.32) of $\bar{C}_p$ and $\tilde{C}$, one remarks that $\bar{C}_p = (3\tilde{C}^{-1})^{N/2}$ and therefore that $\tilde{C}\bar{C}_p^{2p/N(p-1)} = 3(3\tilde{C}^{-1})^{1/(p-1)}$. Since $0 < \tilde{C} < 1$, this implies:

$$\tilde{C}\bar{C}_p^{\frac{2p}{N(p-1)}} \geq 3, \tag{2.35}$$

and (2.34) becomes:

$$\frac{1}{3}\tilde{C}(\bar{C}_p \delta_1^{-1} \delta_p)^{\frac{2p}{N(p-1)}} \geq \delta_p^{-1}\vartheta_2, \quad \forall t \geq 1. \tag{2.36a}$$

On the other hand since, by the definition (2.8), $\delta_p \geq \delta_1 \sup_{t\in(1,\infty)} \vartheta_1(t)^{\frac{N}{2}\left(1-\frac{1}{p}\right)}$, we have also $\tilde{C}(\bar{C}_p\delta_1^{-1}\delta_p)^{2p/N(p-1)} \geq \tilde{C}\bar{C}_p^{2p/N(p-1)}\vartheta_1$ for all $t \geq 1$. As a consequence of (2.35) we get:

$$\frac{1}{3}\tilde{C}(\bar{C}_p \delta_1^{-1} \delta_p)^{\frac{2p}{N(p-1)}} \geq \vartheta_1, \quad \forall t \geq 1. \tag{2.36b}$$

Finally, once again, according to the definition (2.8), $\delta_p \geq (1 + \alpha_p)^{N/2(1-1/p)}\delta_1$, what leads to $\tilde{C}(\bar{C}_p\delta_1^{-1}\delta_p)^{2p/N(p-1)} \geq \tilde{C}\bar{C}_p^{2p/N(p-1)}(1+\alpha_p)$, and hence, with (2.35):

$$\frac{1}{3}\tilde{C}(\bar{C}_p \delta_1^{-1} \delta_p)^{\frac{2p}{N(p-1)}} \geq (1 + \alpha_p) \geq 1. \tag{2.36c}$$

Summing together the three relations (2.36), we get:

$$\tilde{C}(\delta_1^{-1}\bar{C}_p\delta_p)^{\frac{2p}{N(p-1)}} - 1 - \vartheta_1 - \delta_p^{-1}\vartheta_2 \geq 0, \quad \forall t \geq 1.$$

We deduce also, from (2.36c) that $\tilde{C}(\delta_1^{-1}\bar{C}_p\delta_p)^{\frac{2p}{N(p-1)}} - \alpha_p \geq 0$ for all $t > 0$, what allows us to conclude that (2.31) is true for all $t \geq 1$.

**The case $t \in (0,1)$:** We must now establish the estimate (2.31) on the interval $(0,1)$. Since $t^{\alpha_p} > 0$, (2.31) is equivalent to:

$$\gamma_1 \left[ \tilde{C} \left( \bar{C}_p \frac{\delta_p}{\delta_1} \right)^{\frac{2p}{N(p-1)}} - \alpha_p \right] + \tilde{C} t^{\alpha_p} \left( \bar{C}_p \frac{\delta_p}{\delta_1} \right)^{\frac{2p}{N(p-1)}} - t^{\alpha_p} - \vartheta_1 t^{\alpha_p} - \delta_p^{-1} \vartheta_2 t^{\alpha_p} \geq 0.$$

According to (2.36c) we have $\tilde{C}\left(\bar{C}_p\delta_p/\delta_1\right)^{2p/N(p-1)} - 1 \geq 0$, as well as

$$\tilde{C}\left(\bar{C}_p\delta_p/\delta_1\right)^{2p/N(p-1)} - \alpha_p \geq 1.$$

From the definition (2.8) of $\delta_p$, we deduce straightforwardly that $\delta_p \geq \delta_1$. Hence, it remains only to check that:

$$\gamma_1 - \vartheta_1 t^{\alpha_p} - \delta_1^{-1}\vartheta_2 t^{\alpha_p} \geq 0, \quad \forall 0 < t \leq 1, \tag{2.37}$$

what is obvious in view of the definition (2.25) of $\gamma_1$. The proof is then completed for $p < \infty$. ■

## 2.3 Decay rates

We are now in a position to prove the following Proposition, as announced in Application 2:

**Proposition 2.2** *Assume that the initial data $(\mathbf{v}_0, \mathbf{g}_0) \in \mathbf{L}^2(\Omega) \times \mathbb{R}^N$ of system (1.4) are such that $\mathbf{v}_0 \in \mathbf{L}^1(\Omega)$. Then:*

$$\|\mathbf{v}(t)\|_p \leq C[\|\mathbf{v}_0\|_1 + m|\mathbf{g}_0|]t^{-\frac{N}{2}(1-\frac{1}{p})} \quad \textit{and} \quad |\mathbf{g}(t)| \leq C[\|\mathbf{v}_0\|_1 + m|\mathbf{g}_0|]t^{-\frac{N}{2}},$$

*for all $t > 0$ and for all $1 \leq p \leq \infty$. The constant $C > 0$ in these estimates depends on $p$, $m$ and $N$ but is independent of the initial data.*

**Remark 2.2** *The complete asymptotic analysis will show that these decay estimates are sharp. The decay rate of $\mathbf{g}$ is a consequence of the $\mathbf{L}^\infty$ estimate of $\mathbf{v}$, because of the transmission condition $\mathbf{v} = \mathbf{g}$ on the interface $\partial\Omega$.*

**Proof :** As explained in Application 1, we only have to apply Proposition 2.1, setting $\mathbf{V} = \mathbf{0}$, $[\mathbf{U}] = [\mathbf{0}]$. Condition (2.6) is trivially satisfied. The decay property (2.2) ensures that, since $\mathbf{v}_0$ is in $\mathbf{L}^1(\Omega)$, we have $\|\mathbf{v}\|_1 + m|\mathbf{g}| \leq \|\mathbf{v}_0\|_1 + m|\mathbf{g}_0|$. Therefore, in this case, we can set $\delta_1 = 2N(\|\mathbf{v}_0\|_1 + m|\mathbf{g}_0|)$ and the proof of the Proposition is then completed. ■

## 3 The first term in the asymptotic expansion

This section is devoted to the proof of Theorem 1.2.

As we have pointed out in section 1.3, the first momentum $\mathbf{M}_1$, defined by $\mathbf{M}_1 := \int_\Omega \mathbf{v}(t)d\mathbf{x} + m\mathbf{g}(t)$, is constant in time. The role played by this quantity in the description of the large time behavior of $\mathbf{v}$ is made precise in Theorem 1.2. Actually, $\mathbf{M}_1 G(t)$ is the first term in the asymptotic development of $\mathbf{v}$.

In application 2 we have defined $\bar{\mathbf{v}} := \mathbf{v} - \mathbf{M}_1 G$ and $\bar{\mathbf{g}}$, the trace of $\bar{\mathbf{v}}$ on the boundary of $\Omega$. The pair, $(\bar{\mathbf{v}}, \bar{\mathbf{g}})$ solves:

$$\begin{cases} \bar{\mathbf{v}}_t - \Delta\bar{\mathbf{v}} - \mathbf{g}\cdot\nabla\bar{\mathbf{v}} = \boldsymbol{\varepsilon}_1, & \mathbf{x}\in\Omega, \qquad\qquad t>0,\\ \bar{\mathbf{v}}(\mathbf{x},t) = \bar{\mathbf{g}}(t), & \mathbf{x}\in\partial B, \qquad\qquad t>0,\\ m\bar{\mathbf{g}}'(t) = -\displaystyle\int_{\partial\Omega} \mathbf{n}\cdot\nabla\bar{\mathbf{v}}\,d\sigma_x + \boldsymbol{\varepsilon}_2(t), & \qquad\qquad\qquad\quad t>0,\\ \bar{\mathbf{v}}(\mathbf{x},0) := \bar{\mathbf{v}}_0(\mathbf{x}) = \mathbf{v}_0(\mathbf{x}), \quad \mathbf{x}\in\Omega, & \bar{\mathbf{g}}(0) := \bar{\mathbf{g}}_0 = h_1, \end{cases} \tag{3.1}$$

where $\boldsymbol{\varepsilon}_1 := \mathbf{M}_1\mathbf{g}\cdot\nabla G$ and $\boldsymbol{\varepsilon}_2 = -\mathbf{M}_1\varepsilon$ (see (1.17)). The following Proposition concerns the decay rate of $\bar{\mathbf{v}}$ and $\bar{\mathbf{g}}$:

**Proposition 3.1** *Assume that the initial data $(\mathbf{v}_0, \mathbf{g}_0) \in \mathbf{L}^2(\Omega)\times\mathbb{R}^N$ of (1.4) are such that $\mathbf{v}_0 \in \mathbf{L}^1(\Omega)$. Then define, for all $t \geq 0$:*

$$\delta_1(t) := 2N \sup_{s\geq t}\left[\|\bar{\mathbf{v}}(s)\|_1 + m|\bar{\mathbf{g}}(s)|\right], \tag{3.2}$$

*and also set, for all $t \geq 1$ and all $1 < p \leq \infty$, distinguishing the values of the mass $m$ of the ball:*

*When $m = \dfrac{\sigma_N}{N}$:*

$$\delta_p(t) = \delta_1(t)\max\left\{\left(1+\frac{N}{2}\right)^{\frac{N}{2}\left(1-\frac{1}{p}\right)}, \left(\frac{[\|\mathbf{v}(t)\|_1 + m|\mathbf{g}(t)|]}{\delta_1(t)\, t^{\frac{N}{2}\min\{\frac{1}{p}+\frac{2}{N},1-\frac{1}{N}\}}}\right)^{\frac{N(p-1)}{2p+N(p-1)}}\right\}. \tag{3.3a}$$

*When $m \neq \dfrac{\sigma_N}{N}$:*

$$\delta_p(t) = \delta_1(t)\max\left\{\left(1+\frac{N}{2}\right)^{\frac{N}{2}\left(1-\frac{1}{p}\right)}, \left(\frac{[\|\mathbf{v}(t)\|_1 + m|\mathbf{g}(t)|]}{\delta_1(t)\, t^{\frac{N}{2}\min\{\frac{1}{p},1-\frac{1}{N}\}}}\right)^{\frac{N(p-1)}{2p+N(p-1)}}\right\}. \tag{3.3b}$$

*Then, $\delta_1(t)$ is bounded on $[0,\infty)$ and for any $t_0 \geq 1$, the solution $(\bar{\mathbf{v}}, \bar{\mathbf{g}})$ of (3.1) satisfies the following decay properties:*

$$\|\bar{\mathbf{v}}(t)\|_p \leq C\delta_p(t_0)t^{-\frac{N}{2}(1-\frac{1}{p})} \quad \textit{and} \quad |\bar{\mathbf{g}}(t)| \leq C\delta_\infty(t_0)t^{-\frac{N}{2}}, \quad \forall t \geq t_0+1. \tag{3.4}$$

**Remark 3.1** *We shall prove later that* $\|\bar{\mathbf{v}}(t)\|_1 + m|\bar{\mathbf{g}}(t)|$ *goes to 0 as* $t \to \infty$ *and hence that also* $\delta_1(t)$ *and* $\delta_p(t)$ *go to* 0 *as* $t \to \infty$. *Then, choosing* $t_0 = t/2$ *we will be able to improve the decay rate of* $\|\bar{\mathbf{v}}(t)\|_p$ *and* $|\bar{\mathbf{g}}(t)|$.

Let us define

$$\mathcal{K}(t, \mathbf{x}) := \exp\left(\frac{\mathbf{x}^2}{4(t+1)}\right), \tag{3.5}$$

and recall that $K(\mathbf{x}) := \mathcal{K}(0, \mathbf{x})$. Since $K$ is radially symmetric, we will sometimes use the notation $K(r)$ with $r = |\mathbf{x}| \in \mathbb{R}_+$ instead of $K(\mathbf{x})$.

In the sequel, we will perform the proof of the following Proposition, improving the decay rate of $\bar{\mathbf{v}}$ given in Proposition 3.1 for particular values of $p$.

**Proposition 3.2** *Let* $\mathbf{v}_0$ *be in* $\mathbf{L}^2(K, \Omega)$. *Then the solution* $(\mathbf{v}, \mathbf{g})$ *of system* (1.4) *satisfies the estimate:*

$$\textit{When } N = 2: \qquad \|\mathbf{v} - \mathbf{M}_1 G\|_p \leq C|\log(1+t)|t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}, \quad \forall t \geq 1, \tag{3.6a}$$

$$\textit{When } N \geq 3: \qquad \|\mathbf{v} - \mathbf{M}_1 G\|_p \leq Ct^{-\frac{N}{2}\left(1-\frac{1}{p}\right)-\frac{1}{2}}, \quad t \geq 1, \tag{3.6b}$$

*for all* $1 \leq p \leq 2$. *The constant* $C$ *in these estimates depends on* $p$, $N$ *and* $m$.

**Remark 3.2** *Estimates* (3.6) *fit exactly those of the heat equation on the whole space* $\mathbb{R}^N$, *the case* $N = 2$ *being excepted, where a logarithmic term appears in the decay rate. We shall show that this logarithmic term is due to the contribution of the solid mass in the system and that estimates* (3.6) *are sharp when* $p = 2$.

**Proof of Theorem 1.2:** Assuming that Proposition 3.2 holds, let us proceed to complete the proof of Theorem 1.2.

Relation (3.6) with $p = 1$ provides the estimates:

$$\text{When } N = 2: \qquad \|\bar{\mathbf{v}}\|_1 \leq C|\log(1+t)|t^{-\frac{1}{2}}, \quad \forall t \geq 1,$$

$$\text{When } N \geq 3: \qquad \|\bar{\mathbf{v}}\|_1 \leq Ct^{-\frac{1}{2}}, \quad \forall t \geq 1.$$

From Proposition 2.2 we deduce that $|\bar{\mathbf{g}}(t)| \leq Ct^{-\frac{N}{2}}$. Therefore the positive constant $\delta_1(t)$ of Proposition 3.1 can be estimated as follows:

$$\text{When } N = 2: \qquad \delta_1(t) \leq C|\log(1+t)|t^{-\frac{1}{2}}, \quad \forall t \geq 1, \tag{3.7a}$$

$$\text{When } N \geq 3: \qquad \delta_1(t) \leq Ct^{-\frac{1}{2}}, \quad \forall t \geq 1. \tag{3.7b}$$

On the other hand, (3.4) ensures that, for all $t_0 \geq 1$:

$$\|\bar{\mathbf{v}}(t)\|_p \leq C\delta_p(t_0)t^{-\frac{N}{2}\left(1-\frac{1}{p}\right)}, \quad \forall t \geq t_0 + 1, \tag{3.8}$$

the constant $\delta_p(t_0)$ being defined by (3.3).

**The case $m \neq \sigma_N/N$:** since the quantity $\|\mathbf{v}\|_1 + m|\mathbf{g}|$ decreases in time (2.2) and $(1+N/2)^{N/2(1-1/p)} \leq C_N := (1+N/2)^{N/2}$, from (3.3) we deduce that, for all $1 < p \leq \infty$:

$$\delta_p(t) \leq \delta_1(t) \max\left\{ C_N, \left( C\delta_1(t)^{-1} t^{-\frac{N}{2}\min\{\frac{1}{p},1-\frac{1}{N}\}} \right)^{\frac{N(p-1)}{2p+N(p-1)}} \right\}. \tag{3.9}$$

Since $0 \leq \frac{N(p-1)}{2p+N(p-1)} \leq \frac{N}{N+2} \leq 1$, we can assume that the constant $C^{\frac{N(p-1)}{2p+N(p-1)}}$ is independent of $p$ and rewrite the inequality (3.9):

$$\delta_p(t) \leq C_N \delta_1(t)^{\frac{2p}{2p+N(p-1)}} \left[ \max\left\{ \delta_1(t), t^{-\frac{N}{2}\min\{\frac{1}{p},1-\frac{1}{N}\}} \right\} \right]^{\frac{N(p-1)}{2p+N(p-1)}}. \tag{3.10}$$

According to (3.7),

- When $N = 2$:

  $$\begin{aligned} \max\left\{ \delta_1(t), t^{-\frac{N}{2}\min\{\frac{1}{p},1-\frac{1}{N}\}} \right\} \leq &\max\left\{ |\log(1+t)| t^{-\frac{1}{2}}, t^{-\frac{N}{2}\min\{\frac{1}{p},1-\frac{1}{N}\}} \right\} \\ \leq & t^{-\frac{1}{2}} \max\left\{ |\log(1+t)|, t^{\frac{1}{2p}[p-\min\{N,p(N-1)\}]} \right\}, \end{aligned}$$

  and, because $N \geq 2$, basic computations yield

  $$p - \min\{N, p(N-1)\} \geq 0 \Leftrightarrow p \geq N.$$

  Therefore, from (3.10) and (3.7), we deduce for all $t$ large enough:

  – For all $1 < p \leq N$:

  $$\begin{aligned} \delta_p(t) \leq & C\left( |\log(1+t)| t^{-\frac{1}{2}} \right)^{\frac{2p}{2p+N(p-1)}} \left( |\log t| t^{-\frac{1}{2}} \right)^{\frac{N(p-1)}{2p+N(p-1)}} \\ \leq & C|\log(1+t)| t^{-\frac{1}{2}}. \end{aligned} \tag{3.11a}$$

  – For all $N < p \leq \infty$:

  $$\begin{aligned} \delta_p(t) \leq & C\left( |\log(1+t)| t^{-\frac{1}{2}} \right)^{\frac{2p}{2p+N(p-1)}} \left( t^{-\frac{N}{2p}} \right)^{\frac{N(p-1)}{2p+N(p-1)}} \\ \leq & C|\log(1+t)|^{\frac{2p}{2p+N(p-1)}} t^{-\frac{1}{2}+\theta(N,p)}, \end{aligned} \tag{3.11b}$$

  with $\theta(N,p) := \frac{N}{2}\frac{(p-1)(p-N)}{p(2p+N(p-1))}$.

- When $N \geq 3$: the only difference with the case $N = 2$ comes from (3.7), that is to say from the absence of logarithmic term. Consequently, we get the following estimates for $\delta_p(t)$, for all $t$ large enough:

– For all $1 < p \leq N$:

$$\delta_p(t) \leq Ct^{-\frac{1}{2}}. \tag{3.11c}$$

– For all $N < p \leq \infty$:

$$\delta_p(t) \leq Ct^{-\frac{1}{2}+\theta(N,p)}. \tag{3.11d}$$

**The case $m = \sigma_N/N$:** the definition of $\delta_p(t)$ is different and according to (3.3), we must turn (3.9) into:

$$\delta_p(t) \leq \delta_1(t) \max\left\{ C_N, \left( C\delta_1(t)^{-1} t^{-\frac{N}{2}\min\{\frac{1}{p}+\frac{2}{N}, 1-\frac{1}{N}\}} \right)^{\frac{N(p-1)}{2p+N(p-1)}} \right\}.$$

The same kind of computations as above leads to:

- When $N = 2$ and for all $1 < p \leq \infty$ and all $t$ large enough:

$$\delta_p(t) \leq C|\log(1+t)|t^{-\frac{1}{2}}. \tag{3.12a}$$

- When $N \geq 3$ and for all $1 < p \leq \infty$ and all $t$ large enough:

$$\delta_p(t) \leq Ct^{-\frac{1}{2}}. \tag{3.12b}$$

The estimates of Theorem 1.2 arise straightforwardly when combining (3.8) with (3.11) and (3.12) and specifying $t_0 = t/2$. ■

We perform now the proof of Proposition 3.1.
**Proof of Proposition 3.1:** It is quite easy to check that $\delta_1(t)$ is bounded for all $t \geq 0$. Indeed, according to the definition of $\bar{\mathbf{v}}$ and $\bar{\mathbf{g}}$ in Application 2, we have $\|\bar{\mathbf{v}}\|_1 + m|\bar{\mathbf{g}}| \leq \|\mathbf{v}\|_1 + m|\mathbf{g}| + |\mathbf{M}_1|\|G\|_1 + |\mathbf{M}_1|m|J|$. Explicit computations give $\|G\|_1 \leq 1$ and $|J| \leq (4\pi t)^{-N/2}e^{-1/4t}$ and $|\mathbf{M}_1| \leq \|\mathbf{v}(t)\|_1 + m|\mathbf{g}(t)|$. On the other hand, relation (2.2) ensures that $\|\mathbf{v}(t)\|_1 + m|\mathbf{g}(t)| \leq \|\mathbf{v}_0\|_1 + m|\mathbf{g}_0|$ so that $\|\bar{\mathbf{v}}\|_1 + m|\bar{\mathbf{g}}| \leq C\|\mathbf{v}_0\| + m|\mathbf{g}_0|$.

The proof of estimates (3.4) derives from Proposition 2.1. We have $\|\nabla G\|_p \leq Ct^{-N/2(1-1/p)-1/2}$, for all $1 \leq p \leq \infty$, where the constant $C$ does not depend on $p$. On the other hand, Proposition 2.2 ensures that:

$$|\mathbf{g}| \leq C[\|\mathbf{v}_0\|_1 + m|\mathbf{g}_0|]t^{-\frac{N}{2}}, \quad \forall\, t \geq 0. \tag{3.13}$$

Therefore, since $\boldsymbol{\varepsilon}_1 = \mathbf{M}_1\mathbf{g}\cdot\nabla G$, we deduce that:

$$\|\varepsilon_1\|_p \leq C|\mathbf{g}|\|\nabla G\|_p \leq C[\|\mathbf{v}_0\|_1 + m|\mathbf{g}_0|]t^{-\frac{N}{2}\left(2-\frac{1}{p}\right)-\frac{1}{2}}, \quad \forall\, t \geq 0. \tag{3.14}$$

The definition of $\boldsymbol{\varepsilon}_2$ leads to the estimates:

$$\text{When } m = \frac{\sigma_N}{N}: \quad |\boldsymbol{\varepsilon}_2| \leq C|\mathbf{M}_1|t^{-\frac{N}{2}-2} \leq C[\|\mathbf{v}_0\|_1 + m|\mathbf{g}_0|]t^{-\frac{N}{2}-2}. \tag{3.15a}$$

$$\text{When } m \neq \frac{\sigma_N}{N}: \quad |\boldsymbol{\varepsilon}_2| \leq C|\mathbf{M}_1|t^{-\frac{N}{2}-1} \leq C[\|\mathbf{v}_0\|_1 + m|\mathbf{g}_0|]t^{-\frac{N}{2}-1}. \tag{3.15b}$$

Note that the decay rate of the correcting term $\boldsymbol{\epsilon}_2$ is of order $t^{-N/2-2}$ when $m = \sigma_N/N$ and only of order $t^{-N/2-1}$ when $m \neq \sigma_N/N$. That leads to distinguish these two cases in Theorem 1.2.

From (3.14) and (3.15) and according to the definition (2.5) of $\vartheta_2$, we deduce that:

When $m = \dfrac{\sigma_N}{N}$:

$$\vartheta_2 \leq C[\|\mathbf{v}_0\|_1 + m|\mathbf{g}_0|]t^{-\frac{N}{2}\max\left\{\frac{1}{p}+\frac{2}{N},1-\frac{1}{N}\right\}}, \quad \forall 0 < t \leq 1, \tag{3.16a}$$

$$\vartheta_2 \leq C[\|\mathbf{v}_0\|_1 + m|\mathbf{g}_0|]t^{-\frac{N}{2}\min\left\{\frac{1}{p}+\frac{2}{N},1-\frac{1}{N}\right\}}, \quad \forall t \geq 1. \tag{3.16b}$$

When $m \neq \dfrac{\sigma_N}{N}$:

$$\vartheta_2 \leq C[\|\mathbf{v}_0\|_1 + m|\mathbf{g}_0|]t^{-\frac{N}{2}\max\left\{\frac{1}{p},1-\frac{1}{N}\right\}}, \quad \forall 0 < t \leq 1, \tag{3.16c}$$

$$\vartheta_2 \leq C[\|\mathbf{v}_0\|_1 + m|\mathbf{g}_0|]t^{-\frac{N}{2}\min\left\{\frac{1}{p},1-\frac{1}{N}\right\}}, \quad \forall t \geq 1. \tag{3.16d}$$

In model (3.1), $[\mathbf{U}] = [\mathbf{0}]$ and $\mathbf{V} = \mathbf{g}$ with the notations of system (2.3). Thus, $\vartheta_1 = 0$ and according to (3.16), the hypotheses (2.6) and (2.9) are fulfilled with $\alpha_p = N/2 + 1$. Note in particular that $\alpha_p$ is independent of $p$. Therefore, Proposition 2.1 applies and relation (3.4) holds with $t_0 = 1$ for all $1 \leq p \leq \infty$. The constant $\delta_p$ is defined by (2.8) and $\delta_\infty$ by (2.8) specifying $p = \infty$.

To get estimates (3.4) for any $t_0 \geq 1$, remark that the proof above applies for the functions $\bar{v}(t + t_0)$ and $\bar{g}(t + t_0)$ and the initial conditions $\bar{\mathbf{v}}(t_0)$ and $\bar{\mathbf{g}}(t_0)$. Indeed, all the estimates (3.13), (3.14), (3.15) and (3.16) remain valid replacing $[\|\mathbf{v}_0\|_1 + m|\mathbf{g}_0|]$ by $[\|\mathbf{v}(t_0)\|_1 + m|\mathbf{g}(t_0)|]$, because this quantity decreases in time (see (2.2)). According to (3.16), we can simplify the expression of $\delta_p(t_0)$ and turn (2.8) into (3.3). ■

**Proof of Proposition 3.2:** We use the so-called similarity variables (we refer to [10], [11], [12] and [22] for details):

$$\mathbf{y} := \frac{\mathbf{x}}{\sqrt{1+t}}, \quad s := \log(1+t), \tag{3.17a}$$

or equivalently:

$$\mathbf{x} := e^{\frac{s}{2}}\mathbf{y}, \quad t := e^s - 1, \tag{3.17b}$$

together with the rescaled functions:

$$\boldsymbol{\xi}(\mathbf{y}, s) := e^{\frac{s}{2}N}\mathbf{v}(\mathbf{y}e^{\frac{s}{2}}, e^s - 1) \quad \text{and} \quad \boldsymbol{\zeta}(s) := e^{\frac{s}{2}N}\mathbf{g}(e^s - 1). \tag{3.18}$$

Equivalently, we can express $\mathbf{v}$ and $\mathbf{g}$ with respect to $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$:

$$\mathbf{v}(\mathbf{x},t) = (t+1)^{-\frac{N}{2}}\boldsymbol{\xi}\left(\frac{\mathbf{x}}{\sqrt{1+t}}, \log(1+t)\right), \quad \mathbf{g}(t) = (t+1)^{-\frac{N}{2}}\boldsymbol{\zeta}(\log(1+t)).$$

This change of variables maps both fixed domains $B$ and $\Omega$ on the time dependent ones $B_s$ and $\Omega_s := \mathbb{R}^N \setminus B_s$, where $B_s$ stands for the ball of radius $r_s := e^{-s/2}$ centered at the origin. In these new variables, the law of conservation of momentum reads as follows:

$$\mathbf{M}_1 = \int_{\Omega_s} \boldsymbol{\xi}(\mathbf{y},s)d\mathbf{y} + me^{-s\frac{N}{2}}\boldsymbol{\zeta}(s), \quad \forall s \geq 0. \tag{3.19}$$

The vector valued functions $\boldsymbol{\xi}$ and $\boldsymbol{\zeta}$ solve the following system:

$$\begin{cases} \boldsymbol{\xi}_s + L_s\boldsymbol{\xi} - \dfrac{N}{2}\boldsymbol{\xi} - e^{-s\frac{N-1}{2}}\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\xi} = \mathbf{0}, & \mathbf{y}\in\Omega_s, \quad s>0, \\ \boldsymbol{\xi}(\mathbf{y},s) = \boldsymbol{\zeta}(s), \quad \mathbf{y}\in\partial B_s, & s>0, \\ \boldsymbol{\zeta}'(s) - \frac{N}{2}\boldsymbol{\zeta}(s) = -\dfrac{e^{\frac{Ns}{2}}}{m}\displaystyle\int_{\partial\Omega_s} \mathbf{n}\cdot\nabla\boldsymbol{\xi}\,d\sigma_y, & s>0, \\ \boldsymbol{\xi}(\mathbf{y},0) = \mathbf{v}_0(\mathbf{y}), \quad \mathbf{y}\in\Omega, \quad \boldsymbol{\zeta}(0) = \mathbf{h}_1, & \end{cases} \tag{3.20}$$

where the operator $L_s$ is defined componentwise by $L_s\xi := -\Delta\xi - \mathbf{y}/2\cdot\nabla\xi$. Note that the domain $\Omega_s$ where (3.20) holds, evolves in the new time variable $s$. Thus, $L_s$ (which is, apparently, time independent) has to be viewed as a time dependent unbounded operator in $L^2(K,\Omega_s)$ with domain $H^2(K,\Omega_s)\cap H_0^1(K,\Omega_s)$. We will denote merely by $L$ this unbounded operator in $L^2(K,\mathbb{R}^N)$ with domain $H^2(K,\mathbb{R}^N)$ (see [10]). We introduce also:

$$\theta_1(\mathbf{y}) := (4\pi)^{-\frac{N}{2}}\exp\left(\frac{-|\mathbf{y}|^2}{4}\right). \tag{3.21}$$

This function $\theta_1$ corresponds to the heat kernel in similarity variables. In addition, the function $\theta_1$ solves:

$$L\theta_1 - \frac{N}{2}\theta_1 = 0 \quad \text{on } \mathbb{R}^N, \tag{3.22}$$

i.e. it is an eigenfunction associated with the eigenvalue $\lambda_1 := N/2$ of $L$. In fact, $\lambda_1$ is simple and it is the first eigenvalue of $L$, which has a discrete spectrum that can be computed explicitly (see [10]).

The quantity $\mathbf{M}_1\,\theta_1$ is expected to be the first term in the large time expansion of $\boldsymbol{\xi}$. Hence, we are mainly interested in the large time behavior of

$$\bar{\boldsymbol{\xi}}(\mathbf{y},s) := \boldsymbol{\xi}(\mathbf{y},s) - \mathbf{M}_1\theta_1(\mathbf{y}), \quad \mathbf{y}\in\Omega_s \text{ and } \bar{\boldsymbol{\zeta}}(s) := \boldsymbol{\zeta}(s) - \mathbf{M}_1\theta_1(\mathbf{y}), \quad \mathbf{y}\in\partial B_s.$$

These functions play the role of $\bar{\mathbf{v}}$ and $\bar{\mathbf{g}}$ in similarity variables. They are bounded: this is a consequence of the decay properties of $|\mathbf{g}|$ and $\|\mathbf{v}\|_\infty$ in Proposition 2.2. Since $\theta_1$ is also bounded on $\mathbb{R}^N$, one deduces that:

$$|\bar{\boldsymbol{\zeta}}(s)| \leq C \quad \text{and} \quad \|\bar{\boldsymbol{\xi}}\|_\infty \leq C, \quad \forall s > 0. \tag{3.23}$$

Combining (3.20) and (3.22), one deduces that the pair $(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\zeta}})$ solves:

$$\begin{cases} \bar{\boldsymbol{\xi}}_s + L_s\bar{\boldsymbol{\xi}} - \dfrac{N}{2}\bar{\boldsymbol{\xi}} - e^{-s\frac{N-1}{2}}\boldsymbol{\zeta}\cdot\nabla\bar{\boldsymbol{\xi}} = e^{-s\frac{N-1}{2}}\mathbf{M}_1\boldsymbol{\zeta}\cdot\nabla\theta_1, & \mathbf{y}\in\Omega_s, \quad s>0, \\ \bar{\boldsymbol{\xi}}(\mathbf{y},s) = \bar{\boldsymbol{\zeta}}(s), \quad \mathbf{y}\in\partial B_s, & s>0, \\ m\left(\bar{\boldsymbol{\zeta}}(s)e^{-s\frac{N}{2}}\right)' = -\displaystyle\int_{\partial\Omega_s}\mathbf{n}\cdot\nabla\bar{\boldsymbol{\xi}}\,d\sigma_y + e^{-s\frac{N}{2}}\boldsymbol{\rho}(s), & s>0, \\ \bar{\boldsymbol{\xi}}(\mathbf{y},0) = \mathbf{v}_0(\mathbf{y}), \quad \mathbf{y}\in\Omega, \quad \bar{\boldsymbol{\zeta}}(0) = \mathbf{h}_1, \end{cases} \tag{3.24}$$

with

$$\boldsymbol{\rho}(s) := \frac{1}{2}m\mathbf{M}_1(4\pi)^{-\frac{N}{2}}\exp\left(-\frac{r_s^2}{4}\right)\left(-\frac{\sigma_N}{m} + N - \frac{e^{-s}}{2}\right), \tag{3.25}$$

and $r_s = e^{-\frac{s}{2}}$ is the radius of the ball $B_s$. In (3.24-iii), the quantity $e^{-s\frac{N}{2}}\boldsymbol{\rho}(s)$ is a correcting term due to the contribution of $\theta_1$. Remark that this system can also be derived from (3.1) in a straightforward way by making the change of variables (3.17).

From now on, we will work componentwise, using the rules of notation of section 1.1. We shall use in the sequel the notation $(\cdot,\cdot)_s$ for the scalar product of $L^2(K,\Omega_s)$, namely $(f,g)_s := \int_{\Omega_s} f\,g\,K\,d\mathbf{y}$ and $\|\cdot\|_s$ the associated norm. Moreover, $\chi(s)$ stands for $K(r_s)$ and hence:

$$\chi(s) := \exp\left(\frac{e^{-s}}{4}\right) = 1 + C(s)e^{-s}, \quad \forall s>0, \tag{3.26}$$

where $C(s)$ is a positive function such that $0 < C_1 \le C(s) \le C_2 < \infty$, for all $s>0$. Multiplying componentwise the first equation of system (3.24) in the weighted Sobolev space $L^2(K,\Omega_s)$ by $\bar{\xi}$ we obtain:

$$(\bar{\xi}_s,\bar{\xi})_s + (L_s\bar{\xi},\bar{\xi})_s - \frac{N}{2}(\bar{\xi},\bar{\xi})_s - e^{-s\frac{N-1}{2}}(\boldsymbol{\zeta}\cdot\nabla\bar{\xi},\bar{\xi})_s - e^{-s\frac{N-1}{2}}M_1(\boldsymbol{\zeta}\cdot\nabla\theta_1,\bar{\xi})_s = 0, \tag{3.27}$$

for all $s>0$. Integrating by parts, it comes $(L_s\bar{\xi},\bar{\xi})_s = \|\nabla\bar{\xi}\|_s^2 - \int_{\partial\Omega_s}\frac{\partial\bar{\xi}}{\partial\mathbf{n}}\bar{\xi}K\,d\sigma_y$. Then, according to the coupling condition on the interface $\partial\Omega_s$ we can rewrite (3.27) as follows:

$$\begin{aligned}(\bar{\xi}_s,\bar{\xi})_s + \|\nabla\bar{\xi}\|_s^2 - \frac{N}{2}\|\bar{\xi}\|_s^2 - e^{-s\frac{N-1}{2}}(\boldsymbol{\zeta}\cdot\nabla\bar{\xi},\bar{\xi})_s - e^{-s\frac{N-1}{2}}M_1(\boldsymbol{\zeta}\cdot\nabla\theta_1,\bar{\xi})_s \\ + m\chi e^{-\frac{s}{2}N}\bar{\zeta}'\bar{\zeta} - \frac{N}{2}m\chi e^{-\frac{s}{2}N}\bar{\zeta}^2 - e^{-s\frac{N}{2}}\rho\chi\bar{\zeta} = 0.\end{aligned} \tag{3.28}$$

In order to analyze the first term in (3.28) involving the time derivative, we need the following identity:

**Lemma 3.1** *For all function $f\in C^1((0,+\infty),W^{1,1}(\mathbb{R}^N))$,*

$$\frac{d}{ds}\left[\int_{\Omega_s} f(\mathbf{z},s)d\mathbf{z}\right]\Bigg|_{s=s_0} = \int_{\Omega_{s_0}} f_s(\mathbf{y},s_0)d\mathbf{y} + \frac{e^{-\frac{s_0}{2}}}{2}\int_{\partial\Omega_{s_0}} f(\mathbf{y},s_0)d\sigma_y. \tag{3.29}$$

This Lemma derives from the Reynolds formula in fluids mechanic (see for example [1, Lemme 1, page 69] or the complete proof in [16]).

Applying the above Lemma to the function $\bar{\xi}^2 K$ in the domain $\Omega_s$, we deduce that:

$$\frac{1}{2}\frac{d}{ds}\|\bar{\xi}\|_s^2 = (\bar{\xi}_s, \bar{\xi})_s + \frac{e^{-\frac{s}{2}}}{4}\int_{\partial B_s} \bar{\zeta}^2 K d\sigma_y = (\bar{\xi}_s, \bar{\xi})_s + \frac{e^{-\frac{s}{2}N}}{4}\chi\bar{\zeta}^2\sigma_N. \tag{3.30}$$

On the other hand, a simple computation gives the following identity for the term of (3.28) involving the time derivative of $\bar{\zeta}$:

$$\frac{1}{2}\frac{d}{ds}\left[m\chi e^{-\frac{s}{2}N}\bar{\zeta}^2\right] = m\chi e^{-\frac{s}{2}N}\bar{\zeta}\,\bar{\zeta}' - \frac{1}{4}m(\frac{e^{-s}}{2} + N)\chi e^{-\frac{s}{2}N}\bar{\zeta}^2. \tag{3.31}$$

Combining together the relations (3.28), (3.30) and (3.31) and introducing the function $X(s) := \|\bar{\xi}\|_s^2 + m\chi e^{-sN/2}\bar{\zeta}^2$, we get:

$$\begin{aligned} \frac{1}{2}X'(s) - \frac{N}{2}\|\bar{\xi}\|_s^2 - e^{-s\frac{N-1}{2}}(\boldsymbol{\zeta}\cdot\nabla\bar{\xi}, \bar{\xi})_s - e^{-s\frac{N-1}{2}}M_1(\boldsymbol{\zeta}\cdot\nabla\theta_1, \bar{\xi})_s \\ - \frac{1}{4}m\chi e^{-\frac{s}{2}N}\bar{\zeta}^2\left(\frac{\sigma_N}{m} + N - \frac{e^{-s}}{2}\right) - e^{-s\frac{N}{2}}\rho\chi\bar{\zeta} = 0. \end{aligned} \tag{3.32}$$

Taking into account that $\nabla\theta_1 = -(\mathbf{y}/2)\,\theta_1 = -(\mathbf{y}/2)(4\pi)^{-N/2}K^{-1}$, we deduce that $M_1(\boldsymbol{\zeta}\cdot\nabla\theta_1, \bar{\xi})_s = -M_1(4\pi)^{-N/2}(\boldsymbol{\zeta}\cdot(\mathbf{y}/2)K^{-1}, \bar{\xi})_s$. Keeping in mind that $|\boldsymbol{\zeta}|$ is bounded, it comes:

$$|(\boldsymbol{\zeta}\cdot\frac{\mathbf{y}}{2}K^{-1}, \bar{\xi})_s| \le \int_{\Omega_s} |\boldsymbol{\zeta}\cdot\frac{\mathbf{y}}{2}K^{-1}\bar{\xi}K|d\mathbf{y} \le \|\boldsymbol{\zeta}\cdot\frac{\mathbf{y}}{2}K^{-1}\|_s\,\|\bar{\xi}\|_s \le C\|\bar{\xi}\|_s. \tag{3.33}$$

On the other hand, we have the obvious inequalities

$$|(\boldsymbol{\zeta}\cdot\nabla\bar{\xi}, \bar{\xi})_s| \le C\,\|\nabla\bar{\xi}\|_s\,\|\bar{\xi}\|_s \le C\,\|\nabla\bar{\xi}\|_s^2 + C\,\|\bar{\xi}\|_s^2, \quad \forall s > 0. \tag{3.34}$$

Combining (3.32), (3.33) and (3.34), we obtain:

$$\begin{aligned} \frac{1}{2}X'(s) \le -(1 - Ce^{-s\frac{N-1}{2}})\|\nabla\bar{\xi}\|_s^2 + \left(\frac{N}{2} + Ce^{-s\frac{N-1}{2}}\right)\|\bar{\xi}\|_s^2 + Ce^{-s\frac{N-1}{2}}\|\bar{\xi}\|_s \\ - \frac{1}{4}me^{-s\frac{N}{2}}\chi\bar{\zeta}^2\left(\frac{e^{-s}}{2} - N - \frac{\sigma_N}{m}\right) + e^{-s\frac{N}{2}}\rho\chi\bar{\zeta}, \quad \forall s > 0. \end{aligned} \tag{3.35}$$

Taking into account once again the fact that $\bar{\zeta}$ (see (3.23)), $\rho$ and $\chi$ are bounded, we can simplify the above estimate as follows:

$$\frac{1}{2}X'(s) \le -(1-Ce^{-s\frac{N-1}{2}})\|\nabla\bar{\xi}\|_s^2 + \left(\frac{N}{2} + Ce^{-s\frac{N-1}{2}}\right)\|\bar{\xi}\|_s^2 + Ce^{-s\frac{N-1}{2}}\|\bar{\xi}\|_s + Ce^{-s\frac{N}{2}}. \tag{3.36}$$

In order to obtain an ordinary differential inequality for $\|\bar{\xi}\|_s^2 + m\chi e^{-s\frac{N}{2}}\bar{\zeta}^2$ , one needs an estimate for the term $\|\nabla\bar{\xi}\|_s^2$. First of all, let us recall some classical results about the operator $L$: this is a self-adjoint unbounded operator in $L^2(K)$ with domain $D(L) := H^2(K)$. Its eigenvalues are $\lambda_k := (N + k - 1)/2$, $k \in \mathbb{N}^*$ and the first eigenvalue is simple. Its eigenspace, denoted $E_1$, is spanned by $\theta_1$ (we refer to [10] for details). Moreover, we can express the eigenvalues by means of the Rayleigh principle. That reads, for $\lambda_1$ and $\lambda_2$:

$$\inf_{\varphi\in L^2(K)} \frac{\|\nabla\varphi\|^2}{\|\varphi\|^2} = \lambda_1 \quad \text{and} \quad \inf_{\varphi\in E_1^\perp} \frac{\|\nabla\varphi\|^2}{\|\varphi\|^2} = \lambda_2, \tag{3.37}$$

where $\|\cdot\|$ stands for the natural norm of $L^2(K)$. Note that the condition $\varphi \in E_1^\perp$ means precisely that $\int_\Omega \varphi d\mathbf{y} = 0$. Thus, $\lambda_1$ and $\lambda_2$ are the minima of the Reyleigh quotient on $H^1(K)$ and on the subspace of $H^1(K)$ of functions of null mass respectively.

However, we are dealing with $L_s$ on $\Omega_s$ and not with $L$ on $\mathbb{R}^N$. But because of the coupling condition (3.20-ii) on the interface $\partial\Omega_s$, any function of $H^1(\Omega_s)$ can be extended to be in $H^1(\mathbb{R}^N)$ by setting

$$\xi(\mathbf{y}, s) := \zeta(s) \text{ on } B_s. \tag{3.38}$$

When $s$ is large, we are going to show that $\bar{\xi} = \xi - M_1\theta_1$ is "almost" in $E_1^\perp$ since it tends to zero as $t \to \infty$ in $L^1(\mathbb{R}^N)$. This together with the definition of $\lambda_2$ shows that $\|\nabla\bar{\xi}\|_s^2 \geq ((N+1)/2)\|\bar{\xi}\|_s^2$ up to a small correcting term. The task consists in evaluating sharply this correcting term. The ideas we shall apply are quite close of those of [9, Lemma 2]. We state the Lemma in a more general framework, in order to apply it in other cases as well:

**Lemma 3.2** *Let $\Psi$ be a function of $H^1(\Omega_s, K)$ and $\psi(s)$ a real valued bounded function on $(0, \infty)$ such that $\Psi|_{\Omega_s} = \psi$. Suppose furthermore that*

$$\mathcal{M}_1 := \int_{\Omega_s} \Psi d\mathbf{y} + me^{-s\frac{N}{2}}\psi, \tag{3.39}$$

*is a constant. Then $\overline{\Psi} := \Psi - \mathcal{M}_1\theta_1$ satisfies the estimate:*

$$\|\nabla\overline{\Psi}\|_s^2 \geq \frac{N+1}{2}\|\overline{\Psi}\|_s^2 - Ce^{-s\frac{N}{2}}, \quad \forall s > 0. \tag{3.40}$$

**Proof :** We extend $\overline{\Psi}$ to be a function defined in the whole space $\mathbb{R}^N$ by setting:

$$\overline{\Psi}(\mathbf{y}, s) := \psi(s) - \mathcal{M}_1\theta_1(\mathbf{y}), \quad \mathbf{y} \in B_s. \tag{3.41}$$

We introduce then $r_1(s) := \mathcal{M}_1 - (\Psi(s), \theta_1)/\|\theta_1\|^2$. Remark that $r_1(s) = 0$ if and only if $\overline{\Psi} \in E_1^\perp$. According to the expression (3.39) of $\mathcal{M}_1$ and since $\|\theta_1\|^2 =$

$(4\pi)^{-N/2}$, it comes $r_1(s) = \int_{\Omega_s} \Psi d\mathbf{y} + me^{-sN/2}\psi(s) - \int_{\mathbb{R}^N} \Psi d\mathbf{y} = me^{-sN/2}\psi(s) - \int_{B_s} \psi d\mathbf{y} = e^{-sN/2}\psi\,(m - \sigma_N/N)$. Since $|\psi|$ is bounded, it follows that:

$$|r_1(s)| \leq Ce^{-s\frac{N}{2}}, \quad \forall s > 0. \tag{3.42}$$

If we set now $\Psi_1 := \overline{\Psi} + r_1\theta_1$ then $\Psi_1 \in E_1^{\perp}$ and according to (3.37) $\|\nabla\Psi_1\|^2 \geq ((N+1)/2)\|\Psi_1\|^2$. Therefore, we obtain $\|\nabla\overline{\Psi}\|^2 + r_1^2\|\nabla\theta_1\|^2 + 2r_1(\nabla\overline{\Psi}, \nabla\theta_1) \geq ((N+1)/2)(\|\overline{\Psi}\|^2 + r_1^2\|\theta_1\|^2 + 2r_1(\overline{\Psi}, \theta_1))$, that is to say

$$\begin{aligned} \|\nabla\overline{\Psi}\|^2 \geq \frac{N+1}{2}\|\overline{\Psi}\|^2 - r_1^2\left(\|\nabla\theta_1\|^2 - \frac{N+1}{2}\|\theta_1\|^2\right) \\ - 2r_1\left((\nabla\overline{\Psi}, \nabla\theta_1) - \frac{N+1}{2}(\overline{\Psi}, \theta_1)\right). \end{aligned} \tag{3.43}$$

The function $\theta_1$, being an eigenfunction of $L$ associated with the eigenvalue $\lambda_1 = N/2$, satisfies the following classical relations:

$$(\nabla\theta_1, \nabla\varphi) = \lambda_1(\theta_1, \varphi), \quad \forall\varphi \in H^1(K), \qquad \|\nabla\theta_1\|^2 = \lambda_1\|\theta_1\|^2. \tag{3.44}$$

Consequently, we can turn (3.43) into $\|\nabla\overline{\Psi}\|^2 \geq ((N+1)/2)\|\overline{\Psi}\|^2 + (1/2)r_1^2\|\theta_1\|^2 + r_1(\overline{\Psi}, \theta_1)$. Observe that $\overline{\Psi} = \Psi_1 - r_1\theta_1$ and $\Psi_1 \perp \theta_1$. Thus, the inequality above can be rewritten as $\|\nabla\overline{\Psi}\|^2 \geq ((N+1)/2)\|\overline{\Psi}\|^2 - (1/2)r_1^2\|\theta_1\|^2$. We denote $\|\cdot\|_{B_s}$ the scalar product in $L^2(K, B_s)$ and $(\cdot,\cdot)_{B_s}$ the associated norm. We get then

$$\|\nabla\overline{\Psi}\|_s^2 \geq \frac{N+1}{2}\|\overline{\Psi}\|_s^2 + R(s), \tag{3.45}$$

where $R(s) := ((N+1)/2)\|\overline{\Psi}\|_{B_s}^2 - \|\nabla\overline{\Psi}\|_{B_s}^2 - (1/2)r_1^2\|\theta_1\|^2$. Let us now estimate the reminder $R(s)$. Because of the definition (3.41) and since $|\psi|$ is bounded, we obtain $((N+1)/2)\|\overline{\Psi}\|_{B_s}^2 = ((N+1)/2)\int_{B_s}(\psi - \mathcal{M}_1\theta_1)^2 d\mathbf{y} \leq C|B_s| \leq Ce^{-sN/2}$, where $|B_s| = \sigma_N/Ne^{-sN/2}$ is the measure of the ball $B_s$. The same argument shows that $\|\nabla\overline{\Psi}\|_{B_s}^2 = \|\nabla\theta_1\|_{B_s}^2 = \mathcal{M}_1^2\|(\mathbf{y}/2)\theta_1\|_{B_s}^2 \leq Ce^{-s}\int_{B_s}\theta_1^2 d\mathbf{y} \leq Ce^{-s(N+2)/2}$. From inequality (3.42), it follows that $(1/2)r_1^2\|\theta_1\|^2 \leq Ce^{-sN}$. From these three inequalities, we deduce that $|R(s)| \leq Ce^{-sN/2}$. This last estimate together with (3.45) yields the conclusion of the Lemma. ■

One plugs now the estimate (3.40) into (3.36):

$$\begin{aligned} \frac{1}{2}X'(s) \leq -(1 - Ce^{-s\frac{N-1}{2}})\left(\frac{N+1}{2}\|\bar{\xi}\|_s^2 - Ce^{-s\frac{N}{2}}\right) \\ + \left(\frac{N}{2} + Ce^{-s\frac{N-1}{2}}\right)\|\bar{\xi}\|_s^2 + Ce^{-s\frac{N-1}{2}}\|\bar{\xi}\|_s + Ce^{-s\frac{N}{2}}, \quad \forall s > 0, \end{aligned}$$

and after grouping together the terms involving $\|\bar{\xi}\|_s^2$ we obtain that:

$$X'(s) + (1 - Ce^{-s\frac{N-1}{2}})X(s) - Ce^{-s\frac{N-1}{2}}\|\bar{\xi}\|_s \\ \leq (1 - Ce^{-s\frac{N-1}{2}})m\chi e^{-\frac{s}{2}N}\bar{\zeta}^2(s) + Ce^{-s\frac{N}{2}}, \quad \forall s > 0. \quad (3.46)$$

Recalling that $|\bar{\zeta}(s)|$ as well as $|\chi(s)|$ are bounded, the inequality (3.46) can be turned into $X'(s) + (1 - Ce^{-s\frac{N-1}{2}})X(s) - Ce^{-s\frac{N-1}{2}}\|\bar{\xi}\|_s \leq Ce^{-s\frac{N}{2}}$. But, since $\sqrt{X} \geq \|\bar{\xi}\|_s$, it comes:

$$X'(s) + (1 - Ce^{-s\frac{N-1}{2}})X(s) - Ce^{-s\frac{N-1}{2}}\sqrt{X} \leq Ce^{-s\frac{N}{2}}. \quad (3.47)$$

This differential inequality yields the desired result by applying a suitable Gronwall-type inequality.

**Lemma 3.3** *Let $X$ be a non-negative function on $(0, \infty)$ which satisfies:*

$$X'(s) + (1 - C_1e^{-s\frac{N-1}{2}})X(s) - C_2e^{-s\frac{N-1}{2}}(1+s)^\gamma\sqrt{X(s)} \leq C_3s^\alpha e^{-s\beta}, \quad (3.48)$$

*where $\alpha \geq 0$, $\beta > 0$, $\gamma \geq 0$, $C_1$, $C_2$ and $C_3$ are given constants. Then, $X(s)$ satisfies the following decay properties:*

- *The case $N = 2$:*

$$\text{When } 0 < \beta < 1: \quad X(s) \leq C(1+s)^{\alpha+1}e^{-\beta s}, \quad (3.49a)$$

$$\text{When } \beta = 1: \quad X(s) \leq C(1+s)^{\frac{1}{2}\max\{2(\alpha+1),\alpha+2\gamma+3,4\}}e^{-s}, \quad (3.49b)$$

$$\text{When } \beta > 1: \quad X(s) \leq C(1+s)^{2(\gamma+1)}e^{-s}, \quad \forall\, s \geq 0. \quad (3.49c)$$

- *The case $N \geq 3$:*

$$\text{When } 0 < \beta \leq 1: \quad X(s) \leq C(1+s)^{\alpha+1}e^{-\beta s}, \quad (3.49d)$$

$$\text{When } \beta > 1: \quad X(s) \leq Ce^{-s}, \quad \forall\, s \geq 0. \quad (3.49e)$$

The proof is linked with the so-called Bihari-type inequality [15, section 1.3]. We refer to [16] for the proof.

With the estimates of Lemma 3.3 applied to (3.47), we obtain

**When $N = 2$:**

$$\|\bar{\xi}\|_s^2 + m\chi e^{-s\frac{N}{2}}\bar{\zeta}^2 \leq Cs^2e^{-s}, \quad \forall s \geq 1. \quad (3.50)$$

In particular $\|\xi - M_1\theta_1\|_s^2 \leq Cs^2e^{-s}$ and, according to the definition (3.18) of $\xi$, $\int_{\Omega_s} e^{sN/2}|v(e^s - 1, e^{s/2}\mathbf{y}) - M_1e^{-sN/2}\theta_1(e^{s/2}\mathbf{y})|^2K(\mathbf{y})e^{sN/2}d\mathbf{y} \leq Cs^2e^{-s}$, for all $s \geq 1$. Getting back to the variables $\mathbf{x}$ and $t$ and since $d\mathbf{x} = e^{sN/2}d\mathbf{y}$, it follows that $\int_\Omega |v(t,\mathbf{x}) - M_1G(t,\mathbf{x})|^2\mathcal{K}(\mathbf{x},t)d\mathbf{x} \leq C\log^2(1+t)t^{-N/2-1}$, where $\mathcal{K}(\mathbf{x},t)$ is as in (3.5). Hence:

$$\|v(t) - M_1G(t)\|_{L^2(\Omega,\mathcal{K}(t))} \leq C|\log(1+t)|t^{-\frac{N}{4}-\frac{1}{2}}. \quad (3.51a)$$

**When** $N \geq 3$**:**

$$\|\bar{\xi}\|_s^2 + m\chi e^{-s\frac{N}{2}}\bar{\zeta}^2 \leq C\ e^{-s}, \quad \forall s \geq 1. \tag{3.51b}$$

In this case, the logarithmic term does not appear and we find simply:

$$\|v(t) - M_1 G(t)\|_{L^2(\Omega,\mathcal{K}(t))} \leq C t^{-\frac{N}{4}-\frac{1}{2}}, \quad \forall t \geq 1. \tag{3.51c}$$

**Remark 3.3** *Comparing* (3.51) *with the estimates of Proposition 3.1, namely with* $\|\bar{v}\|_2 \leq Ct^{-N/4}$*, we have gained a decay rate of the order of* $|\log(1+t)|t^{-1/2}$ *in dimension* $N = 2$ *and* $t^{-1/2}$ *in dimension* $N \geq 3$*. Moreover, the result is also improved by the presence of the weight* $\mathcal{K}(\mathbf{x}, t)$ *into the norm in* (3.51)*. Note that similar results are true for the pure heat equation. In that case, when subtracting the fundamental solution of an appropriate mass, solutions gain a decay rate of the order of* $t^{-1/2}$ *in any space dimension.*

Let us finally prove that, from the estimates (3.51), one can deduce the relations (3.6) of Proposition 3.2.

Fix $p$ in $[1, 2]$ and $t > 0$. Then, by Hölder's inequality, we obtain:

$$\|v\|_p^p \leq \left(\int_\Omega |v|^2 \mathcal{K}(t) d\mathbf{x}\right)^{\frac{p}{2}} \left(\int_\Omega \mathcal{K}(t)^{-p/(2-p)} d\mathbf{x}\right)^{1-\frac{p}{2}}. \tag{3.52a}$$

A straight computation gives

$$\left(\int_\Omega \mathcal{K}(t)^{-\frac{p}{2-p}} d\mathbf{x}\right)^{1-\frac{p}{2}} \leq \left(\int_{\mathbb{R}^N} \mathcal{K}(t)^{-\frac{p}{2-p}} d\mathbf{x}\right)^{1-\frac{p}{2}} = \left(\frac{4\pi(2-p)}{p}(1+t)\right)^{\frac{N}{2}(1-\frac{p}{2})}$$
$$\leq C t^{\frac{N}{2}(1-\frac{p}{2})}, \quad \forall t > 0. \tag{3.52b}$$

So, (3.52a) provides $\|v\|_p^p \leq C\|v\|^p_{L^2(\Omega,\mathcal{K}(t))} t^{N/2(1-p/2)}$, for all $t > 0$. This relation, together with the estimates (3.51), yields the conclusion of the Proposition 3.2. ∎

# 4 Second term in the asymptotic development

In this section, we will make more precise the conclusions of Theorem 1.2 for $p = 2$, analyzing the large time behavior of:

$$\mathbf{v}_1 := t^{\frac{1}{2}} \bar{\mathbf{v}}(t) = t^{\frac{1}{2}}(\mathbf{v} - \mathbf{M}_1 G) \quad \text{and} \quad \boldsymbol{\xi}_1 := e^{\frac{s}{2}} \bar{\boldsymbol{\xi}} = e^{\frac{s}{2}}(\boldsymbol{\xi} - \mathbf{M}_1 \boldsymbol{\theta}_1),$$

in $\mathbf{L}^2(\Omega)$ and $\mathbf{L}^2(K, \Omega_s)$ respectively.

## 4.1 Identification of the second term

In this subsection, we identify the quantities entering in the second term of the asymptotic expansion and we show that they are well-defined.

Observe that the decay rate of $\bar{\boldsymbol{\xi}}$ when $N=2$, in (3.50), namely the fact that $\|\bar{\boldsymbol{\xi}}\|_s \leq Cse^{-s/2}$, does not guarantee that $\boldsymbol{\xi}_1$ is bounded in $\mathbf{L}^2(\Omega_s, K)$. Nevertheless, $s^{-1}\boldsymbol{\xi}_1$ is bounded. However, for $N \geq 3$, $\boldsymbol{\xi}_1$ itself is bounded in $\mathbf{L}^2(K, \Omega_s)$. This fact will be relevant all along this section.

To begin, define $\boldsymbol{\zeta}_1$ as the trace of $\boldsymbol{\xi}_1$ on $\partial\Omega_s$. Since $(\boldsymbol{\xi}_1, \boldsymbol{\zeta}_1) = e^{\frac{s}{2}}(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\zeta}})$ and $(\bar{\boldsymbol{\xi}}, \bar{\boldsymbol{\zeta}})$ solves system (3.24), we deduce that $(\boldsymbol{\xi}_1, \boldsymbol{\zeta}_1)$ solves:

$$\begin{cases} \boldsymbol{\xi}_{1,s} + L_s\boldsymbol{\xi}_1 - \dfrac{N+1}{2}\boldsymbol{\xi}_1 = e^{-s\frac{N-2}{2}}\boldsymbol{\zeta}\cdot\nabla\boldsymbol{\xi}, \quad \mathbf{y}\in\Omega_s, & s>0, \\ \boldsymbol{\xi}_1(\mathbf{y},s) = \boldsymbol{\zeta}_1(s), \quad \mathbf{y}\in\partial B_s, & s>0, \\ m(\boldsymbol{\zeta}_1'(s) - \frac{N+1}{2}\boldsymbol{\zeta}_1(s))e^{-s\frac{N}{2}} = -\displaystyle\int_{\partial\Omega_s}\mathbf{n}\cdot\nabla\boldsymbol{\xi}_1 d\sigma_y + e^{-s\frac{N}{2}}\boldsymbol{\rho}_1(s), & s>0, \\ \boldsymbol{\xi}_1(\mathbf{y},0) = \mathbf{v}_0(\mathbf{y}) - \mathbf{M}_1\theta_1(\mathbf{y}) \quad \mathbf{y}\in\Omega, \quad \boldsymbol{\zeta}_1(0) = \mathbf{h}_1 - (4\pi)^{\frac{N}{2}}\mathbf{M}_1, \end{cases} \tag{4.1}$$

where $\boldsymbol{\rho}_1(s) := e^{s/2}\boldsymbol{\rho}(s)$ and $\boldsymbol{\rho}(s)$ as in (3.25). We denote $\theta_{2,1}, \theta_{2,2}, \ldots, \theta_{2,N}$ the $N$ eigenfunctions of $L$ associated to $\lambda_2 = (N+1)/2$ that span the eigenspace $E_2$:

$$\theta_{2,i}(\mathbf{y}) := \frac{\partial\theta_1}{\partial y_i}(\mathbf{y}) = -\frac{y_i}{2}\theta_1(\mathbf{y}), \quad \mathbf{y}\in\mathbb{R}^N, \quad \boldsymbol{\theta}_2 := \nabla\theta_1. \tag{4.2}$$

As we shall see, $\boldsymbol{\xi}_1$ behaves for large $s$ as $[\mathbf{M}_2(s)]\boldsymbol{\theta}_2$ where $[\mathbf{M}_2(s)]$ is as follows:

- When $N=2$, $[\mathbf{M}_2(s)] := s[\mathbf{M}_2^1] + [\mathbf{M}_2^2]$, is an affine function with $[\mathbf{M}_2^1]$ and $[\mathbf{M}_2^2]$ two constant matrices that we shall identify.
- When $N \geq 3$, $[\mathbf{M}_2]$ is a constant matrix to be determined.

To shorten notations and avoid distinguishing dimensions $N=2$ and $N\geq 3$, we will sometimes use the notation:

$$\eta_N(s) := \begin{cases} \sqrt{1+s}, & \text{when } N=2, \\ 1, & \text{when } N\geq 3. \end{cases}$$

The fact that $\boldsymbol{\theta}_2$ enters in the large time behavior of the solution $\boldsymbol{\xi}_1$ of (4.1) can easily be motivated. For instance, when dealing with the solutions of $w_s + Lw - ((N+1)/2)w = 0$ in $(0,\infty)\times\mathbb{R}^N$, by the Fourier expansion of the solution $w$ on the basis of eigenfunctions of $L$ it can be easily seen that, when $w$ is of zero mass, the leading term is the projection onto $E_2$. System (4.1) can be viewed as a perturbation of this ideal situation. Its dynamics, although it is essentially of the same nature, is more complex.

When $N=2$, the projection of $\boldsymbol{\xi}_1$ over $\boldsymbol{\theta}_2$ grows linearly with $s>0$ and therefore this case needs a distinguished treatment.

As we shall see, the matrices $[\mathbf{M}_2^1]$, $[\mathbf{M}_2^2]$ ($N=2$) and $[\mathbf{M}_2]$ ($N\geq 3$) entering in the second term of the asymptotic expansion are as follows:

- When $N=2$:

$$[\mathbf{M}_2^1] := \lim_{s\to\infty} 2s^{-1}(4\pi)^{N/2}\int_{\Omega_s} \boldsymbol{\xi}_1\boldsymbol{\theta}_2^T K d\mathbf{y}, \tag{4.3a}$$

$$[\mathbf{M}_2^2] := \lim_{s\to\infty}\left(2(4\pi)^{N/2}\int_{\Omega_s} \boldsymbol{\xi}_1\boldsymbol{\theta}_2^T K d\mathbf{y} - s[\mathbf{M}_2^1]\right). \tag{4.3b}$$

- When $N\geq 3$:

$$[\mathbf{M}_2] := \lim_{s\to\infty} 2(4\pi)^{N/2}\int_{\Omega_s} \boldsymbol{\xi}_1\boldsymbol{\theta}_2^T K d\mathbf{y}. \tag{4.3c}$$

The following Proposition guarantees that the limits above are well-defined.

**Proposition 4.1** *The matrices* $[\mathbf{M}_2^1]$*,* $[\mathbf{M}_2^2]$ *(*$N=2$*) and* $[\mathbf{M}_2]$ *(*$N\geq 3$*) are well defined by relations* (4.3)*, i.e. the limits always exist in* $\mathbb{R}$ *and the identities* (1.14) *of Theorem 1.3 hold.*

*Moreover,* $\eta_N^{-1}|\boldsymbol{\beta}|$ *is bounded and all the generalized integrals in the definitions* (1.14) *are finite.*

**Proof :** Fix $i$ between 1 and $N$ and multiply the main equation of (4.1) in $L^2(\Omega_s,K)$ by $\theta_{2,i}$. It comes, for each component $\xi_1$ of $\boldsymbol{\xi}_1$:

$$(\xi_{1,s},\theta_{2,i})_s + (L_s\xi_1,\theta_{2,i})_s - \frac{N+1}{2}(\xi_1,\theta_{2,i})_s = e^{-s\frac{N-2}{2}}(\boldsymbol{\zeta}\cdot\nabla\xi,\theta_{2,i})_s. \tag{4.4}$$

By Green's formula, it follows that:

$$(L_s\xi_1,\theta_{2,i})_s - (\xi_1,L_s\theta_{2,i})_s = \int_{\partial\Omega_s}\frac{\partial\theta_{2,i}}{\partial\mathbf{n}}\xi_1\chi d\sigma_y - \int_{\partial\Omega_s}\frac{\partial\xi_1}{\partial\mathbf{n}}\theta_{2,i}\chi d\sigma_y. \tag{4.5}$$

By direct computation, and since $\theta_{2,i} = -y_i\theta_1/2$ and $\nabla\theta_1 = -\mathbf{y}\theta_1/2$, we get on $\partial\Omega_s$:

$$\frac{\partial\theta_{2,i}}{\partial\mathbf{n}} = \nabla\theta_{2,i}\cdot\mathbf{n} = -\frac{n_i}{2}\theta_1 - \frac{y_i}{2}\nabla\theta_1\cdot\mathbf{n} = \frac{y_i}{2}\theta_1\left(e^{s/2} - \frac{e^{-s/2}}{2}\right), \tag{4.6}$$

because $\mathbf{n} = -e^{s/2}\mathbf{y}$ on $\partial\Omega_s$. Since $\theta_1$ is radially symmetric, we get with (4.6) $\int_{\partial\Omega_s}\frac{\partial\theta_{2,i}}{\partial\mathbf{n}}d\sigma_y = 0$. On the other hand, taking into account the fact that $\xi_1 = \zeta_1$ is constant on the boundary $\partial\Omega_s$, we deduce that $\int_{\partial\Omega_s}\frac{\partial\theta_{2,i}}{\partial\mathbf{n}}\xi_1\chi d\sigma_y = 0$. Since $L_s\theta_{2,i} = ((N+1)/2)\theta_{2,i}$, (4.5) can be turned into:

$$(L_s\xi_1,\theta_{2,i})_s - \frac{N+1}{2}(\xi_1,\theta_{2,i})_s = -\int_{\partial\Omega_s}\frac{\partial\xi_1}{\partial\mathbf{n}}\theta_{2,i}\chi d\sigma_y. \tag{4.7}$$

But $\theta_1$ is a radially symmetric function and hence $\int_{\partial\Omega_s}\frac{\partial\theta_1}{\partial\mathbf{n}}\theta_{2,i}\chi d\sigma_y = 0$, for all $s>0$. According to the definition of $\xi_1(= e^{s/2}(\xi - M_1\theta_1))$, it follows that the term of the right hand side in (4.7) can be reduced to

$$\int_{\partial\Omega_s}\frac{\partial\xi_1}{\partial\mathbf{n}}\theta_{2,i}\chi d\sigma_y = -\frac{1}{2}e^{\frac{s}{2}}(4\pi)^{-\frac{N}{2}}\int_{\partial\Omega_s}\frac{\partial\xi}{\partial\mathbf{n}}y_i d\sigma_y. \tag{4.8}$$

Plugging (4.7) and (4.8) into (4.4), we obtain

$$(\xi_{1,s}, \theta_{2,i})_s = -\frac{1}{2} e^{\frac{s}{2}} (4\pi)^{-\frac{N}{2}} \int_{\partial\Omega_s} \frac{\partial \xi}{\partial \mathbf{n}} y_i d\sigma_y + e^{-s\frac{N-2}{2}} (\boldsymbol{\zeta} \cdot \nabla \xi, \theta_{2,i})_s. \tag{4.9}$$

The use of Lemma 3.1 yields $\frac{d}{ds}(\xi_1, \theta_{2,i})_s = (\xi_{1,s}, \theta_{2,i})_s + (e^{-s/2}/2) \int_{\partial\Omega_s} \xi_1 \theta_{2,i} \chi d\sigma_y$, but since $\xi_1|_{\partial\Omega_s} = \zeta_1$ is constant for all $s \geq 0$, the last term vanishes and we deduce that $\frac{d}{ds}(\xi_1, \theta_{2,i})_s = (\xi_{1,s}, \theta_{2,i})_s$. We rewrite (4.9) as:

$$\frac{d}{ds}(\xi_1, \theta_{2,i})_s = -\frac{1}{2} e^{\frac{s}{2}} (4\pi)^{-\frac{N}{2}} \int_{\partial\Omega_s} \frac{\partial \xi}{\partial \mathbf{n}} y_i d\sigma_y + e^{-s\frac{N-2}{2}} (\boldsymbol{\zeta} \cdot \nabla \xi, \theta_{2,i})_s. \tag{4.10}$$

The extra regularity for the initial data assumed in the statement of Theorem 1.3 is needed to prove the following Lemma (see the proof in [16]) :

**Lemma 4.1** *Let $\boldsymbol{\xi}$ be solution of* (3.20) *with initial data* $(\boldsymbol{\xi}_0, \boldsymbol{\zeta}_0) \in \mathbf{H}^2(\Omega, K) \times \mathbb{R}^N$ *s.t.* $\boldsymbol{\xi}_0|_{\partial\Omega} = \boldsymbol{\zeta}_0$. *Then, there exists $\alpha_0 > 0$ such that:*

$$\left| \int_{\partial\Omega_s} \frac{\partial \xi}{\partial \mathbf{n}} y_i d\sigma_y \right| \leq C\, e^{-s(1/2+\alpha_0)}, \tag{4.11}$$

*for all $N \geq 2$ and all $s$ large enough.*

Applying this Lemma, we obtain:

$$\left| e^{s/2} \int_{\partial\Omega_s} \frac{\partial \xi}{\partial \mathbf{n}} y_i d\sigma_y \right| \leq C\, e^{-\alpha_0 s}. \tag{4.12}$$

Let us address now the second term of the right hand side of equality (4.10). In view of the explicit form (4.2) of $\theta_{2,i}$, we get $\theta_{2,i} K = -(1/2)(4\pi)^{-N/2} y_i$ and then, integrating by parts, we obtain that:

$$(\boldsymbol{\zeta} \cdot \nabla \boldsymbol{\xi}, \theta_{2,i})_s = \frac{1}{2} \zeta_i (4\pi)^{-\frac{N}{2}} \int_{\Omega_s} \boldsymbol{\xi} d\mathbf{y}, \tag{4.13}$$

where $\zeta_i$ stands for the $i-$th component of the vector $\boldsymbol{\zeta}$. But, in similarity variables, according to (3.19), $\mathbf{M}_1 = \int_{\Omega_s} \boldsymbol{\xi} d\mathbf{y} + m e^{-sN/2} \boldsymbol{\zeta}$, for all $s \geq 0$, so (4.13) can be rewritten as $(\boldsymbol{\zeta} \cdot \nabla \boldsymbol{\xi}, \theta_{2,i})_s = (1/2) \mathbf{M}_1 (4\pi)^{-N/2} \zeta_i - (m/2)(4\pi)^{-N/2} e^{-sN/2} \boldsymbol{\zeta} \zeta_i$. Finally, integrating (4.10) in time from 0 to $s$, we obtain that:

$$\begin{aligned}(4\pi)^{N/2} (\boldsymbol{\xi}_1, \theta_{2,i})_s = -\frac{1}{2} (\mathbf{v}_0, y_i)_{\mathbf{L}^2(\Omega)} - \frac{1}{2} \int_0^s e^{\frac{\alpha}{2}} \left( \int_{\partial\Omega_\alpha} \mathbf{n} \cdot \nabla \boldsymbol{\xi} y_i d\sigma_y \right) d\alpha \\ - \frac{1}{2} \int_0^s e^{-\alpha\frac{N-2}{2}} (m e^{-\alpha\frac{N}{2}} \boldsymbol{\zeta} \zeta_i - \mathbf{M}_1 \zeta_i) d\alpha. \end{aligned} \tag{4.14}$$

Estimate (4.12) ensures that, for all $N \geq 2$:

$$\int_0^\infty \left| e^{\frac{\alpha}{2}} \int_{\partial\Omega_\alpha} \mathbf{n} \cdot \nabla \boldsymbol{\xi} y_i d\sigma_y d\alpha \right| < \infty. \tag{4.15}$$

On the other hand, according to Theorem 1.2 (once again, the estimate below can be slightly improved when $m = \sigma_N/N$ but it is sufficient for our purpose):

$$\left|\mathbf{g}(t) - \mathbf{M}_1(4\pi t)^{-N/2}\right| \leq C \begin{cases} |\log(1+t)|^{\frac{1}{2}} t^{-\frac{5}{4}}, & \text{when } N = 2, \\ t^{-\frac{N}{2}-\frac{1}{2+N}}, & \text{when } N \geq 3, \end{cases} \tag{4.16}$$

which in similarity variables can be rewritten as:

$$\left|\boldsymbol{\zeta}(s) - \mathbf{M}_1(4\pi)^{-\frac{N}{2}}\right| \leq C\eta_N(s)e^{-s\frac{1}{2+N}}. \tag{4.17}$$

Then $\boldsymbol{\zeta} = e^{-s\frac{1}{2+N}}\boldsymbol{\beta}(s) + \mathbf{M}_1(4\pi)^{-\frac{N}{2}}$ with $\boldsymbol{\beta} = \boldsymbol{\beta}(s)$ as in (1.14d). Relation (4.17) means that $\eta_N^{-1}|\boldsymbol{\beta}(s)|$ is bounded for all $s \geq 0$. We obtain then, for all $\alpha > 0$:

$$e^{-\alpha\frac{N-2}{2}}\left(me^{-\alpha\frac{N}{2}}\boldsymbol{\zeta}\zeta_i - \mathbf{M}_1\zeta_i\right) = \frac{e^{-\alpha\frac{N-2}{2}}}{(4\pi)^{\frac{N}{2}}}\Big[-\mathbf{M}_1 M_{1,i} + m(4\pi)^{-\frac{N}{2}}e^{-\alpha\frac{N}{2}}\mathbf{M}_1 M_{1,i}$$
$$-(4\pi)^{\frac{N}{2}}e^{-\alpha\frac{1}{2+N}}\mathbf{M}_1\beta_i + me^{-\alpha\frac{N}{2}-\alpha\frac{1}{2+N}}(\boldsymbol{\beta}M_{1,i} + \mathbf{M}_1\beta_i) + m(4\pi)^{\frac{N}{2}}e^{-\alpha\frac{N}{2}-\alpha\frac{2}{N+2}}\boldsymbol{\beta}\beta_i\Big].$$

Integrating now from 0 to $s$, we obtain:

**When $N = 2$:**

$$\int_0^s me^{-\alpha}\boldsymbol{\zeta}\zeta_i - \mathbf{M}_1\zeta_i d\alpha = -\frac{s}{4\pi}\mathbf{M}_1 M_{1,i} + \frac{m}{(4\pi)^2}\mathbf{M}_1 M_{1,i}(1-e^{-s}) - \int_0^s e^{-\frac{\alpha}{4}}\mathbf{M}_1\beta_i d\alpha$$
$$- \int_0^s e^{-\frac{\alpha}{4}}\mathbf{M}_1\beta_i - \frac{m}{4\pi}e^{-\alpha\frac{5}{4}}(\boldsymbol{\beta}M_{1,i} + \mathbf{M}_1\beta_i) - me^{-\alpha\frac{3}{2}}\boldsymbol{\beta}\beta_i d\alpha. \tag{4.18}$$

Then, combining (4.14) and (4.18), it comes:

$$2(4\pi)^{N/2}\int_{\Omega_s}\boldsymbol{\xi}_1\theta_{2,i}K d\mathbf{y} = \frac{s}{4\pi}M_{1,i}\mathbf{M}_1 - (\mathbf{v}_0, y_i)_{\mathbf{L}^2(\Omega)}$$
$$- \int_0^s e^{\frac{\alpha}{2}}\left(\int_{\partial\Omega_\alpha}\mathbf{n}\cdot\nabla\boldsymbol{\xi}y_i d\sigma_y\right)d\alpha - \frac{m}{(4\pi)^2}\mathbf{M}_1 M_{1,i}(1-e^{-s})$$
$$+ \int_0^s e^{-\frac{\alpha}{4}}\mathbf{M}_1\beta_i d\alpha - \frac{m}{4\pi}e^{-\alpha\frac{5}{4}}(M_{1,i}\boldsymbol{\beta} + \mathbf{M}_1\beta_i)d\alpha - me^{-\alpha\frac{3}{2}}\beta_i\boldsymbol{\beta}d\alpha. \tag{4.19}$$

Then, dividing by $s$, taking into account (4.15) and letting $s$ go to $\infty$, we obtain that the definition (4.3a) of $[\mathbf{M}_2^1]$ leads to the equality (1.14a). We get also:

$$\begin{aligned}\left[2(4\pi)^{N/2}\int_{\Omega_s}\boldsymbol{\xi}_1\theta_2^T K d\mathbf{y}-\frac{s}{4\pi}\mathbf{M}_1\mathbf{M}_1^T\right] &= -\int_{\Omega}\mathbf{v}_0\mathbf{y}^T d\mathbf{y}\\ &-\int_0^s e^{\frac{\alpha}{2}}\left(\int_{\partial\Omega_\alpha}(\mathbf{n}\cdot\nabla\boldsymbol{\xi})\mathbf{y}^T d\sigma_y\right)d\alpha-\frac{m}{(4\pi)^2}\mathbf{M}_1\mathbf{M}_1^T\left(1-e^{-s}\right)\\ &+\int_0^s e^{-\alpha\frac{1}{4}}\mathbf{M}_1\boldsymbol{\beta}^T-\frac{m}{4\pi}e^{-\alpha\frac{5}{4}}(\boldsymbol{\beta}\mathbf{M}_1^T+\mathbf{M}_1\boldsymbol{\beta}^T)-me^{-\alpha\frac{3}{2}}\boldsymbol{\beta}\boldsymbol{\beta}^T d\alpha. \quad (4.20)\end{aligned}$$

When $N \geq 3$: We find the following expression:

$$\begin{aligned}2(4\pi)^{N/2}\int_{\Omega_s}\boldsymbol{\xi}_1\theta_2^T K d\mathbf{y} &= -\int_{\Omega}\mathbf{v}_0\mathbf{y}^T d\mathbf{y}-\int_0^s e^{\frac{\alpha}{2}}\left(\int_{\partial\Omega_\alpha}(\mathbf{n}\cdot\nabla\boldsymbol{\xi})\mathbf{y}^T d\sigma_y\right)d\alpha\\ &-\mathbf{M}_1\mathbf{M}_1^T(4\pi)^{-N}\left[\frac{m}{N-1}\left(1-e^{-s(N-1)}\right)-(4\pi)^{\frac{N}{2}}\frac{2}{N-2}\left(1-e^{-s\frac{N-2}{2}}\right)\right]\\ &+\int_0^s e^{-\alpha\frac{N-2}{2}-\alpha\frac{1}{2+N}}\mathbf{M}_1\boldsymbol{\beta}^T-me^{-\alpha(N-1)-\alpha\frac{2}{2+N}}\boldsymbol{\beta}\boldsymbol{\beta}^T d\alpha\\ &-\int_0^s\frac{m}{(4\pi)^{\frac{N}{2}}}e^{-\alpha(N-1)-\alpha\frac{1}{2+N}}(\boldsymbol{\beta}\mathbf{M}_1^T+\mathbf{M}_1\boldsymbol{\beta}^T)d\alpha. \quad (4.21)\end{aligned}$$

Letting $s \to \infty$ in (4.20) and (4.21), we find that the expressions (1.14b) and (4.3b) and (1.14c) and (4.3c) coincide respectively. That concludes the proof of Proposition 4.1. ■

## 4.2 Proof of Theorem 1.3

We shall proceed in several steps, establishing a sequence of preliminary Lemmas. Theorem 1.3 will then hold immediately.

We recall the definition of $[\mathbf{M}_2(s)]$ (see section 4.1):

$$\begin{aligned}[\mathbf{M}_2(s)] :=& s[\mathbf{M}_2^1]+[\mathbf{M}_2^2], && \text{when } N=2, && (4.22a)\\ [\mathbf{M}_2(s)] :=& [\mathbf{M}_2], && \text{when } N\geq 3, && (4.22b)\end{aligned}$$

$[\mathbf{M}_2^1]$, $[\mathbf{M}_2^2]$ and $[\mathbf{M}_2]$ being three constant matrices defined equivalently by (4.3) or (1.14). Furthermore, one introduces:

$$\bar{\boldsymbol{\xi}}_1 := e^{\frac{s}{2}}(\boldsymbol{\xi}-\mathbf{M}_1\theta_1)-[\mathbf{M}_2(s)]\boldsymbol{\theta}_2, \quad \bar{\boldsymbol{\zeta}}_1 := \bar{\boldsymbol{\xi}}_1, \quad \text{on } \partial B_s, \qquad (4.23a)$$

what reads also, according to the notations of section 1.1:

$$\bar{\xi}_1 := e^{\frac{s}{2}}(\xi-M_1\theta_1)-\mathbf{M}_2(s)\cdot\boldsymbol{\theta}_2, \quad \bar{\zeta}_1 := \bar{\xi}_1, \quad \text{on } \partial B_s. \qquad (4.23b)$$

Remark that $\bar{\boldsymbol{\zeta}}_1$, because of the contribution of $\boldsymbol{\theta}_2$, is non-constant along the boundary $\partial\Omega_s$ whenever $[\mathbf{M}_2(s)] \neq [\mathbf{0}]$. The function $\bar{\boldsymbol{\xi}}_1(\mathbf{y}, s)$ solves for all $s > 0$ and all $\mathbf{y} \in \Omega_s$:

$$\bar{\boldsymbol{\xi}}_{1,s} + L_s\bar{\boldsymbol{\xi}}_1 - \frac{N+1}{2}\bar{\boldsymbol{\xi}}_1 - e^{-s\frac{N-2}{2}}\boldsymbol{\zeta} \cdot \nabla\boldsymbol{\xi} = \begin{cases} -[\mathbf{M}_2^1]\boldsymbol{\theta}_2, & \text{when } N = 2, \\ \mathbf{0}, & \text{when } N \geq 3. \end{cases}$$

Multiplying componentwise by $\bar{\xi}_1$ in $L^2(K, \Omega_s)$, it comes:

$$(\bar{\xi}_{1,s}, \bar{\xi}_1)_s + (L_s\bar{\xi}_1, \bar{\xi}_1)_s - \frac{N+1}{2}\|\bar{\xi}_1\|_s^2 - e^{-s\frac{N-2}{2}}(\boldsymbol{\zeta} \cdot \nabla\xi, \bar{\xi}_1)_s$$
$$= \begin{cases} -\displaystyle\sum_{i=1}^{N} M_{2,i}^1(\theta_{2,i}, \bar{\xi}_1)_s, & \text{when } N = 2, \\ 0, & \text{when } N \geq 3. \end{cases} \tag{4.24}$$

As we did for $\bar{\xi}$ in the proof of Proposition 3.2, we are going to show that $\|\bar{\xi}_1\|_s$ solves an ordinary differential inequality and then apply a Gronwall-type inequality.

**Remark 4.1** *From now on, $a$ and $b$ will denote two positive constants that can change from one line to the other.*

For the first term in (4.24), we have the estimate:

$$\frac{1}{2}\frac{d}{ds}\|\bar{\xi}_1\|_s^2 \leq (\bar{\xi}_{1,s}, \bar{\xi}_1)_s + Cs^b e^{-as}, \tag{4.25}$$

where $C > 0$ and $a$ and $b$ are as specified in Remark 4.1. Indeed, according to Lemma 3.1, we can write:

$$\frac{1}{2}\frac{d}{ds}\|\bar{\xi}_1\|_s^2 - (\bar{\xi}_{1,s}, \bar{\xi}_1)_s = \frac{e^{-\frac{s}{2}}}{4}\int_{\partial\Omega_s}(\bar{\zeta}_1)^2\chi d\sigma_y. \tag{4.26}$$

Since $\xi|_{\partial\Omega_s} = \zeta$ and $\theta_1|_{\partial\Omega_s} = (4\pi)^{-N/2}\chi^{-1}$ are constant and $\theta_{2,i} = -y_i\theta_1/2$, we have $\int_{\partial\Omega_s}\zeta\theta_{2,i}\chi d\sigma_y = -\int_{\partial\Omega_s}(y_i/2)\zeta\theta_1\chi d\sigma_y = 0$ and, as well, $\int_{\partial\Omega_s}\theta_1\theta_{2,i}\chi d\sigma_y = -\int_{\partial\Omega_s}(y_i/2)\theta_1^2\chi d\sigma_y = 0$, for all $i = 1, \ldots, N$. We have also $\int_{\partial\Omega_s}\theta_{2,i}\theta_{2,j}\chi d\sigma_y = (1/4)\int_{\partial\Omega_s}y_iy_j\theta_1^2\chi d\sigma_y = 0$ for $i \neq j$, and we can compute:

$$\int_{\partial\Omega_s}\bar{\zeta}_1^2\chi d\sigma_y = e^s\left(\zeta - \frac{M_1}{(4\pi)^{\frac{N}{2}}\chi}\right)^2\chi\sigma_N e^{-s\frac{N-1}{2}} + \frac{1}{4}\frac{e^{-s\frac{N+1}{2}}}{(4\pi)^N\chi}\sigma_{N,1}|\mathbf{M}_2(s)|^2, \tag{4.27}$$

where $\sigma_{N,1} := \int_{\partial B}x_1^2 d\sigma_x = \int_{\partial B}x_i^2 d\sigma_x$ for any $i = 1, \ldots, N$ due to the symmetry of the ball. Since $0 < C_1 < \chi < C_2$ and taking into account the definition (4.22) of $\mathbf{M}_2(s)$, we deduce that (4.25) holds.

Let us now address the second term of (4.24). Integrating by parts, we get:

$$(L_s\bar{\xi}_1, \bar{\xi}_1)_s = \|\nabla\bar{\xi}_1\|_s^2 - \int_{\partial\Omega_s} \frac{\partial\bar{\xi}_1}{\partial\mathbf{n}}\bar{\xi}_1\chi d\sigma_y. \tag{4.28}$$

This term is the one involving the most important technical difficulties. We shall treat separately the two terms on the right hand side.

**Lemma 4.2** *Let $\bar{\xi}_1$ be defined by (4.23), then, there exist $C > 0$, $a > 0$ and $b \geq 0$ such that:*

$$\|\nabla\bar{\xi}_1\|_s^2 \geq \frac{N+2}{2}\|\bar{\xi}_1\|_s^2 - C\,s^b e^{-as}.$$

**Proof :** The proof follows the same ideas as in the proof of Lemma 3.2, but this time is technically more involved. All along the proof, we extend $\xi$ and $\bar{\xi}_1$ on $B_s$ by setting $\xi := \zeta$ and

$$\bar{\xi}_1 = e^{\frac{s}{2}}(\zeta - M_1\theta_1) - \mathbf{M}_2(s)\cdot\boldsymbol{\theta}_2. \tag{4.29}$$

Denoting by $\mathbb{P}$ the orthogonal projection from $L^2(K)$ onto the subspace $(E_1 \cup E_2)^{\perp}$ where $E_1$ and $E_2$ are the eigenspaces of the operator $L$ associated with the eigenvalues $\lambda_1 = N/2$ and $\lambda_2 = (N+1)/2$ respectively. In particular $E_1$ is spanned by $\theta_1$ and $E_2$ by $\theta_{2,1}, \ldots, \theta_{2,N}$. We have $\bar{\xi}_1 = ((\theta_1, \bar{\xi}_1)/\|\theta_1\|^2)\theta_1 + \sum_{i=1}^{N}((\theta_{2,i}, \bar{\xi}_1)/\|\theta_{2,i}\|^2)\theta_{2,i} + \mathbb{P}(\bar{\xi}_1) = \mathbb{P}(\bar{\xi}_1) - e^{s2}r_1(s)\theta_1 - \mathbf{r}_2(s)\cdot\boldsymbol{\theta}_2$ where

$$r_1(s) = M_1 - \frac{(\xi, \theta_1)}{\|\theta_1\|^2}, \tag{4.30a}$$

$$r_{2,i}(s) = M_{2,i}(s) - e^{s/2}\frac{(\xi, \theta_{2,i})}{\|\theta_{2,i}\|^2}, \quad \forall i = 1, \ldots, N. \tag{4.30b}$$

Therefore, we get:

$$\mathbb{P}(\bar{\xi}_1) = \bar{\xi}_1 + e^{\frac{s}{2}}r_1(s)\theta_1 + \mathbf{r}_2\cdot\boldsymbol{\theta}_2. \tag{4.31}$$

Since the third eigenvalue of $L$ is $\lambda_3 = \frac{N+2}{2}$ and $\mathbb{P}(\bar{\xi}_1) \in (E_1 \cup E_2)^{\perp}$, we have:

$$\|\nabla\mathbb{P}(\bar{\xi}_1)\|^2 \geq \frac{N+2}{2}\|\mathbb{P}(\bar{\xi}_1)\|^2. \tag{4.32}$$

The following relations of orthogonality in $L^2(K)$: $(\theta_1, \theta_{2,i}) = 0$, $(\mathbb{P}(\bar{\xi}_1), \theta_1) = 0$ and $(\mathbb{P}(\bar{\xi}_1), \theta_{2,i}) = 0$ for all $i = 1, \ldots, N$, together with the identities $(\nabla f, \nabla\theta_1) = (N/2)(f, \theta_1)$ and $(\nabla f, \nabla\theta_{2,i}) = (N + 1/2)(f, \theta_{2,i})$ for all $i = 1, \ldots, N$ and for all $f \in H^1(K)$, resulting from the fact that $\theta_1$ and $\theta_{2,i}$ are eigenfunctions of $L$, allow us to expand and simplify (4.32):

$$\|\nabla\bar{\xi}_1\|^2 + e^s r_1^2(s)\|\nabla\theta_1\|^2 + \sum_{i=1}^N r_{2,i}(s)^2\|\nabla\theta_{2,i}\|^2 + 2e^{\frac{s}{2}} r_1(s)(\nabla\bar{\xi}_1,\nabla\theta_1)$$
$$+ 2\sum_{i=1}^N r_{2,i}(s)(\nabla\bar{\xi}_1,\nabla\theta_{2,i}) \geq \frac{N+2}{2}\left[\|\bar{\xi}_1\|^2 + e^s r_1^2(s)\|\theta_1\|^2 + \sum_{i=1}^N r_{2,i}^2(s)\|\theta_{2,i}\|^2 \right.$$
$$\left. +2e^{\frac{s}{2}} r_1(s)(\bar{\xi}_1,\theta_1) + 2\sum_{i=1}^N r_{2,i}(s)(\bar{\xi}_1,\theta_{2,i})\right].$$

Since $\|\nabla\theta_1\|^2 = \frac{N}{2}\|\theta_1\|^2$ and $\|\nabla\theta_{2,i}\|^2 = \frac{N+1}{2}\|\theta_{2,i}\|^2$, we obtain that:

$$\|\nabla\bar{\xi}_1\|^2 + e^s r_1^2(s)\frac{N}{2}\|\theta_1\|^2 + \frac{N+1}{2}\sum_{i=1}^N r_{2,i}(s)^2\|\theta_{2,i}\|^2 + Ne^{\frac{s}{2}} r_1(s)(\bar{\xi}_1,\theta_1)$$
$$+ (N+1)\sum_{i=1}^N r_{2,i}(s)(\bar{\xi}_1,\theta_{2,i}) \geq \frac{N+2}{2}\left[\|\bar{\xi}_1\|^2 + e^s r_1^2(s)\|\theta_1\|^2 + \sum_{i=1}^N r_{2,i}^2(s)\|\theta_{2,i}\|^2 \right.$$
$$\left. +2e^{\frac{s}{2}} r_1(s)(\bar{\xi}_1,\theta_1) + 2\sum_{i=1}^N r_{2,i}(s)(\bar{\xi}_1,\theta_{2,i})\right],$$

that is to say:

$$\|\nabla\bar{\xi}_1\|^2 \geq \frac{N+2}{2}\|\bar{\xi}_1\|^2 + e^s r_1^2(s)\|\theta_1\|^2 + 2e^{\frac{s}{2}} r_1(s)(\bar{\xi}_1,\theta_1)$$
$$+\frac{1}{2}\sum_{i=1}^N r_{2,i}^2(s)\|\theta_{2,i}\|^2 + \sum_{i=1}^N r_{2,i}(s)(\bar{\xi}_1,\theta_{2,i}).$$

Plugging the relation (4.31) of $\bar{\xi}_1$ into the relation above, we get:

$$\|\nabla\bar{\xi}_1\|_s^2 - \frac{N+2}{2}\|\bar{\xi}_1\|_s^2 \geq -\|\nabla\bar{\xi}_1\|_{B_s}^2 - e^s r_1^2(s)\|\theta_1\|^2 - \frac{1}{2}\sum_{i=1}^N r_{2,i}^2(s)\|\theta_{2,i}\|^2, \quad (4.33)$$

since $\|f\|^2 = \|f\|_s^2 + \|f\|_{B_s}^2$ for all function $f \in L^2(K)$ and all $s > 0$. It remains to estimate each term of the right hand side of (4.33).

**First term:** From the definition of $\bar{\xi}_1$ on $B_s$ (see (4.29)), we deduce that $\nabla\bar{\xi}_1 = -e^{s/2}M_1\nabla\theta_1 - \sum_{i=1}^N M_{2,i}(s)\nabla\theta_{2,i}$ on $B_s$ and hence:

$$\|\nabla\bar{\xi}_1\|_{B_s}^2 = e^s M_1^2\|\nabla\theta_1\|_{B_s}^2 + \sum_{i=1}^N M_{2,i}(s)^2\|\nabla\theta_{2,i}\|_{B_s}^2. \quad (4.34)$$

Indeed, straight computations yield $\nabla\theta_1 = -(\mathbf{y}/2)\theta_1 = \boldsymbol{\theta}_2$ and $\nabla\theta_{2,i} = -(\mathbf{e}_i/2)\theta_1 + (y_i/4)\mathbf{y}\theta_1$, where $\mathbf{e}_i$ is the vector whose components are $e_{i,j} = \delta_{ij}$, $j = 1,\dots,N$. Thus $(\nabla\theta_1, \nabla\theta_{2,i})_{B_s} = \int_{B_s}(y_i/2)\,(1 - |\mathbf{y}|^2/2)\,\theta_1^2 K d\mathbf{y} = 0$ because $\theta_1$ and $K$ are radially symmetric functions. We also have, since $\mathbf{e}_i\cdot\mathbf{e}_j = 0$, $(\nabla\theta_{2,i}, \nabla\theta_{2,j})_{B_s} = -(1/4)\int_{B_s} y_i y_j\,(1 - |\mathbf{y}|^2/4)\,\theta_1^2 K d\mathbf{y} = 0$ when $i \neq j$. From $\nabla\theta_1 = -(\mathbf{y}/2)\theta_1$, we deduce that $\int_{B_s}|\nabla\theta_1|^2 K d\mathbf{y} = (1/4)(4\pi)^{-N/2}\int_{B_s}\mathbf{y}^2\theta_1 d\mathbf{y}$. It comes $\int_{B_s}|\nabla\theta_1|^2 K d\mathbf{y} = (1/4)(4\pi)^{-N/2}\int_{B_s}\mathbf{y}^2\theta_1 d\mathbf{y} \leq Ce^{-s(N+2)/2}$. In the same way, we have $\|\nabla\theta_{2,i}\|_{B_s}^2 \leq Ce^{-sN/2}$ for all $i = 1,\dots,N$. Combining (4.34) with the two last inequalities, we obtain:

$$\|\nabla\bar{\xi}_1\|_{B_s}^2 \leq C\begin{cases}(1+s)^2 e^{-s} & \text{when } N = 2,\\ e^{-s\frac{N}{2}} & \text{when } N \geq 3,\end{cases} \tag{4.35}$$

taking into account the definition (4.22) of $\mathbf{M}_2(s)$.

**Second term:** Let us recall (see (4.30a)) that $r_1(s) := M_1 - (\xi,\theta_1)/\|\theta_1\|^2$ which, combined with (3.19), yields $r_1(s) = \int_{\Omega_s}\xi(\mathbf{y},s)d\mathbf{y} + me^{-sN/2}\zeta(s) - (\xi,\theta_1)/\|\theta_1\|^2$. But $(\xi,\theta_1)/\|\theta_1\|^2 = (4\pi)^{N/2}\int_{\Omega_s}\xi\theta_1 K d\mathbf{y} + (\sigma_N/N)e^{-sN/2}\zeta$ and hence $r_1(s) = e^{-sN/2}\zeta(m - \sigma_N/N)$. Since $|\zeta|$ is bounded, we get:

$$e^s r_1^2\|\theta_1\|^2 \leq Ce^{-s(N-1)}. \tag{4.36}$$

**Third term:** According to the definition (4.30b) of $r_{2,i}(s)$, and (4.3) of $M_{2,i}(s)$, we have:
**The case $N = 2$:**

$$\begin{aligned} r_{2,i}(s) =& M_{2,i}(s) - e^{\frac{s}{2}}\frac{(\xi,\theta_{2,i})}{\|\theta_{2,i}\|^2}\\ =& M_{2,i}^1 s + \lim_{\alpha\to\infty}\left(\frac{(\xi_1,\theta_{2,i})_\alpha}{\|\theta_{2,i}\|^2} - \alpha M_{2,i}^1\right) - e^{\frac{s}{2}}\frac{(\xi,\theta_{2,i})}{\|\theta_{2,i}\|^2},\end{aligned} \tag{4.37}$$

according to (4.3b). By definition $\xi_1 = e^{\frac{s}{2}}(\xi - M_1\theta_1)$, hence

$$(\xi_1,\theta_{2,i})_s = e^{\frac{s}{2}}(\xi,\theta_{2,i})_s - M_1 e^{\frac{s}{2}}(\theta_1,\theta_{2,i})_s = e^{\frac{s}{2}}(\xi,\theta_{2,i})_s, \tag{4.38}$$

because $(\theta_1,\theta_{2,i})_s = 0$ for all $s \geq 0$. On the other hand, since $\xi = \zeta$ on $B_s$, we can rewrite (4.37) as follows:

$$r_{2,i}(s) = \left[\lim_{\alpha\to\infty}\left(e^{\alpha/2}\frac{(\xi,\theta_{2,i})_\alpha}{\|\theta_{2,i}\|^2} - \alpha M_{2,i}^1\right) - \left(e^{\frac{s}{2}}\frac{(\xi,\theta_{2,i})_s}{\|\theta_{2,i}\|^2} - sM_{2,i}^1\right)\right.$$
$$\left. - e^{\frac{s}{2}}\frac{(\zeta,\theta_{2,i})_{B_s}}{\|\theta_{2,i}\|^2}\right]. \tag{4.39}$$

The last term vanishes $(\zeta,\theta_{2,i})_{B_s} = \zeta\int_{B_s}\theta_{2,i}K d\mathbf{y} = -\zeta\int_{B_s}(y_i/2)\theta_1 K d\mathbf{y} = 0$, because $\theta_1$ and $K$ are radially symmetric. Identity (4.39) reads equivalently, since $\|\theta_{2,i}\|^2 = (1/2)(4\pi)^{-N/2}$ for all $i = 1, \dots, N$ :

$$\mathbf{r}_2(s) = \left[\lim_{\alpha\to\infty}\left(2(4\pi)^{N/2}e^{\alpha/2}\int_{\Omega_\alpha}\boldsymbol{\xi}_1\boldsymbol{\theta}_2^T d\mathbf{y} - \alpha\mathbf{M}_2^1\right)\right. \\ \left. - \left(2(4\pi)^{N/2}e^{s/2}\int_{\Omega_s}\boldsymbol{\xi}_1\boldsymbol{\theta}_2^T d\mathbf{y} - s\mathbf{M}_2^1\right)\right].$$

According to (4.19) and (4.20), the identity above reads:

$$\mathbf{r}_2(s) = -\int_s^\infty \left(e^{\frac{\alpha}{2}}\int_{\partial\Omega_\alpha}\frac{\partial\xi}{\partial\mathbf{n}}\mathbf{y}d\sigma_y\right)d\alpha - \mathbf{M}_1 M_1(4\pi)^{-2}me^{-s} \\ + \int_s^\infty e^{-\alpha\frac{1}{4}}M_1\boldsymbol{\beta} - \frac{m}{4\pi}e^{-\alpha\frac{5}{4}}(\mathbf{M}_1\beta + M_1\boldsymbol{\beta}) - me^{-\alpha\frac{3}{2}}\boldsymbol{\beta}\beta d\alpha.$$

On the other hand, according to estimate (4.12), l'Hospital's rule yields $\left|\int_s^\infty \left(e^{\alpha/2}\int_{\partial\Omega_\alpha}\frac{\partial\xi}{\partial\mathbf{n}}y_i d\sigma_y\right)d\alpha\right| \leq Ce^{-\alpha_0 s}$. The definition (1.14d) of $\boldsymbol{\beta}$, combined with the estimate (4.17), ensures that $\eta_N^{-1}|\boldsymbol{\beta}|$ is bounded. The other terms are then easy to estimate. In particular, by l'Hospital's rule, we get $\left|\int_s^\infty e^{-\alpha/4}\boldsymbol{\beta}d\alpha\right| \leq C\int_s^\infty\sqrt{1+\alpha}e^{-\alpha/4}d\alpha \leq C\sqrt{1+s}e^{-s/4}$. Finally, we obtain that:

$$|r_{2,i}(s)|^2 \leq Cs^b e^{-a\,s}, \tag{4.40}$$

where $a$ and $b$ are positive constants as in Remark 4.1.

**The case $N \geq 3$:**
The definitions (4.30b) of $r_{2,i}(s)$ and (4.3) of $M_{2,i}(s)$ lead to:

$$r_{2,i}(s) = M_{2,i} - e^{\frac{s}{2}}\frac{(\xi,\theta_{2,i})}{\|\theta_{2,i}\|^2} = \lim_{\alpha\to\infty}\frac{(\xi_1,\theta_{2,i})_\alpha}{\|\theta_{2,i}\|^2} - e^{\frac{s}{2}}\frac{(\xi,\theta_{2,i})}{\|\theta_{2,i}\|^2}.$$

According to (4.38), the identity above reads:

$$r_{2,i}(s) = \left[\lim_{\alpha\to\infty}\left(e^{\alpha/2}\frac{(\xi,\theta_{2,i})_\alpha}{\|\theta_{2,i}\|^2} - e^{\frac{s}{2}}\frac{(\xi,\theta_{2,i})_s}{\|\theta_{2,i}\|^2}\right) - e^{\frac{s}{2}}\frac{(\zeta,\theta_{2,i})_{B_s}}{\|\theta_{2,i}\|^2}\right].$$

The last term vanishes and we can rewrite equivalently:

$$\mathbf{r}_2(s) = \lim_{\alpha\to\infty}2(4\pi)^{N/2}\left(e^{\alpha/2}\int_{\Omega_\alpha}\boldsymbol{\xi}_1\boldsymbol{\theta}_2^T d\mathbf{y} - e^{s/2}\int_{\Omega_s}\boldsymbol{\xi}_1\boldsymbol{\theta}_2^T d\mathbf{y}\right).$$

Taking into account (4.21), one obtains:

$$\mathbf{r}_2(s) = -\int_s^{\infty} e^{\frac{\alpha}{2}} \left( \int_{\partial\Omega_\alpha} \frac{\partial \xi}{\partial \mathbf{n}} \mathbf{y} d\sigma_y \right) d\alpha + \frac{\mathbf{M}_1 M_1}{(4\pi)^N} \left[ \frac{(4\pi)^{\frac{N}{2}}}{N-2} e^{-s\frac{N-2}{2}} - \frac{m}{N-1} e^{-s(N-1)} \right]$$
$$+ \int_s^{\infty} e^{-\alpha\left(\frac{N-2}{2} - \frac{1}{N+2}\right)} M_1 \boldsymbol{\beta} - \frac{m}{(4\pi)^{\frac{N}{2}}} e^{-\alpha\left[(N-1) - \frac{1}{2+N}\right]} (\mathbf{M}_1 \beta + M_1 \boldsymbol{\beta}) d\alpha$$
$$- m \int_s^{\infty} e^{-\alpha\left[(N-1) - \frac{2}{N+2}\right]} \boldsymbol{\beta} \beta d\alpha,$$

and we deduce in a straightforward way that (4.40) holds also when $N \geq 3$. To complete the proof of the Lemma 4.2, it suffices to plug the estimates (4.35), (4.36) and (4.40) into (4.33). ■

We address now the second term of equality (4.28). The same ideas as above (we refer to [16] for details) allows us to prove that:

$$\int_{\partial\Omega_s} \frac{\partial \bar{\xi}_1}{\partial \mathbf{n}} \bar{\xi}_1 \chi d\sigma_y \leq -\frac{1}{2} \frac{d}{ds} \left( m \chi \bar{\zeta}^2 e^{-s\frac{N-2}{2}} \right) + C\, s^b e^{-a\, s}, \tag{4.41}$$

and also:

$$|(\boldsymbol{\zeta} \cdot \nabla \xi, \bar{\xi}_1)_s| \leq C e^{-\frac{s}{2}} \|\nabla \bar{\xi}_1\|_s^2 + C e^{-\frac{s}{2}} \|\bar{\xi}_1\|_s^2$$
$$+ \begin{cases} C(1+s) e^{-\frac{s}{2}} \|\bar{\xi}_1\|_s + C s^b e^{-as}, & \text{when } N = 2, \\ C e^{-\frac{s}{2}} \|\bar{\xi}_1\|_s + C s^b e^{-as}, & \text{when } N \geq 3, \end{cases} \tag{4.42}$$

where $a$ and $b$ are positive constants as in Remark 4.1. We dispose now of all the tools to deduce the decay rate of $\bar{\xi}_1$. We rewrite (4.24) using (4.28), and it comes:

$$(\bar{\xi}_{1,s}, \bar{\xi}_1)_s + \|\nabla \bar{\xi}_1\|_s^2 - \int_{\partial\Omega_s} \frac{\partial \bar{\xi}_1}{\partial \mathbf{n}} \bar{\xi}_1 \chi d\sigma_y - \frac{N+1}{2} \|\bar{\xi}_1\|_s^2$$
$$- e^{-s\frac{N-2}{2}} (\boldsymbol{\zeta} \cdot \nabla \xi, \bar{\xi}_1)_s = \begin{cases} -\sum_{i=1}^{N} M_{i,2}^1 (\theta_{2,i}, \bar{\xi}_1)_s, & \text{when } N = 2, \\ 0, & \text{when } N \geq 3. \end{cases} \tag{4.43}$$

On the other hand, estimate (4.25) and Lemma 4.41 ensure that:

$$\frac{1}{2} X'(s) \leq (\bar{\xi}_{1,s}, \bar{\xi}_1)_s - \int_{\partial\Omega_s} \frac{\partial \bar{\xi}_1}{\partial \mathbf{n}} \bar{\xi}_1 \chi d\sigma_y + C s^b e^{-as},$$

where $X(s) := \|\bar{\xi}_1\|_s^2 + m\chi\bar{\zeta}^2 e^{-s(N-2)/2}$. That yields, together with (4.43):

$$\frac{1}{2}X'(s) \leq -\|\nabla\bar{\xi}_1\|_s^2 + \frac{N+1}{2}\|\bar{\xi}_1\|_s^2 + e^{-s\frac{N-2}{2}}(\boldsymbol{\zeta}\cdot\nabla\xi, \bar{\xi}_1)_s$$
$$+ \begin{cases} -\sum_{i=1}^{N} M_{i,2}^1(\theta_{2,i}, \bar{\xi}_1)_s + Cs^b e^{-as}, & \text{when } N = 2, \\ Cs^b e^{-as}, & \text{when } N \geq 3. \end{cases}$$

The expression (4.31) of $\bar{\xi}_1$ leads to $|(\theta_{2,i}, \bar{\xi}_1)_s| \leq C|r_{2,i}| \leq C(1+s)^{3/2}e^{-s/4}$, according to (4.40). Applying Lemma 4.42, we get

$$\frac{1}{2}X'(s) + \leq -(1 - Ce^{-s\frac{N-1}{2}})\|\nabla\bar{\xi}_1\|_s^2 + \left(\frac{N+1}{2} + Ce^{-s\frac{N-1}{2}}\right)\|\bar{\xi}_1\|_s^2$$
$$+ C\begin{cases} (1+s)e^{-s\frac{N-1}{2}}\|\bar{\xi}_1\|_s + Cs^b e^{-as}, & \text{when } N = 2, \\ e^{-s\frac{N-1}{2}}\|\bar{\xi}_1\|_s + Cs^b e^{-as}, & \text{when } N \geq 3. \end{cases} \tag{4.44}$$

According to Lemma 4.2, we deduce that:

$$-\left(1 - Ce^{-s\frac{N-1}{2}}\right)\|\nabla\bar{\xi}_1\|_s^2 + \left(\frac{N+1}{2} + Ce^{-s\frac{N-1}{2}}\right)\|\bar{\xi}_1\|_s^2$$
$$\leq -\frac{1}{2}\|\bar{\xi}_1\|_s^2 + Ce^{-s\frac{N-1}{2}}\|\bar{\xi}_1\|_s^2 + Cs^b e^{-as}.$$

Combining this last estimate with (4.44), we get:

$$X(s)' + (1 - Ce^{-s\frac{N-1}{2}})X(s)$$
$$\leq \begin{cases} C(1+s)e^{-s\frac{N-1}{2}}\sqrt{X(s)} + Cs^b e^{-as}, & \text{when } N = 2, \\ Ce^{-s\frac{N-1}{2}}\sqrt{X(s)} + Cs^b e^{-as}, & \text{when } N \geq 3. \end{cases} \tag{4.45}$$

This differential inequality fits with the general form of Lemma 3.3, which yields:

$$X(s) \leq Cs^b e^{-as}.$$

To conclude the proof of Theorem 1.3, one has only to rewrite the estimates above in the classical variables $\mathbf{x}$ and $t$, using the formulas (3.17b).

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