Computation of a class of continued fraction constants

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Abstract

We describe a class of algorithms which compute in polynomial– time important constants related to the Euclidean Dynamical System. Our algorithms are based on a method which has been previously introduced by Daudé Flajolet and Vallée in [10] and further used in [13, 32]. However, the authors did not prove the correctness of the algorithm and did not provide any complexity bound. Here, we describe a general framework where the DFV–method leads to a proven polynomial–time algorithm that computes "spectral constants" relative to a class of Dynamical Systems. These constants are closely related to eigenvalues of the transfer operator. Since it acts on an infinite–dimensional space, exact spectral computations are almost always impossible, and are replaced by (proven) numerical approximations. The transfer operator can be viewed as an infinite matrix $\mathcal{M} = (M_{i,j})_{1\leq i,j<\infty}$ which is the limit (in some precise sense) of the sequence of truncated matrices $\mathcal{M}_n := (M_{i,j})_{1\leq i,j<n}$ of order n where exact computations are possible. Using results of [1], we prove that each isolated eigenvalue λ of $\mathcal{M}$ is a limit of a sequence $\lambda_n \in \mathrm{Sp}\mathcal{M}_n$, with exponential speed. Then, coming back to the Euclidean Dynamical System, we compute (in polynomial time) three important constants which play a central rôle in the Euclidean algorithm: (i) the Gauss-Kuzmin-Wirsing constant related to the speed of convergence of the continued fraction algorithm to its limit density; (ii) the Hensley constant which occurs in the leading term of the variance of the number of steps of the Euclid algorithm; (iii) the Hausdorff dimension of the Cantor sets relative to constrained continued fraction expansions.

1 Introduction

When mathematical constants do not admit a closed form, it is of great importance to compute them. The book of Finch [12] provides many instances of this situation. Here, we consider a class of constants which are of great interest in the algorithmics of Dynamical Systems. Since they do not seem to admit a closed form, we are interested in their computability: does there exist an efficient algorithm that computes the first d-digits of the constants? More precisely, we wish to prove that they are polynomial–time computable. We recall that a constant is said to be *polynomial-time computable* if its first d digits can be obtained with $\mathcal{O}(d^r)$ arithmetic operations. Here, we are interested in the computability of "spectral" constants which are closely related to the spectrum of transfer operators associated to these Dynamical Systems.

1.1 The principles of the algorithm.

Consider, in the complex plane, a disk D with center x_0. Consider an operator $\mathbf{G}$ that acts on the space

$$A_\infty(D) := \{f : D \to \mathbf{C}; f \text{ analytic on } D \text{ and continuous on } \overline{D}\}.$$

Then, for $f \in A_\infty(D)$, the Taylor expansions at x_0 of f and $\mathbf{G}[f]$ exist and the operator $\mathbf{G}$ can be viewed as an infinite matrix $\mathbf{M} := (M_{i,j})$ with $0 \leq i, j < \infty$ and

$$M_{i,j} = \text{ the coefficient of } (z-x_0)^i \text{ in } \mathbf{G}[(Z-x_0)^j](z).$$

The truncated matrix $\mathbf{M}_n := (M_{i,j})_{0\leq i,j\leq n}$ is the matrix of order $n+1$ which describes the action of a "truncated" operator on the space $\mathcal{P}_n$ formed with polynomials of degree at most n. More precisely, the truncated matrix $\mathbf{M}_n$ represents the truncated operator $\pi_n \circ \mathbf{G}|_{\mathcal{P}_n}$ where π_n is the projection on $\mathcal{P}_n$ which associates to a function f its Taylor expansion of order n at x_0 i.e.

$$\pi_n[f](z) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}\ (z-x_0)^k. \tag{1.1}$$

Note that the operator $\pi_n \circ \mathbf{G}$ and the matrix $\mathbf{M}_n$ have the same spectrum.

In the case of the Euclidean Dynamical System, Daudé, Flajolet and Vallée introduced in [10] a method for computing (a finite part of) the spectrum of transfer operators, which they further used in [13, 32]. Their method, the so–called DFV–method, has three main steps which we describe in an informal way (See Figure 1).

(i) *Compute the truncated matrix* $\mathbf{M}_n$ *relative to the operator* $\mathbf{G}$.

(ii) *Compute the spectrum* $\mathrm{Sp}\,\mathbf{M}_n$ *of matrix* $\mathbf{M}_n$*, i.e., the set of its eigenvalues,* $\mathrm{Sp}\,\mathbf{M}_n := \{\lambda_n^{(i)} : 0 \leq i \leq n\}$.

*GREYC, university of Caen (France), mail: loick.lhote@info.unicaen.fr

(iii) Relate the set $\operatorname{Sp}\mathbf{M}_n$ *with a (finite) part of* $\operatorname{Sp}\mathbf{G}$.

$$\begin{tikzcd}
\begin{array}{c}\text{operator}\\ \mathbf{G}\end{array} \arrow[r, "\textit{Hard}"] \arrow[d] & \begin{array}{c}\text{eigenvalue}\\ \lambda\end{array} \\
\begin{array}{c}\text{matrix}\\ \mathbf{M_n}\end{array} \arrow[r, "\textit{Easy matrix}", "\text{computations}"'] & \begin{array}{c}\text{eigenvalue}\\ \lambda_n\end{array} \arrow[u, "|\lambda_n - \lambda| \leq K\theta^n"']
\end{tikzcd}$$

Figure 1: The DFV-method for computing eugenvalue approximates.

In the case when the transfer operator has a unique dominant eigenvalue λ, isolated from the remainder of the spectrum, one can expect that it is the same for $\mathbf{M}_n$, with a dominant eigenvalue λ_n. Moreover, the authors of [10] observed that the sequence λ_n seems to converge to λ (when the truncation degree n tends to ∞), with exponential speed. They conjectured the following:

There exist n_0, K, θ *such that, for any* $n \geq n_0$*, one has* $|\lambda_n - \lambda| \leq K\theta^n$.

1.2 Our results. In this paper, we prove that the conjecture is true in a very general framework, as soon as the transfer operator is relative to a Dynamical System which is strongly contracting ($\mathcal{SCDS}$ setting). We also prove that the constant θ is closely related to the contraction ratio of the Dynamical System. It is then exactly computable, and we prove in this way that any eigenvalue λ is polynomial–time computable as soon as the truncated matrix $\mathbf{M}_n$ is computable in polynomial–time (in n) (Theorem 1).

However, if we wish to obtain proven digits for λ, we have to exhibit explicit values of K and n_0. This does not seem possible in the general $\mathcal{SCDS}$ setting but we prove that it is the case when (i) we approximate the dominant eigenvalue and (ii) the transfer operator is normal on a convenient functional space.

The Euclidean Dynamical System belongs to the $\mathcal{SCDS}$ class. It has been deeply studied by Mayer. Adapting his results to our more general setting, we exhibit a class of transfer operators which are normal on convenient Hardy spaces. We then prove that a class of continued fraction constants is polynomial–time computable (Theorem 2), and we are able to exhibit, for each constant of the class, an efficient algorithm which computes d proven digits in time $O(d^4)$. We apply our method to three constants: The Gauss-Kusmin-Wirsing constant γ_G, the Hensley constant γ_H, and the Hausdorff dimension of reals associated to constrained continued fraction expansions (Theorem 3).

1.3 The Euclidean Dynamical System and its transfer operators. Every real number $x \in]0,1]$ admits a continued fraction expansion of the form

$$x = \cfrac{1}{m_1 + \cfrac{1}{m_2 + \cfrac{1}{m_3 + \cfrac{1}{\ddots + \cfrac{1}{m_p + \ldots}}}}}, \tag{1.2}$$

where the m_i form a sequence of positive integers. Ordinary continued fraction (CF) expansions of real numbers are the result of an iterative process which constitutes the continuous counterpart of the standard Euclidean division algorithm. They can be viewed as trajectories of a specific Dynamical System relative to the Gauss map $T : [0,1] \to [0,1]$ defined by

$$T(x) := \frac{1}{x} - \lfloor \frac{1}{x} \rfloor, \quad \text{for } x \neq 0, \quad T(0) = 0\,,$$

(here, $\lfloor x \rfloor$ is the integer part of x). The set $\mathcal{G}$ of the inverse branches of T is

$$\mathcal{G} := \{h : x \mapsto \frac{1}{m+x}; m \geq 1\}. \tag{1.3}$$

The set of inverse branches of T^n is $\mathcal{G}^n$ and it is indexed by the set $\mathbf{N}_*^n$

If f_0 be an initial density on I, repeated applications of the map T modify the density and the successive densities $f_1, f_2, \ldots, f_n, \ldots$ describes the global evolution of the system at time $t = 0, 1, 2 \ldots n, \ldots$. The operator $\mathbf{G}$ such that $f_1 = \mathbf{G}[f_0]$ and more generally $f_n = \mathbf{G}[f_{n-1}] = \mathbf{G}^n[f_0]$ for all n is called the density transformer. It is defined as

$$\begin{aligned}
\mathbf{G}[f](z) &= \sum_{h \in \mathcal{G}} |h'(z)|\; f \circ h(z) \\
&= \sum_{m \geq 1} \frac{1}{(m+z)^2}\, f\left(\frac{1}{m+z}\right). \qquad (1.4)
\end{aligned}$$

It acts on $A_\infty(D)$ (for a convenient disk $A_\infty(D)$, see Section 3.1).

A perturbation of the density transformer, the transfer operator $\mathbf{G}_s$, defined as

$$\begin{aligned}
\mathbf{G}_s[f](z) &= \sum_{h \in \mathcal{G}} |h'(z)|^s\; f \circ h(z) \\
&= \sum_{m \geq 1} \frac{1}{(m+z)^{2s}}\, f\left(\frac{1}{m+z}\right) \qquad (1.5)
\end{aligned}$$

involves a new parameter s. It extends the density transformer since $\mathbf{G}_1 = \mathbf{G}$ and plays a crucial rôle in the analysis of rational trajectories. It acts on $A_\infty(D)$ as soon as $\Re s > 1/2$. Remark that its iterate $\mathbf{G}^n$ of order n involves the set $\mathcal{G}^n$ of the inverse branches of depth n,

$$(1.6)\qquad \mathbf{G}_s^n[f](z) = \sum_{h\in\mathcal{G}^n} |h'(z)|^s\ f\circ h(z).$$

The constrained transfer operator does not not involve the whole set $\mathcal{G}^n$ of the inverse branches of some depth n, but only a subset $\mathcal{A} \subset \mathcal{G}^n$ for some n. It is defined as

$$(1.7)\qquad \mathbf{G}_{s,\mathcal{A}}[f](z) = \sum_{h\in\mathcal{A}} |h'(z)|^s\ f\circ h(z).$$

This is a powerful tool for studying the reals whose CF–expansion only uses the set $\mathcal{A}^\star$. It acts on $A_\infty(D_\mathcal{A})$ (for a convenient disk $D_\mathcal{A}$, see Section 3.1) as soon as $\Re s > \sigma_\mathcal{A}$ for some $\sigma_\mathcal{A}$ which depends on $\mathcal{A}$ (note that $\sigma_\mathcal{A} = -\infty$ if $\mathcal{A}$ is finite).
In the following, if there exists $n \geq 1$ for which $\mathcal{A} = \mathcal{G}^n$, the index $\mathcal{A}$ will be omitted. Consider the disk $D_\mathcal{A}$ and the functional space $A_\infty(D_\mathcal{A})$; for real $s > \sigma_\mathcal{A}$, the operator $\mathbf{G}_{s,\mathcal{A}}$ possesses a unique dominant eigenvalue $\lambda_\mathcal{A}(s)$, positive, isolated from the remainder of the spectrum by a spectral gap $\rho_\mathcal{A}(s)$. These two quantities are essential for describing the action of $\mathbf{G}_{s,\mathcal{A}}$. Since the transfer operator plays a fundamental rôle in the analysis of the underlying Dynamical System, this explains why the two quantities $\lambda_\mathcal{A}(s), \rho_\mathcal{A}(s)$ intervene in the description of our three algorithmic constants, which we now describe.

1.4 Gauss–Kuz'min–Wirsing constant. Around 1800, Gauss [14] studied the evolution of the distribution of the iterates $T^k(x)$. In fact, he introduced an operator closely related to the density transformer $\mathbf{G}$ and he exhibited a density $g(x) := (1/\log 2)(1+x)^{-1}$ (now known as Gauss' density) which he proved to be invariant under the action of T (i.e., $\mathbf{G}[g] = g$). He conjectured that it is a limit density; in other words, he asked whether, for any initial density f, the sequence $\mathbf{G}^n[f]$ tends to g. One century later, Kuz'min [24] (1928) and Lévy [26] (1929) proved this assertion. It was then important to obtain the optimal speed of convergence of $\mathbf{G}^n[f]$ to g. Finally, Babenko [2] and Wirsing [36], around 1975, completely solved the problem and showed that the speed is exponential. The ratio equals the subdominant eigenvalue (which is unique and real) of the density transformer $\mathbf{G}$, and Wirsing proved that there is a unique subdominant eigenvalue, real and negative. This constant called Gauss-Kuz'min-Wirsing constant, and denoted here bu γ_G, does not seem to be related to other arithmetical constants [12]. It was computed in [10] to about 30 decimal places by the DFV–method

$$\gamma_G \approx -0.30396355092701333\ldots$$

Using similar methods, Sebah (unpublished) and Briggs [4] (2003) improved the accuracy to respectively 100 and 385 digits. Since we show in this paper that the DFV–method leads to a (proven) algorithm, we exhibit here a polynomial–time algorithm to compute the Gauss-Kuz'min-Wirsing constant.

1.5 Hensley's constant. The Euclid Algorithm is closely related to the Continued Fraction algorithm. Indeed, if $a_0 = a_1\ m_1 + a_2$ is a division performed by the Euclid Algorithm, then the rationals $x_0 = a_1/a_0$ and $x_1 = a_2/a_1$ are related by $T(x_0) = x_1$, and the execution of the Euclid algorithm on (a_0, a_1) is just the trajectory $\mathcal{T}(a_0/a_1)$ of (a_0/a_1) under the action of T. The complexity of the Euclid Algorithm (i.e., the number P of divisions performed) was first studied in the worst–case by Lamé [25] around 1850. A century later (around 1970), Heilbronn [17] and Dixon [11] determined the average number of steps. Finally, in 1994, Hensley [18], using the transfer operator $\mathbf{G}_s$, showed that the number of steps of the Euclid Algorithm follows asymptotically a Gaussian law. Recently, Baladi and Vallée [3], using deeper results on the transfer operators, obtained an alternative proof of this result, that is both more general and more concise. On the set of pairs (u, v) with $0 \leq u \leq v \leq N$, the asymptotic expressions for the mean and the variance involve the first and second derivatives of the dominant eigenvalue function $\lambda(s)$ at $s = 1$:

$$\begin{aligned} &\mathrm{E}_N[P] &\sim\ & \frac{-2}{\lambda'(1)}\log N, \\ (1.8)\qquad &\mathrm{Var}_N[P] &\sim\ & 2\frac{\lambda''(1) - \lambda'(1)^2}{\lambda'(1)^3}\log N. \end{aligned}$$

The first derivative $\lambda'(1)$ equals the opposite of the entropy of the Euclidean Dynamical System. Since the invariant density (the Gauss density) is explicit, the value $-\lambda'(1)$ admits a closed form, $-\lambda'(1) = \pi^2/(6\log 2)$. The constant that appears in the dominant term of the variance is the so–called Hensley constant, denoted by γ_H. It involves the second derivative $\lambda''(1)$ that does not seem to be related to other arithmetical constants. The Hensley constant was previously computed by the DFV–method in [13],

$$\gamma_H \approx 0.5160624\ldots$$

This paper provides a proven approximation for the Hensley constant.

1.6 Hausdorff dimension and constrained *CF***–expansions.** Consider some integer n and a subset $\mathcal{A}$ of $\mathbb{N}_*^n$. Denote by $R_\mathcal{A}$ the Cantor set of reals in I whose continued fraction expansion is restricted to $\mathcal{A}$,

$$R_\mathcal{A} \quad := \quad \{x \in I \,; x = [q_1, q_2, \ldots, q_k, \ldots], \forall k \geq 0, \\ (q_{kn+1}, q_{kn+2}, q_{kn+n}) \in \mathcal{A}\}.$$

As soon as A is different from $\mathbb{N}_*^n$, the Cantor set $R_\mathcal{A}$ has zero Lebesgue measure, and the Hausdorff dimension provides a precise description of it. In particular, the probability that a rational with numerator and denominator less than N belongs to $R_\mathcal{A}$ is $\Theta(N^{2s_\mathcal{A}-2})$, so that the expected time to obtain a rational $\mathcal{A}$–constrained with numerator and denominator less than N is $\Theta(N^{2-2s_\mathcal{A}})$. When $\mathcal{A}$ is finite, the reals of $R_\mathcal{A}$ are interesting since they are all badly approximable by rationals [31].

If, furthermore, the set $\mathcal{A}$ contains more than one element, the Hausdorff dimension of $R_\mathcal{A}$, denoted by $s_\mathcal{A}$, is a real number of $]0,1[$, and is proven to be the unique real $s \in]0,1[$ for which the dominant eigenvalue function $\lambda_\mathcal{A}(s)$ of the transfer operator $\mathbf{G}_{s,\mathcal{A}}$ equals 1. This is why the Hausdorff dimension belongs to the class of spectral constants.

The Hausdorff dimension of the set $R_\mathcal{A}$ relative to $\mathcal{A} := \{1,2\}$ has been intensively studied. In 1941, Good [15] showed that $0.5194 \leq s_{\{1,2\}} \leq 0.5433$ and in 1982, Bumby [5][6] improved these estimates and obtains $s_{\{1,2\}} = 0.5313 \pm 10^{-4}$. In 1996, Hensley [19] provided a polynomial–time algorithm in the case of a finite set $\mathcal{A}$ and obtained the following estimation

$$s_{\{1,2\}} \approx 0.5312805062772051416.$$

Finally, in 1999, Jenkinson and Pollicott [22] designed a powerful algorithm which computes $s_{\{1,2\}}$ up to 25 digits. Note that it is not a polynomial–time algorithm. The DFV–method has been applied (heuristically) to the case of a general set $\mathcal{A}$ and seemed to be efficient [32]. We propose a proven polynomial–time algorithm based on the DFV–heuristics for any subset $\mathcal{A}_1 \times \mathcal{A}_2 \times \ldots \times \mathcal{A}_n$ of $\mathcal{G}^n$. In the particular case when $\mathcal{A} \subset \mathcal{G}$, it gives rise to proven numerical values of $s_\mathcal{A}$, and it seems to run faster than Hensley's algorithm.

2 Polynomial time algorithms for Strictly Contracting Dynamical Systems.

As explained in Section 1, we wish to prove the DFV–method. The following definition is natural in this context.

Definition 1. [Operator with good truncations.] *Let D be a disk of center x_0, and consider an operator $\mathbf{G}$ that acts on $A_\infty(D)$. Consider the projection π_n defined in (1.1) and the truncated operator $\mathbf{G}_n := \pi_n \circ \mathbf{G}$. The operator $\mathbf{G}$ has good truncations if the following is true: there is $\theta < 1$ such that, for any simple isolated eigenvalue λ of the operator $\mathbf{G}$, there exist a constant $K > 0$, an integer n_0, and a sequence $\lambda_n \in \operatorname{Sp} \mathbf{G}_n$, for which, for any $n \geq n_0$, one has:*

$$|\lambda - \lambda_n| \leq K\theta^n.$$

The constant θ is called the truncature ratio.
If moreover the triple $[\theta, K, n_0]$ is computable, then the truncations are said to be computable.

We are interested in constants that arise in spectral objects relative to complete Dynamical System. A complete Dynamical System is a pair $(\mathcal{I}, T)$ formed with an interval $\mathcal{I}$ and a map $T : \mathcal{I} \to \mathcal{I}$ which is piecewise surjective and of class $\mathcal{C}^2$. We denote by $\mathcal{G}$ the set of the inverse branches of T; then, $\mathcal{G}^k$ is the set of the inverse branches of T^k. It is known that contraction properties of the inverse branches are essential to obtain "good" properties on the Dynamical System. Usually, what is needed is the existence of a disk D which is strictly mapped inside itself by all the inverse branches $h \in \mathcal{G}$ of the system [i.e., $h(\overline{D}) \subset \overline{D}$]. Here, we have to strengthen this hypothesis. This motivates the following definition.

Definition 2. [Strongly contracting dynamical system.] *A complete Dynamical System of the interval $\mathcal{I}$ is said to be strongly contracting ($\mathcal{SCDS}$ in short) when the set $\mathcal{G}$ of the inverse branches fulfills the supplementary condition : For any subset $\mathcal{A} \subset \mathcal{G}$, there exist $x_0 \in \mathcal{I}$ and two open disks of same center x_0, the large disk D_L, and the small disk D_S, with $D_S \subsetneq D_L$ and $\mathcal{I} \subset D_L$, such that any $h \in \mathcal{A}$ is an element of $A_\infty(D_L)$ which strictly maps D_L inside D_S [i.e., $h(\overline{D_L}) \subset \overline{D_S}$]. Remark that (x_0, R_S, R_L) depend on $\mathcal{A}$. The largest possible ratio R_S/R_L between the radii R_S and R_L of the two disks is called the $\mathcal{A}$–contraction ratio.*

A strongly contracting system is said to be extra–contracting ($\mathcal{XSCDS}$ in short) if, for any $\mathcal{A} \subset \mathcal{G}$, there exist an integer $k > 1$ and another disk D_{XL} (the extra–large disk) cocentric with D_S, with $D_L \subset D_{XL}$, such that that any $h \in \mathcal{A}^k$ is an element of $A_\infty(D_{XL})$ which strictly maps D_{XL} inside D_S [i.e., $h(\overline{D_{XL}}) \subset \overline{D_S}$].

We shall prove in the following that many Dynamical Systems relative to Euclidean algorithms belong to the $\mathcal{SCDS}$–setting, and even in the $\mathcal{XSCDS}$–setting.

Definition 3. [Transfer operator] *Let (I, T) be of $\mathcal{SCDS}$–type. Consider an integer $n \geq 1$, a subset $\mathcal{A} \subset \mathcal{G}^n$, a real $\sigma_\mathcal{A} \geq 0$, a sequence $(\alpha_h)_{h\in\mathcal{A}}$ of functions* *of $A_\infty(D_L)$ positive on $D_L \cap \mathbb{R}$, such that, for any s with $\Re s > \sigma_{\mathcal{A}}$, the quantity*

$$\delta(s,\mathcal{A}) := \sum_{h\in\mathcal{A}} \sup_{x\in D_L} |\alpha_h(x)|^{\Re(s)} < \infty.$$

Then, the relation

$$\mathbf{G}_{s,\mathcal{A}}[f] = \sum_{h\in\mathcal{A}} \alpha_h^s\ f\circ h.$$

defines, for $\Re s > \sigma_{\mathcal{A}}$, an operator $\mathbf{G}_{s,\mathcal{A}} : A_\infty(D_S) \to A_\infty(D_L)$ whose norm $\|\mathbf{G}_{s,\mathcal{A}}\|_{D_S,D_L}$ is at most $\delta(s,\mathcal{A})$. Such an operator is called a transfer operator with constraint $\mathcal{A}$.
If moreover the system (I,T) is of $\mathcal{XSCDS}$–type, with an integer $k>1$, the operator $\mathbf{G}^k_{s,\mathcal{A}}$ maps $A_\infty(D_S)$ into $A_\infty(D_{XL})$.

Our first result is as follows:

Theorem 1. [A transfer operator has good truncations.] *In the $\mathcal{SCDS}$–setting, a transfer operator $\mathbf{G}_{s,\mathcal{A}} : A_\infty(D_S) \to A_\infty(D_L)$ satisfies the following:*
(ii) It is compact, and its spectrum is formed with isolated eigenvalues of finite multiplicity, except perhaps at 0.
(ii) for any real s, with $s > \sigma_{\mathcal{A}}$, the operator $\mathbf{G}_{s,\mathcal{A}}$ has a unique dominant eigenvalue $\lambda_{\mathcal{A}}(s)$ simple, positive and isolated from the other eigenvalues by a spectral gap.
(iii) for any real s, with $s > \sigma_{\mathcal{A}}$, the operator $\mathbf{G}_{s,\mathcal{A}}$ has good truncations. The truncature ratio θ satisfies $\theta \le R_S/R_L$ where R_S and R_L are the radii of the optimal pair of disks (D_S, D_L) relative to $\mathcal{A}$.
(iv) In the $\mathcal{XSCDS}$–setting, the truncature ratio satisfies $\theta \le R_S/R_{XL}$ where R_S and R_{XL} are the radii of the optimal pair of disks (D_S, D_{XL}) relative to $\mathcal{A}$,

Now, we prove Theorem 1 and provide explicit constants for K, n_0 and θ. The first two assertions are easily adapted from the works of Mayer [30], and we then focus on the the proof of the third assertion, which is mainly based on two results. We first a well-known result of functional analysis, which says: "When two operators are close with respect to the norm, their spectrums are close too". The second result shows that the strongly contraction property entails that the truncated operators converge in norm to the transfer operator.

2.1 Functional analysis. Denote by $(\mathcal{B}, \|\ \|)$ a complex Banach space and by $\mathbf{G}$ an operator which acts on $\mathcal{B}$. Denote by $\operatorname{Sp}\mathbf{G}$ the spectrum of $\mathbf{G}$. Consider a fixed eigenvalue λ and a circle $C = C(\lambda, r)$ (with center λ and radius $r>0$) that isolates λ from the remainder of the spectrum. This means that r satisfies $r < d(\lambda, \operatorname{Sp}\mathbf{G}\setminus\{\lambda\})$. The constants $\alpha_C(\mathbf{G})$ and $\beta_C(\mathbf{G})$, defined by

$$\alpha_C(\mathbf{G}) := \sup_{z\in C} \|\ (\mathbf{G} - z\mathbf{I})^{-1}\ \|, \tag{2.1}$$

$$\beta_C(\mathbf{G}) := \max\left\{\frac{1}{2\alpha_C(\mathbf{G})}, \frac{1}{2r\alpha_C^2(\mathbf{G})}, \frac{1}{8r^2\alpha_C^3(\mathbf{G})}\right\}, \tag{2.2}$$

play a central rôle in the paper. They first intervene in the following result, which is fundamental here.

Lemma 1. *Let $\mathbf{G}$ and $\widetilde{\mathbf{G}}$ be two operators on the Banach space $(\mathcal{B}, \|\ \|)$. Suppose that λ is a simple and isolated eigenvalue of $\mathbf{G}$, with an eigenfunction ϕ, and consider an isolating circle $C = C(\lambda, r)$. Then, if $\mathbf{G}$ and $\widetilde{\mathbf{G}}$ satisfy $\|\mathbf{G} - \widetilde{\mathbf{G}}\| \le \beta_C(\mathbf{G})$, then the operator $\widetilde{\mathbf{G}}$ has a unique simple isolated eigenvalue $\widetilde{\lambda}$ in C that satisfies*

$$\begin{aligned} |\widetilde{\lambda} - \lambda| &\le 2r\ \cdot \alpha_C(\mathbf{G}) \cdot \frac{\|\widetilde{\mathbf{G}}[\phi] - \mathbf{G}[\phi]\|}{\|\phi\|} \\ &\le 2r\ \cdot \alpha_C(\mathbf{G}) \cdot \|\widetilde{\mathbf{G}} - \mathbf{G}\|. \end{aligned} \tag{2.3}$$

The condition $\|\mathbf{G} - \widetilde{\mathbf{G}}\| \le \beta_C(\mathbf{G})$ implies, via the definition of $\beta_C(\mathbf{G})$, three conditions, with a precise goal for each of them. The first condition ensures that the circle C is also included in the resolvant set of $\widetilde{\mathbf{G}}$. The second condition implies that $\widetilde{\lambda}$ is the unique eigenvalue of $\widetilde{\mathbf{G}}$ in C. Finally, with the last condition, it is possible to relate the two spectral spaces related to λ and $\widetilde{\lambda}$.
This proposition will be also useful for computing the Hensley constant (in Section 4.2).

2.2 Convergence of truncated operators in the $\mathcal{SCDS}$–setting. Consider any operator $\mathbf{G} : A_\infty(D_S) \to A_\infty(D_L)$. We recall that the non-zero eigenvalues of the operator $\pi_n \circ \mathbf{G}$ and the matrix $\mathbf{M}_n$ are the same. Then, according to the previous Lemma, it is sufficient to obtain the convergence of $\pi_n \circ \mathbf{G}$ to $\mathbf{G}$ (in norm). This is the aim of the following lemma, which requires the strong contracting property. A proof restricted to the framework of Continued Fractions can also be found in [20] with slightly different functional spaces.

Lemma 2. *Consider an operator $\mathbf{G} : A_\infty(D_S) \to A_\infty(D_L)$ with norm $\|G\|_{D_S,D_L}$. Then, one has:*

$$\|\pi_n \circ \mathbf{G} - \mathbf{G}\|_{D_S} \le \|G\|_{D_S,D_L} \frac{R_L}{R_L - R_S}\left(\frac{R_S}{R_L}\right)^{n+1}.$$

Suppose furthermore that there exists some disk D_{XL}, with $D_L \subset D_{XL}$, and some k–th iterate of $\mathbf{G}$ which maps $A_\infty(D_S)$ into $A_\infty(D_{XL})$. Then, for any eigenfunction ϕ of $\mathbf{G}$,

$$\frac{\|\pi_n[\phi] - [\phi]\|_{D_S}}{\|\phi\|_{D_S}} \le \frac{\|\phi\|_{D_{XL}}}{\|\phi\|_{D_S}} \cdot \frac{R_{XL}}{R_{XL} - R_S}\left(\frac{R_S}{R_{XL}}\right)^{n+1}.$$

Proof. For $f \in A_\infty(D_S)$, the i–th coefficient a_i of Taylor expansion of $g := \mathbf{G}[f]$ at x_0 satisfies, with the Cauchy formula, the inequality $a_i R_L^i \le \|g\|_{D_L}$, and the Strongly Contracting Property entails

$$\begin{aligned} \|\pi_n[g] - g\|_{D_S} &\le \|g\|_{D_L} \sum_{i>n} \left(\frac{R_S}{R_L}\right)^i \\ &= \frac{R_L}{R_L - R_S} \left(\frac{R_S}{R_L}\right)^{n+1}. \end{aligned} \tag{2.4}$$

Then, the definition of $\|G\|_{D_S,D_L}$ provides the first result. For the second, note that any eigenfunction ϕ belongs to $A_\infty(D_{XL})$, and apply relation (2.4) with R_{XL} instead of R_L.

2.3 The pair $[\theta, K, n_0]$**.** We now return to the transfer operator $\mathbf{G}_{s,\mathcal{A}}$ and we consider a simple isolated eigenvalue λ of $\mathbf{G}_{s,\mathcal{A}}$ together with an isolating circle $C = C(\lambda, r)$. We recall that $\|\mathbf{G}_{s,\mathcal{A}}\|_{D_S,D_L}$ is at most $\delta(s,\mathcal{A})$. Consider first the $\mathcal{SCDS}$–setting. Denote by n_0 the smallest integer n for which

$$\delta(s,\mathcal{A}) \frac{R_L}{R_L - R_S} \left(\frac{R_S}{R_L}\right)^{n+1} \le \beta_C(\mathbf{G}_{s,\mathcal{A}}). \tag{2.5}$$

According to the two previous lemmas, for all $n \ge n_0$, the truncated operator $\pi_n \circ \mathbf{G}_{s,\mathcal{A}}$ (and then the matrix $\mathbf{M}_{s,\mathcal{A},n}$) admits a unique eigenvalue λ_n in the interior of C that satisfies $|\lambda_n - \lambda| \le K\theta^n$ where K and θ are given by

$$K = 2r \cdot \alpha_C(\mathbf{G}_{s,\mathcal{A}}) \cdot \delta(s,\mathcal{A}) \cdot \frac{R_S}{(R_L - R_S)} \quad \text{and}$$

$$\theta = \frac{R_S}{R_L}. \tag{2.6}$$

In the case of the $\mathcal{XSCDS}$–setting, the integer n_0 is the same as previously; however, the constants K and θ can be chosen as

$$K = 2r \cdot \alpha_C(\mathbf{G}_{s,\mathcal{A}}) \cdot \frac{\|\phi\|_{D_{XL}}}{\|\phi\|_{D_S}} \cdot \frac{R_S}{(R_{XL} - R_S)} \quad \text{and}$$

$$\theta = \frac{R_S}{R_{XL}}. \tag{2.7}$$

This ends the proof of Theorem 1. □

2.4 Instances of the $\mathcal{SCDS}$ **setting and applications of Theorem 1.** The work [34] introduces a class of Euclidean algorithms that is called the Fast Class. We mainly consider here three algorithms of this class : the standard algorithm $\mathcal{S}$ (already described in Section 1.3), the centered algorithm $\mathcal{C}$ and the odd algorithm $\mathcal{O}$. All these algorithms give rise to a Dynamical System (I, T), where T is always of the form

$$T(x) := \left| \frac{1}{x} - V\left(\frac{1}{x}\right) \right|,$$

with $V_\mathcal{S}(u)$ is the integer part to u, $V_\mathcal{C}(u)$ is the nearest integer to u, and $V_\mathcal{O}(u)$ is the odd integer nearest to u. The intervals $I_\mathcal{S}$ and $I_\mathcal{O}$ are equal to $[0,1]$, while the interval $I_\mathcal{C}$ equals $[0, 1/2]$. The set of the inverse branches is $\mathcal{G}_\mathcal{S} = \mathcal{G}$ already described in (1.3), and, in the two other cases,

$$\mathcal{G}_\mathcal{C} := \{x \mapsto \frac{1}{m + \epsilon x}; \epsilon = \pm 1, (m, \epsilon) \ge (2, +1)\},$$

$$\mathcal{G}_\mathcal{O} := \{x \mapsto \frac{1}{m + \epsilon x}; \epsilon = \pm 1, m \text{ odd}, (m, \epsilon) \ge (1, +1)\},$$

(here, the order $\ge$ is relative to the lexicographic order.) In each of the three cases, it is proved in [34] that there exists a disk D, with $I \subset D$, which is strictly mapped inside itself by all the inverse branches $h \in \mathcal{G}$ of the system [i.e., $h(\overline{D}) \subset \overline{D}$]. In fact, in each case, these systems are of $\mathcal{SCDS}$ type and the pair (x_0, R_S, R_L) can be chosen as follows:

$$\mathcal{S} : (1, 1, 3/2), \quad \mathcal{C} : (1/4, 5/12, 3/4), \quad \mathcal{O} : (1, 1, 3/2),$$

so that $\theta_\mathcal{S} = \theta_\mathcal{O} = 2/3$ and $\theta_\mathcal{C} = 5/9$.

For each of the three algorithms, Theorem 1 proves that the "spectral constants" are polynomial-time computable. Then, all the eigenvalues $\lambda(s)$ can be computed in polynomial–time. The case $s = 1$ is trivial since $\lambda(1) = 1$, but, the case $s = 2$ is of great interest, since $\lambda(2)$ plays an important role in lattice reduction algorithms [10] and comparison algorithms using the continued fraction expansion [35]. Finding an estimate for $\lambda(2)$ actually motivated the DFV method. The entropy $-\lambda'(1)$ is explicit for the three algorithms, but the associated Hensley's constant is proven to be polynomial–time computable (with Theorem 1 and methods of 4.2). All the previously described constants are related to the spectrum of the classical transfer operator, i.e.,

$$\mathbf{G}_s[f] := \sum_{h \in \mathcal{G}} |h'|^s \cdot f \circ h.$$

Moreover, the actual analysis of Euclidean Algorithms deals with various "costs". The cost of an execution is the sum of the cost c relative to each step of the algorithm and involves a cost c which depends only on the digit produced at each step. Then the analysis of this cost introduces a more general transfer operator,

$$\mathbf{G}_{s,w}[f] = \sum_{h \in \widetilde{\mathcal{G}}} \exp[w\ c(h)]\ |h'|^s\ f \circ h.$$

More precisely, (see [3]), if the cost c is of "moderate growth", the variance of the total cost of an execution can be expressed with the derivatives (of order 1 or 2) of the dominant eigenvalue $\lambda_{s,w}$ of $\mathbf{G}_{s,w}$. If the cost is of "large growth", then the Hausdorff dimension relative to this cost also involves the dominant eigenvalue $\lambda_{s,w}$ ([7]). And the DFV–method can be proven to apply, with exponential rate of convergence.

The list of these possible applications of Theorem 1 is not exhaustive. We now focus on the standard Euclidean dynamical system, where it is possible to provide estimates for the pair $[K, n_0]$.

3 Proven Computation of the constants in the Euclidean case

We come back now to the Euclidean Dynamical System and its (usual) transfer operators $\mathbf{G}_{s,\mathcal{A}}$ relative to $\alpha_h = |h'|$. The formulae (2.5) (2.6) which define θ, n_0 and K involve two kinds of constants. The constants R_S, R_L (and R_{XL}), are closely related to the definition of the operators and are often easy to compute. This entails that the rate θ is in general easy to compute.

Then, we have to find an isolating circle $C(\lambda, r)$, which needs an estimate of λ and a lower bound for the spectral gap around λ. We prove in Lemmas 3 and 4 that it is possible, at least when $\lambda = \lambda_{\mathcal{A}}(s)$ is the dominant eigenvalue,

If we use the $\mathcal{XSCDS}$–setting, we wish to obtain an estimate of the eigenfunction ϕ relative to λ.

On the other hand, even when $C = C(\lambda, r)$ is well-defined, a lower bound for the constant $\beta_C(\mathbf{G})$, given in (2.2) is not easy to compute. It involves, via the definition of $\alpha_C(\mathbf{G})$ in (2.1) an upper bound for the norm of an inverse operator. Such an upper bound is in general hard to compute. However, when $\mathbf{G}$ is normal, an explicit an expression for $\alpha_C(\mathbf{G})$ are known,

$$\alpha_C(\mathbf{G}) = \frac{1}{d(C, \mathrm{Sp}\,\mathbf{G})} \tag{3.1}$$

(we recall that $\mathbf{G}$ is normal when it commutes with its dual $\mathbf{G}^\star$) But the normality is a rare phenomenon, which is difficult to prove. Here, it is not true that the transfer operator $\mathbf{G}_{s,\mathcal{A}}$ is normal on spaces $A_\infty(D)$; however, there exists another functional space (a Hardy space, denoted by $\mathcal{H}_{s,\mathcal{A}}$, which depends on $(s, \mathcal{A})$ where $\mathbf{G}_{s,\mathcal{A}}$ is normal (note that this normality phenomenon does not seem to hold for the two other Dynamical Systems $\mathcal{C}, \mathcal{O}$). Even if the two spaces, the Hardy space and the space $A_\infty(D)$ are different, the associated norms can be compared, and this provides an upper bound for $\alpha_C(\mathbf{G}_{s,\mathcal{A}})$, and a lower bound for $\beta_C(\mathbf{G})$. Finally, the results of this Section lead to the second main result of the paper.

Theorem 2. *The (standard) Euclidean Dynamical System is a system of $\mathcal{XSCDS}$–type. For any $\mathcal{A} \subset \mathcal{G}$, the triple $[K, n_0, \theta]$ used for approximating the dominant eigenvalue $\lambda_{\mathcal{A}}(s)$ can be estimated, and there exists an effective algorithm that computes $\lambda_{\mathcal{A}}(s)$ in polynomial–time.*

The remainder of the Section is devoted to proving Theorem 2.

3.1 Disks D_S, D_L, D_{XL} and truncature ratio.

We recall that we deal with the Euclidean system, where the set of inverse branches $\mathcal{G}$ is defined in (1.3). Here, we consider a subset $\mathcal{A}$ of $\mathcal{G}$ and A denotes the set of indices of $\mathcal{A}$. We denote by $m_{\mathcal{A}}$ the minimum of A. Then the disks D_S, D_L can be chosen as

$$x_0 := \frac{1}{m_{\mathcal{A}}}, \quad R_S = \frac{1}{m_{\mathcal{A}}}, \quad R_L = \frac{1}{m_{\mathcal{A}}} + \frac{m_{\mathcal{A}}}{2}, \qquad \frac{R_S}{R_L} = \frac{2}{2 + m_{\mathcal{A}}^2}.$$

Furthermore, the operator $\mathbf{G}^2_{s,\mathcal{A}}$ maps $A_\infty(D_S)$ into the set of functions which are analytic in the half plane $\{\Re(z) > -m_{\mathcal{A}}\}$. We then can choose for the disk D_{XL} any disk of center x_0 and radius

$$R_{XL} = \frac{1}{m_{\mathcal{A}}} + m_{\mathcal{A}} - \epsilon, \qquad \text{(with } \epsilon > 0\text{), so that} \qquad \frac{R_S}{R_{XL}} = \frac{1}{1 + m_{\mathcal{A}}^2 - \epsilon m_{\mathcal{A}}}.$$

3.2 Estimate for the dominant eigenvalue $\lambda_{\mathcal{A}}(s)$ of $\mathbf{G}_{s,\mathcal{A}}$.

We use the following classical result previously used in [10]:

Let $\mathbf{G}$ be an operator which acts on the space of analytic functions on an interval $[a, b]$. Furthermore, the operator $\mathbf{G}$ is positive (i.e., $\mathbf{G}[f] > 0$ if $f > 0$) and has a unique dominant eigenvalue λ isolated from the remainder of the spectrum by a spectral gap. Suppose that there exist two strictly positive constants c_1 and c_2 and a function f which is strictly positive and analytic on $[a, b]$ and satisfies $c_1\, f(x) \le \mathbf{G}_{s,\mathcal{A}}[f](x) \le c_2 f(x)$, for any $x \in [a, b]$. Then, the dominant eigenvalue λ satisfies $c_1 \le \lambda \le c_2$.

Denote by $m_{\mathcal{A}}$ the minimum of A and by $M_{\mathcal{A}}$ its supremum (possibly infinite if A is infinite). By convention, if $M_{\mathcal{A}} = \infty$, we put $h_{M_{\mathcal{A}}} = 0$. Each LFT $h_{M_{\mathcal{A}}} \circ h_{m_{\mathcal{A}}}$ or $h_{m_{\mathcal{A}}} \circ h_{M_{\mathcal{A}}}$ is called an extremal LFT; it has a unique positive fixed point, denoted by $a_{\mathcal{A}}$ or $b_{\mathcal{A}}$. Then the disk $D_{\mathcal{A}}$ with diameter $[a_{\mathcal{A}}, b_{\mathcal{A}}]$ is the smallest disk which is mapped into itself by all the elements of $\mathcal{A}$. The application of the previous result with the operator $\mathbf{G}_{s,\mathcal{A}}$ and the functions $f = 1$ for the upper bound and $f = 1/(1+\beta x)^{2s}$ for the upper bound provides the estimate for the dominant eigenvalue $\lambda_{\mathcal{A}}(s)$, as a function of the Hurwitz zeta function $\zeta_{\mathcal{A}}$ restricted to A,

$$\zeta_{\mathcal{A}}(s,x) := \sum_{m\in A} \frac{1}{(m+x)^s} \tag{3.2}$$

Lemma 3. *Fix the real $\beta = (-m_{\mathcal{A}} + \sqrt{m_{\mathcal{A}}^2+4})/2$. The dominant eigenvalue $\lambda_{\mathcal{A}}(s)$ admits the following estimates, which involve the Hurwitz zeta function $\zeta\mathcal{A}(s)$ restricted to A [defined in (3.2)]*

$$\zeta_{\mathcal{A}}(2s,\beta) \le \lambda_{\mathcal{A}}(s) \le \zeta_{\mathcal{A}}(2s,0) \tag{3.3}$$

Since the the function $x \to (1+\beta x)^{2s}\zeta_{\mathcal{A}}(2s,\beta+x)$ is increasing, the previous estimates can be improved to

$$(1+\beta a_{\mathcal{A}})\zeta_{\mathcal{A}}(2s,\beta+a_{\mathcal{A}}) \le \lambda_{\mathcal{A}}(s) \le \zeta_{\mathcal{A}}(2s,a_{\mathcal{A}}),$$

where $a_{\mathcal{A}}$ is the fixed point described in the previously.

3.3 Estimate for the spectral gap. In this second step, we determine a lower bound the for spectral gap $\rho_{\mathcal{A}}(s)$ between the eigenvalue $\lambda_{\mathcal{A}}(s)$ and the remainder of the spectrum. For this purpose, we use the trace of transfer operators. Grothendieck introduced the so–called nuclear operators (of order 0) and proves that they possess a trace that can be viewed as a generalisation of the usual (matrix) trace. Our transfer operator is nuclear (of order 0) and its trace equals the sum of all the eigenvalues. In particular, $\mathrm{Tr}\,\mathbf{G}^2$ is just the sum of all the squares of the eigenvalues of $\mathbf{G}$. This entails a relation between $\mathrm{Tr}\,\mathbf{G}^2$, the dominant eigenvalue $\lambda_{\mathcal{A}}(s)$ and one of its subdominant eigenvalue $\mu_{\mathcal{A}}(s)$,

$$\begin{aligned} \mu_{\mathcal{A}}^2(s) &\le \mathrm{Tr}\ \mathbf{G}_{s,\mathcal{A}}^2 - \lambda_{\mathcal{A}}^2(s) \\ \rho_{\mathcal{A}}(s) &\ge \lambda_{\mathcal{A}}(s) - \left(\mathrm{Tr}\ \mathbf{G}_{s,\mathcal{A}}^2 - \lambda_{\mathcal{A}}^2(s)\right)^{1/2}. \end{aligned} \tag{3.4}$$

The operator $\mathbf{G}_{s,\mathcal{A}}^2$ is the sum of operators of the form $\mathbf{L}[f] = |h'|^s \cdot f \circ h$ for $h \in \mathcal{A}^2$. When h is indexed by the pair (i,j), the spectrum of $\mathbf{L}$ is exactly a geometric progression of the form $\{\tau_{i,j}^{-2s-2n} : n \ge 0\}$ with

$$\tau_{i,j} = \frac{1}{2}(ij + (i^2j^2+4ij)^{1/2}+2).$$

Finally, thanks to the additivity of the trace, the trace of $\mathbf{G}_{s,\mathcal{A}}^2$ satisfies

$$\mathrm{Tr}\ \mathbf{G}_{s,\mathcal{A}}^2 = \sum_{i,j\in A} \frac{\tau_{i,j}^{-2s}}{1-\tau_{i,j}^{-2}} \tag{3.5}$$

But we can show by simple calculations that,

$$\mathrm{Tr}\ \mathbf{G}_{s,\mathcal{A}}^2 - \zeta_{\mathcal{A}}(2s,\beta)^2 < \zeta_{\mathcal{A}}(2s,\beta)^2,$$

with β as in the lemma 3. Now, using the relations (3.4, 3.5) yields the following result:

Lemma 4. *The spectral gap satisfies*

$$\begin{aligned} \rho_{\mathcal{A}}(s) &\ge 2r_{\mathcal{A}}(s) \qquad \textit{with} \\ 2r_{\mathcal{A}}(s) &:= \zeta_{\mathcal{A}}(2s,\beta) - \left(\mathrm{Tr}\ \mathbf{G}_{s,\mathcal{A}}^2 - \zeta_{\mathcal{A}}(2s,\beta)^2\right)^{1/2}, \end{aligned}$$

and $\beta = (1/2)(m_{\mathcal{A}} - (m_{\mathcal{A}}^2+4)^{1/2})$.

Remark that the previous estimates can be improved using the improved estimates of $\lambda_{\mathcal{A}}(s)$.
Wirsing [36] has shown that γ_G satisfies $0.3020 \le |\gamma_G| \le 3043$. Since the dominant eigenvalue of $\mathbf{G}_1$ is 1, using the trace one gets $|\gamma_G| - |\mu| \ge 0.18959$ where μ is one of the sub-subdominant eigenvalues of $\mathbf{G}_1$. This improves the previous estimate for the spectral gap around γ_G which was $|\gamma_G| - |\mu| \ge 0.031$.

3.4 Normality on Hardy spaces. As already said, the constant $\alpha_C(\mathbf{G}_{s,\mathcal{A}})$ has a closed form (3.1) as soon as $\mathbf{G}_{s,\mathcal{A}}$ is normal. The transfer operator is not normal on $A_\infty(D_S)$ but it is normal on another space called a Hardy space [21] and denoted $\mathcal{H}_{s,\mathcal{A}}$. For $\rho \in \mathbb{R}$, denote by P_ρ the half-plane $P_\rho = \{z \in \mathbb{C};\ \Re(z) > \rho\}$. The Hardy space $\mathcal{H}_{s,\mathcal{A}}$ is formed with the functions f which are analytic on $P_{-m_{\mathcal{A}}/2}$ and bounded on all the half-planes P_ρ (with $\rho > -m_{\mathcal{A}}/2$) and admit the following integral representation:

$$f(z) = \int_0^{+\infty} t^{s-\frac{1}{2}}\ e^{-tz}\ \phi(t)\ d\nu_{\mathcal{A}}(t),$$

with

$$d\nu_{\mathcal{A}}(t) = \sum_{n\in A} e^{-nt}\ dt \quad \text{and} \quad \phi \in L^2(\nu_{\mathcal{A}}).$$

With the associated norm

$$\|f\|^2_{<s,A>} = \int_0^{+\infty} |\phi(t)|^2 d\nu_{\mathcal{A}}(t),$$

the space $\mathcal{H}_{s,\mathcal{A}}$ is a Banach space.

There exist close relations between $\mathcal{H}_{s,\mathcal{A}}$ and $A_\infty(D_S)$. For $\mathcal{A} = \mathcal{G}$, Babenko [2] and Mayer [28] proved that the behaviour of $\mathbf{G}_s$ is comparable on $\mathcal{H}_{s,\mathcal{A}}$ on $A_\infty(D_L)$. Their methods cannot be easily generalized in the case when $\mathcal{A} \neq \mathcal{G}$. Then, in this case, we provide here a different method which makes a great use of the generalized Laguerre polynomials.

Lemma 5. *For any complex s with $\Re(s) > \max(\sigma_{\mathcal{A}}, 0)$,*
(i) the transfer operator $\mathbf{G}_{s,\mathcal{A}} : \mathcal{H}_{s,\mathcal{A}} \to \mathcal{H}_{s,\mathcal{A}}$ is isomorphic to an integral operator; it is normal and self-adjoint for real values of s. Thus, for real values of s, the spectrum of $\mathbf{G}_{s,\mathcal{A}}$ is real.
(ii) the spectrum of $\mathbf{G}_{s,\mathcal{A}}$ on $\mathcal{H}_{s,\mathcal{A}}$ and the spectrum of $\mathbf{G}_{s,\mathcal{A}}$ on $A_\infty(D_S)$ are the same.
(iii) Let D be an intermediary disk of center x_0 and radius R with $R_S < R < R_L$ and f a function of $A_\infty(D)$. For any subset $\mathcal{A}$ of $\mathcal{G}$, the function $\mathbf{G}_{s,\mathcal{A}}[f]$ belongs to $\mathcal{H}_{s,\mathcal{A}}$.
(iv) Define, for any R (with $R_S < R < R_L$), three constants $\kappa_1, \kappa_2, \kappa_3$ (which depend on $x_0, s, \mathcal{A}, R$),

(3.6)
$$\kappa_1 = \zeta_{\mathcal{A}}(2s, x_0 - R), \qquad \kappa_2 = \Gamma(2s) \cdot \zeta_{\mathcal{A}}(2s, 2(x_0 - R)),$$

(3.7)
$$\kappa_3 = \sum_{j \geq 0} \left(\frac{R_S}{R}\right)^j \left(\frac{j!}{\Gamma(2s+j)} \frac{(\gamma_j R_S)^{2s}}{\gamma_j^{2s} - 1} + \frac{\zeta_{\mathcal{A}}(2s, 0)}{\Gamma(2s)^2}\right)^{1/2}$$

with $\gamma_j = e^{x_0/(j+1)}$.
Then, the following is true:

(3.8) $\|\mathbf{G}_{s,\mathcal{A}}[f]\|_D \leq \kappa_1 \cdot \|f\|_{D_S}$ *for $f \in A_\infty(D)$*
(3.9) $\|f\|_D \leq \kappa_2 \cdot \|f\|_{<s,A>}$ *for $f \in \mathcal{H}_{s,\mathcal{A}}$,*
(3.10) $\|\mathbf{G}_{s,\mathcal{A}}[f]\|_{<s,A>} \leq \kappa_3 \cdot \|f\|_D$ *for $f \in A_\infty(D)$.*

Before proving Lemma 5, we explain how it provides an estimate for $\alpha_C(\mathbf{G}_{s,\mathcal{A}})$. Consider the isolating circle C of center $\lambda_{\mathcal{A}}(s)$ and radius $r_{\mathcal{A}}(s)$ described in Lemma4, and consider a point $z \in C$. The two inclusions

$$\begin{array}{rcll} \mathbf{G}_{s,\mathcal{A}}[A_\infty(D_S)] & \subset & A_\infty(D) & \text{and} \\ \mathbf{G}_{s,\mathcal{A}}[A_\infty(D)] & \subset & \mathcal{H}_{s,\mathcal{A}}, & \end{array}$$

together with the relation

$$z(\mathbf{G}_{s,\mathcal{A}} - z\mathbf{I})^{-1} = (\mathbf{G}_{s,\mathcal{A}} - z\mathbf{I})^{-1}\mathbf{G}_{s,\mathcal{A}} - \mathbf{I}$$

entail the following

(3.11)
$$|z| \; \|(\mathbf{G}_{s,\mathcal{A}} - z\mathbf{I})^{-1}\|_{D_S} \leq \kappa_1 \cdot \|(\mathbf{G}_{s,\mathcal{A}} - z\mathbf{I})^{-1}\|_D + 1,$$

(3.12)
$$|z| \; \|(\mathbf{G}_{s,\mathcal{A}} - z\mathbf{I})^{-1}\|_D \leq \kappa_2 \cdot \kappa_3 \cdot \|(\mathbf{G}_{s,\mathcal{A}} - z\mathbf{I})^{-1}\|_{<s,\mathcal{A}>} + 1.$$

Now, $\mathbf{G}_{s,\mathcal{A}}$ is normal on $\mathcal{H}_{s,\mathcal{A}}$, so that

$$\|(\mathbf{G}_{s,\mathcal{A}} - z\mathbf{I})^{-1}\|_{<s,\mathcal{A}>} = \frac{1}{d(z, \operatorname{Sp} \mathbf{G}_{s,\mathcal{A}})}$$

. Finally, with the formulae (3.11) and (3.12), the inequality

$$\|(\mathbf{G}_{s,\mathcal{A}} - z\mathbf{I})^{-1}\|_{\infty, D_S} \leq \frac{1}{|z|^2} \left(\frac{\kappa_1 \cdot \kappa_2 \cdot \kappa_3}{r_{\mathcal{A}}} + 1\right) + \frac{1}{|z|},$$

holds. Now, the estimate of $\lambda_{\mathcal{A}}(s)$ (given in Lemma 3) yields that any $z \in C$ satisfies

$$|z| \geq \zeta_{\mathcal{A}}(2s, b) - r_{\mathcal{A}}(s) > 0$$

and an upper bound of $\alpha_C(\mathbf{G}_{s,\mathcal{A}})$ follows. Finally, we obtain:

Lemma 6. *Denote by $r_{\mathcal{A}}(s)$ the lower bound of Lemma 4. For any intermediary radius R, with $R_S < R < R_L$ and $s > \max(0, \sigma_{\mathcal{A}})$, there exist constants κ_i defined in Lemma 5 (which depend on $x_0, R, s, \mathcal{A}$), for which*

(3.13)
$$\alpha_C(\mathbf{G}_{s,\mathcal{A}}) \leq \frac{\kappa_1 \cdot \kappa_2 \cdot \kappa_3 + r_{\mathcal{A}}(s)[1 - r_{\mathcal{A}}(s) + \zeta_{\mathcal{A}}(2s, b_{\mathcal{A}})]}{r_{\mathcal{A}}(s)[\zeta_{\mathcal{A}}(2s, b_{\mathcal{A}}) - r_{\mathcal{A}}(s)]^2},$$

Proof of Lemma 5. For (i) and (ii), we refer to the work of Jenkinson, Gonzalez and Urbański [21], and we mainly deal with (iii) and (iv). The strong contraction condition implies the inequality (3.8). The inequality (3.9) is a direct application of the Cauchy-Schwartz inequality with the identity

$$\Gamma(s)\zeta_{\mathcal{A}}(s, z) = \int_0^\infty t^{s-1} \; e^{-zt} \; d\nu_{\mathcal{A}}(t).$$

The proof of inequality (3.10) is more involved and we only explain here its main steps. First Hensley [20] has shown that, for all $j \geq 0$, the function $\mathbf{G}_{s,\mathcal{A}}[(X - x_0)^j]$ is an element of $\mathcal{H}_{s,\mathcal{A}}$ whose integral representation is closely related to generalised Laguerre polynomials $L_j^{(2s-1)}$. The Laguerre polynomials $(L_j^{(p)})$ form an orthogonal basis for the weight $t^p e^{-t}$ on $]0, \infty[$ and they verify the formula

$$L_j^{(p)}(x) = \frac{\Gamma(p+1+j)}{j!} \sum_{k=0} j(-1)^k \binom{j}{k} \frac{x^k}{\Gamma(p+1+k)}.$$

The function $\mathbf{G}_{s,\mathcal{A}}[(X - x_0)^j]$ then satisfy

$$\mathbf{G}_{s,\mathcal{A}}[(X - x_0)^j](z) =$$
$$\int_0^\infty t^{s-1/2} e^{-tz} \left[\frac{(-x_0)^j j!}{\Gamma(2s+j)} t^{s-1/2} L_j^{2s-1}\left(\frac{t}{x_0}\right)\right] d\nu_{\mathcal{A}}(t)$$

We deduce that the norm is given by

$$\|\mathbf{G}_{s,\mathcal{A}}[(X - x_0)^j]\|^2_{<s,\mathcal{A}>} =$$
$$\left[\frac{(-x_0)^j j!}{\Gamma(2s+j)}\right]^2 \int_0^\infty t^{2s-1} L_j^{2s-1}(t) d\nu_{\mathcal{A}}(x_0 t).$$

But the Laguerre polynomials are positive and decreasing on $[0, 2s/(j+1)]$. Using these properties with some relations of orthogonality and splitting the integrand $\int_0^\infty$ into $\int_0^{2s/(j+1)} + \int_{2s/(j+1)}^\infty$, we prove the inequality

$$\forall j \geq 1, \|\mathbf{G}_{s,\mathcal{A}}[(X-x_0)^j]\|_{<s,\mathcal{A}>} \leq K_j \qquad \text{with}$$

$$\frac{K_{j+1}}{K_j} \to R_S.$$

Now, the i–th coefficient c_i of the Taylor expansion of $f \in A_\infty(D)$ at x_0 satisfies $R^j|c_j| \leq \|f\|_D$ and finally

$$\|\mathbf{G}_{s,\mathcal{A}}[f]\|_{<s,\mathcal{A}>} \leq \kappa_3 \cdot \|f\|_D \qquad \text{with} \qquad \kappa_3 = \sum_{j\geq 0} \frac{K_j}{R^j}$$

Note that the previous series converges exponentially fast.

Finally, the constants K and n_0 defined in (2.5) and(2.6) involve the isolating circle C whose center $\lambda_{\mathcal{A}}(s)$ and radius $r_{\mathcal{A}}(s)$ together with $\alpha_C(\mathbf{G}_{s,\mathcal{A}})$. Since all these quantities are computable, this proves Theorem 2.

4 Application of the DFV-Mehtod to three constants

This section applies the previous results and provides (in polynomial–time, via the DFV-Method) proven numerical values for three continued fraction constants: the Gauss-Kuz'min-Wirsing constant, the Hensley constant and the Hausdorff dimension of the Cantor sets $R_{\mathcal{A}}$ with $\mathcal{A} \subset \mathcal{G}$.

In the second step of the DFV–method, we deal with computations on the matrix $\mathbf{M}_{s,\mathcal{A},n}$. First, we have to build the matrix; second we have to find the roots of $\det(\mathbf{M}_{s,\mathcal{A},n} - z\mathbf{I}_n)$. Then we conclude:
For any subset $\mathcal{A} \subset \mathcal{G}$, building matrix $\mathbf{M}_{s,\mathcal{A},n}$ needs $\mathcal{O}(n^3)$ multiplications and additions on reals and $2n+1$ computations of $\zeta_{\mathcal{A}}(s)$ functions; Computing $\mathrm{Sp}\,\mathbf{M}_{s,\mathcal{A},n}$ *needs at most $\mathcal{O}(n^4)$ arithmetical operations (with a bad method).*

In this section, we shall prove the last result of the paper:

Theorem 3. *For the three following constants – the Gauss-Kuz'min-Wirsing constants, the Henley constants, the Hausdorff dimension relative to constraints $\mathcal{A} \subset \mathcal{G}$– it is possible to provide d proven digits in polynomial–time in d.*

4.1 Algorithm for the Gauss-Kuz'min-Wirsing constant.

The Gauss-Kuz'min-Wirsing constant γ_G is the unique subdominant eigenvalue of $\mathbf{G}_1$. It is real. It is possible to estimate a circle C that isolates γ_G together with the associated constant $\alpha_C(\mathbf{G}_1)$. Thus, the DFV-method provides proven numerical values for γ_G. Only one computation of matrix is needed, and the complexity of γ_G is of order four. Numerical results are summarized up in Figure 4.1.

4.2 Algorithm for the Hensley constant.

The Hensley constant (see 1.8) involves the first two derivatives of $\lambda(s)$ at $s = 1$. Since the first derivative $\lambda'(1)$ has a closed form $\lambda'(1) = -\pi^2/(6\log 2)$, it remains to compute the second derivative $\lambda''(1)$. Consider an interval I_h of the form $I_h := [1-h, 1+h]$ and suppose that an estimate $\widetilde{\lambda}$ of λ satisfies

$$\max\Big(|\lambda(1+h) - \widetilde{\lambda}(1+h)|, |\lambda(1-h) - \widetilde{\lambda}(1-h)|\Big) \leq \frac{h^2\epsilon}{3}.$$

Then Taylor's formulae entail the estimate

$$\left|\lambda''(1) - \frac{\widetilde{\lambda}(1-h) + \widetilde{\lambda}(1+h) - 2}{h^2}\right| \leq \frac{2\epsilon}{3} + \frac{h^2}{24}\sup_{I_h}|\lambda^{(4)}|$$

It then suffices to know an upper bound for the fourth derivative $\lambda^{(4)}$ on the interval I_h. The application $s \to \mathbf{G}_s$ is analytic and the derivative $\mathbf{G}'_s$ satisfies $\|\mathbf{G}'_s\|_{D_S} \leq 8$ for $s \geq 0.9$. Then, $\|\mathbf{G}_s - \mathbf{G}_1\|_{D_S} \leq 8|s-1|$. We apply Lemma 3: the circle C of center 1 and radius $r_1 := (1-\gamma_G)/2$ is an isolating circle for $\lambda = 1$. Then, if s satisfies $|s-1| < r_2$ with $r_2 = \beta_C(\mathbf{G})/8$, the operator $\mathbf{G}_s$ admits a unique simple isolated eigenvalue $\lambda(s)$ in C which satisfies $|\lambda(s)-1| \leq r$ with $r := 16\, r_1\, r_2\, \alpha_C(\mathbf{G})$ as soon as $|s-1| \leq r_2$. Since the application $s \to \mathbf{G}_s$ is analytic, the function $s \to \lambda(s)$ is analytic too. The Cauchy formula, applied in the disk of center 1 and radius r_1 yields the upper bound

$$\sup_{[1-h,1+h]}|\lambda^{(4)}| \leq 4!\frac{1+r}{(r_1-h)^4}.$$

Then, γ_H is computable as soon as the two estimates for $\lambda(1+h)$ and $\lambda(1-h)$ are known (asymptotically to within twice the required precision). This thus needs two computations of the step 2 of the DFV–method.

Tabular 4.2 summarizes some numerical results.

4.3 Algorithm for the Hausdorff dimension.

The algorithm uses a classical dichotomy principle and computes a sequence of intervals of length 2^{-k} which contain the Hausdorff dimension $s_{\mathcal{A}}$. Consider the interval $[u_{k-1}, v_{k-1}]$ obtained after $(k-1)$ steps (it is of length $2^{-(k-1)}$and contains $s_{\mathcal{A}}$. Denote by w_k the

digits	time	proven value
10	11s	-0.3036630028
20	1m46	-0.30366300289873265859
30	9m54	-0.303663002898732658597448121901
40	34m710	-0.3036630028987326585974481219015562331108
50	1h41	-0.30366300289873265859744812190155623311087735225365

Figure 2: Gauss-Kuz'min-Wirsing constant

digits	time	proven value
5	2m30s	0.51606
10	7m30s	0.5160624089
15	41mn	0.516062408899991
20	2h33mn	0.51606240889999180681

Figure 3: Hensley constant

middle point of the interval $[u_{k-1}, v_{k-1}]$ and compute an estimate $\widetilde{\lambda}$ of $\lambda_{\mathcal{A}}(w_k)$ within $2^{-(k+1)}$. Now, there are three possible cases:

(i) If $\widetilde{\lambda} - 1 - 2^{-(k+1)} \geq 0$ then $s_{\mathcal{A}} \geq w_k$ and $[u_k, v_k] := [w_k, v_{k-1}]$.

(ii) If $\widetilde{\lambda} - 1 + 2^{-(k+1)} \leq 0$ then $s_{\mathcal{A}} \leq w_k$ and $[u_k, v_k] := [u_{k-1}, w_k]$

(iii) Else, $[u_k, v_k] := [w_k - 2^{-(k+1)}, w_k + 2^{-(k+1)}]$

This algorithm is just a classical binary splitting. The proof that $s_{\mathcal{A}}$ belongs to $[u_k, v_k]$ is based on the strict decrease of λ together with the inequality $|\lambda_{\mathcal{A}}(s) - \lambda_{\mathcal{A}}(s+h)| \geq h$. There are at most $\mathcal{O}(d)$ iterations, each of them of cubic complexity (in d). Thus, the complexity of the algorithm is asymptotically $\mathcal{O}(d^5)$. Numerical results are given in Figure 4.3 for the Cantor set $\mathcal{R}_{\{1,2\}}$.

5 Conclusion

We proved here that the DFV-method gives rise to an algorithm that computes any isolated simple eigenvalue of a transfer operator, in polynomial–time, provided that two conditions are fulfilled: (i) the operator has good truncations and (ii) the matrices $\mathbf{M}_n$ are easy to compute.

However, if we are interested in actual proven numerical values, we need evaluating the parameters that intervene in the design of the algorithm. These parameters are in general difficult to compute, but we solve this difficulty for the transfer operators relative to the Standard Euclidean Algorithm.

The DFV-method can also be used to compute the Hausdorff dimension of the Cantor sets $\mathcal{R}_{\mathcal{A}}$ with $\mathcal{A}$ of the form $\mathcal{A} = \mathcal{A}_1 \times \mathcal{A}_2 \times \ldots \times \mathcal{A}_n$, $\mathcal{A}_i \in \mathcal{H}$. However, the computation of the matrix is more involved since its coefficients deal with more complicated zeta–functions.

Finally, the authors of [10] and Sebah used the DFV-method with $x_0 = 1/2$. This particular choice do not enter our framework since no disk of center 1/2 is strictly mapped into itself. We can use any disk D_L of center $1/2 + \delta$ with radius $1/2 + 2\delta$ diameter $[-\delta, 1 + 3\delta]$ with $\delta > 0$. This leads to truncature ratio R_S/R_{XL} which tends to $1/3 - \epsilon$ as δ tends to zero, which is the convergence rate actually observed by the authors. However, our method of Section 3.4 does not seem to apply there, and we do not know how to obtain an estimate for the pair $[K, n_0]$ in this setting.

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digits	time	proven value of $s_{\{1,2\}}$
5	2m	0.53128
10	8m	0.5312805062
15	25m	0.531280506277205
20	1h	0.53128050627720514162
30	4h26	0.531280506277205141624468647368
40	14h11	0.5312805062772051416244686473684717854930
45	23h10	0.531280506277205141624468647368471785493059109

Figure 4: Hausdorff dimension of $\mathcal{R}_{\{1,2\}}$

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