Hölder-continuous rough paths by Fourier normal ordering

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An explicit rough path construction for continuous paths with arbitrary Hölder exponent

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We construct in this article an explicit geometric rough path over arbitrary d-dimensional paths with finite $1/\alpha$-variation for any $\alpha \in (0,1)$. The method is a rather straightforward extension of that used in a previous article [20] for multi-dimensional fractional Brownian motion. It may be coined as 'Fourier normal ordering' since it consists in a regularization obtained after permuting the order of integration in iterated integrals so that innermost integrals have highest Fourier frequencies. In doing so, there appear non-trivial tree combinatorics, which are best understood by using the Hopf algebra structure of decorated rooted trees. The new feature here (compared to [20]) is the use of Besov norms to prove Hölder continuity.

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0 Introduction

Assume $\Gamma_t = (\Gamma_t(1), \dots, \Gamma_t(d))$ is a smooth d-dimensional path, and $V_1, \dots, V_d : \mathbb{R}^d \to \mathbb{R}^d$ be smooth vector fields. Then (by the classical Cauchy-Lipschitz theorem for instance) the differential equation driven by Γ

$$dy(t) = \sum_{i=1}^{d} V_i(y(t)) d\Gamma_i(t) \tag{0.1}$$

admits a unique solution with initial condition $y(0) = y_0$. The usual way to prove this is by showing (by a functional fixed-point theorem) that iterated integrals

$$y_n \to y_{n+1}(t) := y_0 + \int_0^t \sum_i V_i(y_n(s)) d\Gamma_i(s) \tag{0.2}$$

converge when $n \to \infty$.

Assume now that Γ is only α-Hölder continuous for some $\alpha \in (0,1)$. Then the Cauchy-Lipschitz theorem does not hold any more because one first needs to give a meaning to the above integrals, and in particular to the cornerstone iterated integrals

$$\mathbf{\Gamma}^n(i_1, \dots, i_n) := \int_s^t d\Gamma_{t_1}(i_1) \int_s^{t_1} d\Gamma_{t_2}(i_2) \dots \int_s^{t_{n-1}} d\Gamma_{t_n}(i_n). \tag{0.3}$$

Let $N = \lfloor 1/\alpha \rfloor$, $\lfloor 1/\alpha \rfloor$=entire part of $1/\alpha$. Assume that Γ may be lifted to a *rough path* $\mathbf{\Gamma}$, namely, that there exist a functional $(\mathbf{\Gamma}^1, \dots, \mathbf{\Gamma}^N)$ lying above Γ, where $\mathbf{\Gamma}^1_{ts} = (\delta\Gamma)_{ts} := \Gamma_t - \Gamma_s$ are the two-point increments of Γ, and each $\mathbf{\Gamma}^k = (\mathbf{\Gamma}^k(i_1, \dots, i_k))_{1 \le i_1, \dots, i_k \le d}$, $k \ge 2$ is a *substitute* for the iterated integrals $\int_s^t d\Gamma_{t_1}(i_1) \int_s^{t_1} d\Gamma_{t_2}(i_2) \dots \int_s^{t_{k-1}} d\Gamma_{t_k}(i_k)$ with the following two properties:

(i) (*Hölder continuity) each component of* $\mathbf{\Gamma}^k$, $k = 1, \dots, N$ *is* $k\alpha$*-Hölder continuous, that is to say,* $\sup_{s \in \mathbb{R}} \left(\sup_{t \in \mathbb{R}} \frac{|\mathbf{\Gamma}^k_{ts}(i_1, \dots, i_k)|}{|t-s|^{k\alpha}} \right) < \infty$.

(ii) (multiplicativity) letting $\delta\mathbf{\Gamma}^k_{tus} := \mathbf{\Gamma}^k_{ts} - \mathbf{\Gamma}^k_{tu} - \mathbf{\Gamma}^k_{us}$, *one requires*

$$\delta\mathbf{\Gamma}^k_{tus}(i_1, \dots, i_k) = \sum_{k_1 + k_2 = k} \mathbf{\Gamma}^{k_1}_{tu}(i_1, \dots, i_{k_1}) \mathbf{\Gamma}^{k_2}_{us}(i_{k_1+1}, \dots, i_k). \tag{0.4}$$

If furthermore the following property holds

(iii) (*geometricity*)

$$\mathbf{\Gamma}^{n_1}_{ts}(i_1,\dots,i_{n_1})\mathbf{\Gamma}^{n_2}_{ts}(j_1,\dots,j_{n_2}) = \sum_{\boldsymbol{k}\in \mathrm{Sh}(\boldsymbol{i},\boldsymbol{j})} \mathbf{\Gamma}^{n_1+n_2}(k_1,\dots,k_{n_1+n_2}) \tag{0.5}$$

where $\mathrm{Sh}(\boldsymbol{i},\boldsymbol{j})$ is the subset of permutations of $i_1,\dots,i_{n_1},j_1,\dots,j_{n_2}$ which do not change the orderings of $(i_1,\dots,i_{n_1})$ and $(j_1,\dots,j_{n_2})$,

then $\mathbf{\Gamma}$ is called a *geometric rough path.*

The multiplicativity property implies in particular the following identity for the twice iterated integral $\mathcal{A}_{ts} := \int_s^t d\Gamma_{x_1}(1)\int_s^{x_1} d\Gamma_{x_2}(2)$ (which is a way to measure the area generated by the first two components of Γ) :

$$\mathcal{A}_{ts} = \mathcal{A}_{tu} + \mathcal{A}_{us} + (B_t(1) - B_u(1))(B_u(2) - B_s(2)) \tag{0.6}$$

while the geometric property implies

$$\begin{aligned}&\int_s^t dB_{t_1}(1)\int_s^{t_1} dB_{t_2}(2) + \int_s^t dB_{t_2}(2)\int_s^{t_2} dB_{t_1}(1)\\ &= \left(\int_s^t dB_{t_1}(1)\right)\left(\int_s^t dB_{t_2}(2)\right) = (B_t(1) - B_s(1))(B_t(2) - B_s(2)).\end{aligned} \tag{0.7}$$

If properties (i), (ii) holds, then rough path theory (in the algebraic formalism due to M. Gubinelli [6] which we shall use here) implies that eq. (0.1) admits a unique (local) solution in a certain class of Γ-controlled paths $\mathcal{Q}$ [6, 7, 14] that we shall not define explicitly here. Furthermore, if the vector fields (V_i) are bounded together with their derivatives up to order N, then the so-called *Itô-Lyons map* $\Gamma \to y_\Gamma$ – where y_Γ is the solution of eq. (0.1) – is continuous in the α-Hölder norms [5].

The above problem is particularly relevant when Γ is a random path; it allows for the pathwise construction of stochastic integrals or of solutions of stochastic differential equations driven by Γ. Rough paths are then usually constructed by choosing some appropriate smooth approximation Γ^η, $\eta \overset{>}{\to} 0$ of Γ and proving that the rough path

$$\left(\mathbf{\Gamma}^{1,\eta}_{ts} = \Gamma^\eta_t - \Gamma^\eta_s, \dots, \mathbf{\Gamma}^{N,\eta}_{ts}(i_1,\dots,i_N) = \int_s^t d\Gamma^\eta_{t_1}(i_1)\int_s^{t_1} d\Gamma^\eta_{t_2}(i_2)\dots\int_s^{t_{N-1}} d\Gamma^\eta_{t_N}(i_N)\right) \tag{0.8}$$

converges a.s. for appropriate Hölder norms to a rough path $\mathbf{\Gamma}$ lying above Γ (see [4, 18] in the case of fractional Brownian motion with Hurst index

$\alpha > 1/4$, and [1, 8] for a class of random paths on fractals, or references in [12]).

A general construction of a rough path for deterministic paths has been given – in the original formulation due to T. Lyons – in an article by T. Lyons and N. Victoir [12]. The idea (see also [5]) is to see a rough path over Γ as a Hölder section of the trivial G-principal bundle over $\mathbb{R}$, where G is a free rank-N nilpotent group (or Carnot group), while the underlying path Γ is a section of the corresponding quotient principal G/K-bundle for some normal subgroup K of G; so one is reduced to the problem of finding Hölder-continuous sections $g_t K \to g_t$. Obviously, there is no canonical way to do this in general. This abstract, group-theoretic construction (which uses the axiom of choice) is unfortunately not particularly appropriate for concrete problems, such as the behaviour of solutions of stochastic differential equations for instance.

In a previous paper [20], we constructed an explicit rough path over a d-dimensional fractional Brownian motion $B^\alpha = (B^\alpha(1), \dots, B^\alpha(d))$ with arbitrary Hurst index $\alpha \in (0,1)$ – recall simply that the paths of B^α are a.s. κ-Hölder for every $\kappa < \alpha$. While writing the paper, we realized that our method should be sufficiently general to apply to arbitrary α-Hölder paths. Indeed, the only difference between [20] and the present paper is in the proof of the Hölder estimates, which (in the previous paper) relies heavily on a tool belonging exclusively to the Gaussian realm, namely, the equivalence of L^p-norms due to the hypercontractivity property of the Ornstein-Uhlenbeck process.

Let us explain briefly our method. Since Γ is α-Hölder, the sequence $\left(||2^{|k|\alpha} D(\phi_k)\Gamma||_{L_\infty}||\right)_{k\in\mathbb{Z}}$ is bounded, where $D(\phi_k)$ (a sequence of smooth, compactly supported Fourier multipliers such that $\sum_{k\in\mathbb{Z}} D(\phi_k) \equiv 1$) essentially 'selects' the frequency domain $[2^k, 2^{k+1})$ $(k \geq 1)$, $(-2^{|k|+1}, -2^{|k|}]$ $(k \leq -1)$. The supremum $\sup_{k\in\mathbb{Z}} ||2^{|k|\alpha} D(\phi_k)\Gamma||_{L_\infty}$ defines the norm of a Besov space $B^\alpha_{\infty,\infty}$ which is known to be equivalent to the α-Hölder norm. Consider now the (formal) 'projected' iterated integral

$$\mathcal{P}^{\{\boldsymbol{k}\}}\boldsymbol{\Gamma}^n_{ts}(i_1, \dots, i_n) := \int_s^t d\left(D(\phi_{k_1})\Gamma(i_1)\right)_{x_1} \dots \int_s^{x_{n-1}} d\left(D(\phi_{k_n})\Gamma(i_n)\right)_{x_n}$$

for some $\boldsymbol{k} = (k_1, \dots, k_n) \in \mathbb{Z}^n$.

Assume first $|k_1| \leq \ldots |k_n|$ (i.e. innermost integrals have highest Fourier frequencies). Let $\int^x$ be the formal integral defined (using the Fourier transform $\mathcal{F}$) by

$$\int^x f(y)dy = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\mathcal{F}f)(\xi) d\xi \int^x e^{\mathrm{i}y\xi}\, dy := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\mathcal{F}f)(\xi) \frac{e^{\mathrm{i}x\xi}}{\mathrm{i}\xi}\, d\xi. \tag{0.9}$$

This is well-defined for instance if $\mathrm{supp}(\mathcal{F}f) \cap [-1,1] = \emptyset$ (in order to avoid the singularity at the origin). Then $\mathcal{P}^{\{\boldsymbol{k}\}}\boldsymbol{\Gamma}^n_{ts}(i_1,\ldots,i_n)$ decomposes a the sum of an *increment term*

$$[\delta\mathcal{P}^{\{\boldsymbol{k}\}}\boldsymbol{\Gamma}^n(i_1,\ldots,i_n)(\delta)]_{ts} = [\mathcal{P}^{\{\boldsymbol{k}\}}\boldsymbol{\Gamma}^n(i_1,\ldots,i_n)(\delta)]_t - [\mathcal{P}^{\{\boldsymbol{k}\}}\boldsymbol{\Gamma}^n(i_1,\ldots,i_n)(\delta)]_s, \tag{0.10}$$

and of a *boundary term*, $\mathcal{P}^{\{\boldsymbol{k}\}}\boldsymbol{\Gamma}^n_{ts}(i_1,\ldots,i_n)(\partial)$, where

$$\begin{aligned}&[\mathcal{P}^{\{\boldsymbol{k}\}}\boldsymbol{\Gamma}^n_t(i_1,\ldots,i_n)(\delta)]_t \\ &:= \int^t d\left(D(\phi_{k_1})\Gamma(i_1)\right)_{x_1} \int^{x_1} d\left(D(\phi_{k_2})\Gamma(i_2)\right)_{x_2} \ldots \int^{x_{n-1}} d\left(D(\phi_{k_n})\Gamma(i_n)\right)_{x_n}\end{aligned} \tag{0.11}$$

and

$$\begin{aligned}&\mathcal{P}^{\{\boldsymbol{k}\}}\boldsymbol{\Gamma}^n_{ts}(i_1,\ldots,i_n)(\partial) = \\ &\quad - \sum_{n_1+n_2=n} \left(\int_s^t d\left(D(\phi_{k_1})\Gamma(i_1)\right)_{x_1} \ldots \int_s^{x_{n_1-1}} d\left(D(\phi_{k_{n_1}})\Gamma(i_{n_1})\right)_{x_{n_1}} \right). \\ &\quad . \left(\int^s d\left(D(\phi_{k_{n_1+1}})\Gamma(i_{n_1+1})\right)_{x_{n_1+1}} \int^{x_{n_1+1}} d\left(D(\phi_{k_{n_1+2}})\Gamma(i_{n_1+2})\right)_{x_{n_1+2}} \ldots \right. \\ &\qquad \left. \int^{x_{n-1}} d\left(D(\phi_{k_n})\Gamma(i_n)\right)_{x_n} \right),\end{aligned} \tag{0.12}$$

Note that for $n = 2$, the above decomposition reads $\int_s^t df_1(x_1) \int_s^{x_1} df_2(x_2) = \int_s^t df_1(x_1) \int^{x_1} df_2(x_2) - \int_s^t df_1(x_1) \int^s df_2(x_2)$, which is trivially true, provided $\int^x$ is a well-defined anti-derivative.

Integrals of the form $\int^t df_1(x_1) \int^{x_1} df_2(x_2) \ldots \int^{x_{j-1}} df_j(x_j)$ (called: *skeleton integrals of order j*) appear both in the increment and the boundary term, and may diverge when $j \geq 2$ unless suitably regularized. Note that the increment term (a skeleton integral) may be simply discarded for $n \geq 2$ without disturbing the multiplicative property (ii) since its δ-increment vanishes because of the fundamental (and trivial) identity $\delta \circ \delta(f)_{tus} = 0$. Discarding similarly all skeleton integrals of order $j \geq 2$ leads (once the above

computations have been extended to a general multi-index $\boldsymbol{k}$, see below) to a well-defined rough path over Γ with the desired Hölder continuity properties.

However, as in the previous article [20], we shall use here a more sophisticated *minimal regularization scheme*. Namely, computations show that skeleton integrals are well-defined provided there exists a uniform constant $C \in (0,1)$ such that

$$|\xi_j + \ldots + \xi_n| > C|\xi_n|, \quad j = 1, \ldots, n \tag{0.13}$$

if $\xi_i \in \mathrm{supp}(\phi_{k_i})$, $i = j, \ldots, n$ (essentially because the partial sums $\xi_j + \ldots + \xi_n$ appear in the denominator of the Fourier-transformed skeleton integrals). So the idea is to discard only the $\boldsymbol{k}$-components of the skeleton integrals for which the inequality (0.13) fails [1].

Assume now that Fourier frequencies are *not* increasingly ordered, so that $|k_{\sigma(1)}| \leq \ldots \leq |k_{\sigma(n)}|$ for some non-trivial permutation σ. Then the same regularization procedure should be applied *after* rewriting $\boldsymbol{\Gamma}^n_{ts}(i_1, \ldots, i_n)$ (using Fubini's theorem) as a multiple integral over some finite union of n-simplices, $\int_s^t d\Gamma(i_{\varepsilon(1)})_{x_1} \int_{s_1}^{t_1} d\Gamma(i_{\varepsilon(2)})_{x_2} \ldots \int_{s_{n-1}}^{t_{n-1}} d\Gamma(i_{\varepsilon(n)})_{x_n}$ (note that the two operations *do not* commute). Then the correct generalization of the above splitting of the iterated integral into increment/boundary terms is best understood in terms of the co-product structure of the Hopf algebra of decorated rooted trees, and requires some combinatorial work. Since the combinatorics are exactly the same as in [20], we only recall the definitions and main results and refer to [20] for details.

Let us state our main result. Throughout the paper $\alpha \in (0,1)$ is some fixed constant and $N = \lfloor 1/\alpha \rfloor$.

Main theorem.

Assume $1/\alpha \notin \mathbb{N}$. Let $\Gamma = (\Gamma(1), \ldots, \Gamma(d)) : \mathbb{R} \to \mathbb{R}^d$ be a compactly supported α-Hölder path such that $\mathrm{supp}(\mathcal{F}\Gamma(j)) \cap [-1,1] = \emptyset$, $j = 1, \ldots, d$. Then the functional $(\mathcal{R}\boldsymbol{\Gamma}^1, \ldots, \mathcal{R}\boldsymbol{\Gamma}^N)$ defined in Lemma 2.13 and Corollary 2.14 is an α-Hölder (weak) geometric rough path lying over Γ in the sense of properties (i),(ii),(iii) of the Introduction.

Note that the compact support assumption is essentially void (we simply define the rough path over an arbitrary large compact interval). The support hypothesis for $\mathcal{F}\Gamma$ is no real problem since

$$\tilde{\Gamma} := \Gamma - \mathcal{F}^{-1}(\mathbf{1}_{\mathbb{R}\setminus[-1,1]} \,.\mathcal{F}\Gamma) = \mathcal{F}^{-1}(\mathbf{1}_{[-1,1]} \,.\, \mathcal{F}\Gamma) \tag{0.14}$$

[1] Note in particular that no regularization is needed if $\mathcal{F}(\Gamma(i))$, $i = 1, \ldots, d$ are supported in $\mathbb{R}_+$, which is the case of analytic fractional Brownian motion (see [14]).

is a smooth (C^∞) path. Hence it is straightforward (though a little disturbing for explicit formulas) to deduce a rough path over Γ from a rough path over $\Gamma - \tilde{\Gamma}$. Actually it is not clear even to the author if it is really necessary to cut off the lowest Fourier modes of Γ (it is clearly useless if the rough path may be defined by a limiting procedure through an ultra-violet cut-off – i.e. by discarding highest Fourier modes –, for instance for the so-called analytic approximation fractional Brownian motion with Hurst index $\alpha > 1/4$, see [20], §7.3); somehow the singularities should cancel when one sums up all terms.

The above theorem extends to paths Γ with finite $1/\alpha$-variation. Namely (see [12], [10] or also [5]), a simple change of variable $\Gamma \to \Gamma^\phi := \Gamma \circ \phi^{-1}$ turns Γ into an α-Hölder path, with ϕ defined for instance as $\phi(t) := \sup_{n\geq 1} \sup_{0=t_0\leq\ldots\leq t_n=t} \sum_{j=0}^{n-1} ||\Gamma(t_{j+1}) - \Gamma(t_j)||^{1/\alpha}$. The construction of the above Theorem (applied to Γ^ϕ) yields a family of paths with Hölder regularities $\alpha, 2\alpha, \ldots, N\alpha$ which may alternatively be seen as a G^N-valued α-Hölder path $\mathbf{\Gamma}^\phi$, where G^N is the Carnot (free nilpotent) group of order N equipped with any subadditive homogeneous norm. Then (as proved in [12], Lemma 8) $\mathbf{\Gamma} := \mathbf{\Gamma}^\phi \circ \phi$ has finite $1/\alpha$-variation, which is equivalent to saying that $\mathbf{\Gamma}^n$ has finite $1/n\alpha$-variation for $n = 1, \ldots, N$, and lies above Γ.

Corollary.

Let $\alpha \in (0,1)$ and $\alpha' < \alpha$. Then every α-Hölder path Γ may be lifted to a (strong) α'-Hölder geometric rough path, namely, there exists a sequence of canonical lifts $\mathbf{\Gamma}^{(n)}$ of smooth paths $\Gamma^{(n)}$ converging to $\mathcal{R}\mathbf{\Gamma}$ for the sequence of α'-Hölder norms.

The *canonical lift* of a smooth path is simply the data of its iterated integrals. The set of *strong* α-Hölder geometric rough paths is strictly included in the set of *weak* α-Hölder geometric rough paths; on the other hand, a weak α-Hölder geometric rough path may be seen as a strong α'-Hölder geometric rough path if $\alpha' < \alpha$. This accounts for the loss of regularity in the Corollary (see [5] for a precise discussion). The proviso $1/\alpha \notin \mathbb{N}$ in the statement of the main theorem is a priori needed because $\mathcal{R}\mathbf{\Gamma}^N$ may not be treated in the same way as the lower-order iterated integrals (although we do not know if it is actually necessary). However, if $1/\alpha \in \mathbb{N}$, all one has to do is replace α by a slightly smaller parameter α', so that the Corollary holds even in this case.

Note that the present paper gives unfortunately no explicit way of approximating $\mathcal{R}\mathbf{\Gamma}$ by smooth paths, i.e. of seeing it concretely as a *strong* geometric rough path. We do not know how to answer this natural question at present time.

Notations. We shall denote by $\mathcal{F}$ the Fourier transform, $\mathcal{F} : L^2(\mathbb{R}^l) \to L^2(\mathbb{R}^l), f \to \mathcal{F}(f)(\xi) = \frac{1}{(2\pi)^{l/2}} \int_{\mathbb{R}^l} f(x) e^{-\mathrm{i}\langle x,\xi\rangle}\, dx$. Throughout the article, $\Gamma : \mathbb{R} \to \mathbb{R}^d$ is some α-Hölder path verifying the hypotheses of the main theorem, i.e. Γ is compactly supported, and $\mathrm{supp}(\mathcal{F}\Gamma) \cap [-1,1] = \emptyset$. Also, if $a, b : X \to \mathbb{R}_+$ are functions on some set X such that $a(x) \leq Cb(x)$ for every $x \in X$, we shall write $a \lesssim b$.

1 Hölder and Besov spaces

Definition 1.1 (Hölder norm) *If $f : \mathbb{R}^l \to \mathbb{R}$ is α-Hölder continuous for some $\alpha \in (0,1)$, we let*

$$||f||_{\mathcal{C}^\alpha} := ||f||_\infty + \sup_{x,y\in\mathbb{R}^l} \frac{|f(x) - f(y)|}{||x-y||^\alpha}. \tag{1.1}$$

The space $\mathcal{C}^\alpha = \mathcal{C}^\alpha(\mathbb{R}^l)$ of real-valued α-Hölder continuous functions, provided with the above norm $|| \; ||_{\mathcal{C}^\alpha}$, is a Banach space.

Proposition 1.2 *Let $l \geq 1$. There exists a family of C^∞ functions $\phi_0, (\phi_{1,j})_{j=1,\ldots,4^l-2^l} : \mathbb{R}^l \to [0,1]$, satisfying the following conditions:*

1. $\mathrm{supp}\phi_0 \subset [-2,2]$ *and* $\phi_0\big|_{[-1,1]} \equiv 1$.

2. *Cut $[-2,2]^l$ into 4^l equal hypercubes of volume 1, and remove the 2^l hypercubes included in $[-1,1]^l$. Let $K_1, \ldots, K_{4^l-2^l}$ be an arbitrary enumeration of the remaining hypercubes, and $\tilde{K}_j \supset K_j$ be the hypercube with the same center as K_j, but with edges twice longer. Then* $\mathrm{supp}\phi_{1,j} \subset \tilde{K}_j$, $j = 1, \ldots, 4^l - 2^l$.

3. *Let $(\phi_{k,j})_{k\geq 2, j=1,\ldots,4^l-2^l}$ be the family of dyadic dilatations of $(\phi_{1,j})$, namely,*
$$\phi_{k,j}(\xi_1, \ldots, \xi_l) := \phi_{1,j}(2^{1-k}\xi_1, \ldots, 2^{1-k}\xi_l). \tag{1.2}$$
Then $(\phi_0, (\phi_{k,j})_{k\geq 1, j=1,\ldots,4^l-2^l})$ is a partition of unity subordinated to the covering $[-2,2]^l \cup \left(\cup_{k\geq 1} \cup_{j=1}^{4^l-2^l} 2^{k-1}\tilde{K}_j\right)$, namely,
$$\phi_0 + \sum_{k\geq 1} \sum_{j=1}^{4^l-2^l} \phi_{k,j} \equiv 1. \tag{1.3}$$

Constructed in this almost canonical way, the family of Fourier multipliers $(\phi_0, (\phi_{k,j}))$ is immediately seen to be uniformly bounded for the norm $||\ .\ ||_{S^0}$ defined in Proposition 1.7 below.

If $l = 1$, letting $K_1 = [1, 2]$ and $K_2 = [-2, -1]$, we shall write ϕ_1, resp. ϕ_{-1}, instead of $\phi_{1,1}$, resp. $\phi_{1,2}$, and define $\phi_k(\xi) = \phi_{\mathrm{sgn}(k)}(2^{1-|k|}\xi)$, so that $\sum_{k\in\mathbb{Z}} \phi_k \equiv 1$ and

$$\mathrm{supp}\phi_0 \subset [-2, 2], \quad \mathrm{supp}\phi_k \subset [2^{k-1}, 5.2^{k-1}], \quad \mathrm{supp}\phi_{-k} \subset [-5.2^{k-1}, -2^{k-1}] \quad (k \geq 1). \tag{1.4}$$

In this particular case, such a family is easily constructed from an arbitrary even, smooth function $\phi_0 : \mathbb{R} \to [0, 1]$ with the correct support by setting $\phi_k(\xi) = \mathbf{1}_{\mathbb{R}_+}(\xi).(\phi_0(2^{-k}\xi) - \phi_0(2^{1-k}\xi))$ and $\phi_{-k}(\xi) = \mathbf{1}_{\mathbb{R}_-}(\xi).(\phi_0(2^{-k}\xi) - \phi_0(2^{1-k}\xi))$ for every $k \geq 1$ (see [17], §1.3.3).

In order to avoid setting apart the one-dimensional case, we let $\mathbb{I} := \mathbb{Z}$ if $l = 1$, and $\mathbb{I} = \{0\} \cup \{(k, j) \mid k \geq 1, 1 \leq j \leq 4^l - 2^l\}$ if $l \geq 2$. Also, if $l \geq 2$, we define $|\kappa| = k \geq 1$ if $\kappa = (k, j)$ with $k \geq 1$.

Definition 1.3 *Let $\ell_\infty(L_\infty)$ be the space of sequences $(f_\kappa)_{\kappa\in\mathbb{I}}$ of a.s. bounded functions $f_\kappa \in L_\infty(\mathbb{R}^l)$ such that*

$$||f_\kappa||_{\ell_\infty(L_\infty)} := \sup_{\kappa\in\mathbb{I}} ||f_\kappa||_\infty < \infty. \tag{1.5}$$

Let $\mathcal{S}'(\mathbb{R}^l, \mathbb{R})$ be the dual of the Schwartz space of rapidly decreasing functions on $\mathbb{R}^l$. As well-known, it includes the space of infinitely differentiable slowly growing functions.

Definition 1.4 *Let $B^\alpha_{\infty,\infty}(\mathbb{R}^l) := \{f \in \mathcal{S}'(\mathbb{R}^l, \mathbb{R}) \mid ||f||_{B^\alpha_{\infty,\infty}} < \infty\}$ where*

$$\begin{aligned} ||f||_{B^\alpha_{\infty,\infty}} &:= ||2^{\alpha|\kappa|} D(\phi_\kappa) f||_{\ell_\infty(L_\infty)} \\ &= \sup_{\kappa\in\mathbb{I}} 2^{\alpha|\kappa|} ||D(\phi_\kappa) f||_\infty. \end{aligned} \tag{1.6}$$

Proposition 1.5 *(see [16], §2.2.9)*

For every $\alpha \in (0, 1)$, $B^\alpha_{\infty,\infty}(\mathbb{R}^l) = \mathcal{C}^\alpha(\mathbb{R}^l)$, and the two norms $||\ ||_{\mathcal{C}^\alpha}$ and $||\ ||_{B^\alpha_{\infty,\infty}}$ are equivalent.

We shall sometimes call $||\ ||_{B^\alpha_{\infty,\infty}}$ the *Hölder-Besov norm.*

Definition 1.6 (Fourier multipliers) *Let $m : \mathbb{R}^l \to \mathbb{R}$ be an infinitely differentiable slowly growing function. Then*

$$D(m) : \mathcal{S}'(\mathbb{R}^l) \to \mathcal{S}'(\mathbb{R}^l), \quad \phi \to \mathcal{F}^{-1}(m.(\mathcal{F}\phi)) \tag{1.7}$$

defines a continuous operator.

In other words, m is a Fourier multiplier of $\mathcal{S}'(\mathbb{R}^d)$. Let us particularize to Fourier multipliers of the Besov space $B^{\alpha}_{\infty,\infty}$:

Proposition 1.7 (Fourier multipliers) *(see [16], §2.1.3, p.30)*
Let $\alpha \in (0,1)$ and $m : \mathbb{R}^l \to \mathbb{R}$ be an infinitely differentiable function such that

$$||m||_{S^0} := \sup_{|j|\leq l+5} \sup_{\boldsymbol{\xi}\in\mathbb{R}^l} |(1+||\boldsymbol{\xi}||)^{|j|} m^{(j)}(\boldsymbol{\xi})| < \infty \tag{1.8}$$

where $j = (j_1, \ldots, j_l)$, $|j| = j_1 + \ldots + j_l$ and $m^{(j)} := \partial^{j_1}_{\xi_1} \ldots \partial^{j_l}_{\xi_l} m$. Then there exists a constant C (depending only on α) such that, for every $f \in B^{\alpha}_{\infty,\infty}(\mathbb{R}^l)$,

$$||D(m)f||_{B^{\alpha}_{\infty,\infty}} \leq C||m||_{S^0}||f||_{B^{\alpha}_{\infty,\infty}}. \tag{1.9}$$

The space S^0 contains the space of translation-invariant pseudo-differential symbols of order 0 (see for instance [2], Definition 1.1, or [15]).

2 Combinatorial structures

2.1 From iterated integrals to trees

It was noted already long time ago [3] that iterated integrals could be encoded by trees. The correspondence between trees and itegrated integrals goes simply as follows.

Definition 2.1 *A decorated rooted tree (to be drawn growing* up*) is a finite tree with a distinguished vertex called* root *and edges oriented* downwards *(i.e. directed towards the root), such that every vertex wears an integer label.*

If $\mathbb{T}$ is a decorated rooted tree, we let $V(\mathbb{T})$ be the set of its vertices (including the root), and $\ell : V(\mathbb{T}) \to \mathbb{N}$ be its vertex labeling.

More generally, a decorated rooted forest *is a finite set of decorated rooted trees. If $\mathbb{T} = \{\mathbb{T}_1, \ldots, \mathbb{T}_l\}$ is a forest, then we shall write $\mathbb{T}$ as the formal commutative product $\mathbb{T}_1 \ldots \mathbb{T}_l$.*

Definition 2.2 *Let $\mathbb{T}$ be a decorated rooted tree.*

- *Letting $v, w \in V(\mathbb{T})$, we say that v* connects directly to *w, and write $v \to w$ or equivalently $w = v^-$, if (v, w) is an edge oriented downwards from v to w.*

- *If $v_1 \to v_2 \to \ldots \to v_m$, then we shall write $v_1 \twoheadrightarrow v_m$, and say that v_1* connects to *v_m. By definition, all vertices (except the root) connect to the root. (Note that v^- exists and is unique except if v is the root).*

- *Let $(v_0, \ldots, v_{|V(\mathbb{T})|-1})$ be an ordering of $V(\mathbb{T})$. Assume that $(v_i \twoheadrightarrow v_j) \Rightarrow (i > j)$ (in particular, v_0 is the root). Then we shall say that the ordering is* compatible *with the* tree partial ordering *defined by $\twoheadrightarrow$.*

Definition 2.3 *Let $\Gamma = (\Gamma(1), \ldots, \Gamma(d))$ be a d-dimensional smooth path, and $\mathbb{T}$ a decorated rooted tree such that $\ell : V(\mathbb{T}) \to \{1, \ldots, d\}$. Then $I_{\mathbb{T}}(\Gamma) : \mathbb{R}^2 \to \mathbb{R}$ is the iterated integral defined as*

$$[I_{\mathbb{T}}(\Gamma)]_{ts} := \int_s^t d\Gamma_{x_0}(\ell(v_0)) \int_s^{x_{v_1^-}} d\Gamma_{x_1}(\ell(v_1)) \ldots \int_s^{x_{v_{|V(\mathbb{T})|-1}^-}} d\Gamma_{x_{v_{|V(\mathbb{T})|-1}}}(\ell(v_{|V(\mathbb{T})|-1})) \tag{2.1}$$

where $(v_0, \ldots, v_{|V(\mathbb{T})|-1})$ is any ordering of $V(\mathbb{T})$ compatible with the tree partial ordering.

In particular, if $\mathbb{T}$ is a trunk tree with n vertices (see Fig. 1) – so that the tree ordering is total – we shall write

$$I_{\mathbb{T}}(\Gamma) = I_n^{\ell}(\Gamma), \tag{2.2}$$

where

$$[I_n^{\ell}(\Gamma)]_{ts} := \int_s^t d\Gamma_{x_0}(\ell(0)) \int_s^{x_0} d\Gamma_{x_1}(\ell(1)) \ldots \int_s^{x_{n-2}} d\Gamma_{x_{n-1}}(\ell(n-1)). \tag{2.3}$$

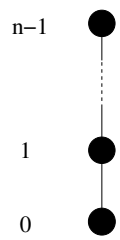

Figure 1: Trunk tree.

The above definition extends by multilinearity to the following

Definition 2.4 *1. Let $\mathbb{T}$ be a tree and $f((x_v)_{v\in\mathbb{T}}) = \sum_{\boldsymbol{k}=(k_v)_{v\in V(\mathbb{T})}} \prod_{v\in V(\mathbb{T})} f_{k_v}^v(x_v)$ be a (converging) sum of smooth functions over some set of multiple*

indices. Then (disregarding the labels)

$$[I_{\mathbb{T}}(f)]_{ts} = \sum_{\boldsymbol{k}} \int_s^t f^{v_0}_{k_{v_0}}(x_{v_0})dx_{v_0} \int_s^{x_{v_1^-}} f^{v_1}_{k_{v_1}}(x_{v_1})dx_{v_1} \dots \int_s^{x_{v^-_{|V(\mathbb{T})|-1}}} f^{v_{|V(\mathbb{T})|-1}}_{k_{v_{|V(\mathbb{T})|-1}}}(x_{v_{|V(\mathbb{T})|-1}}). \tag{2.4}$$

The definition extends straightforwardly to forests.

2. *Let $\mathbb{T} = \mathbb{T}_1.\mathbb{T}_2$ be a forest. Then*

$$[I_{\mathbb{T}}(f)]_{ts} = [I_{\mathbb{T}_1}]_{ts} \,.\, [I_{\mathbb{T}_2}]_{ts}(f), \tag{2.5}$$

where by definition

$$[I_{\mathbb{T}_1}]_{ts} \,.\, [I_{\mathbb{T}_2}]_{ts}(f) = \sum_{\boldsymbol{k}} \left[I_{\mathbb{T}_1}\left(\otimes_{v\in V(\mathbb{T}_1)} f^v_{k_v}\right)\right]_{ts} \left[I_{\mathbb{T}_2}\left(\otimes_{v\in V(\mathbb{T}_2)} f^v_{k_v}\right)\right]_{ts}. \tag{2.6}$$

The above correspondence extends by (multi)linearity to the algebra of decorated rooted trees that we now define.

Definition 2.5 *(i) Let $\mathcal{T}$ be the free commutative algebra over $\mathbb{Z}$ generated by decorated rooted trees. If $\mathbb{T}_1, \mathbb{T}_2, \dots \mathbb{T}_l$ are (decorated rooted) trees, then the product $\mathbb{T}_1 \dots \mathbb{T}_l$ is the forest with connected components $\mathbb{T}_1, \dots, \mathbb{T}_l$.*

(ii) Let $\mathbb{T}' = \sum_{l=1}^L m_l \mathbb{T}_l \in \mathcal{T}$, where $m_l \in \mathbb{Z}$ and each $\mathbb{T}_l = \mathbb{T}_{l,1} \dots \mathbb{T}_{l,L(l)}$ is a forest with labels in the set $\{1, \dots, d\}$, and Γ be a smooth d-dimensional path as above. Then

$$[I_{\mathbb{T}'}(\Gamma)]_{ts} := \sum_{l=1}^L m_l I_{\mathbb{T}_{l,1}}(\Gamma) \dots I_{\mathbb{T}_{l,L}}(\Gamma). \tag{2.7}$$

Consider now a permutation $\sigma : \{0, \dots, n-1\} \to \{0, \dots, n-1\}$. Applying Fubini's theorem yields

$$\begin{aligned} I^\ell_n(\Gamma) &= \int_s^t d\Gamma_{x_0}(\ell(0)) \int_s^{x_0} d\Gamma_{x_1}(\ell(1)) \dots \int_s^{x_{n-2}} d\Gamma_{x_{n-1}}(\ell(n-1)) \\ &= \int_{s_0}^{t_0} d\Gamma_{x_{\sigma(0)}}(\ell(\sigma(0))) \int_{s_1}^{t_1} d\Gamma_{x_{\sigma(1)}}(\ell(\sigma(1))) \dots \int_{s_{n-1}}^{t_{n-1}} d\Gamma_{x_{\sigma(n-1)}}(\ell(\sigma(n-1))), \end{aligned} \tag{2.8}$$

with $s_0 = s$, $t_0 = t$ and $s_j \in \{s\} \cup \{x_{\sigma(i)}, i < j\}$, $t_j \in \{t\} \cup \{x_{\sigma(i)}, i < j\}$ ($j \geq 1$). Now decompose $\int_{s_j}^{t_j} d\Gamma_{x_{\sigma(j)}}(\ell(\sigma(j)))$ into $\left(\int_s^{t_j} - \int_s^{s_j}\right) d\Gamma_{x_{\sigma(j)}}(\ell(\sigma(j)))$ if $s_j \neq s, t_j \neq t$, and $\int_{s_j}^{t} d\Gamma_{x_{\sigma(j)}}(\ell(\sigma(j)))$ into $\left(\int_s^{t} - \int_s^{s_j}\right) d\Gamma_{x_{\sigma(j)}}(\ell(\sigma(j)))$ if $s_j \neq s$. Then $I_n^\ell(\Gamma)$ has been rewritten as a sum of terms of the form

$$\pm \int_s^{\tau_0} d\Gamma_{x_0}(\ell(\sigma(0))) \int_s^{\tau_1} d\Gamma_{x_1}(\ell(\sigma(1))) \ldots \int_s^{\tau_{n-1}} d\Gamma_{x_{n-1}}(\ell(\sigma(n-1))), \quad (2.9)$$

where $\tau_0 = t$ and $\tau_j \in \{t\} \cup \{x_i, i < j\}$, $j = 1, \ldots, n-1$. Note the renaming of variables and vertices from eq. (2.8) to eq. (2.9). Encoding each of these expressions by the forest $\mathbb{T}$ with set of vertices $V(\mathbb{T}) = \{0, \ldots, n-1\}$, label function $\ell \circ \sigma$, roots $\{j = 0, \ldots, n-1 \mid \tau_j = t\}$, and oriented edges $\{(j, j^-) \mid j = 1, \ldots, n-1, \tau_j \neq t, \tau_j = x_{j^-}\}$, yields

$$I_n^\ell(\Gamma) = I_{\mathbb{T}^\sigma}(\Gamma) \quad (2.10)$$

for some $\mathbb{T}^\sigma \in \mathcal{T}$ called **permutation graph associated to** σ. One may also write:

$$I_{\mathbb{T}^\sigma}(\Gamma) = I_{\mathbb{T}^\sigma}\left(\otimes_{v \in \mathbb{T}^\sigma} \Gamma(\ell(\sigma(v)))\right). \quad (2.11)$$

Note that (letting $\mathbb{T}^\sigma = \pm \mathbb{T}_1^\sigma \pm \ldots \pm \mathbb{T}_{J_\sigma}^\sigma$) each forest $\mathbb{T}_j^\sigma$ (and also all its tree components, of course) is by construction provided with a total ordering compatible with its tree structure.

Example 2.6 *Let* $\sigma = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$. *Then*

$$\begin{aligned} &\int_s^t d\Gamma_{x_0}(\ell(0)) \int_s^{t_1} d\Gamma_{x_1}(\ell(1)) \int_s^{t_2} d\Gamma_{x_2}(\ell(2)) = \\ &\quad - \int_s^t d\Gamma_{x_1}(\ell(1)) \int_s^{x_1} d\Gamma_{x_2}(\ell(2)) \int_s^{x_1} d\Gamma_{x_0}(\ell(0)) \\ &\quad + \int_s^t d\Gamma_{x_1}(\ell(1)) \int_s^{x_1} d\Gamma_{x_2}(\ell(2)) \; . \int_s^t d\Gamma_{x_0}(\ell(0)). \quad (2.12) \end{aligned}$$

Hence $\mathbb{T}^\sigma = -\mathbb{T}_1^\sigma + \mathbb{T}_2^\sigma$ *is the sum of a tree and of a forest with two components (see Fig. 4).*

Let us now rewrite these iterated integrals by using Fourier transform.

Definition 2.7 (formal integral) *Let* $f : \mathbb{R} \to \mathbb{R}$ *be a smooth, compactly supported function such that* $\mathrm{supp}(\mathcal{F}f) \cap [-1, 1] = \emptyset$. *Then the formal integral* $\int^t f = -\int_t f$ *of* f *is defined as* $\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} (\mathcal{F}f)(\xi) \frac{e^{\mathrm{i}x\xi}}{\mathrm{i}\xi} \, d\xi$.

Definition 2.8 (skeleton integrals) *Let $\mathbb{T}$ be a tree with $\ell : \mathbb{T} \to \{1, \dots, d\}$ and Γ be a d-dimensional smooth path such that* $\mathrm{supp}(\mathcal{F}\Gamma(j)) \cap [-1, 1] = \emptyset$ *for every $j = 1, \dots, d$. Let $(v_0, \dots, v_{|V(\mathbb{T})|-1})$ be any ordering of $V(\mathbb{T})$ compatible with the tree partial ordering. Then the* skeleton integral *of Γ along $\mathbb{T}$ is by definition*

$$[\mathrm{SkI}_{\mathbb{T}}(\Gamma)]_s = -\int_s d\Gamma_{x_{v_0}}(\ell(v_0)) \int^{x_{v_1^-}} d\Gamma_{x_1}(\ell(v_1)) \dots \int^{x_{v_{|V(\mathbb{T})|-1}^-}} d\Gamma_{x_{v_{|V(\mathbb{T})|-1}}}(\ell(v_{|V(\mathbb{T})|-1})). \tag{2.13}$$

Lemma 2.9 *The following formula holds:*

$$[\mathrm{SkI}_{\mathbb{T}}(\Gamma)]_s = \mathrm{i}^{-|V(\mathbb{T})|} \int \dots \int \prod_{v \in V(\mathbb{T})} d\xi_v \, . \, e^{\mathrm{i}s \sum_{v \in V(\mathbb{T})} \xi_v} \frac{\prod_{v \in V(\mathbb{T})} \mathcal{F}(\Gamma'(\ell(v)))(\xi_v)}{\prod_{v \in V(\mathbb{T})} (\xi_v + \sum_{w \to v} \xi_w)}. \tag{2.14}$$

Proof. We use induction on $|V(\mathbb{T})|$. After stripping the root of $\mathbb{T}$ (denoted by 0) there remains a forest $\mathbb{T}' = \mathbb{T}'_1 \dots \mathbb{T}'_J$, whose roots are the vertices directly connected to 0. Assume

$$[\mathrm{SkI}_{\mathbb{T}'_j}(\Gamma)]_{x_0} = \int \dots \int \prod_{v \in V(\mathbb{T}'_j)} d\xi_v \, . \, e^{\mathrm{i}x_0 \sum_{v \in V(\mathbb{T}'_j)} \xi_v} F_j((\xi_v)_{v \in \mathbb{T}'_j}). \tag{2.15}$$

Note that

$$\mathcal{F}\left(\prod_{j=1}^{J} \mathrm{SkI}_{\mathbb{T}'_j}(\Gamma)\right)(\xi) = \int_{\sum_{v \in V(\mathbb{T}) \setminus \{0\}} \xi_v = \xi} \prod_{v \in V(\mathbb{T}) \setminus \{0\}} d\xi_v \prod_{j=1}^{J} F_j((\xi_v)_{v \in V(\mathbb{T}'_j)}). \tag{2.16}$$

Then

$$\begin{aligned}
[\mathrm{SkI}_{\mathbb{T}}(\Gamma)]_s &= \int^s d\Gamma_{x_0}(\ell(0)) \prod_{j=1}^{J} [\mathrm{SkI}_{\mathbb{T}'_j}(\Gamma)]_{x_0} \\
&= \int_{-\infty}^{+\infty} \frac{d\xi}{\mathrm{i}\xi} e^{\mathrm{i}s\xi} \mathcal{F}\left(\Gamma'(\ell(0)) \prod_{j=1}^{J} \mathrm{SkI}_{\mathbb{T}'_j}(\Gamma)\right)(\xi) \\
&= \int_{-\infty}^{+\infty} d\xi_0 \mathcal{F}(\Gamma'(\ell(0)))(\xi_0) e^{\mathrm{i}s\xi_0} \, . \\
&\qquad \int_{-\infty}^{+\infty} \frac{d\xi}{\mathrm{i}\xi} e^{\mathrm{i}s(\xi - \xi_0)} \int_{\sum_{v \in V(\mathbb{T}) \setminus \{0\}} \xi_v = \xi - \xi_0} d\xi_v \prod_{j=1}^{J} F_j((\xi_v)_{v \in V(\mathbb{T}'_j)})
\end{aligned} \tag{2.17}$$

hence the result. □

Skeleton integrals are the fundamental objects from which regularized rough paths will be constructed in the next sections.

2.2 Coproduct structure and increment-boundary decomposition

Consider for an example the trunk tree $\mathbb{T}^{\mathrm{Id}_n}$ with vertices $n-1 \to n-2 \to \ldots \to 0$ and labels $\ell : \{0, \ldots, n-1\} \to \{1, \ldots, d\}$, and the associated iterated integral (assuming $\Gamma = (\Gamma(1), \ldots, \Gamma(d))$ is a smooth path)

$$[I_n^\ell(\Gamma)]_{ts} = [I_{\mathbb{T}^{\mathrm{Id}_n}}(\Gamma)]_{ts} = \int_s^t d\Gamma_{x_0}(\ell(0)) \ldots \int_s^{x_{n-2}} d\Gamma_{x_{n-1}}(\ell(n-1)). \quad (2.18)$$

Cutting $\mathbb{T}^{\mathrm{Id}_n}$ at some vertex $v \in \{1, \ldots, n-1\}$ produces two trees, $L_v\mathbb{T}^{\mathrm{Id}_n}$ (left or rather bottom part of $\mathbb{T}^{\mathrm{Id}_n}$) and $R_v\mathbb{T}^{\mathrm{Id}_n}$ (right or top part), with respective vertex subsets $\{0, \ldots, v-1\}$ and $\{v, \ldots, n-1\}$. One should actually see the couple $(L_v\mathbb{T}^{\mathrm{Id}_n}, R_v\mathbb{T}^{\mathrm{Id}_n})$ as $L_v\mathbb{T}^{\mathrm{Id}_n} \otimes R_v\mathbb{T}^{\mathrm{Id}_n}$ sitting in the tensor product algebra $\mathcal{T} \otimes \mathcal{T}$. Then multiplicative property (ii) in the Introduction reads

$$[\delta I_{\mathbb{T}^{\mathrm{Id}_n}}(\Gamma)]_{tus} = \sum_{v \in V(\mathbb{T}^{\mathrm{Id}_n}) \setminus \{0\}} [L_v\mathbb{T}^{\mathrm{Id}_n}(\Gamma)]_{tu} [R_v\mathbb{T}^{\mathrm{Id}_n}(\Gamma)]_{us}. \quad (2.19)$$

On the other hand, rewrite $[I_{\mathbb{T}^{\mathrm{Id}_n}}(\Gamma)]_{ts}$ (see eq. (0.11) and (0.12)) as the sum of the increment term

$$\begin{aligned}[\delta I_{\mathbb{T}^{\mathrm{Id}_n}}(\Gamma)(\delta)]_{ts} \quad &= \textstyle\int^t d\Gamma_{x_0}(\ell(0)) \int^{x_0} d\Gamma_{x_1}(\ell(1)) \ldots \int^{x_{n-2}} d\Gamma_{x_{n-1}}(\ell(n-1)) \\ &\quad - \textstyle\int^s d\Gamma_{x_0}(\ell(0)) \int^{x_0} d\Gamma_{x_1}(\ell(1)) \ldots \int^{x_{n-2}} d\Gamma_{x_{n-1}}(\ell(n-1))\end{aligned} \quad (2.20)$$

and of the boundary term

$$\begin{aligned}[I_{\mathbb{T}^{\mathrm{Id}_n}}(\Gamma(\partial)]_{ts} = - \sum_{n_1+n_2=n} \int_s^t d\Gamma_{x_0}(\ell(0)) \ldots \int_s^{x_{n_1-1}} d\Gamma_{x_{n_1}}(\ell(n_1)) \ . \\ . \int^s d\Gamma_{x_{n_1+1}}(\ell(n_1+1)) \int^{x_{n_1+1}} d\Gamma_{x_{n_1+2}}(\ell(n_1+2)) \ldots \int^{x_{n-2}} d\Gamma_{x_{n-1}}(\ell(n-1)).\end{aligned} \quad (2.21)$$

The above decomposition is fairly obvious for $n = 2$ (see Introduction) and obtained by easy induction for general n. Thus (using tree notation this time)

$$[I_{\mathbb{T}^{\mathrm{Id}_n}}(\Gamma)]_{ts} = [\delta \mathrm{SkI}_{\mathbb{T}^{\mathrm{Id}_n}}]_{ts} - \sum_{v \in V(\mathbb{T}^{\mathrm{Id}_n}) \setminus \{0\}} [I_{L_v \mathbb{T}^{\mathrm{Id}_n}}(\Gamma)]_{ts} \,.\, [\mathrm{SkI}_{R_v \mathbb{T}^{\mathrm{Id}_n}}(\Gamma)]_s. \tag{2.22}$$

The above considerations extend to arbitrary trees (or also forests) as follows (see also subsection 2.3).

Definition 2.10 (admissible cuts) *1. Let $\mathbb{T}$ be a tree, with set of vertices $V(\mathbb{T})$ and root denoted by 0. If $\boldsymbol{v} = (v_1, \ldots, v_J)$, $J \geq 1$ is any totally disconnected subset of $V(\mathbb{T}) \setminus \{0\}$, i.e. $v_i \not\twoheadrightarrow v_j$ for all $i, j = 1, \ldots, J$, then we shall say that $\boldsymbol{v}$ is an* admissible cut *of $\mathbb{T}$, and write $\boldsymbol{v} \models V(\mathbb{T})$. We let $R_{\boldsymbol{v}}\mathbb{T}$ be the sub-forest (or sub-tree if $J = 1$) obtained by keeping only the vertices above $\boldsymbol{v}$, i.e. $V(R_{\boldsymbol{v}}\mathbb{T}) = \boldsymbol{v} \cup \{w \in V(\mathbb{T}) : \exists j = 1, \ldots, J, w \twoheadrightarrow v_j\}$, and $L_{\boldsymbol{v}}\mathbb{T}$ be the sub-tree obtained by keeping all other vertices.*

2. Let $\mathbb{T} = \mathbb{T}_1 \ldots \mathbb{T}_l$ be a forest, together with its decomposition into trees. Then an admissible cut of $\mathbb{T}$ is a disjoint union $\boldsymbol{v}_1 \cup \ldots \cup \boldsymbol{v}_l$, $\boldsymbol{v}_i \subset \mathbb{T}_i$, where $\boldsymbol{v}_i$ is either $\emptyset$, $\{0_i\}$ (root of $\mathbb{T}_i$) or an admissible cut of $\mathbb{T}_i$. By definition, we let $L_{\boldsymbol{v}}\mathbb{T} = L_{\boldsymbol{v}_1}\mathbb{T}_1 \ldots L_{\boldsymbol{v}_l}\mathbb{T}_l$, $R_{\boldsymbol{v}}\mathbb{T} = R_{\boldsymbol{v}_1}\mathbb{T}_1 \ldots R_{\boldsymbol{v}_l}\mathbb{T}_l$ (if $\boldsymbol{v}_i = \emptyset$, resp. $\{0_i\}$, then $(L_{\boldsymbol{v}_i}\mathbb{T}_i, R_{\boldsymbol{v}_i}\mathbb{T}_i) := (\mathbb{T}_i, \emptyset)$, resp. $(\emptyset, \mathbb{T}_i)$).

We exclude by convention the two trivial cuts $\emptyset \cup \ldots \cup \emptyset$ and $\{0_1\} \cup \ldots \cup \{0_l\}$.

See Fig. 2 and 3. Defining the co-product operation $\mathcal{T} \to \mathcal{T} \otimes \mathcal{T}$, $\mathbb{T} \to \emptyset \otimes \mathbb{T} + \mathbb{T} \otimes \emptyset + \sum_{\boldsymbol{v} \models V(\mathbb{T})} L_{\boldsymbol{v}}\mathbb{T} \otimes R_{\boldsymbol{v}}\mathbb{T}$ (where $\emptyset$ stands for the empty tree, which is the unity of the algebra) yields a coalgebra structure on $\mathcal{T}$ which makes it (once the antipode – which we do not need here – is defined) a Hopf algebra.

2.3 Definition of the regularized integrals

The regularization procedure is essentially an iteration of Fourier 'projection' operators such as the following one. In the sequel, all trees are implicitly assumed to be equipped with a total ordering compatible with their tree orderings.

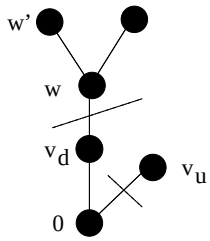

Figure 2: Admissible cut.

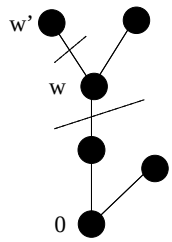

Figure 3: Non-admissible cut.

Definition 2.11 ($\mathcal{P}$-projections) *Let $U \subset \mathbb{Z}^{\mathbb{T}} := \{(k_v)_{v\in V(\mathbb{T})}\}$. Then $\mathcal{P}^U : (B^{\alpha}_{\infty,\infty})^{\otimes\mathbb{T}} \to (B^{\alpha}_{\infty,\infty})^{\otimes\mathbb{T}}$ is the bounded linear operator defined on monomials by*

$$\mathcal{P}^U\left(\otimes_{v\in V(\mathbb{T})} f_v\right) := \sum_{\boldsymbol{k}=(k_v)_{v\in V(\mathbb{T})}\in U} \frac{1}{|\Sigma_{\boldsymbol{k}}|} \cdot \otimes_{v\in V(\mathbb{T})} D(\phi_{k_v}) f_v, \tag{2.23}$$

where $\Sigma_{\boldsymbol{k}}$ is the set of permutations $\sigma : V(\mathbb{T}) \to V(\mathbb{T})$ such that $|k_{\sigma(v)}| = |k_v|$ for every $v \in V(\mathbb{T})$.

In particular, we shall denote by $\mathcal{P}^+$ the operator $\mathcal{P}^U$ defined by

$$U = \mathbb{Z}^{\mathbb{T}}_+ := \{(k_v)_{v\in V(\mathbb{T})} \mid (v < w) \Rightarrow |k_v| \le |k_w|\}. \tag{2.24}$$

In particular, one has the essential decomposition formula:

Lemma 2.12 *Let $\sigma : \{0,\dots,n-1\} \to \{0,\dots,n-1\}$ range over all permutations, and decompose the permutation graphs $\mathbb{T}^\sigma$ as a sum of forests, $\mathbb{T}^\sigma = \sum_{j=1}^{J_\sigma} \varepsilon_{\sigma,j}\mathbb{T}^\sigma_j$, with $\varepsilon_{\sigma,j} \in \{\pm 1\}$. Then:*

$$I^\ell_n(\Gamma) = \sum_\sigma \sum_{j=1}^{J_\sigma} \varepsilon_{\sigma,j} I_{\mathbb{T}^\sigma_j}\left(\mathcal{P}^+\left(\otimes_{v\in V(\mathbb{T}^\sigma_j)}\Gamma(\ell(\sigma(v)))\right)\right). \tag{2.25}$$

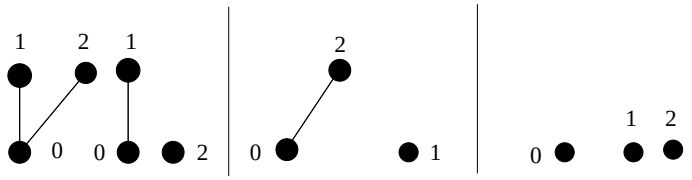

Figure 4: Example 2.6. From left to right: $\mathbb{T}_1^\sigma, \mathbb{T}_2^\sigma$; $L_{\{1\}}\mathbb{T}_1^\sigma \otimes R_{\{1\}}\mathbb{T}_1^\sigma$; $L_{\{1,2\}}\mathbb{T}_1^\sigma \otimes R_{\{1,2\}}\mathbb{T}_1^\sigma$

Proof. Trivial (note that every multi-index $\boldsymbol{k}$ is counted exactly once thanks to the symmetry factor $\frac{1}{|\Sigma_{\boldsymbol{k}}|}$). $\square$

The 'projection' operators $\mathcal{P}^U$ are uniformly bounded by Proposition 1.7. Replacing ϕ_k with $\mathbf{1}_{[\pm 2^k, \pm 2^{k+1}]}$, one would really obtain projections, but unfortunately, smooth Fourier multipliers are required to define the Besov space $B^\alpha_{\infty,\infty}$. One could actually use characteristic functions of intervals (which simplifies a little bit the proofs) by working in $B^\alpha_{p,\infty}$ for p large (see Lizorkin representations in [16], §2.2.4) but at the price of losing some Hölder regularity, namely, $\alpha \to \alpha - 1/p$, see [16], §2.4.1 (1).

By multilinearity, one may split the function

$$(x_0, \ldots, x_{n-1}) \to (\Gamma(i_0) \otimes \ldots \Gamma(i_{n-1}))\,(x_0, \ldots, x_{n-1}) = \Gamma_{x_0}(i_0) \ldots \Gamma_{x_{n-1}}(i_{n-1})$$

into $\sum_{\boldsymbol{k}=(k_0,\ldots,k_{n-1})\in\mathbb{Z}^n} \mathcal{P}^{\{\boldsymbol{k}\}}\left(\Gamma(i_0) \otimes \ldots \Gamma(i_{n-1})\right)$.

Formula (2.22) above generalizes to the following recursive definition of regularized iterated integrals:

Lemma 2.13 (regularization) *Let $\mathbb{T} = \mathbb{T}_1 \ldots \mathbb{T}_l$ be a well-labeled forest, together with its tree decomposition. Let also $\boldsymbol{k} \in \mathbb{Z}^{\mathbb{T}}$ such that $(v < w) \Rightarrow (|k_v| \leq |k_w|)$. Define by induction the regularized integral $\left[\mathcal{R}I_{\mathbb{T}}(\mathcal{P}^{\{\boldsymbol{k}\}}(\otimes_{v\in V(\mathbb{T})}\Gamma(i_v))\right]_{ts}$ (abbreviated in the lemma as $[\mathcal{R}I_{\mathbb{T}}]_{ts}$) by*

$$\prod_{j=1}^{l}\left\{\left[\delta\mathcal{R}\mathrm{SkI}_{\mathbb{T}_j}\right]_{ts} - \sum_{\boldsymbol{v}\models V(\mathbb{T}_j)}\left[\mathcal{R}I_{L_{\boldsymbol{v}}\mathbb{T}_j}\right]_{ts}\left[\mathcal{R}\mathrm{SkI}_{R_{\boldsymbol{v}}\mathbb{T}_j}\right]_s\right\}\left(\mathcal{P}^{\{\boldsymbol{k}\}}(\otimes_{v\in V(\mathbb{T})}\Gamma(i_v)\right) \tag{2.26}$$

where (letting $\boldsymbol{k}\big|_{\mathbb{T}'} = (k_v)_{v\in V(\mathbb{T}')}$ and $f_{\mathbb{T}'} := \mathrm{SkI}_{\mathbb{T}'}(\mathcal{P}^{\{\boldsymbol{k}\big|_{\mathbb{T}'}\}}(\otimes_{v\in V(\mathbb{T}')}\Gamma(i_v))))$ $\mathcal{R}f_{\mathbb{T}'} = f_{\mathbb{T}'}$ or 0 depending on the value of the multi-index $\boldsymbol{k}\big|_{\mathbb{T}'}$, and in particular, $\mathcal{R}f_{\mathbb{T}'} = f_{\mathbb{T}'}$ is $\mathbb{T}'$ is a tree with one vertex only.

Then $[\mathcal{R}I_{\mathbb{T}}]_{ts}$ satisfies the following tree multiplicative property:

$$[\delta\mathcal{R}I_{\mathbb{T}}]_{tus} = \sum_{\boldsymbol{v}\models V(\mathbb{T})} [\mathcal{R}I_{L_{\boldsymbol{v}}\mathbb{T}}]_{tu} \ . \ [\mathcal{R}I_{R_{\boldsymbol{v}}\mathbb{T}}]_{us} \, . \tag{2.27}$$

We shall call $[\mathcal{R}I_{\mathbb{T}_j}(\delta)]_{ts} := \left[\delta\mathcal{R}\mathrm{SkI}_{\mathbb{T}_j}\right]_{ts}$*, resp.* $[\mathcal{R}I_{\mathbb{T}_j}(\partial)]_{ts} := -\sum_{\boldsymbol{v}\models V(\mathbb{T}_j)} \left[\mathcal{R}I_{L_{\boldsymbol{v}}\mathbb{T}_j}\right]_{ts} \left[\mathcal{R}\mathrm{SkI}_{R_{\boldsymbol{v}}\mathbb{T}_j}\right]_s$ *the* increment*, resp.* boundary term *associated to the tree* $\mathbb{T}_j$*.*

Proof. See [20], Lemma 6.14. □

By multilinearity, the above regularization lemma also holds of course for $\mathcal{R}I_{\mathbb{T}}\mathcal{P}^+(\otimes_{v\in V(\mathbb{T})}\Gamma(i_v))$.

Corollary 2.14 *The rough path* $\mathcal{R}\mathbf{\Gamma}^n(\ell(1),\ldots,\ell(n))$*, defined (see Lemma 2.12 for notations) as the sum*

$$\mathcal{R}\mathbf{\Gamma}^n(\ell(1),\ldots,\ell(n)) = \sum_{\sigma}\sum_{j=1}^{J_\sigma} \varepsilon_{\sigma,j}\mathcal{R}I_{\mathbb{T}_j^\sigma}\left(\mathcal{P}^+\left(\otimes_{v\in V(\mathbb{T}_j^\sigma)}\Gamma(\ell(\sigma(v)))\right)\right) \tag{2.28}$$

over all permutations σ*, satisfies the multiplicative property (ii) in the Introduction.*

Proof. Consequence of Lemma 2.12, Lemma 2.13 and [20] Lemma 6.15 which relates the above tree multiplicative property to the usual one (see property (ii) in the Introduction) by combinatorial arguments. The condition $\mathcal{R}f_{\mathbb{T}'} = f_{\mathbb{T}'}$ implies in the end the equality $\mathcal{R}\mathbf{\Gamma}^1 = \mathbf{\Gamma}^1$ (so we have constructed a rough path lying above Γ).

Proposition 2.15 *(see [20], lemma 7.2)*

The rough path $\mathcal{R}\mathbf{\Gamma}$ *satisfies the geometric property (iii) in the Introduction.*

Note that a regularization procedure $\mathcal{R}$ is exactly determined by the choice for each tree $\mathbb{T}$ of a subset $\mathbb{Z}^{\mathbb{T}}_{reg}$ of $\mathbb{Z}^{\mathbb{T}}_{+} := \{(k_v)_{v\in V(\mathbb{T})} \mid (v < w) \Rightarrow |k_v| \leq |k_w|\}$, such that $\mathbb{Z}^{\mathbb{T}}_{reg} = \mathbb{Z}$ if $|V(\mathbb{T})| = 1$. Let us now give an appropriate choice for $\mathbb{Z}^{\mathbb{T}}_{reg}$. Once again, all trees below are implicitly supposed to be equipped with a total ordering compatible with the tree partial ordering as in Definition 2.2. Ultimately, when reconstructing the rough path $\mathcal{R}\mathbf{\Gamma}$ (see Corollary 2.14), the total ordering on trees coming from the decomposition

of $\mathbb{T}^\sigma$ will be induced from the total ordering on the permutation graph $\mathbb{T}^\sigma$ (see subsection 2.1).

We shall need to introduce a little more terminology concerning tree structures (see Fig. 5).

Definition 2.16 *Let $\mathbb{T}$ be a tree.*

(i) A vertex v is a leaf *if no vertex connects to v. The set of leaves above (i.e. connecting to) $v \in V(\mathbb{T})$ is denoted by $Leaf(v)$.*

(ii) Vertices at which 2 or more branches join are called nodes*.*

(iii) The set $Br(v_1 \twoheadrightarrow v_2)$ of vertices from a leaf or a node v_1 to a node v_2 (or to the root) is called a branch *if it does not contain any other node. By convention, $Br(v_1 \twoheadrightarrow v_2)$ includes v_1 and excludes v_2.*

(iv) A node n is called an uppermost node *if no other node is connected to n.*

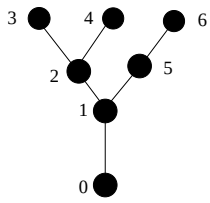

Figure 5: 3,4,6 are leaves; 1, 2 and 5 are nodes, 2 and 5 are uppermost; branches are e.g. $Br(2 \twoheadrightarrow 1) = \{2\}$ or $Br(6 \twoheadrightarrow 1) = \{6, 5\}$; $Leaf(2) = \{3, 4\}$; $w_{max}(2) = 4$.

Definition 2.17 *Let $\mathbb{T}$ be a tree. If $v \in V(\mathbb{T})$, we let $w_{max}(v) := \max\{w \in V(\mathbb{T}) \mid w \twoheadrightarrow v\}$, or simply $w_{max}(v) = v$ if v is a leaf.*

Definition 2.18 *Let $\mathbb{Z}^{\mathbb{T}}_{reg}$ be the set of $V(\mathbb{T})$-uples $\boldsymbol{k} = (k_v)_{v \in V(\mathbb{T})} \in \mathbb{Z}^{\mathbb{T}}$ such that the following conditions are satisfied:*

(i) if $v < w$, then $|k_v| \leq |k_w|$;

(ii) if $v \in V(\mathbb{T})$ and $w \in Leaf(v)$, $k_w.k_v < 0$, then $|k_v| \leq |k_w| - \log_2 10 - \log_2 |V(\mathbb{T})|$;

(iii) if $n \in V(\mathbb{T})$ is a node, then each vertex $w \in \{w_{max}(v) \mid v \to n\}$ such that $k_w.k_{w_{max}(n)} < 0$ satisfies: $|k_w| \leq |k_{w_{max}(n)}| - \log_2 10 - \log_2 |V(\mathbb{T})|$.

Lemma 2.19 *Let* $\boldsymbol{\xi} = (\xi_v)_{v\in\mathbb{T}}$ *such that* $\xi_v \in \mathrm{supp}(\phi_{k_v})$ *for some* $\boldsymbol{k} = (k_v)_{v\in V(\mathbb{T})} \in \mathbb{Z}^{\mathbb{T}}_{reg}$. *Then, for every* $v \in V$,

$$|V(\mathbb{T})| \,.\, |\xi_{w_{max(v)}}| \geq |\xi_v + \sum_{w\twoheadrightarrow v} \xi_w| > \frac{1}{2}|\xi_{w_{max}(v)}|. \tag{2.29}$$

Proof.

The left inequality is trivial. As for the right one, assume first that v is on a terminal branch, i.e. $Leaf(v) = \{w_{max}(v)\}$ is a singleton. Then Definition 2.18 (ii) implies the following: for every vertex v' on the branch between $w_{max}(v)$ and v, i.e. $v' \in Br(w_{max}(v) \twoheadrightarrow v) \cup \{v\}$,

– either $\xi_{v'}$ is of the same sign as $\xi_{w_{max}(v)}$;

– or $|\xi_{v'}| \leq \frac{|\xi_{w_{max}(v)}|}{2|V(\mathbb{T})|}$ since $|\xi_{v'}| \in (2^{|k_{v'}|-1}, 5.2^{|k_{v'}|-1})$ (and similarly for $|\xi_{w_{max}(v)}|$) by the remarks following Proposition 1.2.

Hence $|\xi_v + \sum_{w\twoheadrightarrow v} \xi_w| = |\sum_{v'\in Br(w_{max}(v)\twoheadrightarrow v)\cup\{v\}} \xi_{v'}| > \left(1 - \frac{1}{2}\frac{|\{w:w\twoheadrightarrow v\}|}{|V(\mathbb{T})|}\right)|\xi_{w_{max}(v)}|$.

Consider now what happens at a node n. Let $n^+ := \{v \in V(\mathbb{T}) \mid v \to n\}$. Assume (by induction on the number of vertices) that, for all $v \in n^+$,

$$(1 + |\{w : w \twoheadrightarrow v\}|)\,|\xi_{w_{max}(v)}| \geq |\xi_v + \sum_{w\twoheadrightarrow v} \xi_w| > \left(1 - \frac{1}{2}\frac{|\{w : w \twoheadrightarrow v\}|}{|V(\mathbb{T})|}\right) . |\xi_{w_{max}(v)}|. \tag{2.30}$$

By Definition 2.18 (iii), either $\xi_{w_{max}(v)}.\xi_{w_{max}(n)} > 0$ or $|\xi_{w_{max}(v)}| \leq \frac{|\xi_{w_{max}(n)}|}{2|V(\mathbb{T})|}$. Then (letting w_0 be the element of n^+ such that $w_{max}(v_0) = w_{max}(n)$)

$$\begin{aligned}
(1 + |\{w : w \twoheadrightarrow n\}|)\,|\xi_{w_{max}(n)}| &\geq |\xi_n + \sum_{w\twoheadrightarrow n} \xi_w| = \left|\xi_n + \sum_{v\in n^+}(\xi_v + \sum_{w\twoheadrightarrow v} \xi_w)\right| \\
&\geq \left|\xi_{v_0} + \sum_{w\twoheadrightarrow v_0} \xi_w\right| - \left|\sum_{v\in n^+\setminus\{v_0\}}(\xi_v + \sum_{w\twoheadrightarrow v} \xi_w)\right| - |\xi_n| \\
&> \left(1 - \frac{1}{2}\frac{|\{w : w \twoheadrightarrow n\}|}{|V(\mathbb{T})|}\right) . |\xi_{w_{max}(n)}|.
\end{aligned} \tag{2.31}$$

□

3 Hölder estimates

We shall now prove that the rough path $\mathcal{R}\boldsymbol{\Gamma}^n(\ell(0), \ldots, \ell(n-1))$ satisfies the required Hölder properties by (1) decomposing $\mathcal{R}\boldsymbol{\Gamma}^n(\ell(0), \ldots, \ell(n-1))$

into the sum over all permutations of $\mathcal{R}I_{\mathbb{T}_j^\sigma}\mathcal{P}^+\left(\otimes_{v\in V(\mathbb{T}_j^\sigma)}\Gamma(\ell(\sigma(v)))\right)$ as in Lemma 2.12, and (2) show Hölder regularity with correct exponent of all tree integrals appearing in the recursive definition of Lemma 2.13.

If $\mathbb{T}=\mathbb{T}_j^\sigma$ is one of the forests – or one of the tree components of those forests – appearing in the decomposition of the permutation graph $\mathbb{T}^\sigma$, we shall write $\mathbb{T}\subset\mathbb{T}^\sigma$. The total ordering of $\mathbb{T}$ is inherited from that of $\mathbb{T}^\sigma$ (see subsection 2.1).

3.1 Estimate for the increment term

Lemma 3.1 (Hölder estimate of the increment term) *Let $\mathbb{T}\subset\mathbb{T}^\sigma$ be a tree. Then*

$$||\mathcal{R}\mathrm{SkI}_{\mathbb{T}}\left(\mathcal{P}^+(\otimes_{v\in V(\mathbb{T})}\Gamma(\ell(\sigma(v))))\right)||_{\mathcal{C}^{|V(\mathbb{T})|\alpha}}<\infty. \tag{3.1}$$

Proof. We shall start the computations by adapting the proof of a theorem in [16], §2.6.1 bounding the Hölder-Besov norm of the product of two Hölder functions. We use the shorthand

$$\mathcal{R}\mathrm{SkI}_{\mathbb{T}}=\mathcal{R}\mathrm{SkI}_{\mathbb{T}}\left(\mathcal{P}^+(\otimes_{v\in V(\mathbb{T})}\Gamma(\ell(\sigma(v))))\right). \tag{3.2}$$

By Lemma 2.9,

$$\begin{aligned}\mathcal{R}\mathrm{SkI}_{\mathbb{T}}(x)=&\sum_{\boldsymbol{k}=(k_v)_{v\in V(\mathbb{T})}\in\mathbb{Z}^{\mathbb{T}}_{reg}}\int_{\prod_{v\in V(\mathbb{T})}\mathrm{supp}(\phi_{k_v})}\prod_{v\in V(\mathbb{T})}d\xi_v\ .\\ &.\ e^{\mathrm{i}x\sum_{v\in V(\mathbb{T})}\xi_v}\frac{\prod_{v\in V(\mathbb{T})}\mathcal{F}\left(D(\phi_{k_v})\Gamma'(\ell(\sigma(v))))\right)(\xi_v)}{\prod_{v\in V(\mathbb{T})}(\xi_v+\sum_{w\twoheadrightarrow v}\xi_w)}.\end{aligned} \tag{3.3}$$

Write, for $\boldsymbol{\xi}=(\xi_v)_{v\in V(\mathbb{T})}$,

$$\Theta(\boldsymbol{\xi})=\prod_{v\in V(\mathbb{T})}\frac{\xi_v}{\xi_v+\sum_{w\twoheadrightarrow v}\xi_w} \tag{3.4}$$

and

$$\Theta_1(\boldsymbol{k})=\prod_{v\in V(\mathbb{T})}\frac{2^{|k_v|}}{2^{|k_{w_{max}(v)}|}}. \tag{3.5}$$

Let finally

$$\Theta_{\boldsymbol{k}}(\boldsymbol{\xi}) := \prod_{v\in V(\mathbb{T})} \tilde{\phi}_{k_v}(\xi_v) \ . \ \frac{\Theta(\boldsymbol{\xi})}{\Theta_1(\boldsymbol{k})}, \tag{3.6}$$

where $(\tilde{\phi}_k)_{k\in\mathbb{Z}} : \mathbb{R} \to [0,1]$ is a family of C^∞ functions with uniformly bounded $||\ .\ ||_{S^0}$-norm such that $\tilde{\phi}_k\big|_{\mathrm{supp}(\phi_k)} \equiv 1$ (so $\phi_k\tilde{\phi}_k = \phi_k$) and $\mathrm{supp}\tilde{\phi}_k$ is a little larger than $\mathrm{supp}\phi_k$, say,

$$\mathrm{supp}(\tilde{\phi}_k) \subset [2^{k-2}, 10.2^{k-1}],\ \mathrm{supp}(\tilde{\phi}_{-k}) \subset [-10.2^{k-1}, -2^{k-2}] \quad (k \geq 1) \tag{3.7}$$

(the functions $(\tilde{\phi}_k)_{k\in\mathbb{Z}}$, just as for the $(\phi_k)_{k\in\mathbb{Z}}$, may be constructed simply out of a single function $\tilde{\phi}_0$ with adequate support, see remarks following Proposition 1.2). By Lemma 2.19, $||\Theta_{\boldsymbol{k}}||_{S^0}$ is uniformly bounded in $\boldsymbol{k}$.

Let $k \in \mathbb{Z}$. Apply the operator $D(\phi_k)$ to eq. (3.3): then, letting $\phi_k^*(\boldsymbol{\xi}) := \phi_k(\sum_{v\in V(\mathbb{T})} \xi_v)$,

$$D(\phi_k)\mathcal{R}\mathrm{SkI}_{\mathbb{T}}(x) = \left[\sum_{\boldsymbol{k}\in\mathbb{Z}^{\mathbb{T}}_{reg}} \Theta_1(\boldsymbol{k}) D(\Theta_{\boldsymbol{k}}) D(\phi_k^*) \ . \ \prod_{v\in V(\mathbb{T})} D(\phi_{k_v})\Gamma(\ell(\sigma(v)))\right](\boldsymbol{x}), \tag{3.8}$$

where $\boldsymbol{x} = (x_v)_{v\in V(\mathbb{T})} = (x, \ldots, x)$ is a vector with $|V(\mathbb{T})|$ identical components.

Let $v_{max} := \sup\{v \mid v \in V(\mathbb{T})\}$. Note that $D(\phi_k^*)\ .\ D(\otimes_{v\in V(\mathbb{T})}\phi_{k_v})$ vanishes except if

$$\left(\sum_{v\in V(\mathbb{T})} \mathrm{supp}(\phi_{k_v})\right) \cap \mathrm{supp}(\phi_k) \neq \emptyset, \tag{3.9}$$

which implies by Lemma 2.19

$$|k_{v_{max}} - k| = O(\log_2 |V(\mathbb{T})|) \tag{3.10}$$

(namely, denoting by 0 the root of $\mathbb{T}$, $|V(\mathbb{T})|\ .\ |\xi_{k_{v_{max}}}| \geq |\sum_{v\in V(\mathbb{T})} \xi_{k_v}| = |\xi_{k_0} + \sum_{w\twoheadrightarrow 0} \xi_{k_w}| > \frac{1}{2}|\xi_{k_{v_{max}}}|$ if $\xi_v \in \mathrm{supp}(\phi_{k_v})$ for every v).

Since $\Theta_{\boldsymbol{k}}, \phi_k^* \in S^0(\mathbb{R}^{|V(\mathbb{T})|})$, one gets by Proposition 1.7

$$||D(\phi_k)\mathcal{R}\mathrm{SkI}_{\mathbb{T}}||_\infty \lesssim \sum_{\boldsymbol{k}\in\mathbb{Z}^{\mathbb{T}}_{reg}, k_{v_{max}}=k} \Theta_1(\boldsymbol{k}) \prod_{v\in V(\mathbb{T})} ||D(\phi_{k_v})\Gamma(\ell(\sigma(v)))||_\infty. \tag{3.11}$$

Since Γ is in $\mathcal{C}^\alpha$, one obtains:

$$\begin{aligned} ||D(\phi_k)\mathcal{R}\mathrm{SkI}_{\mathbb{T}}||_\infty &\lesssim \sum_{\boldsymbol{k}\in\mathbb{Z}^{\mathbb{T}}_{reg}\ ,k_{v_{max}}=k} \Theta_1(\boldsymbol{k}) \prod_{v\in V(\mathbb{T})} 2^{-|k_v|\alpha} \\ &\lesssim \sum_{\boldsymbol{k}\in\mathbb{Z}^{\mathbb{T}}_{reg},k_{v_{max}}=k} \prod_{v\in V(\mathbb{T})} 2^{|k_v|(1-\alpha)-|k_{w_{max}(v)}|}. \end{aligned} \tag{3.12}$$

In other words, loosely speaking, each vertex $v \in V(\mathbb{T})$ contributes a factor $2^{|k_v|(1-\alpha)-|k_{w_{max}(v)}|}$ to $||\mathcal{R}\mathrm{SkI}_{\mathbb{T}}||_\infty$. If v is a leaf, then this factor is simply $2^{-|k_v|\alpha}$.

Consider an uppermost node n, i.e. a node to which no other node is connected, together with the set of leaves $\{w_1 < \ldots < w_J\}$ above n. Let $p_j = |V(Br(w_j \twoheadrightarrow n))|$. On the branch number j,

$$\prod_{v\in Br(w_j\twoheadrightarrow n)\setminus\{w_j\}} \sum_{|k_v|\leq|k_{w_j}|} 2^{|k_v|(1-\alpha)-|k_{w_j}|} \lesssim 2^{-|k_{w_j}|\alpha p_j} \tag{3.13}$$

and (summing over $k_{w_1}, \ldots, k_{w_{J-1}}$ and over k_n)

$$\begin{aligned} &2^{-|k_{w_J}|\alpha p_J} \sum_{|k_{w_{J-1}}|\leq|k_{w_J}|} 2^{-|k_{w_{J-1}}|\alpha p_{J-1}} \\ &\quad \left(\ldots\left(\sum_{|k_{w_1}|\leq|k_{w_2}|} 2^{-|k_{w_1}|\alpha p_1}\left(\sum_{|k_n|\leq|k_{w_1}|} 2^{|k_n|(1-\alpha)-|k_{w_J}|}\right)\right)\ldots\right) \\ &\lesssim 2^{-|k_{w_J}|\alpha W(n)}, \end{aligned} \tag{3.14}$$

where $W(n) = p_1 + \ldots + p_J + 1 = |\{v : v \twoheadrightarrow n\}| + 1$ is the *weight* of n.

One may then consider the reduced tree $\mathbb{T}_n$ obtained by shrinking all vertices above n (including n) to *one* vertex with weight $W(n)$ and perform the same operations on $\mathbb{T}_n$. Repeat this inductively until $\mathbb{T}$ is shrunk to one point. In the end, one gets $||D(\phi_k)\mathcal{R}\mathrm{SkI}_{\mathbb{T}}||_\infty \lesssim 2^{-|k|\alpha|V(\mathbb{T})|}$, hence $\mathcal{R}\mathrm{SkI}_{\mathbb{T}} \in \mathcal{C}^{|V(\mathbb{T})|\alpha}$.

□

Remark. Note that the above proof breaks down for the non-regularized quantitities, since the function Θ may be much larger than the expression Θ_1 (it is not even bounded, actually). For instance, the Lévy area of fractional Brownian motion diverges below the barrier $\alpha = 1/4$, see [4], [18], [19]. The proof in [20] (using Fourier series instead of Fourier integrals) is easily rewritten in terms of Besov norms since the estimates for random

Fourier series in section 2, essentially taken from [9], are also of Besov type, see [20], Lemma 2.1, eq. (2.4) for instance). For well-behaved paths Γ with very regular, polynomially decreasing Fourier components, the unregularized integrals are probably well-defined at least for $\alpha > 1/2$ (but this is uninteresting of course since Young's integral converges), otherwise the case is not even clear.

3.2 Estimate for the boundary term

Lemma 3.2 (Hölder regularity of the boundary term) *Let $\mathbb{T} \subset \mathbb{T}^\sigma$ be a tree. Then the regularized boundary term $\mathcal{R}I_{\mathbb{T}}\left(\mathcal{P}^+(\otimes_{v\in V(\mathbb{T})}\Gamma(\ell(\sigma(v))))\right)(\partial)$ is $|V(\mathbb{T})|\alpha$-Hölder.*

Proof.
Once again we use the shorthand

$$\mathcal{R}I_{\mathbb{T}}(\partial) := \mathcal{R}I_{\mathbb{T}}\left(\mathcal{P}^+(\otimes_{v\in V(\mathbb{T})}\Gamma(\ell(\sigma(v))))\right)(\partial). \tag{3.15}$$

Apply repeatedly Lemma 2.13 to $\mathbb{T}$: in the end, $[\mathcal{R}I_{\mathbb{T}}(\partial)]_{ts}$ appears as a sum of 'skeleton-type' terms of the form (see Fig. 6)

$$\begin{aligned} A_{ts} &:= [\delta\mathcal{R}\mathrm{SkI}_{L\mathbb{T}}]_{ts}\ . \\ &[\mathcal{R}\mathrm{SkI}_{R_{\boldsymbol{v}_1}\mathbb{T}}]_s[\mathcal{R}\mathrm{SkI}_{R_{\boldsymbol{v}_2}\circ L_{\boldsymbol{v}_1}(\mathbb{T})}]_s\ldots[\mathcal{R}\mathrm{SkI}_{R_{\boldsymbol{v}_l}\circ L_{\boldsymbol{v}_{l-1}}\circ\ldots\circ L_{\boldsymbol{v}_1}(\mathbb{T})}]_s, \end{aligned} \tag{3.16}$$

where $\boldsymbol{v}_1 = (v_{1,1} < \ldots < v_{1,J_1}) \models V(\mathbb{T})$, $\boldsymbol{v}_2 \models V(L_{\boldsymbol{v}_1}\mathbb{T})$, $\ldots$, $\boldsymbol{v}_l = (v_{l,1},\ldots,v_{l,J_l}) \models V(L_{\boldsymbol{v}_{l-1}}\circ\ldots\circ L_{\boldsymbol{v}_1}(\mathbb{T}))$ and $L\mathbb{T} := L_{\boldsymbol{v}_l}\circ\ldots\circ L_{\boldsymbol{v}_1}(\mathbb{T})$.

<u>First step.</u>

Let $U[\boldsymbol{k}] \subset \prod_{j=1}^{J_1}\mathbb{Z}_{reg}^{R_{v_{1,j}}\mathbb{T}}$ such that $\boldsymbol{k} = (k_{v_{1,1}},\ldots,k_{v_{1,J_1}})$ (with $|k_{v_{1,1}}| \leq \ldots \leq |k_{v_{1,J_1}}|$) is fixed. Then (see proof of Lemma 3.1) each vertex v contributes a factor $2^{|k_v|(1-\alpha)-|k_{w_{max}}(v)|} \leq 2^{-|k_v|\alpha}$, hence

$$\begin{aligned} ||\mathcal{P}^{U[\boldsymbol{k}]}\mathcal{R}\mathrm{SkI}_{R_{\boldsymbol{v}_1}\mathbb{T}}||_\infty &\lesssim \prod_{v\in\boldsymbol{v}_1}\left[2^{-|k_v|\alpha}\sum_{|k_w|\geq|k_v|,w\in R_v\mathbb{T}\setminus\{v\}}2^{-|k_w|\alpha}\right] \\ &\lesssim \prod_{v\in\boldsymbol{v}_1}2^{-|k_v|\alpha|V(R_v\mathbb{T})|}. \end{aligned} \tag{3.17}$$

<u>Second step.</u>

More generally, let $B_s[\boldsymbol{k}]$ be the expression obtained by $\mathcal{P}$-projecting $[\mathcal{R}\mathrm{SkI}_{R_{\boldsymbol{v}_1\mathbb{T}}}]_s[\mathcal{R}\mathrm{SkI}_{R_{\boldsymbol{v}_2}\circ L_{\boldsymbol{v}_1}(\mathbb{T})}]_s\ldots[\mathcal{R}\mathrm{SkI}_{R_{\boldsymbol{v}_l}\circ L_{\boldsymbol{v}_{l-1}}\circ\ldots\circ L_{\boldsymbol{v}_1}(\mathbb{T})}]_s$ onto the sum of terms with fixed value of the indices $\boldsymbol{k}=(k_{v_{l,1}},\ldots,k_{v_{l,J_l}})$. Then

$$||B_s[\boldsymbol{k}]||_\infty \lesssim \prod_{v\in\boldsymbol{v}_l} 2^{-|k_v|\alpha|V(R_v\mathbb{T})|} \tag{3.18}$$

(proof by induction on l).

<u>Third step.</u>
We define

$$A_s(x) := [\mathcal{R}\mathrm{SkI}_{L\mathbb{T}}]_x\,.\,[\mathcal{R}\mathrm{SkI}_{R_{\boldsymbol{v}_1\mathbb{T}}}]_s[\mathcal{R}\mathrm{SkI}_{R_{\boldsymbol{v}_2}\circ L_{\boldsymbol{v}_1}(\mathbb{T})}]_s\ldots[\mathcal{R}\mathrm{SkI}_{R_{\boldsymbol{v}_l}\circ L_{\boldsymbol{v}_{l-1}}\circ\ldots\circ L_{\boldsymbol{v}_1}(\mathbb{T})}]_s \tag{3.19}$$

(see eq. (3.16)), so that $A_{ts}=A_s(t)-A_s(s)$, and show that $\sup_{s\in\mathbb{R}}||x\to A_s(x)||_{B^\alpha_{\infty,\infty}}<\infty$.

Let $V(L\mathbb{T})=\{w_1<\ldots<w_{max}\}$. Fix $s\in\mathbb{R}$ and $K\in\mathbb{Z}$. By definition,

$$(D(\phi_K)A_x)(s) = D(\phi_K)\sum_{\boldsymbol{k}=(k_{v_{l,1}},\ldots,k_{v_{l,J_l}})}\ \sum_{((k_w)_{w\in V(L\mathbb{T})})\in S_{\boldsymbol{k}}}\int_{\prod_{v\in V(L\mathbb{T})}\mathrm{supp}(\phi_{k_v})}\prod_{v\in V(L\mathbb{T})}d\xi_v\ . \tag{3.20}$$

$$.\ e^{\mathrm{i}x\sum_{v\in V(L\mathbb{T})}\xi_v}\ \frac{\prod_{w\in V(L\mathbb{T})}\mathcal{F}\left(D(\phi_{k_w})\Gamma'(\ell(\sigma(w)))\right)(\xi_w)}{\prod_{w\in V(L\mathbb{T})}(\xi_w+\sum_{w'\twoheadrightarrow w,w'\in V(L\mathbb{T})}\xi_{w'})}B_s[\boldsymbol{k}] \tag{3.21}$$

where indices in $S_{\boldsymbol{k}}$ satisfy in particular the following conditions:

(i) $|\xi_w+\sum_{w'\twoheadrightarrow w,w'\in V(L\mathbb{T})}\xi_{w'}|>\frac{1}{2}\max\{|\xi_{w'}|:w'\twoheadrightarrow w,w'\in V(L\mathbb{T})\}$;

(ii) $\left(\sum_{w\in V(L\mathbb{T})}\mathrm{supp}(\phi_{k_w})\right)\cap(\mathrm{supp}(\phi_K))\neq\emptyset$ (see eq. (3.9));

(iii) for every $w\in V(L\mathbb{T})$, $|k_w|\leq|k_{w_{max}}|$; and

(iv) for every $w\in V(L\mathbb{T})$, $|k_w|\leq|k_v|$ for every $v\in R(w):=\{v=v_{l,1},\ldots,v_{l,J_l}\mid v\to w\}$ (note that $R(w)$ may be empty). See Fig. 6.

Note that $|k_{w_{max}}-K|=O(\log_2|V(L\mathbb{T})|)$ by (ii) (see eq. (3.10)). Hence conditions (ii) and (iii) above are more or less equivalent to fixing $k_{w_{max}}\simeq K$ and letting $(k_w)_{w\in V(L\mathbb{T})\setminus\{w_{max}\}}$ range over some subset of $[-|K|,|K|]\times\ldots\times[-|K|,|K|]$.

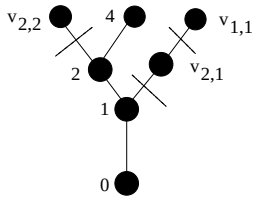

Figure 6: Here $V(L\mathbb{T}) = \{0, 1, 2, 4\}$, $R(0) = R(4) = \emptyset$, $R(1) = \{v_{2,1}\}$, $R(2) = \{v_{2,2}\}$.

If $w \in L\mathbb{T}$, split $R(w)$ into $R(w)_> \cup R(w)_<$, where $R(w)_{\gtrless} := \{v \in R(w) \mid v \gtrless w_{max}\}$. Summing over indices corresponding to vertices in or above $R\mathbb{T}_> := \{v = v_{l,1}, \ldots, v_{l,J_l} \mid v > w_{max}\} = \cup_{w \in L\mathbb{T}} R(w)_>$, one gets by eq. (3.18) a quantity bounded up to a constant by

$$\prod_{v \in R\mathbb{T}_>} \sum_{|k_v| \geq |K|} 2^{-|k_v|\alpha |V(R_v\mathbb{T})|} \lesssim 2^{-|K|\alpha \sum_{v \in R\mathbb{T}_>} |V(R_v\mathbb{T})|}. \tag{3.22}$$

Let $w \in L\mathbb{T} \setminus \{w_{max}\}$ such that $R(w)_< \neq \emptyset$ (note that $R(w_{max})_< = \emptyset$). Let $R(w)_< = \{v_{i_1} < \ldots < v_{i_j}\}$. Then (summing over (k_v), v in or above $R(w)_<$)

$$\begin{aligned} &2^{-|k_w|\alpha} \sum_{|k_{v_{i_1}}| = |k_w|}^{\infty} \sum_{|k_{v_{i_2}}| = |k_{v_{i_1}}|}^{\infty} \cdots \sum_{|k_{v_{i_j}}| = |k_{v_{i_{j-1}}}|}^{\infty} \\ &2^{-|k_{v_{i_1}}\alpha |V(R_{v_{i_1}}\mathbb{T})|} \ldots 2^{-|k_{v_{i_j}}|\alpha |V(R_{v_{i_j}}\mathbb{T})|} \lesssim 2^{-|k_w|\alpha(1 + \sum_{v \in R(w)_<} |V(R_v\mathbb{T})|)}. \end{aligned} \tag{3.23}$$

In other words, each vertex $w \in L\mathbb{T}$ 'behaves' as if it had a weight $1 + \sum_{v \in R(w)_<} |V(R_v\mathbb{T})|$. Hence (by the same method as in the proof of Lemma 3.1), letting $R\mathbb{T}_< := \cup_{w \in L\mathbb{T}} R(w)_<$,

$$||D(\phi_K)A_s||_\infty \lesssim 2^{-|K|\alpha(|V(L\mathbb{T})| + \sum_{v \in R\mathbb{T}_<} |V(R_v\mathbb{T})|)} \cdot 2^{-|K|\alpha \sum_{v \in R\mathbb{T}_>} |V(R_v\mathbb{T})|} = 2^{-|K|\alpha |V(\mathbb{T})|}. \tag{3.24}$$

The above tree estimates easily yield the $n\alpha$-Hölder estimates for $\mathcal{R}I_{\mathbb{T}_j^\sigma}\left(\mathcal{P}^+\left(\otimes_{v \in V(\mathbb{T}_j^\sigma)} \Gamma(\ell(\sigma(v)))\right)\right)$ (see Lemma 2.12) if $\mathbb{T}_j^\sigma = \mathbb{T}_{j,1}^\sigma \ldots \mathbb{T}_{j,J}^\sigma$ is a forest with several components, since $[\mathcal{R}I_{\mathbb{T}_j^\sigma}]_{ts}$ is a $\mathcal{P}$-projection of $[\mathcal{R}I_{\mathbb{T}_{j,1}^\sigma}]_{ts} \ldots [\mathcal{R}I_{\mathbb{T}_{j,J}^\sigma}]_{ts}$ (the total ordering of $\mathbb{T}_j^\sigma$ being stronger than its partial ordering induced by the total ordering of its tree components).

□

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