Wavelets techniques for pointwise anti-Holderian irregularity

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Abstract: In this paper, we introduce a notion of weak pointwise Hölder regularity, starting from the definition of the pointwise anti-Hölder irregularity. Using this concept, a weak spectrum of singularities can be defined as for the usual pointwise Hölder regularity. We build a class of wavelet series satisfying the multifractal formalism and thus show the optimality of the upper bound. We also show that the weak spectrum of singularities is disconnected from the casual one (denoted here strong spectrum of singularities) by exhibiting a multifractal function made of Davenport series whose weak spectrum differs from the strong one.
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1 Introduction

The concept of Hölderian regularity has been introduced to study nowhere differentiable functions (several examples are given in [33, 44]). An archetype of such functions is maybe the Weierstraß function

$$W_H(x) = \sum_{n=0}^{+\infty} a^{-nH} \cos(2\pi a^n x) \quad (0 < H < 1)$$

exhaustively studied by Hardy in [24]. He proved that for every $a > 1$, this function is nowhere differentiable. More precisely, the function W_H satisfies the two following conditions on $[0, 1]^2$,

$$|W_H(y) - W_H(x)| \leq C_1 |x - y|^H$$

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and

$$\sup_{(u,v)\in[x,y]^2} |W_H(u) - W_H(v)| \geq C_2|x-y|^H$$

for two constants C_1, C_2 that do not depend on x or y. The first inequality gives the regularity of W_H, which is said uniformly Hölder with exponent H on $[0,1]$. The second one reflects the irregularity of the function; in particular, W_H is nowhere differentiable. One says, following [49], that W_H is uniformly anti-Hölder with exponent H on $(0,1)$.

An increasing interest has been paid to functions f that are both uniform Hölder and uniformly anti-Hölder with exponent H, since these two properties ensure that the box-counting dimension of the graph of f is equal to $2-H$ (see e.g. [22]). Canonical Weierstraß functions, i.e. functions of the form

$$f(x) = \sum_n b^{-n\alpha} g(b^n x)$$

where g is 1-periodic, $1 < b < \infty$ and $0 < \alpha < 1$, have been extensively studied from the irregularity point of view by many authors (see [34, 43, 47, 12, 13, 26, 27, 21, 33]). Other well-known examples of such functions are provided by sample paths of Gaussian fields, generalizing the fractional Brownian motion. Irregularity properties, such as law of the iterated logarithm, are established using fine results concerning the regularity of the local time of the studied fields (see [10, 23, 2]). In particular, in this class of examples are included the so-called index-α Gaussian fields studied in [2], or more generally non locally deterministic Gaussian fields (see e.g. [11, 46]) and strongly non locally deterministic Gaussian fields (see [50, 51, 52]).

In this paper, we focus on the pointwise anti-Hölderian irregularity, which is the pointwise counterpart of the concept of uniform anti-Hölderian irregularity. Our main goal is to answer quite natural questions: Can we overstep the usual framework of functions both uniform Hölder and uniformly anti-Hölder? More precisely, can we give some explicit examples of functions for which the pointwise anti-Hölderian behavior is different from point to point? What are the main characteristics of such a behavior?

In the case of the usual pointwise regularity, multifractal functions provide examples of functions for which the Hölder exponent vary from point to point. So, we naturally tend to be interested in defining some multifractal functions for this notion of pointwise anti-Hölderian irregularity. We also need suitable tools to describe the multifractal behavior of such functions. It raises the problem of the related multifractal formalism. Indeed, let us recall that, in general settings, it is not possible to estimate the regularity index (which will be defined hereafter) of a function at a given point. The relevant information is then the "size" of the sets of points where the regularity is the same. This "size" is mathematically formalized as the Hausdorff dimension. The function that associates the dimension of the set of points sharing the same regularity index with this index is referred to as the spectrum of singularities. The goal of any multifractal formalism is to provide a method which allows to estimate this spectrum of singularities from numerically computable quantities derived from the signal. The same problem arises when dealing with pointwise anti-Hölderian irregularity.

Section 2 is devoted to the definitions related to the Hölder regularity. In Section 3 we investigate the structure of the irregularity exponent and define, by means of wavelet series, functions with prescribed irregularity exponent. In Section 4, we recall already known results about the multifractal formalism for the pointwise anti-Hölderian irregularity. Section 5 is devoted to the question of the validity of this multifractal formalism: Using multifractal measures, we define a class of wavelet series for which the multifractal formalism holds. In the last section, we compare the two concepts of multifractal functions: The usual one and this new one related to anti-Hölderianity. We show that the two notions are clearly disconnected. Indeed, we exhibit an example of Davenport series which is multifractal for the usual pointwise regularity but monofractal for the pointwise irregularity.

2 Pointwise Hölderian regularity

We start by giving the definitions of the pointwise Hölderian regularity and anti-Hölderian irregularity. The concept of anti-Hölderian functions with exponent H has been introduced by C.Tricot in [49]; he formalized a notion already used for investigating Weierstraß-type functions or sample paths properties of locally non deterministic Gaussian fields. Anti-Hölderian functions with exponent H were only defined in the case $H \in (0,1)$. A consistent definition is given here for H larger than 1. Since the anti-Hölderian condition is stronger than just negating the Hölderian condition, a weaker Hölderian regularity is obtained by negating the anti-Hölderian condition. Finally, discrete wavelet transform and multiresolution analysis are particularly efficient tools to study the Hölderian regularity of a function (see e.g. [31]). The main results binding the regularity of a function and its wavelet coefficients are briefly reviewed at the end of this section.

Let us point out that the anti-Hölderian irregularity condition has also been considered in the measure setting (see e.g. [15]); a review of this measure-based irregularity framework is presented in Section 5.1.

2.1 Weak and strong pointwise Hölderian regularity

We recall first the definition of the Hölderian regularity; this definition naturally leads to a notion of Hölderian irregularity. One will talk about Hölderian and anti-Hölderian functions. Finally, a weaker definition of pointwise smoothness is obtained by negating the condition related to the anti-Hölderian functions.

Definition 1 Let $f : \mathbf{R}^d \to \mathbf{R}^{d'}$ be a locally bounded function, let $x_0 \in \mathbf{R}^d$ and $\alpha \geq 0$; $f \in C^\alpha(x_0)$ if there exist $C, R > 0$ and a polynomial P of degree less than α such that

$$\|f(x) - P(x)\|_{L^\infty(B(x_0,r))} \leq Cr^\alpha, \quad \forall r \leq R. \tag{1}$$

Such a function is said Hölderian of exponent α at x_0. The lower Hölder exponent of f at x_0 is

$$\underline{h}_f(x_0) = \sup\{\alpha : f \in C^\alpha(x_0)\}.$$

A function f is uniformly Hölderian of exponent α ($f \in C^\alpha(\mathbf{R}^d)$) if there exists $C > 0$ such that (1) is satisfied for any $x_0 \in \mathbf{R}^d$ and $R = \infty$; f is uniformly Hölderian if there exists $\varepsilon > 0$ such that $f \in C^\varepsilon(\mathbf{R}^d)$.

Recall that the lower Hölder exponent is simply denoted Hölder exponent in the literature. Since we are interested in introducing another concept of pointwise Hölderian regularity, the accustomed notation h is replaced here by $\underline{h}$

The irregularity of a function can be studied through the notion of anti-Hölderianity. Recall that the finite differences of arbitrary order are defined as follows,

$$\Delta_h^1 f(x) = f(x+h) - f(x), \quad \Delta_h^{n+1} f(x) = \Delta_h^n f(x+h) - \Delta_h^n f(x).$$

We use the following notation,

$$B_h(x_0, r) = \{x : [x, x + ([\alpha] + 1)h] \subset B(x_0, r)\}.$$

Since condition (1) is equivalent to

$$\sup_{|h| \leq r} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(B_h(x_0,r))} \leq Cr^\alpha, \quad \forall r \leq R \tag{2}$$

(see e.g. [20, 18, 35]), the next definition is intuitive.

Definition 2 Let $f : \mathbf{R}^d \to \mathbf{R}^{d'}$ be a locally bounded function, let $x_0 \in \mathbf{R}^d$ and $\alpha \geq 0$; $f \in I^\alpha(x_0)$ if there exist $C, R > 0$ such that

$$\sup_{|h| \leq r} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(B_h(x_0,r))} \geq Cr^\alpha, \quad \forall r \leq R. \tag{3}$$

Such a function is said anti-Hölderian of exponent α at x_0. Let us notice that the Whitney theorem asserts that $I^{\alpha_1}(x_0) \subset I^{\alpha_2}(x_0)$ if $\alpha_1 \leq \alpha_2$ (more precisely, it is a direct consequence of Proposition 1 of [17]). The upper Hölder exponent (or irregularity exponent) of f at x_0 is

$$\overline{h}_f(x_0) = \inf\{\alpha : f \in I^\alpha(x_0)\}.$$

We will say that f is strongly Hölderian of exponent α at x_0 ($f \in C_s^\alpha(x_0)$) if $f \in C^\alpha(x_0) \cap I^\alpha(x_0)$.

It follows from the definitions that if a function f is anti-Hölderian with exponent α, it cannot be Hölderian with exponent β if $\beta > \alpha$. We thus have the following relation between the lower and upper exponents of f: $\underline{h}_f \leq \overline{h}_f$.

The statement (3) is stronger than just negating the Hölderian regularity since such a negation only yields the existence, for any $C > 0$, of a subsequence $(r_n)_n$ (depending on C) for which

$$\sup_{|h| \leq r_n} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(B_h(x_0, r_n))} \geq C r_n^\alpha.$$

We are naturally led to the following definition.

Definition 3 Let $f : \mathbf{R}^d \to \mathbf{R}^{d'}$ be a locally bounded function, let $x_0 \in \mathbf{R}^d$ and $\alpha \geq 0$; $f \in C_w^\alpha(x_0)$ if $f \notin I^\alpha(x_0)$, i.e. for any $C > 0$ there exists a decreasing sequence $(r_n)_n$ such that

$$\sup_{|h| \leq r_n} \|\Delta_h^{[\alpha]+1} f\|_{L^\infty(B_h(x_0, r_n))} \leq C r_n^\alpha, \quad \forall n \in \mathbf{N}. \tag{4}$$

Such a function is said weakly Hölderian of exponent α at x_0.

Roughly speaking, a function is weakly Hölderian of exponent α at x_0 if for any $C > 0$, one can bound the oscillation of f over $B(x_0, r_n)$ by $C r_n^\alpha$ for a remarkable decreasing subsequence $(r_n)_n$ of scales, whereas for an Hölderian function, the oscillation of f over $B(x_0, r)$ has to be bounded at each scale $r > 0$ by $C r^\alpha$, for some $C > 0$.

2.2 Hölderian regularity and wavelet coefficients

Here, we review the wavelet criterion for strong Hölderian regularity and irregularity.

Let us briefly recall some definitions and notations (for more precisions, see e.g. [19, 39, 37]). Under some general assumptions, there exists a function ϕ and $2^d - 1$ functions $(\psi^{(i)})_{1 \leq i < 2^d}$, called wavelets, such that $\{\phi(x-k)\}_{k \in \mathbf{Z}^d} \cup \{\psi^{(i)}(2^j x - k) : 1 \leq i < 2^d, k \in \mathbf{Z}^d, j \in \mathbf{Z}\}$ form an orthogonal basis of $L^2(\mathbf{R}^d)$. Any function $f \in L^2(\mathbf{R}^d)$ can be decomposed as follows,

$$f(x) = \sum_{k \in \mathbf{Z}^d} C_k \phi(x-k) + \sum_{j=1}^{+\infty} \sum_{k \in \mathbf{Z}^d} \sum_{1 \leq i < 2^d} c_{j,k}^{(i)} \psi^{(i)}(2^j x - k),$$

where

$$c_{j,k}^{(i)} = 2^{dj} \int_{\mathbf{R}^d} f(x) \psi^{(i)}(2^j x - k)\, dx,$$

and

$$C_k = \int_{\mathbf{R}^d} f(x) \phi(x-k)\, dx.$$

Let us remark that we do not choose the L^2 normalization for the wavelets, but rather an L^∞ normalization, which is better fitted to the study of the Hölderian regularity. Hereafter, the wavelets are always supposed to belong to C^r with $r > \alpha$ and the functions $\{\partial^s \phi\}_{|s|\leq r}$, $\{\partial^s \psi^{(i)}\}_{|s|\leq r}$ are assumed to have fast decay.

A dyadic cube of scale j is a cube of the form

$$\lambda = [\frac{k_1}{2^j}, \frac{k_1+1}{2^j}) \times \cdots \times [\frac{k_d}{2^j}, \frac{k_d+1}{2^j}),$$

where $k = (k_1, \ldots, k_d) \in \mathbf{Z}^d$. In the sequel, we denote $|\lambda|$ the scale of a dyadic cube $|\lambda|$. From now on, wavelets and wavelet coefficients will be indexed with dyadic cubes λ. Since i takes $2^d - 1$ values, we can assume that it takes values in $\{0,1\}^d - (0, \ldots, 0)$; we will use the following notations:

- $\lambda = \lambda(i,j,k) = \frac{k}{2^j} + \frac{i}{2^{j+1}} + [0, \frac{1}{2^{j+1}})^d$,
- $c_\lambda = c_{j,k}^{(i)}$,
- $\psi_\lambda = \psi^{(i)}(2^j x - k)$,
- $e_\lambda = k/2^j$.

The pointwise Hölderian regularity of a function is closely related to the decay rate of its wavelet leaders.

Definition 4 The wavelet leaders are defined by

$$d_\lambda = \sup_{\lambda' \subset \lambda} |c_{\lambda'}|.$$

Two dyadic cubes λ and λ' are adjacent if they are at the same scale and if $\mathrm{dist}(\lambda, \lambda') = 0$. We denote by 3λ the set of 3^d dyadic cubes adjacent to λ and by $\lambda_j(x_0)$ the dyadic cube of side 2^{-j} containing x_0. Then

$$d_j(x_0) = \sup_{\lambda \subset 3\lambda_j(x_0)} d_\lambda.$$

The following theorem (Theorem 1 of [31]) allows to "nearly" characterize the Hölderian regularity by a decay condition on d_j as j goes to infinity.

Theorem 1 *Let $\alpha > 0$; if $f \in C^\alpha(x_0)$, then there exists $C > 0$ such that*

$$d_j(x_0) \leq C 2^{-\alpha j}, \quad \forall j \geq 0. \tag{5}$$

Conversely, if (5) holds and if f is uniformly Hölder, then there exist $C, R > 0$ and a polynomial P of degree less than α such that

$$\|f(x) - P(x)\|_{L^\infty(B(x_0,r))} \leq C r^\alpha \log \frac{1}{r}, \quad \forall r \leq R.$$

To give necessary and sufficient conditions concerning the irregularity, we suppose that the wavelets are compactly supported and belong to $C^{[\alpha]+1}(\mathbf{R}^d)$; such wavelets are constructed in [19]. The result relies on the following lemma.

Lemma 1 *Let $f \in L^\infty_{\mathrm{loc}}(\mathbf{R}^d)$; the two following assertions are then equivalent:*

1. *there exists some $\beta > 1$ such that, for any $C > 0$, there exists a non decreasing sequence of integers (j_n) such that*
$$\sup_{j \le j_n} \{2^{j([\alpha]+1)} d_j(x_0)\} \le C \frac{2^{j_n([\alpha]+1-\alpha)}}{j_n^\beta}, \tag{6}$$
2. *there exists some $\beta > 1$ such that, for any $C > 0$, there exists a strictly increasing sequence of integers (j_n) such that, for any λ,*
$$|c_\lambda| \le C(\theta(2^{-|\lambda|}) + \theta(|x_0 - e_\lambda|)), \tag{7}$$
where θ is a non decreasing function such that, if $j \in \{j_n, \dots, j_{n+1} - 1\}$ for some $n \in \mathbf{N}$,
$$\theta(2^{-j}) = \inf(\frac{2^{j_{n+1}([\alpha]+1-\alpha)} 2^{-j([\alpha]+1)}}{j_{n+1}^\beta}, \frac{2^{-j_n\alpha}}{j_n^\beta}).$$

Proof. Let us suppose that Property (7) holds. For any j, we have $\theta(2 \cdot 2^{-j}) \le 2^{[\alpha]+1}\theta(2^{-j})$. Moreover, if $\lambda' \subset 3\lambda_j(x_0)$, one has $|x_0 - e_{\lambda'}| \le 4d 2^{-|\lambda'|} = 2^{-(|\lambda'|-2-\log_2 d)}$. Therefore,

$$\begin{aligned} 2^{j([\alpha]+1)} d_j(x_0) &= 2^{j([\alpha]+1)} \sup_{\lambda' \subset 3\lambda_j(x_0)} |c_{\lambda'}| \\ &\le C 2^{j([\alpha]+1)} \sup_{\lambda' \subset 3\lambda_j(x_0)} \theta(2^{-|\lambda'|})(1 + 2^{(2+\log_2 d)([\alpha]+1)}) \\ &\le C 2^{j([\alpha]+1)} 2^{[\alpha]+1} (1 + 2^{(2+\log_2 d)([\alpha]+1)}) \theta(2^{-j}), \end{aligned}$$

Since $2^{([\alpha]+1)j}\theta(2^{-j}) \le 2^{j_n([\alpha]+1-\alpha)}/j_n^\beta$ for any $j \le j_n$, inequality (6) is satisfied.

Conversely, if inequality (6) holds, we have $d_{j_n}(x_0) \le C 2^{-j_n\alpha}/j_n^\beta$ and therefore, since the sequence $(d_j(x_0))$ is non increasing, $d_j(x_0) \le C 2^{-j_n\alpha}/j_n^\beta$ for any $n \in \mathbf{N}$ and any $j \ge j_n$. Moreover, we have

$$d_j(x_0) \le 2^{-j([\alpha]+1)} \sup_{j' \le j_{n+1}} \{2^{j'([\alpha]+1)} d_{j'}(x_0)\} \le C \frac{2^{j_{n+1}([\alpha]+1-\alpha)}}{j_n^\beta} 2^{-j([\alpha]+1)}$$

for any $n \in \mathbf{N}$, and any $j \le j_{n+1}$. These relations imply that the inequality $d_j(x_0) \le C\theta(2^{-j})$ is valid for any $n \in \mathbf{N}$ and any $j \in \{j_n \cdots, j_{n+1} - 1\}$. Let us now fix λ' and set $j = \sup\{m : \lambda' \subset 3\lambda_m(x_0)\}$. By definition of j, one has $|c_{\lambda'}| \le d_j(x_0) \le C\theta(2^{-j})$.

If $j = |\lambda'|$ or if $j = |\lambda'| + 1$, using the fact that θ is non decreasing, one gets

$$|c_{\lambda'}| \le C\theta(2 \cdot 2^{-j}) \le C 2^{[\alpha]+1}\theta(2^{-|\lambda'|}) \le C 2^{[\alpha]+1}(\theta(2^{-|\lambda'|}) + \theta(|x_0 - e_{\lambda'}|)).$$

If $j < |\lambda'|$, $\theta(2^{-j})$ is larger than $\theta(2^{-|\lambda'|})$. Nevertheless, one has $2^{-j-1} \le |x_0 - e_{\lambda'}|$ and thus

$$\begin{aligned} |c_{\lambda'}| &\le d_j(x_0) \le C\theta(2^{-j}) \le C\theta(|x_0 - e_{\lambda'}|) \\ &\le C 2^{[\alpha]+1}(\theta(2^{-|\lambda'|}) + \theta(|x_0 - e_{\lambda'}|)). \end{aligned}$$

In any case, property (7) is recovered.
This Lemma is indeed a two-microlocal characterization of Property (6). We need it to prove the following wavelet characterization of pointwise irregularity:

Theorem 2 *Let* $\alpha > 0$ *and* $f \in L^{\infty}_{\text{loc}}(\mathbf{R}^d)$*;*

1. *if there exists* $C > 0$ *such that*
$$d_j(x_0) \geq C 2^{-j\alpha}, \tag{8}$$
for any $j \geq 0$*, then* $f \in I^{\alpha}(x_0)$*,*

2. *conversely, suppose that* f *is uniformly Hölder; if* $f \in I^{\alpha}(x_0)$ *then, for any* $\beta > 1$*, there exists* $C > 0$ *such that*
$$\sup_{j' \leq j} 2^{j'([\alpha]+1)} d_{j'}(x_0) \geq C \frac{2^{j([\alpha]+1-\alpha)}}{j^{\beta}}, \tag{9}$$
for any $j \geq 0$*.*

Proof. The first statement is a direct consequence of Theorem 4 of [17]. Let us prove the second statement by contrapositive. Assume that Property (9) does not hold, which is equivalent to assume that Property (6) is satisfied. We use Lemma 1 to prove that inequality (4) is satisfied for $r_n = C 2^{-j_n}$. Let $x \in \mathbf{R}^d$ such that $[x, x + ([\alpha]+1)h] \subset B(x_0, 2^{-j_n})$. We have
$$\Delta_h^M f(x) = \sum_k C_k \Delta_h^M \phi_k(x) + \sum_\lambda c_\lambda \Delta_h^M \psi_\lambda(x) = \sum_{j \geq 0} \Delta_h^M f_j,$$
where $f_0(x) = \sum_k C_k \phi_k(x)$ and $f_j(x) = \sum_{i,k} c_{j,k}^{(i)} \psi^{(i)}(2^j x - k)$ if $j \geq 1$. Since f is assumed to be uniformly Hölder, there exists some $\varepsilon > 0$ such that
$$\max(\sup_k |C_k|, \sup_j (2^{j\varepsilon} \sup_\lambda |c_\lambda|)) < \infty.$$
Let $\alpha' > \alpha$ and define $J_n = [\frac{\alpha' j_n}{\varepsilon}] + 1$. The proof of Theorem 1 of [31] yields the following inequality
$$\left| \sum_{j \geq J_n} \Delta_h^{[\alpha]+1} f_j(x) \right| \leq C 2^{-J_n \varepsilon_0} \leq C 2^{-j_n \alpha'}$$
for n sufficiently large.

Since the wavelets are assumed to be compactly supported, there exists ℓ_0 such that $\operatorname{supp}(\psi^{(i)}) \subset (-2^{\ell_0}, 2^{\ell_0})$. Let us give an upper bound of $\sum_{j=0}^{j_n+\ell_0} \Delta_h^{[\alpha]+1} f_j(x)$. If $|x_0 - e_\lambda| \geq 2^{-j+\ell_0+1}$ then $|x - e_\lambda| \geq 2^{-j+\ell_0}$ and $\psi_\lambda(x) = 0$. Therefore
$$\sum_{j=0}^{j_n+\ell_0} \Delta_h^{[\alpha]+1} f_j(x) = \sum_{j=0}^{j_n+\ell_0} \sum_{\lambda,\, |x_0 - e_\lambda| \leq 2^{-j+\ell_0+1}} c_\lambda \Delta_h^{[\alpha]+1} \psi_\lambda(x).$$

The regularity of the wavelets implies that $\sum_{j=0}^{j_n+\ell_0}\sum_{\lambda,\,|x_0-\mu_\lambda|\le 2^{-j+\ell_0+1}} c_\lambda\psi_\lambda(x)$ belongs to $C^{[\alpha]+1}(\mathbf{R}^d)$. Hence

$$\begin{aligned}
&|\sum_{j=0}^{j_n+\ell_0}\sum_{\lambda,\,|x_0-e_\lambda|\le 2^{-j+\ell_0+1}} c_\lambda\Delta_h^{[\alpha]+1}\psi_\lambda(x)| \\
&\le\ |h|^{[\alpha]+1}\sum_{j=0}^{j_n+\ell_0}\|\sum_{\lambda,\,|x_0-e_\lambda|\le 2^{-j+\ell_0+1}} c_\lambda\psi_\lambda(x)\|_{C^{[\alpha]+1}(\mathbf{R}^d)}
\end{aligned}$$

We now use Lemma 1 and the wavelet characterization of the spaces $C^{[\alpha]+1}(\mathbf{R}^d)$ to deduce that

$$\begin{aligned}
&\|\sum_{\lambda,\,|x_0-e_\lambda|\le 2^{-j+\ell_0+1}} c_\lambda\psi_\lambda(x)\|_{C^{[\alpha]+1}(\mathbf{R}^d)} \\
&\le\ \sup_{\lambda,\,|x_0-e_\lambda|\le 2^{-j+\ell_0+1}}(2^{j[\alpha]+1}|c_\lambda|) \\
&\le\ C2^{j[\alpha]+1}(1+2^{(\ell_0+1)([\alpha]+1)})\theta(2^{-j})).
\end{aligned}$$

This leads to the following upper bound,

$$\begin{aligned}
|\sum_{j=0}^{j_n+\ell_0}\Delta_h^{[\alpha]+1}f_j(x)| &\le |h|^{[\alpha]+1}C(1+2^{(\ell_0+1)([\alpha]+1)})\sum_{j=0}^{j_n}2^{j([\alpha]+1)}\theta(2^{-j}) \\
&= |h|^{[\alpha]+1}C(1+2^{(\ell_0+1)([\alpha]+1)})2^{j_n([\alpha]+1-\alpha)} \\
&\le C(1+2^{(\ell_0+1)([\alpha]+1)})2^{-j_n\alpha}.
\end{aligned}$$

Let us now give an upper bound of $\sum_{j=j_n+\ell_0}^{J_n}\Delta_h^{[\alpha]+1}f_j(x)$. In that case, let us remark that if $|x_0-e_\lambda|\ge 2^{-j_n+1}$ then $|x-e_\lambda|\ge 2^{-j+\ell_0+1}$ and $\psi_\lambda(x)=0$. We have

$$\begin{aligned}
&|\sum_{j=j_n+\ell_0}^{J_n}\Delta_h^{[\alpha]+1}f_j(x)| \\
&=\ |\sum_{j=j_n+\ell_0+1}\sum_{\lambda,\,|x_0-e_\lambda|\le 2^{-j_n+1}} c_\lambda\Delta_h^{[\alpha]+1}\psi_\lambda(x)| \\
&\le\ ([\alpha]+1)\sum_{j=j_n+\ell_0+1}^{J_n}\sum_{\lambda,\,|x_0-e_\lambda|\le 2^{-j_n+1}}|c_\lambda|\sup_j\sup_{x\in\mathbf{R}^d}\sum_{|\lambda|=2^{-j}}|\psi_\lambda(x)| \\
&\le\ C([\alpha]+1)\left(\sup_j\sup_{x\in\mathbf{R}^d}\sum_{|\lambda|=2^{-j}}|\psi_\lambda(x)|\right)\sum_{j=j_n+\ell_0+1}^{J_n}\sum_{\lambda,\,|x_0-e_\lambda|\le 2^{-j_n+1}}\theta(2^{-j_n}) \\
&\le\ C([\alpha]+1)\left(\sup_j\sup_{x\in\mathbf{R}^d}\sum_{|\lambda|=2^{-j}}|\psi_\lambda(x)|\right)J_n\frac{2^{-j_n\alpha}}{j_n^\beta}
\end{aligned}$$

$$\leq \quad C([\alpha]+1)\left(\sup_j \sup_{x\in\mathbf{R}^d} \sum_{|\lambda|=2^{-j}} |\psi_\lambda(x)|\right) 2^{-j_n\alpha}$$

for n sufficiently large. Gathering these relations, we obtain $f \in C^\alpha_w(x_0)$.

Note that we do not have a wavelet characterization of the property $\overline{h}_f(x_0) = \alpha$. Indeed, it is proved in [18] that, even up to a logarithmic correction, neither condition (8) is necessary, nor condition (9) is sufficient. Nevertheless, one can characterize the stronger property $\underline{h}_f(x_0) = \overline{h}_f(x_0) = \alpha$ using wavelets. Indeed, Theorems 1 and 2 lead to the following corollary that we will use in the sequel.

Corollary 1 *Let $\alpha > 0$ and suppose that f is uniformly Hölder. We have $\underline{h}_f(x_0) = \overline{h}_f(x_0) = \alpha$ if and only if*

$$\lim_{j\to\infty} \frac{\log d_j(x_0)}{-j\log 2} = \alpha.$$

Proof. The first point of Theorem 2 implies that if $\lim_j \log d_j(x_0)/-j\log 2 = \alpha$, we have $\underline{h}_f(x_0) = \overline{h}_f(x_0) = \alpha$. Let us prove the converse result. Assume that for any $\varepsilon > 0$, we have $f \in C^{\alpha-\varepsilon}(x_0) \cap I^{\alpha+\varepsilon}(x_0)$. For any $\beta > 1$, the preceding Theorems imply the existence of a constant $C > 0$ such that $d_j(x_0) \leq C2^{-j(\alpha-\varepsilon)}$ and $\sup_{j'\leq j} 2^{j'([\alpha]+1)} d_{j'}(x_0) \geq 2^{j([\alpha]+1-\alpha-\varepsilon)} j^{-\beta}/C$, for any $j \geq 0$. Let $a > \varepsilon + 2\varepsilon/([\alpha]+1-\alpha+\varepsilon)$ and $b = (\beta+\varepsilon)/([\alpha]+1-\alpha+\varepsilon)$. The previous relations lead to

$$\sup_{j'\leq j(1-a)-b\log_2 j} 2^{j'([\alpha]+1)} d_{j'}(x_0) < C\frac{2^{j([\alpha]+1-\alpha-\varepsilon)} - j\varepsilon}{j^\beta} < C^{-1}\frac{2^{j([\alpha]+1-\alpha-\varepsilon)}}{j^\beta}$$

and

$$\sup_{j(1-a)-b\log_2 j\leq j'\leq j} 2^{j'([\alpha]+1)} d_{j'}(x_0) \geq C^{-1}\frac{2^{j([\alpha]+1-\alpha-\varepsilon)}}{j^\beta} \geq C^{-1}\frac{2^{j([\alpha]+1-\alpha-\varepsilon)}}{j^\beta},$$

for any $j \geq 2\log_2(C)/\varepsilon$. Since the sequence $(d_j(x_0))$ is non increasing, we have

$$2^{j([\alpha]+1)} d_{j(1-a)-b\log_2 j}(x_0) \geq C^{-1}\frac{2^{j([\alpha]+1-\alpha-\varepsilon)}}{j^\beta},$$

for any $j \geq 2\log_2(C)/\varepsilon$. By setting $\ell = j(1-a) - b\log_2 j$, the preceding relation can be rewritten

$$2^{\frac{1}{1-a}(\ell+2b\log_2 \ell)([\alpha]+1)} d_\ell(x_0) \geq C^{-1}\frac{2^{j([\alpha]+1-\alpha-\varepsilon)}}{j^\beta} \geq C^{-1}\frac{2^{\frac{\ell}{1-a}([\alpha]+1-\alpha-\varepsilon)}}{(\frac{\ell}{1-a})^\beta}.$$

for any $\ell \geq 2\log_2(C)(1-a)/\varepsilon$. Using the relation $d_j(x_0) \leq C2^{-j(\alpha-\varepsilon)}$, we then obtain

$$\alpha - \varepsilon \leq \liminf_{j\to\infty} \frac{\log d_j(x_0)}{-j\log 2} \leq \limsup_{j\to\infty} \frac{\log d_j(x_0)}{-j\log 2} < \alpha + \varepsilon.$$

Since this inequality holds for any $\varepsilon > 0$, the required result follows.

3 Construction of functions with prescribed lower and upper Hölder exponents

In this Section, we investigate in detail the structure of the irregularity exponent of a continuous function. In Section 3.1, we first prove, considering a Weierstrass-type function, that it is possible to construct a continuous function with prescribed pointwise Hölder exponent H provided that H satisfies "good properties". In Section 3.2 we focus on describing all the functions which are both the classical Hölder exponent and irregularity exponent of a continuous function. Finally, we study the case where classical pointwise Hölder exponent and irregularity exponent may differ. In this special case, we give a sufficient condition and a necessary condition for a couple of functions $(\underline{H}, \overline{H})$ to be respectively the pointwise Hölder exponent and the irregularity exponent of a continuous function.

3.1 A generic Weierstraß function with prescribed Hölder exponents

In the same spirit as in [1], we consider the Weierstraß-type function

$$W(t) = \sum_{j=0}^{+\infty} \lambda^{-jH(t)} \sin(2\pi\lambda^j t). \tag{10}$$

and study its pointwise regularity.

Proposition 1 *Let H be a β-Hölderian function from $[0,1]$ to $[a,b] \subset (0,1]$, satisfying $\sup_{t\in[0,1]} |H(t)| < \beta$. If W is a function of the form (10), where λ is an integer larger than 1, then*

$$W \in C_s^{H(t)}(t) = C^{H(t)}(t) \cap I^{H(t)}(t), \quad \forall t \in [0,1].$$

The proof of this proposition relies on the two following lemma, analogous to Lemma 14 and Proposition 15 of [33].

Lemma 2 *Let $\lambda > 1$ and $(f_j)_{j\in\mathbf{N}}$ a sequence of bounded and Lipschitz functions on $\mathbf{R}$ for which there exists $C > 0$ such that*

$$\|f_j\|_\infty + \|f_j'\|_\infty \leq C.$$

The function $f(t) = \sum_{j=0}^{+\infty} \lambda^{-jH(t)} f_j(\lambda^j t)$ belongs to $C^{H(t)}(t)$, for any $t \in [0,1]$.

Proof. Let $t \in [0,1]$. For any j, $|f_j(t) - f_j(s)|$ is bounded by $C|s-t|$ or C. Let $j_0(t) = [-\log|s-t|/(H(t)\log\lambda)]$. We have

$$|f(t) - f(s)| \quad \leq \quad C \sum_{j=0}^{j_0-1} |\lambda^j t - \lambda^j s| \lambda^{-jH(t)} + C \sum_{j\geq j_0} \lambda^{-jH(t)}$$

$$
\begin{aligned}
&+C\sum_{j=0}^{+\infty}|\lambda^{-jH(t)}-\lambda^{-jH(s)}| \\
\leq\ & C|t-s|C\lambda^{j_0(1-H(t))}+C\lambda^{-j_0H(t)} \\
&+C|\log\lambda||t-s|^{\beta}\sum_{j=0}^{+\infty}j\lambda^{-ja} \\
\leq\ & C|t-s|^{H(t)}
\end{aligned}
$$

using the mean value theorem, β-Hölderianity of H and the fact that $H([0,1])\subset(0,1]$.

Lemma 3 *Let $(f_j)_{j\in\mathbf{N}}$ be a sequence of 1-periodic C-Lipschitz functions from* $\mathbf{R}$ *to* $\mathbf{R}$ *and*

$$f(t)=\sum_{j=0}^{\infty}\lambda^{-jH(t)}f_j(\lambda^j t),$$

where λ is an integer larger than 1. Assume that there exists $\ell\in\{-\lambda-1,\cdots,\lambda-1\}$ and an integer J such that

$$D=\inf_{j\geq J}|f_j(\frac{\ell}{\lambda})-f_j(0)|>0.$$

If $\dfrac{C\ell}{\lambda(\lambda-1)}\leq D$ *then*

$$f\in I^{H(t)}(t),\quad\forall t\in[0,1].$$

Proof. Since

$$f(t)-f(s)=\sum_{j=0}^{\infty}\lambda^{-jH(t)}(f_j(\lambda^j t)-f_j(\lambda^j s))+\sum_{j=0}^{\infty}(\lambda^{-jH(t)}-\lambda^{-jH(s)})f_j(s)$$

Proposition 15 of [33] yields that for some $C_0>0$,

$$|\sum_{j=0}^{\infty}\lambda^{-jH(t)}(f_j(\lambda^j t)-f_j(\lambda^j s))|\geq C_0|t-s|^{H(t)}.$$

Moreover

$$|\sum_{j=0}^{\infty}(\lambda^{-jH(t)}-\lambda^{-jH(s)})f_j(s)|\leq|t-s|^{\beta}$$

with $\beta>H(t)$. It provides the required conclusion.
By applying Lemma 2 and Lemma 3 to $f_j=sin(2\pi\cdot)$, $C=1$, $\ell=[\lambda/4]$ and $D=|\sin(\ell/\lambda)|$, Proposition 1 is then straightforward.

3.2 Wavelet series-defined functions with similar lower and upper Hölder exponents at any point

The aim of this section is to prove a result analogous to Theorem 1 of [1]. Here we extend the results stated in [1] since we give a characterization of functions which are both the lower and the upper exponent of a continuous function and thus satisfy a stronger property than in Theorem 1 of [1].

Theorem 3 *Let f a continuous nowhere differentiable function defined on $[0,1]$ with similar lower and upper Hölder exponent at any point. There exists a sequence of continuous functions $(H_j)_{j\in\mathbf{N}}$ from $[0,1]$ to $[0,1]$ such that*

$$\underline{h}(t) = \overline{h}(t) = \lim_{j\to\infty} H_j(t) \quad \forall t.$$

Conversely, if H is a function from $[0,1]$ to $[0,1]$ such that

$$H(t) = \lim_{j\to\infty} H_j(t),$$

where the $(H_j)_{j\in\mathbf{N}}$ are continuous functions, then there exists a continuous function f defined on $[0,1]$ such that

$$\underline{h}(t) = \overline{h}(t) = H(t), \quad \forall t.$$

The first part of Theorem 3 is straightforward. If one sets

$$\omega_x(t) = \sup_{|h|\le t} |f(x+h) - f(x)|,$$

the sequence $(H_j)_{j\in\mathbf{N}}$ defined by

$$H_j(t) = \frac{\log(\omega_t(2^{-j}) + 2^{-2j})}{-j\log 2}$$

satisfies the required conditions, since

$$\underline{h}(t) = \overline{h}(t) = \lim_{j\to\infty} \frac{\log \omega_t(2^{-j})}{-j\log 2}.$$

Let us prove the converse assertion by means of wavelet series. We will need the following wavelet criterium for the pointwise regularity (see [1, 31]):

Proposition 2 *Let $\alpha > 0$ and assume that there exists $C > 0$ such that for any $j \in \mathbf{N}$,*

$$\sup_k |c_{j,k}| \le C 2^{-\frac{j}{\log j}}, \tag{11}$$

and

$$d_j(t) \le C 2^{-j\alpha}.$$

Then for any $\varepsilon > 0$, $f \in C^{\alpha-\varepsilon}(t)$.

Remark 1 This Proposition is a reformulation of Proposition 4 of [1] in terms of wavelet coefficients. Let us point out that in [1], f is expanded in the Schauder basis. Then for any J (Lemma 1 of [1]),

$$\sum_{j\leq J}\sum_{k} c_{j,k}\Lambda_{j,k}$$

is a continuous piecewise affine function coinciding with f on dyadic numbers. In the case of wavelet basis, this property does not hold. In order to prove Proposition 2, we thus need to assume (11) and use different arguments.

Proof. Since $\sup_k |c_{j,k}| \leq C2^{-\frac{j}{\log j}}$, the wavelet series converge uniformly on any compact. Let $(f_j)_{j\geq -1}$ be defined as follows,

$$f_{-1}(x) = \sum_k C_k\varphi(x-k), \quad f_j(x) = \sum_k c_{j,k}\psi_{j,k}(x) \quad (\forall j \geq 0)$$

As in [31], if $|\beta| \leq [\alpha]$, the series $\sum_j \partial^\beta f_j$ converges absolutely. Now, for any $j \geq -1$, let us define

$$P_{j,t}(x) = \sum_{\beta\leq[\alpha]} \frac{(x-t)^\beta}{\beta!}\partial f_j(t).$$

If j_0 is the number such that

$$2^{-j_0-1} \leq |x-t| \leq 2^{-j_0},$$

and j_1 satisfies $2^{-\frac{j_1}{\log j_1}} \leq 2^{-\alpha j_0}$, then as proved in [31],

$$\sum_{j\leq j_0} |f_j(x) - P_{j,t}(x)| \leq Cj_0|x-t|^\alpha$$

and

$$\sum_{j\geq j_0} |P_{j,t}(x)| \leq C|x-t|^\alpha.$$

Moreover,

$$\sum_{j=j_0}^{j_1} |f_j(x)| \leq Cj_1|x-t|^\alpha.$$

Since for any ε and j_0 sufficiently large,

$$j_1 \leq C_\varepsilon 2^{\varepsilon j_0},$$

one has

$$\sum_{j=j_0}^{j_1} |f_j(x)| \leq C_\varepsilon|x-t|^{\alpha-\varepsilon}.$$

Since

$$\sum_{j\geq j_1}|f_j(x)|\leq C\sum_{j\geq j_1}2^{-\frac{j}{\log j}}\leq C\log^2(j_1)2^{-\frac{j_1}{\log j_1}}\leq |x-t|^{\alpha-\varepsilon},$$

the proposition follows.
We will also use a slightly modified version of Lemma 2 of [1]:

Proposition 3 *Let H a function from $[0,1]$ to $[0,1]$ such that $H(t)=\lim_j H_j(t)$, where $(H_j)_{j\in\mathbf{N}}$ is a sequence of continuous functions. There exists a sequence $(P_j)_{j\in\mathbf{N}}$ of polynomials such that*

$$\begin{cases} H(t)=\lim\limits_{j\to\infty}P_j(t), & \forall t\in[0,1],\\ \|P_j'\|_\infty\leq j, & \forall j\in\mathbf{N}.\end{cases} \tag{12}$$

We can now define a wavelet series with the desired properties:

Proposition 4 *Let H be a function from $[0,1]$ to $[0,1]$ such that $H(t)=\lim_j H_j(t)$, where the $(H_j)_{j\in\mathbf{N}}$ is a sequence of continuous functions. Let $(P_j)_{j\in\mathbf{N}}$ be a sequence of polynomials satisfying the relations (12). For any $(j,k)\in\mathbf{N}\times\{0,\cdots,2^j-1\}$, set*

$$H_{j,k}=\max(\frac{1}{\log j},P_j(\frac{k}{2^j})).$$

The function f defined as

$$f(x)=\sum_{j\in\mathbf{N}}\sum_{k=0}^{2^j-1}2^{-jH_{j,k}}\psi_{j,k}(x). \tag{13}$$

satisfies the following relations,

$$\underline{h}(t)=\overline{h}(t)=H(t),\quad\forall t\in[0,1].$$

Let us look at a particular case.

Remark 2 If H is a continuous function, the wavelet series

$$\sum_{j\in\mathbf{N}}\sum_{k=0}^{2^j-1}2^{-jH_{j,k}}\psi_{j,k}(x),$$

with $H_{j,k}=\max(1/\log j,H(2^{-j}k))$ has H both as lower and upper Hölder exponents.
Proof. If $\lambda'=\lambda'(j',k')\subset 3\lambda_j(t)$, then

$$|\frac{k'}{2^{j'}}-t|\leq C_1 2^{-j}.$$

Since the sequence $(P_j)_{j\in\mathbf{N}}$ satisfies equalities (12), for any $\varepsilon > 0$, there exists an integer j_0 such that for any $j \geq j_0$ and $\lambda' \subset 3\lambda_j(t)$,

$$|H_{j',k'} - H(t)| \leq |P'_j(t) - P'_j(\frac{k'}{2^{j'}})| + |H(t) - P'_j(t)| \leq j2^{-j} + |H(t) - P'_j(t)| \leq \varepsilon.$$

Then for any $j \geq j_0$,

$$\max(2^{-\frac{j'}{\log j'}}, 2^{-j'\varepsilon}2^{-j'H(t)}) \leq |c'_\lambda| \leq \max(2^{-\frac{j'}{\log j'}}, 2^{j'\varepsilon}2^{-j'H(t)}).$$

We deduce that

$$\lim_j \frac{\log d_j(t)}{-j\log 2} = H(t), \quad \forall t \in [0,1]$$

and therefore

$$\overline{h}(t) = \underline{h}(t) = H(t), \quad \forall t \in [0,1].$$

3.3 Wavelet series-defined functions with different lower and upper Hölder exponents

Our main goal is to prove the following Theorem.

Theorem 4 *Let f a continuous nowhere differentiable function defined on $[0,1]$. There exists a sequence of continuous functions $(H_j)_{j\in\mathbf{N}}$ such that*

$$\underline{h}(t) = \liminf_{j\to\infty} H_j(t), \quad \overline{h}(t) = \limsup_{j\to\infty} H_j(t), \quad \forall t \in [0,1].$$

Conversely, let $(\underline{H}, \overline{H})$ a couple of functions from $[0,1]$ to $[a,b] \subset (0,1)$ such that

$$\underline{H}(t) = \liminf_{j\to\infty} H_j(t), \quad \overline{H}(t) = \limsup_{j\to\infty} H_j(t),$$

where the $(H_j)_{j\in\mathbf{N}}$ is a sequence of continuous functions. There exists a uniform Hölder function f from $[0,1]$ to $\mathbf{R}$ such that

$$\underline{h}(t) = \underline{H}(t) \leq \overline{H}(t) = \overline{h}(t), \quad \forall t \in [0,1].$$

Remark 3 The second assertion of Theorem 4 is much weaker than the corresponding one of Theorem 3. In order to ensure the existence of a function f with prescribed lower and upper exponents at any point, we need stronger assumptions on $\underline{H}$ and $\overline{H}$: indeed we assume that these functions take values in $[a,b] \subset (0,1)$ (and not in $(0,1)$).

Remark 4 Let $\underline{H}$ and $\overline{H}$ be two continuous functions from $[0,1]$ to $[a,b] \subset (0,1)$ satisfying

$$\forall t \in [0,1],\ \underline{H}(t) \leq \overline{H}(t).$$

Define the sequence $(H_j)_{j\in\mathbf{N}}$ as

$$H_{2j} = \underline{H}, \quad H_{2j+1} = \overline{H}, \quad \forall j \in \mathbf{N}.$$

Theorem 4 provides the existence of a function f with lower and upper exponent $\underline{H}$ and $\overline{H}$ at any point.

The proof of the direct part of Theorem 4 is exactly similar to this of Theorem 3 and is left to the reader. In order to prove the converse assertion we first need the following lemma.

Lemma 4 *Let $(\underline{H}, \overline{H})$ a couple of functions defined from $[0,1]$ to $[a,b] \subset (0,1)$ such that*

$$\underline{H}(t) = \liminf_{j\to\infty} H_j(t), \quad \overline{H}(t) = \limsup_{j\to\infty} H_j(t),$$

where the H_j are continuous functions from $[0,1]$ to $[0,1]$. For any $(a',b') \in (0,1)^2$ such that $a' < a < b < b'$, there exists a sequence $(P_j)_{j\in\mathbf{N}}$ of polynomials from $[0,1]$ to $[a',b']$ and for any $t \in [0,1]$, there exists a strictly increasing sequence of integers $(j_n(t))_{n\in\mathbf{N}}$ depending on t such that the three following properties hold simultaneously

- $\forall t \in [0,1]$,

$$\underline{H}(t) = \liminf_{j\to\infty} P_j(t), \quad \overline{H}(t) = \limsup_{j\to\infty} P_j(t), \tag{14}$$

- $\forall j \in \mathbf{N}$,

$$\|P_j'\|_\infty \le j, \tag{15}$$

- $\forall t \in [0,1]$ $\forall n \in \mathbf{N}$, $\forall j \in \{j_n(t), \cdots, j_{n+1}(t)-1\}$,

$$jP_j(t) \ge \sup(j_n(t)(\overline{H}(t)-\varepsilon), j + j_{n+1}(t)(\overline{H}(t)-\varepsilon-1)). \tag{16}$$

Proof. Lemma 2 of [1] implies that there exists a sequence of polynomials $(Q_\ell)_{j\in\mathbf{N}}$ such that Conditions (14) and (15) both hold. Moreover in the construction of [1], one may assume

$$\|Q_\ell - H_\ell\|_\infty \le \frac{1}{\ell}.$$

Then, for ℓ sufficiently large, Q_ℓ is onto $[a',b'] \subset (0,1)$. Set $\beta_1 = [b'/a'] + 1$, $\beta_2 = [(1-a')/(1-b')] + 1$, $\beta = \beta_1\beta_2$ and define the sequence $(P_j)_{j\in\mathbf{N}}$ as follows

$$\forall \ell \in \mathbf{N},\ \forall \beta^\ell + 1 \le j \le \beta^{\ell+1}, \quad P_j = Q_\ell.$$

We now prove that the sequence $(P_j)_{j\in\mathbf{N}}$ satisfies the required properties. Let $\varepsilon > 0$, $t \in [0,1]$, $(\ell_n(t))_{n\in\mathbf{N}}$ a sequence such that

$$Q_{\ell_n(t)} \ge \overline{H}(t) - \varepsilon$$

and set $j_n(t) = \beta^{\ell_n(t)}\beta_2$. For any integer j we distinguish three cases.

- If there exists $n \in \mathbf{N}$ such that $j_n \leq \ell \leq \beta_1 j_n$, then $P_j = Q_{\ell_n} \geq \overline{H} - \varepsilon$. Hence
$$jP_j \geq j(\overline{H} - \varepsilon) \geq j_n(\overline{H} - \varepsilon),$$
and
$$j(1 - P_j) \leq j(1 - \overline{H} + \varepsilon) \leq j\beta_1(1 - \overline{H} + \varepsilon)j_{n+1}(1 - \overline{H} + \varepsilon).$$

- If there exists $n \in \mathbf{N}$ such that $\beta_1 j_n \leq j \leq j_{n+1}/\beta_2$, then $P_j = Q_\ell$ with $\ell_n + 1 \leq \ell \leq \ell_{n+1} - 1$. Then
$$jP_j \geq ja' \geq \frac{j}{\beta_1}\overline{H} \geq j_n\overline{H},$$
and
$$j(1 - P_j) \leq j(1 - a') \leq j\beta_2(1 - \overline{H}) \leq j_{n+1}(1 - \overline{H}).$$

- If there exists $n \in \mathbf{N}$ such that $j_{n+1}/\beta_2 \leq j \leq j_{n+1}$, then $P_j = Q_{j_{n+1}} \geq \overline{H} - \varepsilon$. Hence
$$jP_j \geq j(\overline{H} - \varepsilon) \geq j_n(\overline{H} - \varepsilon),$$
and
$$j(1 - P_j) \leq j(1 - \overline{H} + \varepsilon) \leq j_{n+1}(1 - \overline{H} + \varepsilon).$$

Thus, in any case we obtain the required property.
Now we prove the following proposition

Proposition 5 *Let $(\underline{H}, \overline{H})$ a couple of functions from $[0,1]$ to $[a,b] \subset (0,1)$ such that*
$$\underline{H}(t) = \liminf_{j\to\infty} H_j(t), \quad \overline{H}(t) = \limsup_{j\to\infty} H_j(t),$$
where the $(H_j)_{j\in\mathbf{N}}$ is a sequence of continuous functions. Let $(P_j)_{j\in\mathbf{N}}$ a sequence of polynomials satisfying Properties (14),(15) and (16) and consider the wavelet series defined by
$$f(x) = \sum_{j\in\mathbf{N}} \sum_{k=0}^{2^j-1} 2^{-jP_j(\frac{k}{2^j})}.$$
Then for any $t \in [0,1]$,
$$\underline{h}(t) = \underline{H}(t) < \overline{h}(t) = \overline{H}(t).$$

Proof. The assumption $P_j([0,1]) \subset [a', b'] \subset (0,1)$ implies that f is uniform Hölder. Since for any j, P_j satisfies Properties (14), (15) and (16), for any $\lambda \in 3\lambda_j(x_0)$, we have
$$2^{-jP_j(t)}2^{-j^2 2^{-j}} \leq |c_\lambda| \leq 2^{-jP_j(t)}2^{j^2 2^{-j}}.$$

Thus, for any $\varepsilon > 0$, there exists j_0 sufficiently large such that

$$\forall j \geq j_0,\ 2^{-jP_j(t)}2^{-\varepsilon j} \leq d_j(x_0) \leq 2^{-jP_j(t)}2^{\varepsilon j}.$$

By definition of $\underline{H}$,

$$\liminf_{j\to\infty} \frac{\log d_j(t)}{-\log 2} = \liminf_{j\to\infty} P_j(t) = \underline{H}(t).$$

Hence,

$$\underline{h}(t) = \underline{H}(t).$$

In the same way,

$$\overline{h}(t) \leq \overline{H}(t).$$

We now use Properties (14), (15) and (16). There exists a strictly increasing sequence of integers $(j_n)_{n\in\mathbf{N}}$ such that (14), (15) and (16) hold. Then, $\forall n \in \mathbf{N}$, $\forall j \in \{j_n, \cdots, j_{n+1}\}$,

$$d_j(t) \leq 2^{-jP_j(t)}2^{\varepsilon j} \leq \inf(2^{-j_n(\overline{H}(t)-\varepsilon)}, 2^{-j_{n+1}(\overline{H}(t)-\varepsilon-1)-j}).$$

The wavelet criteria then provides

$$\overline{h}(t) \geq \overline{H}(t).$$

4 Weak multifractal formalism

The aim of the multifractal analysis is to study "irregularly irregular" functions, i.e. functions whose Hölder exponent can jump from point to point. From a practical point of view, the numerical computation of the pointwise Hölder exponent of a signal is completely instable, and is indeed quite meaningless, especially for signals whose pointwise Hölder exponent can take very different values. Leaving this utopian view, one rather wishes to get global informations about the pointwise regularity: What are the values taken by the Hölder exponent? What is the "size" of the set of points E_h where the Hölder exponent takes a given value h? First of all, one has to define this notion of size. Since the sets under consideration can be dense or negligible, by "size", we cannot mean "Lebesgue measure". The "fractal dimensions" are more fitted for this purpose. Once the right definition of dimension has been chosen, one still has to determine the spectrum of singularities of the function, i.e. the dimension of the sets E_h. This is the purpose of the multifractal formalism. Naturally, all these definitions can be transposed for the upper Hölder exponent.

4.1 A notion of dimension

In multifractal analysis, the notion of dimension which is mainly used is the Hausdorff dimension. We recall here its definition.

The Hausdorff dimension is defined through the Hausdorff measure (see [22] for more details). The best covering of a set $E \subset \mathbf{R}^d$ with sets subordinated to a diameter ε can be estimated as follows,

$$\mathcal{H}^\delta_\varepsilon(E) = \inf\{\sum_{i=1}^{\infty} |E_i|^\delta : E \subset \bigcup_{i=1}^{\infty} E_i, |E_i| \leq \varepsilon\},$$

where for any i, $|E_i|$ denotes the diameter of E_i.
Clearly, $\mathcal{H}^\delta_\varepsilon$ is an outer measure. The Hausdorff measure is defined from $\mathcal{H}^\delta_\varepsilon$ as ε goes to 0.

Definition 5 The outer measure $\mathcal{H}^\delta$ defined as

$$\mathcal{H}^\delta(E) = \sup_{\varepsilon > 0} \mathcal{H}^\delta_\varepsilon(E)$$

is a metric outer measure. Its restriction to the σ-algebra of the $\mathcal{H}^\delta$-measurable sets defines the Hausdorff measure of dimension δ.

Since the outer measure $\mathcal{H}^\delta$ is metric, the algebra includes the Borelian sets. The Hausdorff measure is decreasing as δ goes to infinity. Moreover, $\mathcal{H}^\delta(E) > 0$ implies $\mathcal{H}^{\delta'}(E) = \infty$ if $\delta' < \delta$. The following definition is thus meaningful.

Definition 6 The Hausdorff dimension $\dim_{\mathcal{H}}(E)$ of a set $E \subset \mathbf{R}^d$ is defined as follows

$$\dim_{\mathcal{H}}(E) = \sup\{\delta : \mathcal{H}^\delta(E) = \infty\}.$$

With this definition, $\dim_{\mathcal{H}}(\emptyset) = -\infty$.

4.2 From the strong multifractal formalism to the weak multifractal formalism

We first review the wavelet leaders based multifractal formalism as defined by Jaffard [31], which is one of the two methods allowing to recover, in some particular cases, the entire spectrum of singularities (the second one is the wavelet transform of the maxima of the modulus method, introduced by Arneodo and his collaborators [6]). Other multifractal formalisms only give, at best, the increasing part of the spectrum (see [30]). These considerations on the strong Hölderian regularity can be transposed to the weak one.

The lower spectrum of singularities allows to characterize globally the regularity of a function through the lower Hölder exponents.

Definition 7 Let f be a locally bounded function; its lower isoHölder sets are the sets

$$\underline{E}_H = \{x : \underline{h}(x) = H\}.$$

The lower spectrum of singularities of f is the function

$$\underline{d} : \mathbf{R}^+ \cup \{\infty\} \to \mathbf{R}^+ \cup \{-\infty\} \quad H \mapsto \dim_{\mathcal{H}}(\underline{E}_H),$$

It is not always possible to compute the lower spectrum of singularities of a function. A multifractal formalism is a method that is expected to yield the function $\underline{d}$ through the use of a Legendre transform. These formalisms are variants of a seminal derivation which was proposed by Parisi and Frisch [45]. The wavelet leaders method (WLM) uses wavelet coefficients instead of L^p norms, which are meaningless for negative values of p. The partition function is defined as follows

$$S(j,p) = 2^{-j} \sum_{\lambda:|\lambda|=2^{-dj}} d_\lambda^p.$$

By setting,

$$\omega(p) = \liminf_{j\to\infty} \frac{\log S(j,p)}{\log 2^{-j}}, \tag{17}$$

the spectrum of singularities $\underline{d}(h)$ is expected to be equal to

$$\inf_p \{hp - \omega(p) + d\}. \tag{18}$$

The heuristic argument leading to the previous method is the following. The contribution of the dyadic cubes of side 2^{-j} containing a point whose lower Hölder exponent is h to the sum $\sum d_\lambda^p$ can be estimated as follows. By Theorem 1, the lower Hölder exponent $\underline{h}(x)$ of a function at x is

$$\underline{h}(x) = \liminf_{\substack{j\to\infty \\ \lambda' \subset 3\lambda_j(x)}} \frac{\log d_{\lambda'}}{\log 2^{-j}},$$

which allows us to write $d_\lambda \sim 2^{-hj}$. Moreover, the number of these dyadic intervals should be about $2^{d(h)j}$, each of volume 2^{-dj}. Hence, the contribution is $2^{(d(h)-d-hp)j}$. The dominating contribution is the one corresponding to the value h associated with the biggest exponent; by writing the equality (17) as $\sum d_\lambda^p \sim 2^{-\omega(p)j}$, one can expect the following relation, $-\omega(p) = \sup_h\{d(h) - d - hp\}$. As $-\omega$ is a convex function, if d is concave, then $-\omega$ and $-d$ are convex conjugate functions, so that $d(h) = \inf_p\{hp - \omega(p) + d\}$. Let us remark that the preceding argument is far from being a mathematical proof; see [3] and [33] for a comparison between the WLM and other multifractal formalisms.

Although it can be shown that formula (18) allows to recover the spectrum of singularities under additional assumptions (see [29, 30, 5] for instance), the validity does not hold in complete generality. Indeed, the only result valid in the general case is the following inequality [30, 31],

$$\underline{d}(h) \leq \inf_{p\in\mathbf{R}_*} \{hp - \omega(p) + d\}. \tag{19}$$

Once the lower spectrum of singularities has been introduced, the upper spectrum of singularities can be defined in a totally analogous way. A relation similar to the inequality (19) holds.

The weak multifractal formalism is defined as follows.

Definition 8 Let f be a locally bounded function; its upper isoHölder sets are the sets

$$\overline{E}_H = \{x : \overline{h}(x) = H\}.$$

The upper spectrum of singularities of f is the function

$$\overline{d} : \mathbf{R}^+ \cup \{\infty\} \to \mathbf{R}^+ \cup \{-\infty\} \quad H \mapsto \dim_{\mathcal{H}}(\overline{E}_H).$$

The following theorem which can be found in [4] gives an upper bound for the upper spectrum of singularities.

Theorem 5 *Let f a uniform Hölder function. The following inequality holds*

$$\overline{d}(h) \leq \inf_{p \in \mathbf{R}^*} \{hp - \omega(p) + d\}. \tag{20}$$

Definition 9 Let f a uniform Hölder function and $h > 0$; if (20) is an equality, i.e.

$$\forall h > 0, \quad \overline{d}(h) = \inf_{p \in \mathbf{R}^*} \{hp - \omega(p) + d\}$$

then function f is said to obey the weak multifractal formalism.

5 Construction of a class of wavelet series obeying the weak multifractal formalism

The aim of this Section is to exhibit a class of multifractal functions for pointwise irregularity. This question is in fact a non trivial one. A quite natural approach to solve this problem is to consider multifractal functions for the usual pointwise regularity.

Let us point out that if we want to define wavelet series that are multifractal both for the strong and weak Hölderian regularity point of views, we have to take into account that, except in the case where the lower and upper exponents coincide, there is no wavelet criteria for the pointwise irregularity.

In the same spirit as Barral and Seuret in [14], we will define wavelets series built from a multifractal measure μ on $[0,1[^d$ in the following way,

$$F_\mu(x) = \sum_{j \geq 0} \sum_{\substack{|\lambda| = 2^{-j}, \\ \lambda \subset [0,1]^d}} 2^{-j(s_0 - \frac{d}{p_0})} \mu(\lambda)^{\frac{1}{p_0}} \psi_\lambda(x), \tag{21}$$

$\forall x \in [0,1]^d$, where the wavelets ψ^i belongs to the Schwartz class on $\mathbf{R}^d$ with all moments vanishing. This class of examples also proves that upper and lower spectra may coincide: Under specific assumptions detailed in section 5.3, we can obtain a class of functions obeying both the strong and the weak multifractal formalisms.

We begin by recalling some basic facts about the multifractal analysis of measures.

5.1 Some results about multifractal analysis of measures

Following Barral and Seuret, we adapt here the usual multifractal formalism of [15]. The main difference lies in the definition of the isoHölder sets, since we just need a multifractal formalism associated with a dyadic grid.
We first give some slightly modified versions of the usual definitions of lower and upper exponents of a given Borel measure μ at a point x_0. For any $\sigma \in \{-1,0,1\}^d$ and any dyadic cube $\lambda = \prod_\ell [\frac{k_\ell}{2^j}, \frac{k_\ell}{2^j+1}[$, let us set

$$\lambda^\sigma = \prod_\ell [\frac{k_\ell + \sigma_\ell}{2^j}, \frac{k_\ell + 1 + \sigma_\ell}{2^j}[\quad \text{and} \quad \mu^\sigma(\lambda) = \mu(\lambda^\sigma).$$

We also define the quantities

$$\underline{\alpha}^\sigma_\mu(x_0) = \liminf_{j\to\infty} \frac{\log \mu^\sigma(\lambda_j(x_0))}{-j\log(2)}, \quad \overline{\alpha}^\sigma_\mu(x_0) = \limsup_{j\to\infty} \frac{\log \mu^\sigma(\lambda_j(x_0))}{-j\log(2)},$$

and, in case of existence,

$$\alpha^\sigma_\mu(x_0) = \lim_{j\to\infty} \frac{\log \mu^\sigma(\lambda_j(x_0))}{-j\log(2)}.$$

We will be concerned by the estimate of the Hausdorff dimension of the following isoHölder sets

$$\widetilde{E}_\alpha(\mu) = \{x \in [0,1[^d,\ \alpha_\mu(x) = \min_{\sigma\in\{-1,0,1\}^d} \{\alpha^\sigma_\mu(x)\} = \alpha\}.$$

The mapping

$$d_\mu : \alpha \geq 0 \mapsto \dim_{\mathcal{H}}(\widetilde{E}_\alpha(\mu)),$$

will be called the multifractal spectrum of the Borel measure μ. Recall that in the framework of [15], the following isoHölder sets are used

$$E_\alpha(\mu) = \{x \in [0,1[^d,\ \lim_{j\to+\infty} \frac{\log \mu(\lambda_j(x))}{-j\log(2)} = \alpha\}.$$

Unfortunately, these isoHölder sets are not adapted to the study of the pointwise regularity of wavelet series F_μ. Indeed, starting from $\lim_j \log \mu(\lambda_j(x))/-j\log(2)$, we cannot deduce the value of the upper pointwise Hölder exponent of the function F_μ at x using wavelet criteria.

We now recall well known results about upper bound of the upper multifractal spectrum, which can be found in [15]. For any q in $\mathbf{R}$, set

$$\tau(q) = \liminf_{j\to+\infty} \frac{1}{\log|\lambda|} \log \sideset{}{^*}\sum_{|\lambda|=2^{-j}} \mu(\lambda)^q,$$

where $\sideset{}{^*}\sum$ means that the sum is taken over those λ such that $\mu(\lambda) > 0$.
As usual, τ^* denotes the Legendre transform of the function τ, that is

$$\forall \alpha \geq 0, \quad \tau^*(\alpha) = \inf_{q\in\mathbf{R}} \{\alpha q - \tau(q)\}.$$

Remark that, since $\alpha > 0$,

$$\widetilde{E}_\alpha(\mu) \subset E_\alpha(\mu).$$

Using this inclusion, an upper bound for the multifractal spectrum of any Borel measure can be obtained from [15]:

Proposition 6 *Let $\alpha \geq 0$ and μ a Borel measure. One has*

$$\dim_{\mathcal{H}}(\widetilde{E}_\alpha(\mu)) \leq \tau^*(\alpha).$$

Moreover, if $\tau^*(\alpha) < 0$ then $\widetilde{E}_\alpha(\mu) = \emptyset$.

Definition 10 Let $\alpha_0 \geq 0$. One says that the Borel measure μ obeys the multifractal formalism at $\alpha = \alpha_0$ for the sets $\widetilde{E}_\alpha(\mu)$ if $\dim_{\mathcal{H}}(\widetilde{E}_{\alpha_0}(\mu)) = \tau^*(\alpha_0)$.

5.2 Wavelet series and multifractal measures

We want to define a wavelet series of the form (21) obeying the weak multifractal formalism for functions. First, we give an explicit relationship between the wavelet series F_μ and the measure μ from the multifractal point of view.

5.2.1 A transference theorem

Theorem 6 *Let μ a Borel measure and s_0, p_0 two positive real numbers. Let F_μ be the wavelet series defined by equality (21). If the measure μ obeys the multifractal formalism at $\alpha_0 \geq 0$ for the sets $\widetilde{E}_\alpha(\mu)$, then F_μ obeys both the strong and weak multifractal formalisms at*

$$H = s_0 - \frac{d}{p_0} + \frac{\alpha_0 d}{p_0},$$

and

$$\underline{d}(H) = \overline{d}(H) = d_\mu(\alpha_0).$$

Remark 5 Let us notice, as in [14], that if $x_0 \notin supp(\mu)$, there exists some j_0 such that,

$$\forall j \geq j_0, \quad d_j(x_0) = 0.$$

Thus, in this special case, $\overline{h}_{F_\mu}(x_0) = +\infty$.

Proof. The proof mimics the one of Theorem 1 of [14]. It relies on the following Lemma:

Lemma 5 *For any $\alpha \geq 0$, the following inclusion holds:*

$$\widetilde{E}_\alpha(\mu) \subset \underline{E}_H(F_\mu) \cap \overline{E}_H(F_\mu),$$

where $H = s_0 - \dfrac{d}{p_0} + \dfrac{\alpha d}{p_0}$.

Proof. Remark that, for all λ

$$d_\lambda = 2^{-j(s_0-\frac{d}{p_0})}\mu(\lambda)^{\frac{1}{p_0}}.$$

where $j = -\log(|\lambda|)/\log(2)$.
For any given x_0,

$$d_j(x_0) = 2^{-j(s_0-\frac{d}{p_0})} \max_{\sigma\in\{-1,0,1\}^d}\{\mu^\sigma(\lambda_j(x_0))\}^{\frac{1}{p_0}}.$$

Assume that $x_0 \in \widetilde{E}_\alpha(\mu)$. Since for any $\sigma \in \{-1,0,1\}^d$, $\alpha^\sigma(x_0) \geq \alpha$, for all $\varepsilon > 0$ there exists an integer $j_0(\varepsilon,\sigma)$ such that

$$\forall j \geq j_0(\varepsilon,\sigma), \quad \mu^\sigma(\lambda_j(x_0)) \leq 2^{-j\frac{d}{p_0}(\alpha-\varepsilon)}.$$

Hence,

$$\forall j \geq \max_\sigma\{j_0(\varepsilon,\sigma)\}, \quad d_j(x_0) \leq 2^{-j(s_0-\frac{d}{p_0})}2^{-j\frac{d}{p_0}(\alpha-\varepsilon)} \leq 2^{-j(H-\varepsilon)}$$

and

$$\liminf_{j\to+\infty}\frac{\log d_j(x_0)}{\log 2^{-j}} \geq H.$$

Furthermore, since for some $\sigma_0 \in \{-1,0,1\}^d$ we have $\alpha^{\sigma_0}(x_0) \leq \alpha$, for all $\varepsilon > 0$ there exists an integer $j_0(\varepsilon)$ such that

$$\forall j \geq j_0(\varepsilon), \quad \mu^{\sigma_0}(\lambda_j(x_0)) \geq 2^{-jd(\alpha+\varepsilon)}.$$

Then,

$$\forall j \geq j_0(\varepsilon), \quad d_j(x_0) \geq 2^{-j(H+\varepsilon)}$$

and

$$\limsup_{j\to+\infty}\frac{\log d_j(x_0)}{\log 2^{-j}} \leq H.$$

Hence, using Corollary 1,

$$\widetilde{E}_\alpha(\mu) \subset \{x \in [0,1]^d,\, \underline{h}_{F_\mu}(x) = \overline{h}_{F_\mu}(x) = H\} = \underline{E}_H(F_\mu) \cap \overline{E}_H(F_\mu).$$

Since μ obeys the multifractal formalism at $\alpha = \alpha_0$ for the sets $\widetilde{E}_\alpha(\mu)$,

$$\dim_{\mathcal{H}}(\widetilde{E}_{\alpha_0}(\mu)) = \tau^*(\alpha_0) \leq \dim_{\mathcal{H}}(\overline{E}_H(F_\mu))$$

with $H = s_0 - \dfrac{d}{p_0} + \dfrac{\alpha_0 d}{p_0}$.
Remark then that for any $p \in \mathbf{R}$,

$$\omega_f^*(p) = p(s_0 - \frac{d}{p_0}) - \tau(\frac{p}{p_0}).$$

Since for any $0 < H < \infty$ and any locally bounded function one has

$$\dim_{\mathcal{H}}(\overline{E}_H(F_\mu)) \leq \inf_{p \in \mathbf{R}} \{pH - \omega_f^*(p) + d\} = \tau^*(\alpha_0) = \dim_{\mathcal{H}}(\widetilde{E}_{\alpha_0}(\mu)),$$

one can conclude that F_μ obeys the weak multifractal formalism. A similar approach proves that F_μ also obeys the strong multifractal formalism.

5.3 A class of wavelet series obeying both the strong and the weak multifractal formalisms

The aim of this section is to exhibit a class of multifractal measures obeying the multifractal formalism at any $\alpha \geq 0$ for the sets $\widetilde{E}_\alpha(\mu)$, yielding an example of wavelet series satisfying both the strong and weak multifractal formalisms. To this end we first give some examples of multifractal measures obeying the multifractal formalism for sets $\widetilde{E}_\alpha$ using Theorem 2 of [14] for sets $\widetilde{E}_\alpha$. Indeed, even if we consider slightly different iso–Hölder sets Theorem 2 still holds : the proof is exactly the same that this of [14].
We give two canonical examples of measures satisfying the conditions above.

5.3.1 Quasi-Bernoulli measures

Let b an integer larger than 2. Let us recall that a Borel positive measure on $[0,1]^d$ μ is said quasi-Bernoulli if for some $C > 0$ and for any $v, w \in (\mathcal{A}^d)^*$,

$$\frac{1}{C}\mu(I_v)\mu(I_w) \leq \mu(I_{vw}) \leq C\mu(I_v)\mu(I_w).$$

A classical example of quasi-Bernoulli measures is the well-known example of multinomial measures:

Example 1 Let b an integer larger than 2 and let $(m_0, \cdots, m_{b-1}) \in (0,1)^b$ such that,

$$\sum_{i=0}^{b-1} m_i = 1;$$

we can construct a sequence of probability measures $(\mu_n)_{n \in \mathbf{N}}$ on $[0,1)^d$ as follows. For any integer n, define a probability measure μ_n on $[0,1)^d$ such that for any $w \in \mathcal{A}^{nd}$,

$$\mu_n(I_w) = \prod_{\ell=1}^{nd} m_{w_\ell}.$$

This sequence has a weak limit μ called multinomial measure of base b with weight $(m_0, \cdots, m_{b-1})$.
By construction, any multinomial measure is quasi-Bernoulli.

In the following, we consider only continuous quasi-Bernoulli measure, that is without atom. Recall that any continuous quasi-Bernoulli measure satisfies the assumptions of Theorem 2 of [14] and thus obeys the multifractal formalism at any $\alpha > 0$ for the sets $\widetilde{E}_\alpha$.

Hence we have the following Theorem.

Theorem 7 *If μ is a continuous quasi-Bernoulli measure, then the wavelet series F_μ defined by (21) obeys both the strong and the weak multifractal formalisms at any $H > s_0 - d/p_0$.*

5.3.2 The case of b-adic random multiplicatives cascades

Let b an integer larger than 2 and $d = 1$. Canonical random cascades were introduced by Mandelbrot in [38] and their multifractal properties have been widely studied, mainly in the setting of b-adic grid (see e.g. [36, 28, 16, 42, 7, 8]). We first recall the construction of these measures. Let W a non negative random variable, not almost surely constant, satisfying $\mathbf{E}(W) = 1/b$. We thus consider $(W_w)_{w\in\mathcal{A}^*}$ a sequence of independent copies of W and μ_n the random measure whose density with respect to the Lebesgue measure on any dyadic interval is constant and equals

$$b^n W_{w_1} \cdots W_{w_1 \cdots w_n}.$$

Almost surely, this sequence of measures converges weakly to a measure μ as n goes to infinity. Recall that if μ is a b-adic random multiplicative cascade, then almost surely on J, μ satisfies the assumptions of Theorem 2 of [14] and thus obeys the multifractal formalism for any $q \in J$ at $\alpha = \widetilde{\tau}'(q)$ for the sets $\widetilde{E}_\alpha$.

Then similarly, to the case of quasi-Bernoulli measure, we have the following result,

Theorem 8 *Let W be an almost surely positive random variable. Let μ a b-adic random multiplicative cascade such that $\widetilde{\tau}'(1) = -1 - \log_b(\mathbf{E}(W)) > 0$. Then the wavelet series F_μ defined by (21) obeys almost surely both the strong and the weak multifractal formalisms at any $H > 0$.*

6 A multifractal function whose lower and upper spectra of singularities differ

Although similar results hold for both the strong and the weak multifractal formalisms, there is no direct relation between $\underline{d}$ and $\overline{d}$. We introduce here a function defined as p-adic Davenport series whose upper multifractal spectrum is reduced to two single points, while its lower multifractal spectrum is linear on an interval.

A p-adic Davenport series ($p \geq 2$) is a series of the form

$$f(x) = \sum_{j=0}^{\infty} a_j \{p^j t\},$$

where $\{x\}$ is the sawtooth function

$$\{x\} = x - [x] - \frac{1}{2}.$$

We will assume here that $(a_j)_j \in l^1$, so that the series is normally convergent. The function f is thus continuous at every non p-adic rational number and has left and right limit at every p-adic rational kp^{-l} ($k \wedge p = 1$) with a jump of amplitude $\sum_{m \geq l} a_m$. Recent results on Davenport series can be found in [32]. Let $\beta > 1$; the functions f_β we will study is defined by

$$f_\beta(x) = \sum_{l \in \mathbf{N}} \frac{\{2^l x\}}{2^{l\beta}}.$$

6.1 The lower spectrum of singularities of f_β

The functions f_β are derived from the famous Lévy's function (which can be seen as a special case, where $\beta = 1$). The properties of the lower spectrum of singularities of this function have already been investigated in [30]; Propositions 8 and 9 together can be seen as a generalization of Proposition 4 of [30]. Proposition 12 of [32] implies that $\underline{d}$ is linear on $[0, \beta]$.

To determine explicitly the lower isoHölder sets of f_β, we will use the following notations. Let $p \in \mathbf{N}$, $p > 1$; for a sequence of integers $(x_l)_{l \in \mathbf{N}}$ satisfying $0 \leq x_j < p$, we will write

$$(0; x_1, \ldots, x_l, \ldots)_p \tag{22}$$

to denote one expansion in basis p of the real number

$$x = \sum_{l \in \mathbf{N}} \frac{x_l}{p^l}.$$

If there is no k such that $x_l = p - 1$ for all $l \geq k$, (22) is the proper expansion of x in basis p. If $(0; x_1, \ldots)_p$ is the proper expansion of x, we define

$$\theta_p(x) = \inf\{l : x_l \neq 0\} - 1.$$

Let $\delta(k) = \sup\{l : \forall l' \leq l, x_{k+l'} = x_k\}$ and let $(m_l)_{l \in \mathbf{N}}$ be the sequence defined recursively, $m_1 = \inf\{l : x_l = 0 \text{ or } x_l = p - 1\}$, $m_k = \inf\{l \geq m_{k-1} + \delta(m_{k-1}) : x_l = 0 \text{ or } x_l = p - 1\}$ ($k > 1$). One also defines the sequence $(\delta_k)_{k \in \mathbf{N}}$ by $\delta_k = \delta(m_k)$. Finally, $\rho_p(x) = \limsup_{k \to \infty} \delta_k / m_k$; if x is a p-adic rational, one sets $\rho_p(x) = \infty$. The number $\rho_p(x)$ defines, in some way, the rate of approximation of the number x by p-adic rationals, since we have the following obvious result.

Proposition 7 *If x is not a p-adic rational, the equation (depending on k and l)*

$$|x - \frac{k}{p^l}| \leq (\frac{1}{p^l})^\phi \qquad (k \wedge p = 1)$$

has an infinity of solutions if and only if $\phi \leq \rho_p(x) + 1$.

We will denote by $\phi(x)$ the critical exponent $\phi(x) = \rho_2(x)+1$. The lower Hölder exponents of f_β only depend on ϕ.

Proposition 8 *The lower Hölder exponents of f_β are given by*

$$\underline{h}(x) = \frac{\beta}{\phi(x)}.$$

Proof. As a corollary of Theorem 21 of [33], we have the following equalities: if x is not a dyadic rational,

$$\underline{h}(x) = \liminf_{j \to \infty} \frac{-\beta j}{\log_2 \operatorname{dist}(x, 2^{-j}\mathbf{Z})}; \tag{23}$$

otherwise, $\underline{h}(x) = 0$. We can suppose that $x \in (0,1)$ is not a dyadic rational. For a given $j \in \mathbf{N}$, let $\varepsilon_j = \operatorname{dist}(x, 2^{-j}\mathbf{Z})$. One has $\theta_2(\varepsilon_j) = j+1+\delta(j+1)$ and thus $\varepsilon_j \sim 2^{-(j+\delta(j+1)+1)}$. Then (23) can be rewritten

$$\underline{h}(x) = \liminf_{j \to \infty} \frac{\beta j}{j+1+\delta(j+1)} = \frac{\beta}{1+\rho_2(x)}.$$

The lower isoHölder sets are now characterized.

Corollary 2 *The lower isoHölder sets of the function f_β are the sets*

$$\underline{E}_H = \{x : \phi(x) = \frac{\beta}{H}\} \quad (0 < H \leq \beta).$$

The set $\underline{E}_0$ is the set of the dyadic rationals.

To conclude this study on the strong Hölder regularity, we have the following result.

Proposition 9 *The lower spectrum of singularities of f_β is*

$$\underline{d}(h) = \begin{cases} \frac{h}{\beta} & \text{if } h \in [0, \beta] \\ -\infty & \text{otherwise} \end{cases}.$$

Proof. The main idea is the same as in Proposition 4 of [30]. If $\alpha \geq 1/\beta$, let

$$F_\alpha = \limsup_{j \to \infty} \bigcup_k [k2^{-j} - 2^{-j\alpha\beta}, k2^{-j} + 2^{-j\alpha\beta}].$$

Using (23), $\underline{h}(x) = H$ means

$$x \in \bigcap_{\gamma > H} F_{1/\gamma} - \bigcup_{\gamma < H} F_{1/\gamma}. \tag{24}$$

Clearly, $\dim_{\mathcal{H}}(F_\alpha) \leq 1/\alpha\beta$; let us show that the converse inequality holds.

Let $(j_l)_{l \in \mathbf{N}}$ be a sequence satisfying $j_l = 2^{j_{l-1}}$, let

$$I_k(l) = [k2^{-j_l} - 2^{-j_l \alpha\beta}, k2^{-j_l} + 2^{-j_l \alpha\beta}]$$

and

$$G_\alpha = \bigcap_l \bigcup_k I_k(l).$$

A probability measure μ supported by G_α can be obtained as follows. If $l = 1$, we put on each interval $I_k(1)$ the same mass 2^{-j_1}. If each of these intervals contains n intervals of type $I_k(2)$, on each of these intervals, we put the measure $2^{-j_1}/n$. This construction can be iterated to obtain, at the limit, a probability measure μ supported by G_α. One easily checks that

$$\mu([x-h, x+h]) \leq Ch^{1/\alpha\beta} \quad \forall x \in G_\alpha.$$

Moreover, Proposition 4.9 of [22] implies that

$$\mathcal{H}^{1/\alpha\beta}(G_\alpha) > 0$$

and thus, since $G_\alpha \subset F_\alpha$,

$$\dim_{\mathcal{H}}(F_\alpha) = 1/\alpha\beta,$$

which, thanks to (24), is sufficient to conclude.

6.2 The upper spectrum of singularities of f_β

We show here that from the weak Hölder regularity point of view, the function f_β only displays two kinds of singularities: it is discontinuous at dyadic rationals and has an upper Hölder exponent equal to β at non dyadic rationals.

Let

$$\Omega_\alpha = \liminf_{j \to \infty} \{x \in \mathbf{R} : \exists k \in \mathbf{Z} \text{ such that } |x - \frac{k}{2^j}| \leq 2^{-\alpha j}\}.$$

We have the following relation between the sets Ω_α and $\overline{h}(x_0)$.

Proposition 10 *If $\alpha > 1$, then*

$$x_0 \notin \Omega_\alpha \Rightarrow \overline{h}(x_0) \geq \frac{\beta}{\alpha}.$$

Proof. Let $\varepsilon > 0$ be a given real number. We want to prove that for any $C > 0$, there exists a strictly decreasing sequence $(r_n)_n$ of real positive numbers, such that

$$\sup_{h>0} \|\Delta_h^{[\beta/\alpha]+1} f_\beta(x)\|_{L^\infty(B_h(x_0,r_n))} \leq C r_n^{\beta/\alpha}.$$

One has

$$\Delta_h^{[\beta/\alpha]+1} f_\beta(x) = \sum_{l\in\mathbf{N}_0} \frac{1}{2^{l\beta}} \sum_{m=0}^{M} (-1)^m \binom{M}{m} \{2^l(x+mh)\}. \quad (25)$$

If $x_0 \notin \Omega_\alpha$, then there exists a strictly increasing sequence of integers $(j_n)_n$ such that

$$|x_0 - \frac{k}{2^{j_n}}| \geq 2^{-\alpha j_n}, \quad \forall n \in \mathbf{N},\ k \in \mathbf{Z}.$$

Let $r_n = 2^{-\alpha j_n}$; the interval $[x, x + ([\beta/\alpha]+1)h] \subset B(x_0, r_n)$ does not contain any dyadic rational of the form $k2^{-l}$, with $l \leq j_n - 1$. This implies

$$\sum_{l=0}^{j_n-1} \frac{1}{2^{l\beta}} \sum_{m=0}^{M} (-1)^m \binom{M}{m} \{2^l(x+mh)\} = 0.$$

Relation (25) leads to the following inequality,

$$\begin{aligned} |\Delta_h^{[\beta/\alpha]+1} f_\beta(x)| &= |\sum_{l=j_n}^{\infty} \frac{1}{2^{l\beta}} \sum_{m=0}^{M} (-1)^m \binom{M}{m} \{2^l(x+mh)\}| \\ &\leq C' \sum_{l=j_n}^{\infty} \frac{1}{2^{l\beta}} \leq C' r_n^{\beta/\alpha}. \end{aligned}$$

Let now n_0 be an integer such that $C' r_n^{\beta/\alpha} \leq C r_n^{\beta/\alpha-\varepsilon}$ for all $n \geq n_0$. We have $\overline{h}(x_0) \geq \beta/\alpha - \varepsilon$, which is sufficient to conclude.
The sets Ω_α are explicitly known whenever $\alpha \geq 1$.

Proposition 11 *If $\alpha > 1$, then*

$$\Omega_\alpha = \{\frac{k}{2^j} : (k,j) \in \mathbf{Z} \times \mathbf{N}\}.$$

Moreover, $\Omega_1 = \mathbf{R}$.

Proof. The case $\alpha = 1$ is trivial. Let $\alpha > 1$ and let $x \in (0,1)$ be a non dyadic rational. If $x = (0; x_1, \ldots)_2$, one has, for j sufficiently large,

$$\min_{k\in\mathbf{Z}} |x - \frac{k}{2^j}| \geq 2^{-n(j)+1},$$

where $n(j)$ is the first index greater or equal to j such that $x_{n(j)} = 1$. We then have, for $j = n(j) - 1$ sufficiently large,

$$\min_{k\in\mathbf{Z}} |x - \frac{k}{2^j}| \geq 2^{-j} > 2^{-\alpha j}.$$

Therefore, $x \notin \Omega_\alpha$.
Thus, we have a lower bound for the upper Hölder exponent.

Corollary 3 *If x_0 is not a dyadic rational, then*

$$\overline{h}(x_0) \geq \beta.$$

Let us now prove the converse inequality. The following proposition is similar to Lemma 1 of [30].

Proposition 12 *Let f be a function defined on $\mathbf{R}$, continuous everywhere except on a dense countable set of points and admitting a left and a right limit at every point. Let also $x_0 \in \mathbf{R}$ be a point of continuity of f and $(r_n)_n$ a sequence of points of discontinuity converging to x_0. Finally, let s_n ($n \in \mathbf{N}$) be the jump of f at r_n. If there exists a strictly increasing function ψ satisfying $\psi(0) = 0$ such that*

$$\exists r_0 : \Big(\forall r \leq r_0, \exists r_n : \big(|r_n - x_0| \leq r,\ |s_n| \geq \psi(r)\big)\Big),$$

then

$$\overline{h}(x_0) \leq \limsup_{r \to 0} \frac{\log \psi(r)}{\log r}.$$

Proof. Let $\alpha > 1$; if $f \in C^\alpha_w(x_0)$, for any $C > 0$ there exists a sequence $(t_n)_n$ such that

$$\sup_{h>0} \|\Delta_h^{[\alpha]+1} f(x)\|_{L^\infty(B_h(x_0,t_n))} \leq C t_n^\alpha.$$

Let $(r_n)_n$ a sequence such that, for any integer n sufficiently large,

$$|r_n - x_0| < t, \quad |s_n| \geq \psi(t_n).$$

Let F_n be the function defined on $\mathbf{R}$ by

$$F_n(h) = (f(r_n + h), \ldots, f(r_n + ([\alpha]+1)h)).$$

Since F_n has only a countable set of discontinuities, one can find h arbitrarily close to zero such that $[r_n, r_n + ([\alpha]+1)h] \subset B(x_0, t_n)$ and such that F_n is continuous at h. Therefore,

$$\begin{aligned}
\psi(t_n) &\leq s_n = |f(r_n^+) - f(r_n^-)| \\
&\leq |\sum_{k=1}^{[\alpha]+1} (-1)^k \binom{[\alpha]+1}{k} f(r_n + kh) + f(r_n^-)| \\
&\quad + |\sum_{k=1}^{[\alpha]+1} (-1)^k \binom{[\alpha]+1}{k} f(r_n + kh) + f(r_n^+)| \\
&\leq 2(|\Delta_h^{[\alpha]+1} f(r_n^-)| + |\Delta_h^{[\alpha]+1} f(r_n^+)|) \leq 4C t_n^\alpha.
\end{aligned}$$

This inequality implies

$$\frac{\log \psi(t_n)}{\log t_n} \geq \alpha + \frac{\log 4C}{\log t_n}.$$

As t_n tends to zero,

$$\limsup_{r\to 0} \frac{\log \psi(r)}{\log r} \geq \alpha,$$

and the result follows.
Since f_β is continuous except at dyadic rationals and since $\Omega_1 = \mathbf{R}$, Proposition 12 and Corollary 3 imply the following result.

Theorem 9 *If x_0 is not a dyadic rational,*

$$\overline{h}(x_0) = \beta,$$

if x_0 is a dyadic rational, $\overline{h}(x_0) = 0$.

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