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Abstract: In the first part, we consider general singular control problems for random fields given by a stochastic partial differential equation (SPDE). We show that under some conditions the optimal singular control can be identified with the solution of a coupled system of SPDE and *a kind of reflected backward* SPDE (RBSPDE).In the second part, existence and uniqueness of solutions of RBSPDEs are established, which is of independent interest.

Key-words: Stochastic partial differential equations (SPDEs), singular control of SPDEs, maximum principles, comparison theorem for SPDEs, reflected SPDEs, optimal stopping of SPDEs

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CENTRE DE RECHERCHE

PARIS - ROCQUENCOURT

Contrôle Singulier d'Équations aux Dérivés Partielles Stochastiques et Équations aux Dérivés Partielles Stochastiques Rétrogrades avec Réflexion

Résumé : On considère des problèmes de contrôle singulier d'Équations aux Dérivés Partielles Stochastiques (EDPS) et l'on prouve un principe du maximum stochastique pour le le contrôle optimal singulier, faisant intervenir un système couplé d'EDPS et d'EDPS réfléchies. Des résultats d'existence et d'unicité pour les EDPS réfléchie sont démontrés dans la deuxième partie du papier.

Mots-clés : Équations aux Dérivés Partielles Stochastiques (EDPS), contrôle singulier, arrêt optimal

1 Introduction

Let B_t, $t \geq 0$ be an m-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$. Let D be a bounded smooth domain in $\mathbb{R}^d$. Fix $T > 0$ and let $\phi(\omega, x)$ be an $\mathcal{F}_T$-measurable $H = L^2(D)$-valued random variable. Let

$$k : [0,T] \times D \times \mathbb{R} \times \mathbb{R}^m \to \mathbb{R}$$

be a given measurable mapping and $L(t,x) : [0,T] \times D \to \mathbb{R}$ a given continuous function. Consider the problem to find $\mathcal{F}_t$-adapted random fields $u(t,x) \in \mathbb{R}, Z(t,x) \in \mathbb{R}^m, \eta(t,x) \in \mathbb{R}^+$ left-continuous and increasing w.r.t. t, such that

$$\begin{aligned} du(t,x) &= -Au(t,x)dt - k(t,x,u(t,x),Z(t,x))dt + Z(t,x)dB_t, t \in (0,T) \\ &\quad -\eta(dt,x), t \in (0,T), \qquad (1.1) \\ u(t,x) &\geq L(t,x), \\ \int_0^T &\int_D (u(t,x) - L(t,x))\eta(dt,x) = 0, \\ u(T,x) &= \phi(x) \quad a.s, \qquad (1.2) \end{aligned}$$

where A is a second order linear partial differential operator. This is a backward stochastic partial differential equation (BSPDE) with reflection.

The maximum principle method for solving a stochastic control problem for stochastic partial differential equations involves a BSPDE for the adjoint processes $p(t,x), q(t,x)$.See [ØPZ].

The purpose of this paper is twofold: (i) We study a class of singular control problems for SPDEs and prove a maximum principle for the solution of such problems. This maximum principle leads to a kind of reflected backward stochastic partial differential equations. (ii) We study backward stochastic *partial* differential equations (BSPDEs) with reflection. This means that we solve the BSPDE with the constraint that the solution must stay in a pre-described region.

2 Singular control of SPDEs

Suppose the state equation is an SPDE of the form

$$\begin{aligned} dY(t,x) = \{AY(t,x) + b(t,x,Y(t,x))\}dt + \sigma(t,x,Y(t,x))dB(t) \\ + \lambda(t,x,Y(t,x))\xi(dt,x) \; ; \; (t,x) \in [0,T] \times D \qquad (2.1) \end{aligned}$$

$$\begin{cases} Y(0,x) & = y_0(x) \; ; \; x \in D \\ Y(t,x) & = y_1(t,x) \; ; \; (t,x) \in (0,T) \times \partial D. \end{cases} \tag{2.2}$$

Here A is a given linear second order partial differential operator.

The *performance functional* is given by

$$\begin{aligned} J(\xi) = E\Big[\int_D \int_0^T f(t,x,Y(t,x))dtdx + \int_D g(x,Y(T,x))dx \\ + \int_D \int_0^T h(t,x,Y(t,x))\xi(dt,x)\Big], \end{aligned} \tag{2.3}$$

where $f(t,x,y)$, $g(x,y)$ and $h(t,x,y)$ are bounded measurable functions which are differentiable in the argument y and continuous w.r.t. t.

We want to maximize $J(\xi)$ over all $\xi \in \mathcal{A}$, where $\mathcal{A}$ is a given family of adapted processes $\xi(t,x)$, which are non-decreasing and left-continuous w.r.t. t for all x, $\xi(0,x) = 0$. We call $\mathcal{A}$ the set of admissible singular controls. Thus we want to find $\xi^* \in \mathcal{A}$ (called an optimal control) such that

$$\sup_{\xi \in \mathcal{A}} J(\xi) = J(\xi^*)$$

Define the *Hamiltonian* H by

$$\begin{aligned} H(t,x,y,p,q)(dt,\xi(dt,x)) = \{f(t,x,y) + b(t,x,y)p + \sigma(t,x,y)q\}dt \\ + \{\lambda(t,x,y)p + h(t,x,y)\}\xi(dt,x). \end{aligned} \tag{2.4}$$

To this Hamiltonian we associate the following *backward* SPDE (BSPDE) in the unknown process $(p(t,x), q(t,x))$:

$$\begin{aligned} dp(t,x) = -\left\{A^* p(t,x)dt + \frac{\partial H}{\partial y}(t,x,Y(t,x),p(t,x),q(t,x))(dt,\xi(dt,x))\right\} \\ + q(t,x)dB(t) \; ; \; (t,x) \in (0,T) \times D \end{aligned} \tag{2.5}$$

with boundary/terminal values

$$p(T,x) = \frac{\partial g}{\partial y}(x,Y(T,x)) \; ; \; x \in D \tag{2.6}$$

$$p(t,x) = 0 \; ; \; (t,x) \in (0,T) \times \partial D. \tag{2.7}$$

Here A^* denotes the adjoint of the operator A.

Theorem 2.1 (Sufficient maximum principle for singular control of SPDE) *Let $\hat{\xi} \in \mathcal{A}$ with corresponding solutions $\hat{Y}(t,x)$, $\hat{p}(t,x)$, $\hat{q}(t,x)$. Assume that*

$$y \to h(x,y) \text{ is concave} \tag{2.8}$$

and

$$(y,\xi) \to H(t,x,y,\hat{p}(t,x),\hat{q}(t,x))(dt,\xi(dt,x)) \\ \text{is concave.} \tag{2.9}$$

Assume that

$$E[\int_D(\int_0^T \{(Y^\xi(t,x)-\hat{Y}(t,x))^2\hat{q}^2(t,x)+\hat{p}^2(t,x)(\sigma(t,x,Y^\xi(t,x))-\sigma(t,x,\hat{Y}(t,x))^2\} \\ dt)dx] < \infty, \quad \text{for all} \quad \xi \in \mathcal{A}. \tag{2.10}$$

Moreover, assume that the following maximum condition holds:

$$\hat{\xi}(dt,x) \in \operatorname*{argmax}_{\xi\in\mathcal{A}} H(t,x,\hat{Y}(t,x),\hat{p}(t,x),\hat{q}(t,x))(dt,\xi(dt,x)), \tag{2.11}$$

i.e.

$$\{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x)+h(t,x,\hat{Y}(t,x))\}\xi(dt,x) \\ \leq \{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x)+h(t,x,\hat{Y}(t,x))\}\hat{\xi}(dt,x) \text{ for all } \xi \in \mathcal{A}. \tag{2.12}$$

Then $\hat{\xi}$ is an optimal singular control.

Proof of Theorem 2.1 Choose $\xi \in \mathcal{A}$ and put $Y = Y^\xi$. Then by (2.3) we can write

$$J(\xi) - J(\hat{\xi}) = I_1 + I_2 + I_3, \tag{2.13}$$

where

$$I_1 = E\left[\int_0^T \int_D \left\{f(t,x,Y(t,x)) - f(t,x,\hat{Y}(t,x))\right\} dxdt\right] \tag{2.14}$$

$$I_2 = E\left[\int_D \left\{g(x,Y(T,x)) - g(x,\hat{Y}(T,x))\right\} dx\right] \tag{2.15}$$

$$I_3 = E\left[\int_0^T \int_D \left\{h(t,x,Y(t,x))\xi(dt,x) - h(t,x,\hat{Y}(t,x))\hat{\xi}(dt,x)\right\}\right]. \tag{2.16}$$

By our definition of H we have

$$
\begin{aligned}
I_1 = E\Bigg[\int_0^T \int_D \{&H(t,x,Y(t,x),\hat{p}(t,x),\hat{q}(t,x))(dt,\xi(dt,x)) \\
&-H(t,x,\hat{Y}(t,x),\hat{p}(t,x),\hat{q}(t,x))(dt,\hat{\xi}(dt,x))\Big\} \\
&- \int_0^T \int_D \{b(t,x,Y(t,x)) - b(t,x,\hat{Y}(t,x)\}\hat{p}(t,x)dxdt \\
&- \int_0^T \int_D \{\sigma(t,x,Y(t,x)) - \sigma(t,x,\hat{Y}(t,x))\}\hat{q}(t,x)dxdt \\
&- \int_0^T \int_D \hat{p}(t,x)\{\lambda(t,x,Y(t,x))\xi(dt,x) - \lambda(t,x,\hat{Y}(t,x))\hat{\xi}(dt,x)\}dx \\
&- \int_0^T \int_D \{h(t,x,Y(t,x))\xi(dt,x) - h(t,x,\hat{Y}(t,x))\hat{\xi}(dt,x)\}dx\Bigg]. \qquad (2.17)
\end{aligned}
$$

By (2.10) and concavity of g we have, with $\tilde{Y} = Y - \hat{Y}$,

$$
\begin{aligned}
I_2 &\le E\left[\int_D \frac{\partial g}{\partial y}(x,\hat{Y}(T,x))(Y(T,x)-\hat{Y}(T,x))dx\right] = E\left[\int_D \hat{p}(T,x)\tilde{Y}(T,x)dx\right] \\
&= E\Bigg[\int_D \int_0^T \tilde{Y}(t,x)d\hat{p}(t,x)dx + \int_D \int_0^T \hat{p}(t,x)d\tilde{Y}(t,x)dx \\
&\quad + \int_D \int_0^T \{\sigma(t,x,Y(t,x)) - \sigma(t,x,\hat{Y}(t,x))\}\hat{q}(t,x)dtdx\Bigg] \\
&= E\Bigg[\int_D \int_0^T \tilde{Y}(t,x)\left\{-A^*\hat{p}(t,x)dt - \frac{\partial H}{\partial y}(t,x,\hat{Y},\hat{p},\hat{q})(dt,\hat{\xi}(dt,x))\right\}dx \\
&\quad + \int_D \int_0^T \hat{p}(t,x)\{A\tilde{Y}(t,x) + b(t,x,Y(t,x)) - b(t,x,\hat{Y}(t,x))\}dtdx \\
&\quad + \int_D \int_0^T \hat{p}(t,x)\{\lambda(t,x,Y(t,x))\xi(dt,x) - \lambda(t,x,\hat{Y}(t,x))\hat{\xi}(dt,x)\}dx \\
&\quad + \int_D \int_0^T \{\sigma(t,x,Y(t,x)) - \sigma(t,x,\hat{Y}(t,x))\}\hat{q}(t,x)dtdx\Bigg]. \qquad (2.18)
\end{aligned}
$$

Using integration by parts we get, since $\tilde{Y}(t,x) = \hat{p}(t,x) = 0$ for all $(t,x) \in (0,T) \times \partial D$,

$$
\int_D \tilde{Y}(t,x)A^*\hat{p}(t,x)dx = \int_D \hat{p}(t,x)A\tilde{Y}(t,x)dx. \qquad (2.19)
$$

Hence, combining (2.13)-(2.19) and concavity of H,

$$\begin{aligned}
J(\xi) - J(\hat{\xi}) \leq E \Bigg[\int_D \int_0^T & \{H(t,x,Y(t,x),\hat{p}(t,x),\hat{q}(t,x))(dt,\xi(dt,x)) \\
& -H(t,x,\hat{Y}(t,x),\hat{p}(t,x),\hat{q}(t,x))(dt,\hat{\xi}(dt,x)) - \tilde{Y}(t,x)\frac{\partial H}{\partial y}(t,x,\hat{Y},\hat{p},\hat{q})(dt,\hat{\xi}(dt,x))\Big\} dx \Bigg] \\
& \leq \left[\int_D \int_0^T \nabla_\xi H(t,x,\hat{Y}(t,x),\hat{p}(t,x),\hat{q}(t,x))(\xi(dt,x) - \hat{\xi}(dt,x))dx \right] \\
& = E\left[\int_D \int_0^T \{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x))\}(\xi(dt,x) - \hat{\xi}(dt,x))dx \right] \\
& \leq 0 \text{ by (2.12).}
\end{aligned}$$

This proves that $\hat{\xi}$ is optimal. □

For $\xi \in \mathcal{A}$ we let $\mathcal{V}(\xi)$ denote the set of adapted processes $\zeta(t,x)$ of finite variation w.r.t. t such that there exists $\delta = \delta(\xi) > 0$ such that $\xi + y\zeta \in \mathcal{A}$ for all $y \in [0,\delta]$.

Proceeding as in [ØS] we prove the following useful result:

Lemma 2.2 *The inequality (2.12) is equivalent to the following two variational inequalities:*

$$\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x)) \leq 0 \textit{ for all } \quad t,x \tag{2.20}$$

$$\{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x))\}\hat{\xi}(dt,x) = 0 \textit{ for all } t,x \tag{2.21}$$

Proof. (i). Suppose (2.12) holds. Choosing $\xi = \hat{\xi} + y\zeta$ with $\zeta \in \mathcal{V}(\hat{\xi})$ and $y \in (0, \delta(\hat{\xi}))$ we deduce that

$$\{\lambda(s,x,\hat{Y}(s,x))\hat{p}(s,x) + h(s,x,\hat{Y}(s,x))\}\zeta(ds,x) \leq 0; (s,x) \in (0,T) \times D \tag{2.22}$$

for all $\zeta \in \mathcal{V}(\hat{\xi})$.

In particular, this holds if we fix $t \in (0,T)$ and put

$$\zeta(ds,x) = a(\omega)\delta_t(ds)\phi(x); (s,x,\omega) \in (0,T) \times D \times \Omega,$$

where $a(\omega) \geq 0$ is $\mathcal{F}_t$- measurable and bounded, $\phi(x) \geq 0$ is bounded, deterministic and $\delta_t(ds)$ denotes the Dirac measure at t. Then we get

$$\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x)) \leq 0 \text{ for all } \quad t,x \tag{2.23}$$

which is (2.20).

On the other hand, clearly $\zeta(dt,x) := \hat{\xi}(dt,x) \in \mathcal{V}(\hat{\xi})$ and this choice of ζ in (2.22) gives

$$\{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x))\}\hat{\xi}(dt,x) \leq 0; (t,x) \in (0,T) \times D \tag{2.24}$$

Similarly, we can choose $\zeta(dt,x) = -\hat{\xi}(dt,x) \in \mathcal{V}(\hat{\xi})$ and this gives

$$\{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x))\}\hat{\xi}(dt,x) \leq 0; (t,x) \in (0,T) \times D \tag{2.25}$$

combining (2.24) and (2.25) we get

$$\{\lambda(t,x,\hat{Y}(t,x))\hat{p}(t,x) + h(t,x,\hat{Y}(t,x))\}\hat{\xi}(dt,x) = 0$$

which is (2.21). Together with (2.23) this proves (i).

(ii). Conversely, suppose (2.20) and (2.21) hold. Since $\xi(dt,x) \geq 0$ for all $\xi \in \mathcal{A}$ we see that (2.12) follows. □

We may formulate what we have proved as follows:

Theorem 2.3 *(Sufficient maximum principle II) Suppose the conditions of Theorem 2.1 hold. Suppose $\xi \in \mathcal{A}$, and that ξ together with its corresponding processes $Y^{\xi}(t,x), p^{\xi}(t,x), q^{\xi}(t,x)$ solve the coupled SPDE-RBSPDE system consisting of the SPDE* (2.1)-(2.2) *together with the reflected backward SPDE (RBSPDE) given by*

$$\begin{aligned}
dp^{\xi}(t,x) = &-\left\{A^* p^{\xi}(t,x) + \frac{\partial f}{\partial y}(t,x,Y^{\xi}(t,x)) + \frac{\partial b}{\partial y}(t,x,Y^{\xi}(t,x))p^{\xi}(t,x)\right. \\
&\left. + \frac{\partial \sigma}{\partial y}(t,x,Y^{\xi}(t,x))q^{\xi}(t,x)\right\} dt \\
&- \left\{\frac{\partial \lambda}{\partial y}(t,x,Y^{\xi}(t,x))p^{\xi}(t,x) + \frac{\partial h}{\partial y}(t,x,Y^{\xi}(t,x))\right\} \xi(dt,x)\ ;\ (t,x) \in [0,T] \times D \\
&\lambda(t,x,Y^{\xi}(t,x))p^{\xi}(t,x) + h(t,x,Y^{\xi}(t,x)) \leq 0\ ;\ \textit{for all } t,x,a.s. \\
&\{\lambda(t,x,Y^{\xi}(t,x))p^{\xi}(t,x) + h(t,x,Y^{\xi}(t,x))\}\xi(dt,x) = 0\ ;\ \textit{for all} \quad t,x,a.s. \\
&p(T,x) = \frac{\partial g}{\partial y}(x,Y^{\xi}(T,x))\ ;\ x \in D \\
&p(t,x) = 0\ ;\ (t,x) \in (0,T) \times \partial D.
\end{aligned}$$

Then ξ maximizes the performance functional $J(\xi)$.

The concavity conditions of Theorem 2.1 are not always satisfied in applications, and it is of interest to have a maximum principle which does not need these assumptions. Moreover, it is useful to have a version which is of so called "necessary type". To this end, we first prove some auxiliary results:

Lemma 2.4 *Let $\xi(dt,x) \in \mathcal{A}$ and choose $\zeta(dt,x) \in \mathcal{V}(\xi)$. Define the derivative process*

$$\mathcal{Y}(t,x) = \lim_{y\to 0^+} \frac{1}{y}(Y^{\xi+y\zeta}(t,x) - Y^{\xi}(t,x)). \tag{2.26}$$

Then $\mathcal{Y}$ satisfies the SPDE

$$\begin{aligned} d\mathcal{Y}(t,x) &= A\mathcal{Y}(t,x)dt + \mathcal{Y}(t,x)[\frac{\partial b}{\partial y}(t,x,Y(t,x))dt \\ &\quad + \frac{\partial \sigma}{\partial y}(t,x,Y(t,x))dB(t) + \frac{\partial \lambda}{\partial y}(t,x,Y(t,x))\xi(dt,x)] \\ &\quad + \lambda(t,x,Y(t,x))\zeta(dt,x)\ ; \quad (t,x) \in [0,T] \times D \\ \mathcal{Y}(t,x) &= 0; \quad (t,x) \in (0,T) \times \partial D \\ \mathcal{Y}(0,x) &= 0\ ; \quad x \in D \end{aligned} \tag{2.27}$$

Proof. This follows from the equation (2.1)-(2.2) for $Y(t,x)$. We omit the details. □

Lemma 2.5 *Let $\xi(dt,x) \in \mathcal{A}$ and $\zeta(dt,x) \in \mathcal{V}(\xi)$. Put $\eta = \xi + y\zeta; y \in [0,\delta(\xi)]$. Assume that*

$$\begin{aligned} E[\int_D (\int_0^T &\{(Y^{\eta}(t,x) - Y^{\xi}(t,x))^2 q^2(t,x) \\ &+ p^2(t,x)(\sigma(t,x,Y^{\eta}(t,x)) - \sigma(t,x,Y^{\xi}(t,x)))^2\}dt)dx] < \infty \ \ \text{for all } y \in [0,\delta(\xi)], \end{aligned} \tag{2.28}$$

where $(p(t,x),\ q(t,x))$ is the solution of (2.5)-(2.7) corresponding to $Y^{\xi}(t,x)$. Then

$$\begin{aligned} &\lim_{y\to 0^+} \frac{1}{y}(J(\xi + y\zeta) - J(\xi)) \\ &\quad = E[\int_D (\int_0^T \{\lambda(t,x,Y(t,x))p(t,x) + h(t,x,Y(t,x))\}\zeta(dt,x))dx]. \end{aligned} \tag{2.29}$$

Proof. By (2.3) and (2.26), we have

$$\begin{aligned} &\lim_{y\to 0^+} \frac{1}{y}(J(\xi + y\zeta) - J(\xi)) \\ &\quad = E[\int_D \{\int_0^T \frac{\partial f}{\partial y}(t,x,Y(t,x))\mathcal{Y}(t,x)dt + \frac{\partial g}{\partial y}(x,Y(T,x))\mathcal{Y}(T,x)\}dx \\ &\quad + \int_D \int_0^T \frac{\partial h}{\partial y}(t,x,Y(t,x))\mathcal{Y}(t,x)\xi(dt,x)dx + \int_D \int_0^T h(t,x,Y(t,x))\zeta(dt,x)dx]. \end{aligned} \tag{2.30}$$

By (2.4) and (2.27) we obtain

$$E[\int_D \int_0^T \frac{\partial f}{\partial y}(t,x,Y(t,x))\mathcal{Y}(t,x)dtdx]$$
$$= E[\int_D (\int_0^T \mathcal{Y}(t,x)\{\frac{\partial H}{\partial y}(dt,\xi(dt,x)) - p(t,x)\frac{\partial b}{\partial y}(t,x)dt$$
$$-q(t,x)\frac{\partial \sigma}{\partial y}(t,x)dt - (p(t,x)\frac{\partial \lambda}{\partial y}(t,x) + \frac{\partial h}{\partial y}(t,x))\xi(dt,x)\})dx, \tag{2.31}$$

where we have used the abbreviated notation

$$\frac{\partial H}{\partial y}(dt,\xi(dt,x)) = \frac{\partial H}{\partial y}(t,x,Y(t,x),p(t,x),q(t,x))(dt,\xi(dt,x))$$

etc.

By the Itô formula and (2.5), (2.28) we see that

$$E[\int_D \frac{\partial g}{\partial y}(x)\mathcal{Y}(T,x)dx]$$
$$= E[\int_D p(T,x)\mathcal{Y}(T,x)dx]$$
$$= E[\int_D (\int_0^T \{p(t,x)d\mathcal{Y}(t,x) + \mathcal{Y}(t,x)dp(t,x)\} + [p(\cdot,x),\mathcal{Y}(\cdot,x)](T))dx]$$
$$= E[\int_D (\int_0^T [p(t,x)\{A\mathcal{Y}(t,x)dt + \mathcal{Y}(t,x)\frac{\partial b}{\partial y}(t,x)dt$$
$$+ \mathcal{Y}(t,x)\frac{\partial \lambda}{\partial y}(t,x)\xi(dt,x) + \lambda(t,x)\zeta(dt,x)\}$$
$$+ \mathcal{Y}(t,x)\{-A^*p(t,x)dt - \frac{\partial H}{\partial y}(dt,\xi(dt,x))\}$$
$$+ \mathcal{Y}(t,x)\frac{\partial \sigma}{\partial y}(t,x)q(t,x)]dt)dx], \tag{2.32}$$

where $[p(\cdot,x),\mathcal{Y}(\cdot,x)](t)$ denotes the covariation process of $p(\cdot,x)$ and $\mathcal{Y}(\cdot,x)$.

Since $p(t,x) = \mathcal{Y}(t,x) = 0$ for $x \in \partial D$, we deduce that

$$\int_D p(t,x)A\mathcal{Y}(t,x)dx = \int_D A^*p(t,x)\mathcal{Y}(t,x)dx. \tag{2.33}$$

Therefore, substituting (2.31) and (2.32) into (2.30), we get

$$\lim_{y\to 0^+} \frac{1}{y}(J(\xi + y\zeta) - J(\xi))$$
$$= E[\int_D (\int_0^T \{\lambda(t,x)p(t,x) + h(t,s)\}\zeta(dt,x))dx].$$

□

We can now state our necessary maximum principle:

Theorem 2.6 *[Necessary maximum principle]*
(i) Suppose $\xi^* \in \mathcal{A}$ *is optimal, i.e.*

$$\max_{\xi \in \mathcal{A}} J(\xi) = J(\xi^*). \tag{2.34}$$

Let $Y^*, (p^*, q^*)$ *be the corresponding solution of (2.1)-(2.2) and (2.5)-(2.7), respectively, and assume that* (2.28) *holds with* $\xi = \xi^*$. *Then*

$$\lambda(t, x, Y^*(t, x))p^*(t, x) + h(t, x, Y^*(t, x)) \leq 0 \quad \textit{for all } t, x \in [0, T] \times D, a.s. \tag{2.35}$$

and

$$\{\lambda(t, x, Y^*(t, x))p^*(t, x) + h(t, x, Y^*(t, x))\}\xi^*(dt, x) = 0 \quad \textit{for all } t, x \in [0, T] \times D, a.s. \tag{2.36}$$

(ii) Conversely, suppose that there exists $\hat{\xi} \in \mathcal{A}$ *such that the corresponding solutions* $\hat{Y}(t, x), (\hat{p}(t, x), \hat{q}(t, x))$ *of (2.1)-(2.2) and (2.5)-(2.7), respectively, satisfy*

$$\lambda(t, x, \hat{Y}(t, x))\hat{p}(t, x) + h(t, x, \hat{Y}(t, x)) \leq 0 \quad \textit{for all } t, x \in [0, T] \times D, a.s. \tag{2.37}$$

and

$$\{\lambda(t, x, \hat{Y}(t, x))\hat{p}(t, x) + h(t, x, \hat{Y}(t, x))\}\hat{\xi}(dt, x) = 0 \quad \textit{for all } t, x \in [0, T] \times D, a.s. \tag{2.38}$$

Then $\hat{\xi}$ *is a directional sub-stationary point for* $J(\cdot)$*, in the sense that*

$$\lim_{y \to 0^+} \frac{1}{y}(J(\hat{\xi} + y\zeta) - J(\hat{\xi})) \leq 0 \quad \textit{for all } \zeta \in \mathcal{V}(\hat{\xi}). \tag{2.39}$$

Proof. This is proved in a similar way as in Theorem 2.4 in [ØS]. For completeness we give the details:

(i) If $\xi \in \mathcal{A}$ is optimal, we get by Lemma 2.5

$$\begin{aligned} 0 &\geq \lim_{y \to 0^+} \frac{1}{y}(J(\xi + y\zeta) - J(\xi)) \\ &= E[\int_D \int_0^T \{\lambda(t, x)p(t, x) + h(t, x)\}\zeta(dt, x)dx], \quad \text{for all } \zeta \in \mathcal{V}(\xi). \end{aligned} \tag{2.40}$$

In particular, this holds if we choose ζ such that

$$\zeta(ds, x) = a(\omega)\delta_t(s)\phi(x) \tag{2.41}$$

for some fixed $t \in [0,T]$ and some bounded $\mathcal{F}_t$-measurable random variable $a(\omega) \geq 0$ and some bounded, deterministic $\phi(x) \geq 0$, where $\delta_t(s)$ is Dirac measure at t. Then (2.40) gets the form

$$E[\int_D \{\lambda(t,x)p(t,x) + h(t,x)\}a(\omega)\phi(x)dx] \leq 0.$$

Since this holds for all such $a(\omega), \phi(x)$ we deduce that

$$\lambda(t,x)p(t,x) + h(t,x) \leq 0 \quad \text{for all } t, x, a.s. \tag{2.42}$$

Next, if we choose $\zeta(dt,x) = \xi(dt,x) \in \mathcal{V}(\xi)$, we get from (2.40)

$$E[\int_D \int_0^T \{\lambda(t,x)p(t,x) + h(t,x)\}\xi(dt,x)dx] \leq 0. \tag{2.43}$$

On the other hand, we can also choose $\zeta(dt,x) = -\xi(dt,x) \in \mathcal{V}(\xi)$, and this gives

$$E[\int_D \int_0^T \{\lambda(t,x)p(t,x) + h(t,x)\}\xi(dt,x)dx] \geq 0. \tag{2.44}$$

Combining (2.43) and (2.44) we get

$$E[\int_D \int_0^T \{\lambda(t,x)p(t,x) + h(t,x)\}\xi(dt,x)dx] = 0. \tag{2.45}$$

Combining (2.42) and (2.45) we see that

$$\{\lambda(t,x)p(t,x) + h(t,x)\}q(dt,x) = 0 \quad \text{for all } t, x, a.s. \tag{2.46}$$

as claimed. This proves (i).

(ii) Conversely, suppose $\hat{\xi} \in \mathcal{A}$ is as in (ii). Then (2.39) follows from Lemma 2.5.

□

3 Existence and Uniqueness

In this section, we will prove the main existence and uniqueness result for reflected backward stochastic partial differential equations. For notational simplicity, we choose the operator A to be the Laplacian operator Δ. However, our methods work equally well for general second order differential operators like

$$A = \frac{1}{2}\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j}),$$

where $a = (a_{ij}(x)) : D \to \mathbb{R}^{d\times d}$ $(d > 2)$ is a measurable, symmetric matrix-valued function which satisfies the uniform elliptic condition

$$\lambda|z|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x)z_i z_j \leq \Lambda|z|^2,\ \forall\, z \in \mathbb{R}^d \text{ and } x \in D$$

for some constant λ, $\Lambda > 0$

First we will establish a comparison theorem for BSPDEs, which is of independent interest. Consider two backward SPDEs:

$$\begin{aligned} du_1(t,x) &= -\Delta u_1(t)dt - b_1(t, u_1(t,x), Z_1(t,x))dt + Z_1(t,x)dB_t, t \in (0,T) \\ u_1(T,x) &= \phi_1(x) \quad a.s. \end{aligned} \tag{3.1}$$

$$\begin{aligned} du_2(t,x) &= -\Delta u_2(t)dt - b_2(t, u_2(t,x), Z_2(t,x))dt + Z_2(t,x)dB_t, t \in (0,T) \\ u_2(T,x) &= \phi_2(x) \quad a.s. \end{aligned} \tag{3.2}$$

From now on, if $u(t,x)$ is a function of (t,x), we write $u(t)$ for the function $u(t,\cdot)$.

The following result is a comparison theorem for backward stochastic partial differential equations.

Theorem 3.1 *(Comparison theorem for BSPDEs) Suppose $\phi_1(x) \leq \phi_2(x)$ and $b_1(t,u,z) \leq b_2(t,u,z)$. Then we have $u_1(t,x) \leq u_2(t,x), x \in D$, a.e. for every $t \in [0,T]$.*

Proof. For $n \geq 1$, define functions $\psi_n(z)$, $f_n(x)$ as follows (see [DP1]).

$$\psi_n(z) = \begin{cases} 0 & \text{if } z \leq 0, \\ 2nz & \text{if } 0 \leq z \leq \frac{1}{n}, \\ 2 & \text{if } z > \frac{1}{n}. \end{cases} \tag{3.3}$$

$$f_n(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ \int_0^x dy \int_0^y \psi_n(z)dz & \text{if } x > 0. \end{cases} \tag{3.4}$$

We have

$$f_n'(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ nx^2 & \text{if } x \leq \frac{1}{n}, \\ 2x - \frac{1}{n} & \text{if } x > \frac{1}{n}. \end{cases} \tag{3.5}$$

Also $f_n(x) \uparrow (x^+)^2$ as $n \to \infty$. For $h \in K := L^2(D)$, set

$$F_n(h) = \int_D f_n(h(x))dx.$$

F_n has the following derivatives for $h_1, h_2 \in K$,

$$F_n'(h)(h_1) = \int_D f_n'(h(x))h_1(x)dx, \tag{3.6}$$

$$F_n''(h)(h_1, h_2) = \int_D f_n''(h(x))h_1(x)h_2(x)dx. \tag{3.7}$$

Applying Ito's formula we get

$$\begin{aligned}
& F_n(u_1(t) - u_2(t)) \\
= \;& F_n(\phi_1 - \phi_2) + \int_t^T F_n'(u_1(s) - u_2(s))(\Delta(u_1(s) - u_2(s)))ds \\
& + \int_t^T F_n'(u_1(s) - u_2(s))(b_1(s, u_1(s), Z_1(s)) - b_2(s, u_2(s), Z_2(s)))ds \\
& - \int_t^T F_n'(u_1(s) - u_2(s))(Z_1(s) - Z_2(s))dB_s \\
& -\frac{1}{2}\int_t^T F_n''(u_1(s) - u_2(s))(Z_1(s) - Z_2(s), Z_1(s) - Z_2(s))ds \\
=: \;& I_n^1 + I_n^2 + I_n^3 + I_n^4 + I_n^5,
\end{aligned} \tag{3.8}$$

where,

$$\begin{aligned}
I_n^2 &= \int_t^T F_n'(u_1(s) - u_2(s))(\Delta(u_1(s) - u_2(s)))ds \\
&= \int_t^T \int_D f_n'(u_1(s,x) - u_2(s,x))(\Delta(u_1(s,x) - u_2(s,x)))dxds \\
&= -\int_t^T f_n''(u_1(s,x) - u_2(s,x))|\nabla(u_1(s,x) - u_2(s,x)|^2 dxds \le 0,
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
I_n^5 &= -n\int_t^T \int_D \chi_{\{0 \le u_1(s,x) - u_2(s,x) \le \frac{1}{n}\}}(u_1(s,x) - u_2(s,x))|Z_1(s,x) - Z_2(s,x)|^2 dxds \\
&\quad - \int_t^T \int_D \chi_{\{u_1(s,x) - u_2(s,x) > \frac{1}{n}\}}|Z_1(s,x) - Z_2(s,x)|^2 dxds.
\end{aligned} \tag{3.10}$$

For I_n^3, we have

$$
\begin{aligned}
I_n^3 &= \int_t^T \int_D f_n'(u_1(s,x)-u_2(s,x))(b_1(s,u_1(s,x),Z_1(s,x))-b_2(s,u_2(s,x),Z_2(s,x)))dxds \\
&= \int_t^T \int_D f_n'(u_1(s,x)-u_2(s,x))(b_1(s,u_1(s,x),Z_1(s,x))-b_2(s,u_1(s,x),Z_1(s,x)))dxds \\
&\quad + \int_t^T \int_D f_n'(u_1(s,x)-u_2(s,x))(b_2(s,u_1(s,x),Z_1(s,x))-b_2(s,u_2(s,x),Z_1(s,x)))dxds \\
&\quad + \int_t^T \int_D f_n'(u_1(s,x)-u_2(s,x))(b_2(s,u_2(s,x),Z_1(s,x))-b_2(s,u_2(s,x),Z_2(s,x)))dxds \\
&\leq \int_t^T \int_D f_n'(u_1(s,x)-u_2(s,x))(b_2(s,u_2(s,x),Z_1(s,x))-b_2(s,u_2(s,x),Z_2(s,x)))dxds \\
&\quad +C\int_t^T \int_D ((u_1(s,x)-u_2(s,x)^+)^2 dxds := I_{n,1}^3 + I_{n,2}^3, \qquad (3.11)
\end{aligned}
$$

where the Lipschiz condition of b and the assumption $b_1 \leq b_2$ have been used.

$I^3_{n,1}$ can be estimated as follows:

$$\begin{aligned}
I^3_{n,1} &\leq C\int_t^T\int_D f_n'(u_1(s,x)-u_2(s,x))|Z_1(s,x)-Z_2(s,x)|dxds \\
&= C\int_t^T\int_D \chi_{\{0\leq u_1(s,x)-u_2(s,x)\leq\frac{1}{n}\}}n(u_1(s,x)-u_2(s,x))^2|Z_1(s,x)-Z_2(s,x)|dxds \\
&\quad +C\int_t^T\int_D \chi_{\{u_1(s,x)-u_2(s,x)>\frac{1}{n}\}}[2(u_1(s,x)-u_2(s,x))-\frac{1}{n}]|Z_1(s,x)-Z_2(s,x)|dxds \\
&\leq C\int_t^T\int_D \chi_{\{u_1(s,x)-u_2(s,x)>\frac{1}{n}\}}(2(u_1(s,x)-u_2(s,x))-\frac{1}{n})^2dxds \\
&\quad +\int_t^T\int_D \chi_{\{u_1(s,x)-u_2(s,x)>\frac{1}{n}\}}|Z_1(s,x)-Z_2(s,x)|^2dxds \\
&\quad +\frac{1}{4}C^2\int_t^T\int_D \chi_{\{0\leq u_1(s,x)-u_2(s,x)\leq\frac{1}{n}\}}n(u_1(s,x)-u_2(s,x))^3dxds \\
&\quad +\int_t^T\int_D \chi_{\{0\leq u_1(s,x)-u_2(s,x)\leq\frac{1}{n}\}}n(u_1(s,x)-u_2(s,x))^2|Z_1(s,x)-Z_2(s,x)|^2dxds \\
&\leq C'\int_t^T\int_D ((u_1(s,x)-u_2(s,x))^+)^2dxds \\
&\quad +\int_t^T\int_D \chi_{\{u_1(s,x)-u_2(s,x)>\frac{1}{n}\}}|Z_1(s,x)-Z_2(s,x)|^2dxds \\
&\quad +\int_t^T\int_D \chi_{\{0\leq u_1(s,x)-u_2(s,x)\leq\frac{1}{n}\}}n(u_1(s,x)-u_2(s,x))^2|Z_1(s,x)-Z_2(s,x)|^2dxds
\end{aligned} \tag{3.12}$$

(3.10),(3.11) and (3.12) imply that

$$I^3_n+I^5_n\leq C\int_t^T\int_D((u_1(s,x)-u_2(s,x))^+)^2dxds \tag{3.13}$$

Thus it follows from (3.8), (3.9) and (3.13) that

$$\begin{aligned}
&F_n(u_1(t)-u_2(t)) \\
\leq\ & F_n(\phi_1-\phi_2)+C\int_t^T\int_D((u_1(s,x)-u_2(s,x))^+)^2dxds \\
&-\int_t^T F_n'(u_1(s)-u_2(s))(Z_1(s)-Z_2(s))dB_s
\end{aligned} \tag{3.14}$$

Take expectation and let $n\to\infty$ to get

$$E[\int_D((u_1(t,x)-u_2(t,x))^+)^2dx]\leq\int_t^T dsE[\int_D((u_1(s,x)-u_2(s,x))^+)^2dx] \tag{3.15}$$

Gronwall's inequality yields that

$$E[\int_D ((u_1(t,x) - u_2(t,x))^+)^2 dx] = 0, \tag{3.16}$$

which completes the proof of the theorem. ■

Remark. After this paper was written we became aware of the paper [MYZ], where a similar comparison theorem is proved. However, the theorems are not identical and the proofs are quite different.

We now proceed to prove existence and uniqueness of the reflected BSPDEs. Let $V = W_0^{1,2}(D)$ be the Sobolev space of order one with the usual norm $||\cdot||$. Consider the reflected backward stochastic partial differential equation:

$$\begin{aligned} du(t) &= -\Delta u(t)dt - b(t, u(t,x), Z(t,x))dt + Z(t,x)dB_t, t \in (0,T) \quad (3.17)\\ &\quad -\eta(dt,x), t \in (0,T), \quad (3.18)\\ u(t,x) &\geq L(t,x),\\ \int_0^T &\int_D (u(t,x) - L(t,x))\eta(dt,x)dx = 0,\\ u(T,x) &= \phi(x) \quad a.s. \quad (3.19) \end{aligned}$$

Theorem 3.2 *Assume that $E[|\phi|_K^2] < \infty$. and that*

$$|b(s,u_1,z_1) - b(s,u_1,z_1)| \leq C(|u_1 - u_2| + |z_1 - z_2|).$$

Let $L(t,x)$ be a measurable function which is differentiable in t and twice differentiable in x such that

$$\int_0^T \int_D L'(t,x)^2 dxdt < \infty, \int_0^T \int_D |\Delta L(t,x)|^2 dxdt < \infty.$$

Then there exists a unique $K \times L^2(D, \mathbb{R}^m) \times K$-valued progressively measurable process $(u(t,x), Z(t,x), \eta(t,x))$ such that

$$\begin{array}{ll} (i) & E[\int_0^T ||u(t)||_V^2 dt] < \infty, \quad E[\int_0^T |Z(t)|_{L^2(D,R^m)}^2 dt] < \infty.\\ (ii) & \eta \text{ is a } K\text{-valued continuous process, non-negative and nondecreasing in } t \text{ and } \eta(0,x) = 0.\\ (iii) & u(t,x) = \phi(x) + \int_t^T \Delta u(t,x)ds + \int_t^T b(s, u(s,x), Z(s,x))ds - \int_t^T Z(s,x)dB_s\\ & \quad +\eta(T,x) - \eta(t,x); \quad 0 \leq t \leq T,\\ (iv) & u(t,x) \geq L(t,x) \quad a.e. \quad x \in D, \forall t \in [0,T].\\ (v) & \int_0^T \int_D (u(t,x) - L(t,x))\eta(dt,x)dx = 0 \end{array} \tag{3.20}$$

where $u(t)$ stands for the K-valued continuous process $u(t,\cdot)$ and (iii) is understood as an equation in the dual space V^* of V.

For the proof of the theorem, we introduce the penalized BSPDEs:

$$\begin{aligned} du^n(t) &= -\Delta u^n(t)dt - b(t,u^n(t,x),Z^n(t,x))dt + Z^n(t,x)dB_t \\ &\quad -n(u^n(t,x)-L(t,x))^-dt, \quad t\in(0,T) \qquad (3.21) \\ u^n(T,x) &= \phi(x) \quad a.s. \qquad (3.22) \end{aligned}$$

According to [ØPZ], the solution (u^n,Z^n) of the above equation exists and is unique. We are going to show that the sequence (u^n,Z^n) has a limit, which will be a solution of the equation (3.20). First we need some a priori estimates.

Lemma 3.3 *Let (u^n,Z^n) be the solution of equation (3.21). We have*

$$\sup_n E[\sup_t |u^n(t)|_K^2] < \infty, \qquad (3.23)$$

$$\sup_n E[\int_0^T ||u^n(t)||_V^2] < \infty, \qquad (3.24)$$

$$\sup_n E[\int_0^T |Z^n(t)|^2_{L^2(D,R^m)}] < \infty. \qquad (3.25)$$

Proof. Take a function $f(t,x)\in C_0^{2,2}([-1,T+1]\times D)$ satisfying $f(t,x)\geq L(t,x)$. Applying Itô's formula, it follows that

$$\begin{aligned} |u^n(t)-f(t)|_K^2 &= |\phi-f(T)|_K^2 + 2\int_t^T < u^n(s)-f(s),\Delta u^n(s) > ds \\ &\quad +2\int_t^T < u^n(s)-f(s), b(s,u^n(s),Z^n(s)) > ds \\ &\quad -2\int_t^T < u^n(s)-f(s), Z^n(s) > dB_s \\ &\quad +2n\int_t^T < u^n(s)-f(s),(u^n(s)-L(s))^- > ds - \int_t^T |Z^n(s)|^2_{L^2(D,\mathbb{R}^m)}ds \\ &\quad +2\int_t^T < u^n(s)-f(s), f'(s) > ds, \quad a.s. \qquad (3.26) \end{aligned}$$

where $<,>$ denotes the inner product in K. Now we estimate each of the terms on the right hand side.

$$\begin{aligned}
& 2\int_t^T < u^n(s) - f(s), \Delta u^n(s) > ds \\
= & -2\int_t^T ||u^n(s)||_V^2 ds + 2\int_t^T < \frac{\partial f(s)}{\partial x}, \frac{\partial u^n(s)}{\partial x} > ds \\
\leq & -\int_t^T ||u^n(s)||_V^2 ds + \int_t^T ||f(s)||_V^2 ds
\end{aligned} \tag{3.27}$$

$$\begin{aligned}
& 2\int_t^T < u^n(s) - f(s), b(s, u^n(s), Z^n(s)) > ds \\
= & \; 2\int_t^T < u^n(s) - f(s), b(s, u^n(s), Z^n(s)) - b(s, f(s), Z^n(s)) > ds \\
& +2\int_t^T < u^n(s) - f(s), b(s, f(s), Z^n(s)) - b(s, f(s), 0) > ds \\
& +2\int_t^T < u^n(s) - f(s), b(s, f(s), 0) > ds \\
\leq & \; C\int_t^T |u^n(s) - f(s)|_H^2 ds + \frac{1}{2}\int_t^T |Z^n(s)|^2_{L^2(D,\mathbb{R}^m)} ds \\
& +C\int_t^T |b(s, f(s), 0)|_H^2 ds
\end{aligned} \tag{3.28}$$

$$\begin{aligned}
& 2n\int_t^T < u^n(s) - f(s), (u^n(s) - L(s))^- > ds \\
= & \; 2n\int_t^T \int_D (u^n(s,x) - f(s,x))\chi_{\{u^n(s,x) \leq L(s,x)\}}(L(s,x) - u^n(s,x)) dsdx \geq 0
\end{aligned} \tag{3.29}$$

Substituting (3.27),(3.28) and (3.29) into (3.26) we obtain

$$\begin{aligned}
& |u^n(t) - f(t)|_K^2 + \int_t^T ||u^n(s)||_V^2 ds + \frac{1}{2}\int_t^T |Z^n(s)|^2_{L^2(D,\mathbb{R}^m)} ds \\
\leq & \; |\phi - f(T)|_K^2 + C\int_t^T |u^n(s) - f(s)|_K^2 ds + C\int_t^T |b(s, f(s), 0)|_K^2 ds \\
& + \int_t^T ||f(s)||_V^2 ds - 2\int_t^T < u^n(s) - f(s), Z^n(s) > dB_s
\end{aligned} \tag{3.30}$$

Take expectation and use the Gronwall inequality to obtain

$$\sup_n \sup_t E[|u^n(t)|_K^2] < \infty \tag{3.31}$$

$$\sup_n E[\int_0^T ||u^n(t)||_V^2] < \infty \tag{3.32}$$

$$\sup_n E[\int_0^T |Z^n(t)|^2_{L^2(D,\mathbb{R}^m)} dt] < \infty \tag{3.33}$$

By virtue of (3.33), (3.31) can be further strengthened to (3.23). Indeed, by Burkholder inequality,

$$\begin{aligned}
& E\Big[2 \sup_{v\le t\le T} |\int_t^T < u^n(s) - f(s), Z^n(s) > dB_s|\Big] \\
\le\ & CE\Big[(\int_v^T |u^n(s) - f(s)|_K^2 |Z^n(s)|^2_{L^2(D,\mathbb{R}^m)} ds)^{\frac{1}{2}}\Big] \\
\le\ & CE\Big[\sup_{v\le s\le T} (|u^n(s) - f(s)|_K)(\int_v^T |Z^n(s)|^2_{L^2(D,\mathbb{R}^m)} ds)^{\frac{1}{2}}\Big] \\
\le\ & \frac{1}{2}E\Big[\sup_{v\le s\le T} (|u^n(s) - f(s)|_K^2)\Big] + CE\Big[\int_v^T |Z^n(s)|^2_{L^2(D,\mathbb{R}^m)} ds\Big] \qquad (3.34)
\end{aligned}$$

With (3.34), taking superum over $t \in [v,T]$ on both sides of (3.26) we obtain (3.23). ■

We need the following estimates.

Lemma 3.4 *Suppose the conditions in Theorem 3.2 hold. Then there is a constant C such that*

$$E[\int_0^T \int_D ((u^n(t,x) - L(t,x))^-)^2 dx dt] \le \frac{C}{n^2}. \tag{3.35}$$

Proof. Let f_m be defined as in the proof of Theorem 3.1. Then $f_m(x) \uparrow (x^+)^2$ and $f_m'(x) \uparrow 2x^+$ as $m \to \infty$. For $h \in K$, set

$$G_m(h) = \int_D f_m(-h(x)) dx.$$

It is easy to see that for $h_1, h_2 \in K$,

$$G_m'(h)(h_1) = -\int_D f_m'(-h(x)) h_1(x) dx, \tag{3.36}$$

$$G_n''(h)(h_1,h_2) = \int_D f_m''(-h(x)) h_1(x) h_2(x) dx. \tag{3.37}$$

Applying Itô's formula we get

$$\begin{aligned}
& G_m(u^n(t) - L(t)) \\
= \quad & G_m(\phi - L(T)) + \int_t^T G'_m(u^n(s) - L(s))(\Delta u^n(s)))ds \\
& + \int_t^T G'_m(u^n(s) - L(s))(b(s, u^n(s), Z^n(s)))ds \\
& + n\int_t^T G'_m(u^n(s) - L(s))((u^n(s) - L(s))^-)ds \\
& + \int_t^T G'_m(u^n(s) - L(s))(L'(s))ds \\
& - \int_t^T G'_m(u^n(s) - L(s))(Z^n(s))dB_s \\
& - \frac{1}{2}\int_t^T G''_m(Z^n(s), Z^n(s))ds \\
=: \quad & I_m^1 + I_m^2 + I_m^3 + I_m^4 + I_m^5 + I_m^6 + I_m^7.
\end{aligned} \tag{3.38}$$

Now,

$$\begin{aligned}
& I_m^2 = \int_t^T G'_m(u^n(s) - L(s))(\Delta u^n(s)))ds \\
= \quad & -\int_t^T \int_D f'_m(L(s,x) - u^n(s,x))(\Delta(u^n(s,x) - L(s,x)))dxds \\
& -\int_t^T \int_D f'_m(L(s,x) - u^n(s,x))(\Delta L(s,x))dxds \\
\leq \quad & -\int_t^T \int_D f''_m(L(s,x) - u^n(s,x))|\nabla(u^n(s,x) - L(s,x))|^2 dxds \\
& + \frac{1}{4}n\int_t^T \int_D f'_m(L(s,x) - u^n(s,x))^2 xds \\
& + \frac{C}{n}\int_t^T \int_D (\Delta L(s,x))^2 dxds,
\end{aligned} \tag{3.39}$$

$$\begin{aligned}
I_m^3 \quad = \quad & -\int_t^T \int_D f'_m(L(s,x) - u^n(s,x))b(s, u^n(s,x), Z^n(s,x))dxds \\
\leq \quad & \frac{1}{4}n\int_t^T \int_D f'_m(L(s,x) - u^n(s,x))^2 ds \\
+ \quad & \frac{C}{n}\int_t^T \int_D (b(s, u^n(s,x), Z^n(s,x)))^2 dxds,
\end{aligned} \tag{3.40}$$

$$
\begin{aligned}
I_m^5 &= -\int_t^T \int_D f'_m(L(s,x) - u^n(s,x))(L'(s,x))dxds \\
&\leq \frac{1}{4}n \int_t^T \int_D f'_m(L(s,x) - u^n(s,x))^2 ds \\
&+ \frac{C}{n} \int_t^T \int_D (L'(s,x))^2 dxds. \qquad (3.41)
\end{aligned}
$$

Combining (3.38)–(3.41) and taking expectation we obtain

$$
\begin{aligned}
& E[G_m(u^n(t) - L(t))] \\
\leq\ & E[G_m(\phi - L(T))] + \frac{3}{4}n \int_t^T \int_D f'_m(L(s,x) - u^n(s,x))^2 ds \\
& + \frac{C}{n} E[\int_t^T \int_D (L'(s,x))^2 dxds] + \frac{C}{n} E[\int_t^T \int_D (\Delta L(s,x))^2 dxds] \\
& + \frac{C}{n} E[\int_t^T \int_D (b(s,u^n(s,x), Z^n(s,x)))^2 dxds] \\
& - nE[\int_t^T \int_D f'_m(L(s,x) - u^n(s,x))((u^n(s,x) - L(s,x))^-)ds]. \qquad (3.42)
\end{aligned}
$$

Letting $m \to \infty$ we conclude that

$$
\begin{aligned}
& E[\int_D ((u^n(t,x) - L(t,x))^-)^2 dx] \\
\leq\ & \frac{3}{4} nE[\int_t^T \int_D ((u^n(s,x) - L(s,x))^-)^2 dxds] \\
& - nE[\int_t^T \int_D ((u^n(s,x) - L(s,x))^-)^2 dxds] + \frac{C'}{n}, \qquad (3.43)
\end{aligned}
$$

where the Lipschiz condition of b and Lemma 3.3 have been used. In particular we have

$$
E[\int_t^T \int_D ((u^n(s,x) - L(s,x))^-)^2 dxds] \leq \frac{C'}{n^2}. \qquad (3.44)
$$

■

Lemma 3.5 *Let (u^n, Z^n) be the solution of equation (3.21). We have*

$$
\lim_{n,m\to\infty} E[\sup_{0\leq t\leq T} |u^n(t) - u^m(t)|_K^2] = 0, \qquad (3.45)
$$

$$\lim_{n,m\to\infty} E[\int_0^T ||u^n(t)-u^m(t)||_V^2 dt] = 0. \tag{3.46}$$

$$\lim_{n,m\to\infty} E[\int_0^T |Z^n(t)-Z^m(t)|^2_{L^2(D,\mathbb{R}^m)} dt] = 0. \tag{3.47}$$

Proof. Applying Itô's formula, it follows that

$$\begin{aligned}
& |u^n(t)-u^m(t)|_K^2 \\
= & \ 2\int_t^T < u^n(s)-u^m(s), \Delta(u^n(s)-u^m(s)) > ds \\
& +2\int_t^T < u^n(s)-u^m(s), b(s,u^n(s),Z^n(s)) - b(s,u^m(s),Z^m(s)) > ds \\
& -2\int_t^T < u^n(s)-u^m(s), Z^n(s)-Z^m(s) > dB_s \\
& +2\int_t^T < u^n(s)-u^m(s), n(u^n(s)-L(s))^- - m(u^m(s)-L(s))^- > ds \\
& -\int_t^T |Z^n(s)-Z^m(s)|^2_{L^2(D,\mathbb{R}^m)} ds
\end{aligned} \tag{3.48}$$

Now we estimate each of the terms on the right side.

$$\begin{aligned}
& 2\int_t^T < u^n(s)-u^m(s), \Delta(u^n(s)-u^m(s)) > ds \\
= & -2\int_t^T ||u^n(s)-u^m(s)||_V^2 ds.
\end{aligned} \tag{3.49}$$

By the Lipschitz continuity of b and the inequality $ab \le \varepsilon a^2 + C_\varepsilon b^2$, one has

$$\begin{aligned}
& 2\int_t^T < u^n(s)-u^m(s), b(s,u^n(s),Z^n(s)) - b(s,u^m(s),Z^m(s)) > ds \\
\le & \ C\int_t^T |u^n(s)-u^m(s)|_K^2 ds + \frac{1}{2}\int_t^T |Z^n(s)-Z^m(s)|^2_{L^2(D,\mathbb{R}^m)} ds.
\end{aligned} \tag{3.50}$$

In view of (3.44),

$$
\begin{aligned}
& 2E[\int_t^T < u^n(s)-u^m(s), n(u^n(s)-L(s))^- - m(u^m(s)-L(s))^- > ds] \\
= \ & 2nE[\int_t^T < u^n(s)-L(s), (u^n(s)-L(s))^- > ds] \\
& +2mE[\int_t^T < L(s)-u^n(s), (u^m(s)-L(s))^- > ds] \\
& +2mE[\int_t^T < u^m(s)-L(s), (u^m(s)-L(s))^- > ds] \\
& +2nE[\int_t^T < L(s)-u^m(s), (u^n(s)-L(s))^- > ds] \\
\leq \ & 2mE[\int_t^T < L(s)-u^n(s), (u^m(s)-L(s))^- > ds] \\
& +2nE[\int_t^T < L(s)-u^m(s), (u^n(s)-L(s))^- > ds] \\
\leq \ & 2mE[\int_t^T \int_D (u^n(s,x)-L(s,x))^-(u^m(s,x)-L(s,x))^- dxds] \\
& +2nE[\int_t^T \int_D (u^m(s,x)-L(s,x))^-(u^n(s,x)-L(s,x))^- dxds] \\
\leq \ & 2m(E[\int_t^T \int_D ((u^n(s,x)-L(s,x))^-)^2 dxds])^{\frac{1}{2}}(E[\int_t^T \int_D ((u^m(s,x)-L(s,x))^-)^2 dxds])^{\frac{1}{2}} \\
& +2n(E[\int_t^T \int_D ((u^n(s,x)-L(s,x))^-)^2 dxds])^{\frac{1}{2}}(E[\int_t^T \int_D ((u^m(s,x)-L(s,x))^-)^2 dxds])^{\frac{1}{2}} \\
\leq \ & C'(\frac{1}{n}+\frac{1}{m}).
\end{aligned} \tag{3.51}
$$

It follows from (3.48) and (3.49) that

$$
\begin{aligned}
& E[|u^n(t)-u^m(t)|_K^2] + \frac{1}{2}E[\int_t^T |Z^n(s)-Z^m(s)|^2_{L^2(D,\mathbb{R}^m)} ds] \\
& +E[\int_t^T ||u^n(s)-u^m(s)||_V^2 ds] \\
\leq \ & C\int_t^T E[|u^n(s)-u^m(s)|_K^2]ds + C'(\frac{1}{n}+\frac{1}{m}).
\end{aligned} \tag{3.52}
$$

Application of the Gronwall inequality yields

$$
\lim_{n,m\to\infty}\{E[|u^n(t)-u^m(t)|_K^2]+\frac{1}{2}E[\int_t^T |Z^n(s)-Z^m(s)|^2_{L^2(D,\mathbb{R}^m)} ds]\} = 0, \tag{3.53}
$$

$$\lim_{n,m\to\infty} E[\int_t^T ||u^n(s)-u^m(s)||_V^2 ds] = 0. \tag{3.54}$$

By (3.53) and the Burkholder inequality we can further show that

$$\lim_{n,m\to\infty} E[\sup_{0\le t\le T} |u^n(t)-u^m(t)|_K^2] = 0. \tag{3.55}$$

The proof is complete. ■

Proof of Theorem 3.2. From Lemma 3.5 we know that $(u^n, Z^n), n \geq 1$, forms a Cauchy sequence. Denote by $u(t,x)$, $Z(t,x)$ the limit of u^n and Z^n. Put

$$\bar{\eta}^n(t,x) = n(u^n(t,x) - L(t,x))^-$$

Lemma 3.4 implies that $\bar{\eta}^n(t,x)$ admits a non-negative weak limit, denoted by $\bar{\eta}(t,x)$, in the following Hilbert space:

$$\bar{K} = \{h; \quad \text{h is a K-valued adapted process such that} \quad E[\int_0^T |h(s)|_K^2 ds] < \infty\}$$

with inner product

$$< h_1, h_2 >_{\bar{K}} = E[\int_0^T \int_D h_1(t,x)h_2(t,x)dtdx].$$

Set $\eta(t,x) = \int_0^t \bar{\eta}(s,x)ds$. Then η is a continuous K-valued process which is increasing in t. Keeping Lemma 3.5 in mind and letting $n \to \infty$ in (3.21) we obtain

$$\begin{aligned}
& u(t,x) \\
= \; & \phi(x) + \int_t^T \Delta u(t,x)ds + \int_t^T b(s,u(s,x),Z(s,x))ds - \int_t^T Z(s,x)dB_s \\
& + \eta(T,x) - \eta(t,x); \quad 0 \le t \le T.
\end{aligned} \tag{3.56}$$

Recall from Lemma 3.4 that

$$E[\int_t^T \int_D ((u^n(s,x) - L(s,x))^-)^2 dxds] \le C' \frac{1}{n^2}$$

By the Fatou Lemma, this implies that $E[\int_t^T \int_D ((u(s,x) - L(s,x))^-)^2 dxds] = 0$. In view of the continuity of u in t, we conclude $u(t,x) \ge L(t,x)$ a.e. in

x, for every $t \geq 0$. Combining the strong convergence of u^n and the weak convergence of $\bar{\eta}^n$, we also have

$$\begin{aligned}
& E[\int_0^T \int_D (u(s,x) - L(s,x))\eta(dt,x)dx] \\
= & E[\int_0^T \int_D (u(s,x) - L(s,x))\bar{\eta}(t,x)dtdx] \\
\leq & \lim_{n\to\infty} E[\int_0^T \int_D (u^n(s,x) - L(s,x))\bar{\eta}^n(t,x)dtdx] \leq 0 \qquad (3.57)
\end{aligned}$$

Hence,

$$\int_0^T \int_D (u(s,x) - L(s,x))\eta(dt,x)dx = 0, \quad a.s.$$

We have shown that (u, Z, η) is a solution to the reflected BSPDE (3.17).

Uniqueness. Let (u_1, Z_1, η_1), (u_2, Z_2, η_2) be two such solutions to equation (3.20). By Itô's formula, we have

$$\begin{aligned}
& |u_1(t) - u_2(t)|_K^2 \\
= & 2\int_t^T < u_1(s) - u_2(s), \Delta(u_1(s) - u_2(s)) > ds \\
& +2\int_t^T < u_1(s) - u_2(s), b(s, u_1(s), Z_1(s)) - b(s, u_2(s), Z_2(s)) > ds \\
& -2\int_t^T < u_1(s) - u_2(s), Z_1(s) - Z_2(s) > dB_s \\
& +2\int_t^T < u_1(s) - u_2(s), \eta_1(ds) - \eta_2(ds) > \\
& - \int_t^T |Z_1(s) - Z_2(s)|^2_{L^2(D,\mathbb{R}^m)} ds \qquad (3.58)
\end{aligned}$$

Similar to the proof of Lemma 3.5, we have

$$2\int_t^T < u_1(s) - u_2(s), \Delta(u_1(s) - u_2(s)) > ds \leq 0, \qquad (3.59)$$

and

$$\begin{aligned}
& 2\int_t^T < u_1(s) - u_2(s), b(s, u_1(s), Z_1(s)) - b(s, u_2(s), Z_2(s)) > ds \\
\leq & C\int_t^T |u_1(s) - u_2(s)|_K^2 ds + \frac{1}{2}\int_t^T |Z_1(s) - Z_2(s)|^2_{L^2(D,\mathbb{R}^m)} ds \qquad (3.60)
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
& 2E[\int_t^T < u_1(s) - u_2(s), \eta_1(ds) - \eta_2(ds) >] \\
= \; & 2E[\int_t^T \int_D (u_1(s,x) - L(s,x))\eta_1(ds,x)dx] \\
& -2E[\int_t^T \int_D (u_1(s,x) - L(s,x))\eta_2(ds,x)dx] \\
& +2E[\int_t^T \int_D (u_2(s,x) - L(s,x))\eta_2(ds,x)dx] \\
& -2E[\int_t^T \int_D (u_2(s,x) - L(s,x))\eta_1(ds,x)dx] \\
\leq \; & 0 \qquad (3.61)
\end{aligned}
$$

Combining (3.58)—(3.61) we arrive at

$$
\begin{aligned}
& E[|u_1(t) - u_2(t)|_K^2] + \frac{1}{2}E[\int_t^T |Z_1(s) - Z_2(s)|^2_{L^2(D,\mathbb{R}^m)} ds] \\
\leq \; & C \int_t^T E[|u_1(s) - u_2(s)|_K^2] ds. \qquad (3.62)
\end{aligned}
$$

Appealing to Gronwall inequality, this implies

$$u_1 = u_2, \quad Z_1 = Z_2$$

which further gives $\eta_1 = \eta_2$ from the equation they satisfy. □

4 Link to optimal stopping

In this section, we provide a link between the solution of a reflected backward stochastic partial differential equation and an optimal stopping problem. Let $u(t,x)$ be the solution of the following reflected BSPDE.

$$
\begin{aligned}
& u(t,x) \\
= \; & \phi(x) + \int_t^T \frac{1}{2}\Delta u(t,x) ds + \int_t^T k(s,x,u(s,x),Z(s,x)) ds - \int_t^T Z(s,x) dB_s \\
& +\eta(T,x) - \eta(t,x); \quad 0 \leq t \leq T, \\
& u(t,x) \geq L(t,x), \\
& \int_0^T \int_D (u(s,x) - L(s,x))\eta(dt,x)dx = 0 \quad a.s. \qquad (4.1)
\end{aligned}
$$

Let $\mathcal{S}_{t,T}$ be the set of all stopping times τ satisfying $t \leq \tau \leq T$. For $\tau \in \mathcal{S}_{t,T}$, define

$$R_t(\tau, x) = \int_t^\tau P_{s-t}k(s,x)ds + P_{\tau-t}L(\tau,x)\chi_{\{\tau<T\}} + P_{\tau-t}\phi(x)\chi_{\{\tau=T\}},$$

where $k(s,\cdot) = k(s,\cdot,u(s,\cdot),Z(s,\cdot))$ and P_t denotes the semigroup generated by the Laplacian operator $\frac{1}{2}\Delta$, i.e.

$$P_t f(x) = (2\pi t)^{-d/2} \int_{\mathbb{R}^d} f(y) exp(-\frac{|y-x|^2}{2t})dy; f \in L^1(\mathbb{R}^d).$$

Here, and in the following we will use the simplified notation $P_t k(s,x) = (P_t k(s,\cdot))(x)$ etc.

Theorem 4.1 *$u(t,x)$ is the value function of the the optimal stopping problem associated with $R_t(\tau,x)$, i.e.,*

$$u(t,x) = esssup_{\tau \in \mathcal{S}_{t,T}} E[R_t(\tau,x)|\mathcal{F}_t] \tag{4.2}$$

Proof. Observe that u admits the following mild representation:

$$\begin{aligned} & u(t,x) \\ = \; & P_{T-t}\phi(x) + \int_t^T P_{s-t}(k(s,u(s,x),Z(s,x)))ds - \int_t^T P_{s-t}(Z(s,x))dB_s \\ & + \int_t^T P_{s-t}\eta(ds,x); \quad 0 \leq t \leq T. \end{aligned} \tag{4.3}$$

This implies that for any stopping time τ with $t \leq \tau \leq T$, we have

$$\begin{aligned} & u(t,x) \\ = \; & P_{\tau-t}(u(\tau,x)) + \int_t^\tau P_{s-t}(k(s,x))ds - \int_t^\tau P_{s-t}(Z(s,x))dB_s \\ & + \int_t^\tau P_{s-t}\eta(ds,x); \quad 0 \leq t \leq \tau. \end{aligned} \tag{4.4}$$

Since $\eta(s,x)$ is increasing in s and $u(s,x) \geq L(s,x)$ for $s < T$, it follows that

$$u(t,x) \geq R_t(\tau,x) - \int_t^\tau P_{s-t}(Z(s,x))dB_s \tag{4.5}$$

Take conditional expectation with respect to $\mathcal{F}_t$ on both sides to get

$$\begin{aligned} & u(t,x) \\ \geq \; & E[R_t(\tau,x)|\mathcal{F}_t] - E[\int_t^\tau P_{s-t}(Z(s,x))dB_s|\mathcal{F}_t] \\ = \; & E[R_t(\tau,x)|\mathcal{F}_t]. \end{aligned} \tag{4.6}$$

As τ is arbitrary, we obtain

$$u(t,x) \geq ess \sup_{\tau \in \mathcal{S}_{t,T}} E[R_t(\tau,x)|\mathcal{F}_t] \tag{4.7}$$

Now, define

$$\hat{\tau}_t = \inf\{s \in [t,T)|u(s) = L(s)\} \wedge T$$

From the property of η, it is not increasing on the interval $[t, \hat{\tau}_t]$. Therefore, $\int_t^{\hat{\tau}_t} P_{s-t}\eta(ds,x) = 0$. Thus we have from (4.4) that

$$\begin{aligned} & u(t,x) \\ = \; & P_{\hat{\tau}_t - t}(u(\hat{\tau}_t),x) + \int_t^{\hat{\tau}_t} P_{s-t}(k(s,x))ds - \int_t^{\hat{\tau}_t} P_{s-t}(Z(s,x))dB_s \\ = \; & R_t(\hat{\tau}_t,x) - \int_t^{\hat{\tau}_t} P_{s-t}(Z(s,x))dB_s. \end{aligned} \tag{4.8}$$

Taking conditional expectation yields that

$$u(t,x) = E[R_t(\hat{\tau}_t,x)|\mathcal{F}_t]$$

Combining this with (4.7) we obtain the theorem. ■

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