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Optimal multiple stopping problem and financial applications

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Abstract: In their paper [2], Carmona and Touzi have studied an optimal multiple stopping time problem in a market where the price process is continuous. In this paper, we generalize their results when the price process is allowed to jump. Also, we generalize the problem associated to the valuation of swing options to the context of jump diffusion processes. Then we relate our problem to a sequence of ordinary stopping time problems. We characterize the value function of each ordinary stopping time problem as the unique viscosity solution of the associated Hamilton-Jacobi-Bellman Variational Inequality (HJBVI in short).

Key-words: Optimal multiple stopping, swing option, jump diffusion process, Snell envelop, viscosity solution.

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Problèmes de temps d'arrêt optimal multiple et applications financières.

Résumé : Dans ce travail, on généralise les résultats de Carmona et Touzi [2] pour les processus avec sauts. On montre que résoudre un problème de temps d'arrêt optimal multiple revient à résoudre une suite de problème de temps d'arrêt optimal classique. On caractérise la fonction valeur de chaque problème de temps d'arrêt optimal ordinaire comme l'unique solution de viscosité de l'inéquation variationnelle d'Hamilton Jacobi Bellman. On montre l'existence d'un temps d'arrêt optimal multiple pour l'évaluation d'une option swing dans le cas d'une diffusion avec sauts. On montre que la fonction valeur anisi que le pay-off de chaque problème de temps d'arrêt optimal ordinaire sont lipschitziens en espaces et höldériens en temps. On montre que chaque fonction valeur est l'unique solution de viscosité associée à l'inéquation variationnelle d'Hamilton Jacobi Bellman.

Mots-clés : temps d'arrêt optimal, Option Swing, processus de diffusion avec sauts, enveloppe de Snell, solution de viscosité.

1 Introduction

2 Introduction

Optimal stopping problems in general setting was the object of many works. Maingueneau [12] and El Karoui [6] characterized the optimal stopping time as the beginning of the set where the process is equal to its Snell envelop.
In the Markovian context, Pham [14] studied the valuation of American options when the risky assets are modeled by a jump diffusion process. He showed that the last problem is equivalent to an optimal stopping problem which leads to a parabolic integrodifferential free boundary problem. For details we refer to El Karoui [6], when the reward process is non-negative, right continuous, $\mathbb{F}$-adapted and left continuous in expectation and its supremum is bounded in L^p, $p > 1$, Karatzas and Shreve [9] in the continuous setting and Peskir and Shiryaev [13] in the Markovian context.
Carmona and Touzi [2] introduced the problem of optimal multiple stopping time where the underlying process is continuous. They characterized the optimal multiple stopping time as the solution of a sequence of ordinary stopping time problems. As an application, they studied the valuation of swing options. The latter products are defined as American options with many exercise rights. In fact, the holder of a swing option has the right to exercise it or not at many times under the condition that he respects the refracting time which separates two successive exercises. The consumption in the energy market is not simple, in fact it depends on Foreign parameters like temperature and weather. When the temperature has a high variation, the power consumption has a sharp increase and price follow. Although these spikes of consumption are infrequent, they have a large financial impact, so pricing swing options must take them into account. Bouzguenda and Mnif [1] generalized the valuation of the swing option where the reward process is allowed to jump.
Kobylanski et al. [10] studied an optimal multiple stopping time problem. They showed that such a problem is reduced to compute an optimal one stopping time problem where the new reward function is no longer a right continuous left limited (RCLL) process but a family of positive random variables which satisfy some compatibility properties.
In the present paper, we present a generalisation of the classical theory of optimal stopping introduced by El Karoui. We relate our multiple stopping time problem to a sequence of ordinary stopping time problems, we prove the existence of an optimal multiple stopping time. In Bouzguenda and Mnif [1] we assume that the expectation of the Snell envelop variation is equal to zero for every predictable time. Such assumption is checked when the process is modeled by the exponential of Lévy process. In the present paper, we get rid of this assumption. As in El Karoui [6], we assume that the state process is non-negative, right continuous, $\mathbb{F}$-adapted and left continuous in expectation and its supremum is bounded in L^p, $p > 1$ and so we can apply our result for a general jump diffusion process. We characterize the value function of each ordinary problem as the unique viscosity solution of the associated HJBVI. Such characterization is important in the sense that if we propose a monotonous consistent and stable numerical scheme, then it converges to the unique viscosity solution of the associated HJBVI. This part is postponed in future research. Bouzguenda and

Mnif [1] solved numerically the sequence of optimal stopping problem by using Malliavin Calculus to approximate the conditional expectation, but they didn't obtain a convergence result. In our case such convergence result is possible thanks to the powerfull tool of viscosity solutions.
This paper is organized as follows, in section 2 we formulate the problem. In section 3 we provide the existence of a multiple optimal stopping time. In section 4 we study the valuation of swing options in the jump diffusion case. The regularity of the value function is studied in section 5. In the last section we prove that each value function is the unique viscosity solution of the associated HJBVI.

3 Problem Formulation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ a filtration which satisfies the usual conditions, i.e. an increasing right continuous family of sub-σ-algebras of $\mathbb{F}$ such that $\mathcal{F}_0$ contains all the $\mathbb{P}$-null sets. Let $T \in (0, \infty)$ be the option maturity time i.e. the time of expiration of our right to stop the process or exercise, $\mathcal{S}$ the set of $\mathbb{F}$-stopping times with values in $[0, T]$ and $\mathcal{S}_\sigma = \{\tau \in \mathcal{S} \ ; \ \tau \geq \sigma\}$ for every $\sigma \in \mathcal{S}$.
We shall denote by $\delta > 0$ the refracting period which separates two successive exercises. We also fix $\ell \geq 1$ the number of rights we can exercise. Now, we define by $\mathcal{S}_\sigma^{(\ell)}$ the set:

$$\mathcal{S}_\sigma^{(\ell)} := \left\{ \begin{array}{l} (\tau_1, ..., \tau_\ell) \in \mathcal{S}^\ell, \ \tau_1 \in \mathcal{S}_\sigma, \ \tau_i - \tau_{i-1} \geq \delta \ on \ \{\tau_{i-1} + \delta \leq T\} \ a.s, \\ \tau_i = (T_+) \ on \ \{\tau_{i-1} + \delta > T\} \ a.s, \ \forall \ i = 2, ..., \ell \end{array} \right\} \quad (1)$$

Let $X = (X_t)_{t\geq 0}$ be a non-negative, right continuous and $\mathbb{F}$-adapted process. We assume that X satisfies the integrability condition :

$$E\left[\bar{X}^p\right] < \infty \quad \text{for some} \quad p > 1, \quad \text{where} \quad \bar{X} = \sup_{0\leq t\leq T} X_t. \quad (2)$$

we assume that :

$$X_t = 0, \quad \forall t > T. \quad (3)$$

We introduce the following optimal multiple stopping problem :

$$Z_0^{(\ell)} \quad := \quad \sup_{(\tau_1,...,\tau_\ell)\in\mathcal{S}_0^{(\ell)}} E\left[\sum_{i=1}^{\ell} X_{\tau_i}\right]. \quad (4)$$

It consists in computing the maximum expected reward $Z_0^{(\ell)}$ and finding the optimal exercise strategy $(\tau_1, ..., \tau_\ell) \in \mathcal{S}_0^{(\ell)}$ at which the supremum in (4) is attained, if such a strategy exists.

Remark 3.1 *Notice that Assumption* (2) *guaranties the finiteness of* $Z_0^{(1)}$. *As it is easily seen that* $Z_0^{(\ell)} \leq \ell Z_0^{(1)}$, *every* $Z_0^{(k)}$, $k \geq 1$, *will also be finite.*

To solve the optimal multiple stopping problem, we define inductively the sequence :

$$Y^{(0)} = 0 \quad \text{and} \quad Y_t^{(i)} = \operatorname{ess\,sup}_{\tau\in\mathcal{S}_t} E\left[X_\tau^{(i)}|\mathcal{F}_t\right], \quad \forall \ t \geq 0, \quad \forall \ i = 1, ..., \ell, \quad (5)$$

where the i-th exercise reward process $X^{(i)}$ is given by :

$$X_t^{(i)} \quad = \quad X_t + E\left[Y_{t+\delta}^{(i-1)}|\mathcal{F}_t\right] \quad \text{for } 0 \leq t \leq T-\delta \tag{6}$$

and

$$X_t^{(i)} \quad = \quad X_t \quad \text{ for } \ t > T-\delta.$$

Notation .1 *Note that the constants which appear in this paper are generic constants and could change from line to line.*

4 Existence of an optimal multiple stopping time

In this section, we shall prove that $Z_0^{(\ell)}$ can be computed by solving inductively ℓ single optimal stopping problems sequentially. This result is proved in [2] under the assumption that the process X is continuous a.s.. As it is proved by El Karoui [6, Theorem 2.18, p.115], the existence of the optimal stopping strategy for a right continuous, non-negative and $\mathbb{F}$-adapted process X requires assumption (2) in addition to the left continuity in expectation of the process X, i.e. for all $\tau \in \mathcal{S}$, $(\tau_n)_{n\geq 0}$ an increasing sequence of stopping times such that $\tau_n \uparrow \tau$, $E\left[X_{\tau_n}\right] \to E\left[X_\tau\right]$.

Definition 4.1 *For all stopping time τ, we said that $\theta^* \in \mathcal{S}_\tau$ is an optimal stopping time for $Y_\tau^{(i)}$, for $i = 1, ..., \ell$ if*

$$Y_\tau^{(i)} = E\left[X_{\theta^*}^{(i)}|\mathcal{F}_\tau\right] \ \textit{a.s.}.$$

In Lemmas 4.2, 4.3 and Proposition 4.8 we show that the i-th exercise reward process $X^{(i)}$ satisfies the conditions required to solve the i-th optimal stopping problem.

Lemma 4.2 *Suppose that the non-negative $\mathbb{F}$-adapted and right continuous process X satisfies condition (2). Then, for all $i = 1, ..., \ell$, the process $X^{(i)}$ satisfies :*

$$E\left[\left(\bar{X}^{(i)}\right)^p\right] < \infty, \quad p > 1, \ \textit{where} \quad \bar{X}^{(i)} = \sup_{0\leq t\leq T} X_t^{(i)},$$

Proof. We proceed by induction on i.
For $i = 1$ we have that $X^{(1)} = X$ so by assumption (2) we have that $E[\bar{X}^{(1)^p}] < \infty$.
Let $2 \leq i \leq \ell$, let us assume that $E[\bar{X}^{(i-1)^p}] < \infty$, we will show that $E[\bar{X}^{(i)^p}] < \infty$. We have that for all $\tau \in \mathcal{S}_0$, $0 \leq t \leq T$

$$E[X_\tau^{(i-1)}|\mathcal{F}_t] \leq E\left[\sup_{0\leq s\leq T} X_s^{(i-1)}|\mathcal{F}_t\right] := \hat{X}_t^{(i-1)}$$

then $Y_t^{(i-1)} = \operatorname{ess\,sup}_{\tau\in\mathcal{S}_t} E\left[X_\tau^{(i-1)}|\mathcal{F}_t\right] \leq \hat{X}_t^{(i-1)}$, hence

$$E\left[\sup_{0\leq t\leq T}(Y_t^{(i-1)})^p\right] \leq E\left[\sup_{0\leq t\leq T}(\hat{X}_t^{(i-1)})^p\right]. \tag{7}$$

We have that $(\hat{X}_t^{(i-1)})_{t\geq 0}$ is a martingale. Hence the Doob's L^p inequality and Jensen inequality show:

$$\begin{aligned} E\left[\sup_{0\leq t\leq T}(\hat{X}_t^{(i-1)})^p\right] &\leq \left(\frac{p}{p-1}\right)^p \sup_{0\leq t\leq T} E\left[(\hat{X}_t^{(i-1)})^p\right] \\ &\leq \left(\frac{p}{p-1}\right)^p E\left[(\bar{X}^{(i-1)})^p\right] \end{aligned} \tag{8}$$

From (7), (8) and the induction assumption we deduce that:

$$E\left[\sup_{0\leq t\leq T}(Y_t^{(i-1)})^p\right] \leq \left(\frac{p}{p-1}\right)^p E\left[(\bar{X}^{(i-1)})^p\right] < \infty. \tag{9}$$

The last inequality added to the supermartingale property of $Y^{(i-1)}$ and the fact that $\left(\sup_{0\leq t\leq T} Y_t^{(i-1)}\right)^p = \sup_{0\leq t\leq T}(Y_t^{(i-1)})^p$ give

$$\begin{aligned} E\left[(\bar{X}^{(i)})^p\right]^{1/p} &\leq E\left[\left(\sup_{0\leq t\leq T}\left(X_t + E\left[Y_{t+\delta}^{(i-1)}|\mathcal{F}_t\right]\right)\right)^p\right]^{1/p} \\ &\leq E\left[\left(\bar{X} + \sup_{0\leq t\leq T} E\left[Y_{t+\delta}^{(i-1)}|\mathcal{F}_t\right]\right)^p\right]^{1/p} \\ &\leq C\left(E\left[\bar{X}^p\right]^{1/p} + E\left[\left(\sup_{0\leq t\leq T} E\left[Y_{t+\delta}^{(i-1)}|\mathcal{F}_t\right]\right)^p\right]^{1/p}\right) \\ &\leq C\left(E\left[\bar{X}^p\right]^{1/p} + E\left[\sup_{0\leq t\leq T}(Y_t^{(i-1)})^p\right]^{1/p}\right) < \infty, \end{aligned} \tag{10}$$

where C is a positive constant. Inequality (10) implies that: $E\left[(\bar{X}^{(i)})^p\right] < \infty$.
□

Lemma 4.3 *Suppose that the non-negative $\mathbb{F}$-adapted process X is right continuous, then for all $i = 1, ..., \ell$, the process $X^{(i)}$ is non-negative $\mathbb{F}$-adapted and right continuous.*

Proof. The process $X^{(i)}$ is non-negative and $\mathbb{F}$-adapted since it is the case for the process X. Let us prove by induction that for all $i = 1, ..., \ell$, $X^{(i)}$ is right continuous.
For $i = 1$ we have that $X^{(1)} = X$, which is right continuous.
Let $1 \leq i \leq \ell - 1$, assume that $X^{(i)}$ is right continuous and let us prove that $X^{(i+1)}$ is right continuous. We have that $X_t^{(i+1)} = X_t + E\left[Y_{t+\delta}^{(i)}|\mathcal{F}_t\right]$, since X is a right continuous, it suffices to prove that $\left(E\left[Y_{t+\delta}^{(i)}|\mathcal{F}_t\right]\right)$ is right continuous. By Dellacherie and Meyer [5, p.119], it suffices to prove that $Y^{(i)}$ is right continuous. We have that $X^{(i)}$ is non-negative and $\mathbb{F}$-adapted process, also by Lemma 4.2 we have that it is bounded in $L^1(\mathbb{P})$ and by the induction assumption it is right continuous, so by El Karoui [6, Theorem 2.15, p.113] we obtain that $Y^{(i)}$ is right continuous. We deduce then (by Dellacherie and Meyer) that the optional projection $\left(E\left[Y_{t+\delta}^{(i)}|\mathcal{F}_t\right]\right)$ of $Y^{(i)}$ is right continuous and then $X^{(i+1)}$ is also.

We conclude then the right continuity of $X^{(i)}$ for $i = 1, ..., \ell$. $\square$
To prove the existence of optimal stopping time for problem (5) we start by giving the definition of a closed under pairwise maximisation family and proving that
$\left(E\left[X_\tau^{(i-1)}|\mathcal{F}_t\right], \tau \in \mathcal{S}_t\right)$ is such a family.

Definition 4.4 *A family $(X_i)_{i\in I}$ of random variables is said to be closed under pairwise maximisation if for all $i, j \in I$, there exists $k \in I$ such that $X_k \geq X_i \vee X_j$.*

Lemma 4.5 *Let $t \in [0, T]$, the family $\left(E\left[X_\tau^{(i-1)}|\mathcal{F}_t\right], \tau \in \mathcal{S}_t\right)$ is closed under pairwise maximisation.*

Proof. Let $\tau_1, \tau_2 \in \mathcal{S}_t$ and $\tilde{X}_{t,\tau_j}^{(i-1)} := E\left[X_{\tau_j}^{(i-1)}|\mathcal{F}_t\right]$ $(j = 1, 2)$. We define the stopping time

$$\tau = \tau_1 \mathbf{1}_{\{\tilde{X}_{t,\tau_1}^{(i-1)} \geq \tilde{X}_{t,\tau_2}^{(i-1)}\}} + \tau_2 \mathbf{1}_{\{\tilde{X}_{t,\tau_1}^{(i-1)} < \tilde{X}_{t,\tau_2}^{(i-1)}\}}.$$

We have that $\tau \in \mathcal{S}_t$ and $E\left[X_\tau^{(i-1)}|\mathcal{F}_t\right] \geq E\left[X_{\tau_j}^{(i-1)}|\mathcal{F}_t\right]$, for $j = 1, 2$. We deduce then that $\left(E\left[X_\tau^{(i-1)}|\mathcal{F}_t\right], \tau \in \mathcal{S}_t\right)$ is closed under pairwise maximisation. $\square$

Lemma 4.6 *For $i = 1, ..., \ell$, for all $t \in [0, T]$*

$$E\left[Y_t^{(i)}\right] = \sup_{\tau\in\mathcal{S}_t} E\left[X_\tau^{(i)}\right]. \tag{11}$$

Proof. We have that $\left(E\left[X_\tau^{(i)}|\mathcal{F}_t\right], \tau \in \mathcal{S}_t\right)$ is closed under pairwise maximisation, then by Theorem 8.1 (see Appendix) there exists a sequence (τ_n) of stopping times in $\mathcal{S}_t$ such that $\left(E\left[X_{\tau_n}^{(i)}|\mathcal{F}_t\right]\right)_n$ is a non-decreasing sequence of random variables satisfying

$$Y_t^{(i)} := \operatorname{ess\,sup}_{\tau\in\mathcal{S}_t} E\left[X_\tau^{(i)}|\mathcal{F}_t\right] = \lim_{n\to\infty} E\left[X_{\tau_n}^{(i)}|\mathcal{F}_t\right]. \tag{12}$$

By the last equality we obtain that

$$E\left[\lim_{n\to\infty} E\left[X_{\tau_n}^{(i)}|\mathcal{F}_t\right]\right] \geq E\left[E\left[X_\tau^{(i)}|\mathcal{F}_t\right]\right], \quad \forall \tau \in \mathcal{S}_t$$

then

$$\sup_{n\in\mathbb{N}} E\left[X_{\tau_n}^{(i)}\right] = \lim_{n\to\infty} E\left[X_{\tau_n}^{(i)}\right] \geq E\left[X_\tau^{(i)}\right], \quad \forall \tau \in \mathcal{S}_t,$$

and so $\sup_{n\in\mathbb{N}} E\left[X_{\tau_n}^{(i)}\right] \geq \sup_{\tau\in\mathcal{S}_t} E\left[X_\tau^{(i)}\right]$. The converse inequality is clear since $\sup_{n\in\mathbb{N}} E\left[X_{\tau_n}^{(i)}\right] \leq \sup_{\tau\in\mathcal{S}_t} E\left[X_\tau^{(i)}\right]$. By applying the expectation to the equation (12) and using the monotone convergence Theorem, we deduce

$$\begin{aligned} E\left[Y_t^{(i)}\right] &= E\left[\lim_{n\to\infty} E\left[X_{\tau_n}^{(i)}|\mathcal{F}_t\right]\right] \\ &= \lim_{n\to\infty} E\left[X_{\tau_n}^{(i)}\right] \\ &= \sup_{n\in\mathbb{N}} E\left[X_{\tau_n}^{(i)}\right] \\ &= \sup_{\tau\in\mathcal{S}_t} E\left[X_\tau^{(i)}\right], \end{aligned} \tag{13}$$

and so equation (11) is proved. □

Our aim now is to prove the left continuity in expectation of $X^{(i)}$, for $i = 1, ..., \ell$.

Definition 4.7 *A process X is said to be left continuous along stopping times in expectation (LCE) if for any $\tau \in \mathcal{S}$ and for any sequence $(\tau_n)_{n\geq 0}$ of stopping times such that $\tau_n \uparrow \tau$ a.s. one has $\lim_{n\to\infty} E[X_{\tau_n}] = E[X_\tau]$.*

In the following proposition we prove that $X^{(i)}$ is LCE, it relies on a result of Kobylanski et al. [10].

Proposition 4.8 *Suppose that the non-negative $\mathbb{F}$-adapted process X is right continuous, left continuous in expectation (LCE) and satisfies condition* (2). *Then, for all $i = 1, ..., \ell$, the process $X^{(i)}$ is LCE.*

Proof. We proceed by induction.
For $i = 1$, we have that $X^{(1)} = X$, so it is left continuous in expectation.
Let $1 \leq i \leq \ell - 1$, assume that $X^{(i)}$ is LCE and we will show that $X^{(i+1)}$ is LCE.
We begin by proving that $Y^{(i)}$ is LCE.
Let $\tau \in \mathcal{S}$ and (τ_n) be a sequence of stopping times such that $\tau_n \uparrow \tau$ a.s.. Note that by the supermartingale property of $Y^{(i)}$ we have

$$E\left[Y_{\tau_n}^{(i)}\right] \geq E\left[Y_\tau^{(i)}\right], \quad \forall n \in \mathbb{N}. \tag{14}$$

We have that $X^{(i)}$ is non-negative, $\mathbb{F}$-adapted, bounded in $L^1(\mathbb{P})$ and LCE then by El Karoui [6]

$$\theta_n^{(i)} = \inf\{t \geq \tau_n, X_t^{(i)} = Y_t^{(i)}\}$$

is an optimal stopping time of

$$Y_{\tau_n}^{(i)} = \operatorname{ess}\sup_{\tau\in\mathcal{S}_{\tau_n}} E\left[X_\tau^{(i)}|\mathcal{F}_{\tau_n}\right] \tag{15}$$

and then by the Definition 4.1

$$E\left[Y_{\tau_n}^{(i)}\right] = E\left[X_{\theta_n^{(i)}}^{(i)}\right],$$

moreover, it is clear that $\left(\theta_n^{(i)}\right)_i$ is a nondecreasing sequence of stopping times dominated by T. Let us define $\bar{\theta}^{(i)} = \lim_{n\to\infty} \uparrow \theta_n^{(i)}$. Note that $\bar{\theta}^{(i)}$ is a stopping time. Also, as for each n, $\theta_n^{(i)} \geq \tau_n$ a.s., it follows that $\bar{\theta}^{(i)} \in \mathcal{S}_\tau$ a.s.. Therefore, since $X^{(i)}$ is LCE and since $\theta_n^{(i)}$ is an optimal stopping time of (15), we obtain

$$\begin{aligned} E\left[Y_\tau^{(i)}\right] &= \sup_{\theta \in \mathcal{S}_\tau} E\left[X_\theta^{(i)}\right] \\ &\geq E\left[X_{\bar{\theta}^{(i)}}^{(i)}\right] \\ &= \lim_{n\to\infty} E\left[X_{\theta_n^{(i)}}^{(i)}\right] \\ &= \lim_{n\to\infty} E\left[Y_{\tau_n}^{(i)}\right]. \end{aligned} \tag{16}$$

By inequalities (14) and (16) we deduce that $Y^{(i)}$ is LCE.
Let τ be a stopping time and $(\tau_n)_n$ a sequence of stopping times such that $\tau_n \uparrow \tau$ a.s., we have that

$$E[X_{\tau_n}^{(i+1)}] = E[X_{\tau_n}] + E\left[Y_{\tau_n+\delta}^{(i)}\right].$$

Sending n to ∞, we obtain that $E[X_{\tau_n}^{(i+1)}] \to E[X_\tau^{(i+1)}]$, and then $X^{(i+1)}$ is LCE.
We conclude then that for all $i = 1, ..., \ell$, $X^{(i)}$ is LCE. □
Let us set:

$$\tau_1^* = \inf\{t \geq 0 \;;\; Y_t^{(\ell)} = X_t^{(\ell)}\} \tag{17}$$

We immediately see that $\tau_1^* \leq T$ a.s. ($Y_T^{(\ell)} = X_T^{(\ell)}$). Next, for $2 \leq i \leq \ell$, we define

$$\tau_i^* = \inf\{t \geq \delta + \tau_{i-1}^* \;;\; Y_t^{(\ell-i+1)} = X_t^{(\ell-i+1)}\}\mathbf{1}_{\{\delta+\tau_{i-1}^* \leq T\}} + (T_+)\mathbf{1}_{\{\delta+\tau_{i-1}^* > T\}} \tag{18}$$

Clearly, $\vec{\tau}^* := (\tau_1^*, ..., \tau_\ell^*) \in \mathcal{S}_0^{(\ell)}$.
Since for all $i = 1, ..., \ell$, $X^{(\ell-i+1)}$ is a non-negative right continuous $\mathbb{F}$-adapted process that satisfies the integrability condition $E\left[\operatorname{ess\,sup}_{\tau\in\mathcal{S}} X_\tau^{(\ell-i+1)}\right] < \infty$ and which is LCE along stopping times, then for El Karoui [6] we have the existence of optimal stopping time which is the objective of the following Theorem.

Theorem 4.9 *(Existence of optimal stopping time) For each $\tau \in \mathcal{S}$ there exists an optimal stopping time for $Y_\tau^{(\ell-i+1)}$, $i = 1, ..., \ell$. Moreover τ_i^* is the minimal optimal stopping time for $Y_{\tau_{i-1}^*+\delta}^{(\ell-i+1)}$ (by convention $\tau_0^* + \delta = 0$).*
We have also that

$$E\left[Y_{\tau_i^*}^{(\ell-i+1)}\right] = \sup_{\tau\in\mathcal{S}_{\tau_{i-1}^*+\delta}} E\left[X_\tau^{(\ell-i+1)}\right]$$

and the stopped supermartingale $\{Y_{t\wedge\tau_i^*}^{(\ell-i+1)}, \tau_{i-1}^* + \delta \leq t \leq T\}$ is a martingale.

By Theorem 4.10 we generalize Theorem 1 of [2] to right-continuous price processes.

Theorem 4.10 *Let us assume that the non-negative, $\mathbb{F}$-adapted process X is right continuous, left continuous in expectation and satisfies condition* (2)*. Then,*

$$Z_0^{(\ell)} = Y_0^{(\ell)} = E\left[\sum_{i=1}^{\ell} X_{\tau_i^*}\right]$$

where $(\tau_1^*, ..., \tau_\ell^*)$ represents the optimal exercise strategy.

Proof. From Theorem 4.9, τ_i^* is an optimal stopping time for the problem

$$Y^{(\ell-i+1)}_{\tau^*_{i-1}+\delta} = \text{ess} \sup_{\tau \in \mathcal{S}_{\tau^*_{i-1}+\delta}} E\left[X_\tau^{(\ell-i+1)} | \mathcal{F}_{\tau^*_{i-1}+\delta}\right] \tag{19}$$

Let $\vec{\tau} = (\tau_1, ..., \tau_\ell)$ be an arbitrary element in $\mathcal{S}_0^{(\ell)}$. For ease of notation, we set $\bar{\tau}_i := \tau_{\ell-i+1}$.

A) Let us prove that, for all $1 \leq i \leq \ell$

$$E\left[\sum_{j=1}^{\ell} X_{\tau_j}\right] \leq E\left[X^{(i)}_{\bar{\tau}_i} + \sum_{j=i+1}^{\ell} X_{\bar{\tau}_j}\right]. \tag{20}$$

We prove this result by induction. For $i = 1$ we have that

$$E\left[\sum_{j=1}^{\ell} X_{\tau_j}\right] \leq E\left[X^{(1)}_{\bar{\tau}_1} + \sum_{j=2}^{\ell} X_{\bar{\tau}_j}\right],$$

since $X^{(1)} \equiv X$ and for all $\ell \geq 1$ we have $\sum_{j=1}^{\ell} X_{\tau_j} = \sum_{j=1}^{\ell} X_{\bar{\tau}_j}$. We conclude then that the inequality (20) is true for $i = 1$.
Let $1 \leq i \leq \ell - 1$, suppose that (20) is true for i. We prove that it is true for $i + 1$.
We have that

$$\begin{aligned} E\left[\sum_{j=1}^{\ell} X_{\tau_j}\right] &= E\left[\sum_{j=1}^{\ell} X_{\bar{\tau}_j}\right] \\ &= E\left[\sum_{j=1}^{i} X_{\bar{\tau}_j} + \sum_{j=i+1}^{\ell} X_{\bar{\tau}_j}\right], \end{aligned} \tag{21}$$

so by the assumption (20) we have that

$$E[X^{(i)}_{\bar{\tau}_i}] \geq E\left[\sum_{j=1}^{i} X_{\bar{\tau}_j}\right]. \tag{22}$$

Let us prove that

$$E[X^{(i)}_{\bar{\tau}_i}] \leq E[X^{(i+1)}_{\bar{\tau}_{i+1}} - X_{\bar{\tau}_{i+1}}]. \tag{23}$$

- If $T-\delta < \bar{\tau}_{i+1} \leq T$

then $X^{(i+1)}_{\bar{\tau}_{i+1}} = X_{\bar{\tau}_{i+1}}$. We have that $\tau_{\ell-i} = \bar{\tau}_{i+1} > T-\delta$ so $\bar{\tau}_i = \tau_{\ell-i+1} \geq \delta + \tau_{\ell-i} > T$, then $X^{(i)}_{\bar{\tau}_i} = 0$, see assumption (3), and then

$$E[X^{(i)}_{\bar{\tau}_i}] = 0 = E[X^{(i+1)}_{\bar{\tau}_{i+1}} - X_{\bar{\tau}_{i+1}}]. \tag{24}$$

- If $0 \leq \bar{\tau}_{i+1} \leq T-\delta$

$$\begin{aligned} E[X^{(i)}_{\bar{\tau}_i}] &= E\left[E\left[X^{(i)}_{\bar{\tau}_i}|\mathcal{F}_{\bar{\tau}_{i+1}+\delta}\right]\right] \\ &\leq E\left[Y^{(i)}_{\bar{\tau}_{i+1}+\delta}\right] \quad ,\bar{\tau}_i = \tau_{\ell-i+1} \geq \tau_{\ell-i} + \delta = \bar{\tau}_{i+1} + \delta \\ &= E[X^{(i+1)}_{\bar{\tau}_{i+1}} - X_{\bar{\tau}_{i+1}}]. \end{aligned} \tag{25}$$

Then the inequality (23) holds in both cases.
In view of (23) and in addition with (22) it gives that

$$E\left[\sum_{j=1}^{i} X_{\bar{\tau}_j}\right] \leq E[X^{(i+1)}_{\bar{\tau}_{i+1}} - X_{\bar{\tau}_{i+1}}].$$

In addition with (21) we obtain that

$$\begin{aligned} E\left[\sum_{j=1}^{\ell} X_{\tau_j}\right] &\leq E[X^{(i+1)}_{\bar{\tau}_{i+1}}] + E\left[\sum_{j=i+1}^{\ell} X_{\bar{\tau}_j}\right] - E[X_{\bar{\tau}_{i+1}}] \\ &= E[X^{(i+1)}_{\bar{\tau}_{i+1}}] + E\left[\sum_{j=i+2}^{\ell} X_{\bar{\tau}_j}\right] \end{aligned}$$

and then (20) is true for $i+1$. We conclude then that (20) is true for all $0 \leq i \leq \ell$.

B) Now using (20) with $i = \ell$, the definition of $Y^{(\ell)}_0$ and Theorem 4.9, we can see that:

$$E\left[\sum_{j=1}^{\ell} X_{\tau_j}\right] \leq E[X^{(\ell)}_{\tau_1}] \leq Y^{(\ell)}_0 = E[X^{(\ell)}_{\tau^*_1}]. \tag{26}$$

We have that $E[X^{(\ell)}_{\tau^*_1}] = E\left[X_{\tau^*_1} + E[Y^{(\ell-1)}_{\tau^*_1+\delta}|\mathcal{F}_{\tau^*_1}]\right]$, then

$$E\left[\sum_{j=1}^{\ell} X_{\tau_j}\right] \leq Y^{(\ell)}_0 = E\left[X_{\tau^*_1}\right] + E[Y^{(\ell-1)}_{\tau^*_1+\delta}]. \tag{27}$$

By Theorem 4.9 and equation (6) we have that for all $i = 1, ..., \ell-1$

$$\begin{aligned} E[Y^{(\ell-i)}_{\tau^*_i+\delta}] &= E[X^{(\ell-i)}_{\tau^*_{i+1}}] \\ &= E[X_{\tau^*_{i+1}}] + E[Y^{(\ell-i-1)}_{\tau^*_{i+1}+\delta}] \end{aligned} \tag{28}$$

By equations (27), (28) and the assumption that $Y^{(0)} = 0$, we obtain that

$$E\left[\sum_{i=1}^{\ell} X_{\tau_i}\right] \leq Y_0^{(\ell)} \leq E[X_{\tau_1^*} + ... + X_{\tau_\ell^*}].$$

We have that $(\tau_1, ..., \tau_\ell)$ is an arbitrary element in $\mathcal{S}_0^{(\ell)}$ so

$$Z_0^{(\ell)} = \sup_{\tau \in \mathcal{S}_0^{(\ell)}} E\left[\sum_{i=1}^{\ell} X_{\tau_i}\right] \leq Y_0^{(\ell)} \leq E[X_{\tau_1^*} + ... + X_{\tau_\ell^*}]. \tag{29}$$

By the definition of $Z_0^{(\ell)}$ we have that $E[X_{\tau_1^*} + ... + X_{\tau_\ell^*}] \leq Z_0^{(\ell)}$, which joined to inequality (29) prove the optimality of the stopping times vector $(\tau_1^*, ..., \tau_\ell^*)$ for the problem $Z_0^{(\ell)}$ together with the equality $Z_0^{(\ell)} = Y_0^{(\ell)}$. □

5 Swing Options in the jump diffusion Model

In this section, we consider a jump diffusion model. We prove that conditions ensuring the existence of an optimal stopping time vector for the optimal multiple stopping time problem are satisfied. Then, we give the solution to the valuation and a vector of optimal stopping times of the swing option under the risk neutral probability measure for general jump diffusion processes.

5.1 The jump diffusion Model

We consider two assets (S^0, X), where S^0 is the bond and X is a risky asset. The dynamics of S^0 is given by $dS_t^0 = rS_t^0 dt$, where $r > 0$ is the interest rate. We assume that the financial market is incomplete, i.e. there are many equivalent martingale measures. We denote by $P_{t,x}$ the historical probability measure when $X_t = x$ and by $Q_{t,x}$ an equivalent martingale measure. To alleviate notations, we omit the dependence of the probability measure $Q_{t,x}$ on the parameters t and x, we denote it by Q, and the expectation under Q by E^Q.

We define two $\mathbb{F}$-Q adapted processes, a standard Brownien motion W and a homogeneous Poisson random measure v with intensity measure $q(ds, dz) = ds \times m(dz)$, m is the Lévy measure on $\mathbb{R}$ of v and $\tilde{v}(ds, dz) := (v - q)(ds, dz)$ is called the compensated jump martingale random measure of v. The process $X = (X_t)_{0 \leq t \leq T}$ evolves according to the following stochastic differential equation:

$$dX_s = b(s, X_{s^-})ds + \sigma(s, X_{s^-})dW_s + \int_{\mathbb{R}} \gamma(s, X_{s^-}, z)\tilde{v}(ds, dz), \quad X_t = x, \tag{30}$$

where b, σ, and γ are continuous functions with respect to (t, x). The Lévy measure m is a positive, σ-finite measure on $\mathbb{R}$, such that

$$\int_{\mathbb{R}} m(dz) \quad < \quad +\infty. \tag{31}$$

Furthermore, we shall make the following assumptions:
there exists $K > 0$ such that for all $t, s \in [0, T]$, x, y and $z \in \mathbb{R}$,

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| + |\gamma(t, x, z) - \gamma(t, y, z)| \leq K|x - y| \tag{32}$$

Notice that the continuity of b, σ and γ with respect to (t,x) and the Lipschitz condition (32) implies the global linear condition

$$|b(t,x)| + |\sigma(t,x)| + |\gamma(t,x,z)| \leq K(1+|x|). \tag{33}$$

Assumptions on b, σ and γ ensure that there exists a unique càdlàg adapted solution to (30) with an initial condition such that

$$E^Q\left[\sup_{s\in[0,T]} |X_s|^2\right] < \infty.$$

We shall also use the notation $X_s^{t,x}$ for X_s whenever we need to emphasize the dependence of the process X on its initial condition.

Remark 5.1 *The process X solution of (30) is quasi-left continuous. It means that $\Delta X_\tau = 0$ a.s on the set $\{\tau < \infty\}$, for every predictable time τ.*

5.2 Formulation of the Optimal Multiple Stopping Time Problem

By the definition of swing option, the valuation of such option is equivalent to solving an optimal multiple stopping time problem. Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}^+$ be a Lipschitz payoff function with linear growth.
The value function of the swing option problem with ℓ exercise rights and refraction time $\delta > 0$ is given by:

$$v^{(\ell)}(x) \quad = \sup_{(\tau_1,\ldots,\tau_\ell)\in\ \mathcal{S}_0^{(\ell)}} E^Q\left[\sum_{i=1}^{\ell} e^{-r\tau_i}\phi(X_{\tau_i}^{0,x})\right]. \tag{34}$$

To solve the problem (34), we define inductively the sequence of ℓ single optimal stopping time problems. We know by a result of El Karoui-Lepeltier-Millet [7] that the Q-viable American option price process is function only of the current price of the underlying stock and of the option expiry time, it is given by :

$$v^{(k)}(t,x) \quad = \quad \operatorname{ess\,sup}_{\tau\in\ \mathcal{S}_t} E^Q\left[e^{-r(\tau-t)}\phi^{(k)}(\tau,X_\tau)|X_t = x\right], \qquad v^{(0)} \equiv 0 \tag{35}$$

where $k = 1, ..., \ell$ and

$$\phi^{(k)}(t,x) = \phi(x) + e^{-r\delta}E^Q\left[v^{(k-1)}(t+\delta, X_{t+\delta})|X_t = x\right], \qquad \forall\ 0 \leq t \leq T-\delta \tag{36}$$

$$\phi^{(k)}(t,x) = \phi(x), \qquad \forall\ T-\delta < t \leq T.$$

Assume that for all $t > T$, $\phi^{(k)}(t,X_t) = 0$.
It is well-known that the process $\left(e^{-rt}v^{(k)}(t,X_t)\right)_{t\in[0,T]}$ is the Snell envelop of the process $\left(e^{-rt}\phi^{(k)}(t,X_t)\right)_{t\in[0,T]}$. To apply the general result, Theorem 4.10, on the optimal multiple stopping time problem obtained in the previous section, we have to show that section 3 conditions are satisfied. This is proved in Proposition 5.2, where the reward process is then given by $U_t := e^{-rt}\phi(X_t)$.

Proposition 5.2 *The $\mathbb{F}$-adapted process U satisfies the following conditions :*

$$U \text{ is right continuous,} \tag{37}$$
$$E^Q\left[\bar{U}^2\right] < \infty \text{ where } \bar{U} = \sup_{0\le t\le T} U_t \tag{38}$$
$$U \text{ is left continuous in expectation,} \tag{39}$$

Proof. Since ϕ is a continuous function and X is right continuous, then the process $(U_t)_{t\in[0,T]} = (e^{-rt}\phi(X_t))_{t\in[0,T]}$ is right continuous.
We have that $\phi(X_s) \le K(1+|X_s|)$, then

$$\begin{aligned} E^Q\left[\left(\sup_{0\le s\le T} \phi(X_s)\right)^2\right] &\le E^Q\left[K\left(1+\sup_{0\le s\le T}|X_s|\right)^2\right] \\ &\le K\left(1+E^Q\left[\sup_{0\le s\le T}|X_s|^2\right]\right), \end{aligned}$$

we obtain then that $E^Q\left[\bar{U}^2\right] < \infty$.
Since X is quasi left continuous, i.e. $\lim_{n\longrightarrow+\infty} X_{\tau_n} = X_\tau$ a.s. on $\{\tau<\infty\}$, where $(\tau_n)_{n\in\mathbb{N}}$ is an increasing sequence of stopping times with $\lim_{n\to\infty}\tau_n = \tau$ a.s., then from the continuity of the payoff function ϕ we deduce that it is also the case for the process U. From the growth condition of ϕ, we have

$$\forall\ 0\le t\le T,\quad U_t \le K(1+|X_t|) \text{ a.s. and } E^Q[|X_t|^2] \le E^Q\left[\sup_{0\le s\le T}|X_s|^2\right] < \infty. \tag{40}$$

By the dominated convergence theorem, we deduce that $\lim_{n\to\infty} E^Q[U_{\tau_n}] = E^Q[U_\tau]$ and then U is left continuous in expectation. □
Let us set

$$\theta_1^* = \inf\{t\ge 0\ ;\ v^{(\ell)}(t,X_t) = \phi^{(\ell)}(t,X_t)\}.$$

For $2\le k\le \ell$, we define

$$\theta_k^* = \inf\{t\ge \delta+\theta_{k-1}^*; v^{(\ell-k+1)}(t,X_t) = \phi^{(\ell-k+1)}(t,X_t)\}\mathbf{1}_{\{\delta+\theta_{k-1}^*\le T\}} + (T_+)\mathbf{1}_{\{\delta+\theta_{k-1}^*>T\}}.$$

Our aim is to relate problem (34) to the sequence of problems defined in (35). The next theorem allows us to use Theorem 4.9 in the Markovian context.

Theorem 5.3 *For each $k=1,...,\ell$, there exists an optimal stopping time for problem (35). Moreover θ_k^* is the minimal optimal stopping time of the problem*

$$v^{(\ell-k+1)}(\theta_{k-1}^*+\delta, X_{\theta_{k-1}^*+\delta}) = ess\sup_{\tau\in\ \mathcal{S}_{\theta_{k-1}^*+\delta}} E^Q\left[e^{-r(\tau-\theta_{k-1}^*-\delta)}\phi^{(\ell-k+1)}(\tau,X_\tau)|\mathcal{F}_{\theta_{k-1}^*+\delta}\right].$$

Also we have

$$v^{(\ell)}(x) = E^Q\left[\sum_{i=1}^{\ell} e^{-r\theta_i^*}\phi(X_{\theta_i^*})\right]$$

Proof. By Proposition 5.2 we have that U is left continuous in expectation and satisfies $E^Q[\bar{U}^2]<\infty$, so by Proposition 4.8 we obtain that for each $k=1,...,\ell$, $U^{(k)} := (e^{-rt}\phi^{(k)}(t,X_t))$ is also left continuous in expectation. In addition, it is a non negative right continuous $\mathbb{F}$-adapted process and satisfies $E^Q\left[(\bar{U}^{(k)})^2\right] < \infty$, see Lemma 4.2. By Theorem 4.9 we obtain the first required result, and by Theorem 4.10, we obtain the second. □

6 Properties of the value functions

In this section we study the regularity of the sequence of the payoff functions defined by (36) and the sequence of value functions defined by (35).

Lemma 6.1 *for all $k = 1, ..., \ell$, there exists $K > 0$ such that for all $(t, x) \in [0, T] \times \mathbb{R}$*

$$|\phi^{(k)}(t,x)| \leq K(1+|x|) \qquad \textit{and} \qquad |v^{(k)}(t,x)| \leq K(1+|x|)$$

Proof. We proceed by induction on k.
For $k = 1$, by the linear growth of ϕ we have that

$$\phi^{(1)}(t,x) := \phi(t,x) \leq K(1+|x|). \tag{41}$$

and

$$\begin{aligned} v^{(1)}(t,x) &= \sup_{\tau \in \ \mathcal{S}_t} E^Q\left[e^{-r(\tau-t)}\phi(X_\tau^{t,x})\right] \\ &\leq K \sup_{\tau \in \ \mathcal{S}_t} E^Q\left[1+|X_\tau^{t,x}|\right] \\ &\leq K(1+|x|), \end{aligned} \tag{42}$$

where the last inequality is deduced by Lemma 3.1 of Pham [15, p.9].
Let $1 \leq k \leq \ell - 1$, suppose that there exists $K > 0$ such that for all $(t, x) \in [0, T] \times \mathbb{R}$, $|\phi^{(k)}(t, x)| \leq K(1 + |x|)$ and $|v^{(k)}(t, x)| \leq K(1 + |x|)$, then

$$\begin{aligned} \phi^{(k+1)}(t,x) &= \phi(x) + e^{-r\delta}E^Q\left[v^{(k)}(t+\delta, X_{t+\delta}^{t,x})\right] \\ &\leq K(1+|x|) + KE^Q\left[1+|X_{t+\delta}^{t,x}|\right] \\ &\leq K(1+|x|). \end{aligned} \tag{43}$$

$$\begin{aligned} v^{(k+1)}(t,x) &= \sup_{\tau \in \ \mathcal{S}_t} E^Q\left[e^{-r(\tau-t)}\phi^{(k)}(X_\tau^{t,x})\right] \\ &\leq K \sup_{\tau \in \ \mathcal{S}_t} E^Q\left[1+|X_\tau^{t,x}|\right] \\ &\leq K(1+|x|), \end{aligned} \tag{44}$$

Which proves the desired result. □

Proposition 6.2 *for all $k = 1, ..., \ell$, there exists $K > 0$ such that for all $t \in [0, T]$, $x, y \in \mathbb{R}$*

$$|\phi^{(k)}(t,x) - \phi^{(k)}(t,y)| \leq K|x-y| \qquad \textit{and} \qquad |v^{(k)}(t,x) - v^{(k)}(t,y)| \leq K|x-y|.$$

Proof. We proceed by induction on k.
For $k = 1$, we have that

$$|\phi^{(1)}(t,x) - \phi^{(1)}(t,y)| := |\phi(x) - \phi(y)| \leq K|x-y| \tag{45}$$

and

$$
\begin{aligned}
|v^{(1)}(t,x)-v^{(1)}(t,y)| &= \left|\sup_{\tau\in\ \mathcal{S}_t} E^Q\left[e^{-r(\tau-t)}\phi(X_\tau^{t,x})\right]-\sup_{\tau\in\ \mathcal{S}_t} E^Q\left[e^{-r(\tau-t)}\phi(X_\tau^{t,y})\right]\right|\\
&\leq \sup_{\tau\in\ \mathcal{S}_t} E^Q\left[|\phi(X_\tau^{t,x})-\phi(X_\tau^{t,y})|\right]\\
&\leq \sup_{\tau\in\ \mathcal{S}_t} E^Q\left[|X_\tau^{t,x}-X_\tau^{t,y}|\right]\\
&\leq K|x-y|. \qquad (46)
\end{aligned}
$$

Let $1\leq k\leq \ell-1$, suppose that there exists $K>0$ such that for all $t\in[0,T]$, $x,y\in\mathbb{R}$, $|\phi^{(k)}(t,x)-\phi^{(k)}(t,y)|\leq K|x-y|$ and $|v^{(k)}(t,x)-v^{(k)}(t,y)|\leq K|x-y|$, then

$$
\begin{aligned}
|\phi^{(k+1)}(t,x)-\phi^{(k+1)}(t,y)| &\leq |\phi(x)-\phi(y)|+Ke^{-r\delta}E^Q[|X_{t+\delta}^{t,x}-X_{t+\delta}^{t,y}|]\\
&\leq K|x-y|, \qquad (47)
\end{aligned}
$$

so $\phi^{(k+1)}$ is Lipschitz with respect to x. Let us prove that it is also for $v^{(k+1)}$, we have

$$
\begin{aligned}
|v^{(k+1)}(t,x)-v^{(k+1)}(t,y)| &\leq \left|\sup_{\tau\in\ \mathcal{S}_t} E^Q\left[\phi^{(k+1)}(\tau,X_\tau^{t,x})\right]-\sup_{\tau\in\ \mathcal{S}_t} E^Q\left[\phi^{(k+1)}(\tau,X_\tau^{t,y})\right]\right|\\
&\leq \sup_{\tau\in\ \mathcal{S}_t} E^Q\left[|\phi^{(k+1)}(\tau,X_\tau^{t,x})-\phi^{(k+1)}(\tau,X_\tau^{t,y})|\right]\\
&\leq K\sup_{\tau\in\ \mathcal{S}_t} E^Q\left[|X_\tau^{t,x}-X_\tau^{t,y}|\right]\\
&\leq K|x-y| \qquad (48)
\end{aligned}
$$

then $v^{(k+1)}$ is Lipschitz with respect to x.
We conclude then that for all $k=1,...,\ell$, $\phi^{(k)}$ and $v^{(k)}$ are both Lipschitz with respect to x. □
To prove the following theorem, we need to recall the Dynamic Programming Principle.

Proposition 6.3 *[15]**(Dynamic Programming Principle)** For all $(t,x)\in[0,T]\times\mathbb{R}$, $h\in\mathcal{S}_t$, $k=1,...,\ell$ we have*

$$v^{(k)}(t,x)=\sup_{\tau\in\ \mathcal{S}_t} E^Q\left[\mathbf{1}_{\{\tau<h\}}e^{-r(\tau-t)}\phi^{(k)}(\tau,X_\tau^{t,x})+\mathbf{1}_{\{\tau\geq h\}}e^{-r(h-t)}v^{(k)}(h,X_h^{t,x})\right].$$

Theorem 6.4 *For $k=1,...,\ell$ and for all $t<s\in[0,T]$, $x\in\mathbb{R}$, there exists a constant $C>0$ such that*

$$\left|\phi^{(k)}(t,x)-\phi^{(k)}(s,x)\right|\leq C(1+|x|)\sqrt{s-t} \qquad (49)$$

and

$$\left|v^{(k)}(s,x)-v^{(k)}(t,x)\right|\leq C(1+|x|)\sqrt{s-t}. \qquad (50)$$

Proof. Let us prove this theorem by induction.
For $k=1$, we have that $\phi^{(1)}(t,x)=\phi^{(1)}(s,x)=\phi(x)$, then the inequality (49) is true for $k=1$.

Let $0 \leq t < s \leq T$, by the Dynamic Programming Principle, with $h = s$ we obtain

$$\begin{aligned} v^{(1)}(t,x) - v^{(1)}(s,x) = \sup_{\tau \in \mathcal{S}_t} E^Q \Big[& \mathbf{1}_{\{\tau < s\}} e^{-r(\tau - t)} \phi(X_\tau^{t,x}) + \mathbf{1}_{\{\tau \geq s\}} e^{-r(s-t)} v^{(1)}(s, X_s^{t,x}) \\ & - \mathbf{1}_{\{\tau \geq s\}} v^{(1)}(s,x) - \mathbf{1}_{\{\tau < s\}} v^{(1)}(s,x) \Big] \\ \leq \sup_{\tau \in \mathcal{S}_t} E^Q \Big[& \mathbf{1}_{\{\tau < s\}} e^{-r(\tau - t)} |\phi(X_\tau^{t,x}) - \phi(x)| \\ & + \mathbf{1}_{\{\tau < s\}} (\phi(x) - v^{(1)}(s,x)) + \mathbf{1}_{\{\tau \geq s\}} e^{-r(s-t)} |v^{(1)}(s, X_s^{t,x}) - v^{(1)}(s,x)| \\ & + \mathbf{1}_{\{\tau \geq s\}} |e^{-r(s-t)} - 1| |v^{(1)}(s,x)| + \mathbf{1}_{\{\tau < s\}} |e^{-r(\tau - t)} - 1| |v^{(1)}(s,x)| \Big]. \end{aligned}$$

From the Lipschitz property of ϕ and $v^{(1)}$ and Lemma 3.1 of Pham [15], we have

$$\begin{aligned} E^Q \big[\mathbf{1}_{\{\tau < s\}} e^{-r(\tau - t)} |\phi(X_\tau^{t,x}) - \phi(x)| \big] &\leq C E^Q \big[\mathbf{1}_{\{\tau < s\}} |X_\tau^{t,x} - x| \big] \\ &\leq C(1 + |x|)\sqrt{s - t} \end{aligned} \tag{51}$$

and

$$\begin{aligned} E^Q \big[\mathbf{1}_{\{\tau \geq s\}} e^{-r(s-t)} |v^{(1)}(s, X_s^{t,x}) - v^{(1)}(s,x)| \big] &\leq C E^Q \big[|X_s^{t,x} - x| \big] \\ &\leq C(1 + |x|)\sqrt{s - t}. \end{aligned} \tag{52}$$

From the growth condition of $v^{(1)}$ and since $0 \leq 1 - e^{-rh} \leq r\sqrt{h}, \forall h \in [0, \infty)$, we have

$$E^Q \big[\mathbf{1}_{\{\tau \geq s\}} |e^{-r(s-t)} - 1| |v^{(1)}(s,x)| + \mathbf{1}_{\{\tau < s\}} |e^{-r(\tau - t)} - 1| |v^{(1)}(s,x)| \big] \leq C(1 + |x|)\sqrt{s - t}. \tag{53}$$

By noting that $\phi(x) \leq v^{(1)}(s,x)$ and by inequalities (51), (52) and (53) we deduce that

$$|v^{(1)}(t,x) - v^{(1)}(s,x)| \leq C(1 + |x|)\sqrt{s - t}. \tag{54}$$

Let $1 \leq k \leq \ell - 1$, suppose that (49) and (50) are true for k and let us prove that there are true for $k + 1$.
By the induction assumption we obtain

$$\begin{aligned} \left| \phi^{(k+1)}(t,x) - \phi^{(k+1)}(s,x) \right| &= \left| e^{-r\delta} E^Q [v^{(k)}(t + \delta, X_{t+\delta}^{t,x}) - v^{(k)}(s + \delta, X_{s+\delta}^{s,x})] \right| \\ &\leq E^Q [|v^{(k)}(t + \delta, X_{t+\delta}^{t,x}) - v^{(k)}(s + \delta, X_{t+\delta}^{t,x})|] \\ &+ E^Q [|v^{(k)}(s + \delta, X_{t+\delta}^{t,x}) - v^{(k)}(s + \delta, X_{s+\delta}^{s,x})|] \\ &\leq C \left((1 + E^Q [|X_{t+\delta}^{t,x}|]) \sqrt{s - t} + E^Q [|X_{t+\delta}^{t,x} - X_{s+\delta}^{s,x}|] \right) \\ &\leq C(1 + |x|)\sqrt{s - t} \end{aligned} \tag{55}$$

where the last inequality is deduced by Lemma 8.2 (see Appendix).
By the Dynamic Programming Principle, with $h = s$ we obtain

$$\begin{aligned}
v^{(k+1)}(t,x) - v^{(k+1)}(s,x) = \sup_{\tau\in\mathcal{S}_t} E^Q\Big[&\mathbf{1}_{\{\tau<s\}}e^{-r(\tau-t)}\phi^{(k+1)}(\tau,X_\tau^{t,x})\\
&+\mathbf{1}_{\{\tau\geq s\}}e^{-r(s-t)}v^{(k+1)}(s,X_s^{t,x})\\
&-\mathbf{1}_{\{\tau\geq s\}}v^{(k+1)}(s,x)-\mathbf{1}_{\{\tau<s\}}v^{(k+1)}(s,x)\Big]\\
\leq \sup_{\tau\in\mathcal{S}_t} E^Q\Big[&\mathbf{1}_{\{\tau<s-t\}}e^{-r(\tau-t)}|\phi^{(k+1)}(\tau,X_\tau^{t,x})-\phi^{(k+1)}(s,x)|\\
&+\mathbf{1}_{\{\tau<s\}}e^{-r(\tau-t)}(\phi^{(k+1)}(s,x)-v^{(k+1)}(s,x))\\
&+\mathbf{1}_{\{\tau\geq s\}}e^{-r(s-t)}|v^{(k+1)}(s,X_s^{t,x})-v^{(k+1)}(s,x)|\\
&+\mathbf{1}_{\{\tau\geq s\}}|e^{-r(s-t)}-1||v^{(k+1)}(s,x)|\\
&+\mathbf{1}_{\{\tau<s\}}|e^{-r\tau}-1||v^{(k+1)}(s,x)|\Big]
\end{aligned}$$

From the Lipschitz property of $\phi^{(k+1)}$ and $v^{(k+1)}$ and Lemma 3.1 of Pham [15], we have

$$\begin{aligned}
E^Q\big[\mathbf{1}_{\{\tau<s\}}e^{-r(\tau-t)}|\phi^{(k+1)}(\tau,X_\tau^{t,x})-\phi^{(k+1)}(s,x)|\big] &\leq CE^Q\big[\mathbf{1}_{\{\tau<s\}}|X_\tau^{t,x}-x|\big]\\
&\leq C(1+|x|)\sqrt{s-t} \qquad (56)
\end{aligned}$$

and

$$\begin{aligned}
E^Q\big[\mathbf{1}_{\{\tau\geq s\}}e^{-r(s-t)}|v^{(k+1)}(s,X_s^{t,x})-v^{(k+1)}(s,x)|\big] &\leq CE^Q\big[\mathbf{1}_{\{\tau<s\}}|X_s^{t,x}-x|\big]\\
&\leq C(1+|x|)\sqrt{s-t}. \qquad (57)
\end{aligned}$$

From the growth condition of $v^{(k+1)}$ and since $0\leq 1-e^{-rh}\leq r\sqrt{h}, \forall h\in[0,\infty)$, we have

$$E^Q\big[\mathbf{1}_{\{\tau\geq s\}}|e^{-r(s-t)}-1||v^{(k+1)}(s,x)|+\mathbf{1}_{\{\tau<s\}}|e^{-r(\tau)}-1||v^{(k+1)}(s,x)|\big]\leq C(1+|x|)\sqrt{s-t}. \tag{58}$$

By noting that $\phi^{(k+1)}(s,x)\leq v^{(k+1)}(s,x)$ and by the inequalities (56), (57) and (58) we deduce that

$$|v^{(k+1)}(t,x)-v^{(k+1)}(s,x)|\leq C(1+|x|)\sqrt{s-t}. \tag{59}$$

We can then conclude that (49) and (50) are true for all $k = 1, ..., \ell$. □

7 Viscosity solutions and comparison theorem

The aim of this section is to characterize the value function as the unique viscosity solution of the associated HJBVI defined by

$$\begin{aligned}
\min\{rv^{(k)}(t,x)-\frac{\partial v^{(k)}}{\partial t}(t,x)-A(t,x,\frac{\partial v^{(k)}}{\partial x}(t,x),\frac{\partial^2 v^{(k)}}{\partial x^2}(t,x))-B(t,x,\frac{\partial v^{(k)}}{\partial x}(t,x),v^{(k)});\\
v^{(k)}(t,x)-\phi^{(k)}(t,x)\}=0, \quad \forall (t,x)\in[0,T)\times\mathbb{R}
\end{aligned} \tag{60}$$

$$v^{(k)}(T,x)=\phi(x), \quad \forall x\in\mathbb{R} \tag{61}$$

where, for $t \in [0,T]$, $x \in \mathbb{R}$, $p \in \mathbb{R}$, $M \in \mathbb{R}$ the operator:

$$A(t,x,p,M) \quad := \quad \frac{1}{2}\sigma^2(t,x)M + b(t,x)p, \tag{62}$$

and for $\varphi \in \mathcal{C}^{1,2}([0,T] \times \mathbb{R})$, we define:

$$B(t,x,\frac{\partial \varphi}{\partial x}(t,x),\varphi) := \int_{\mathbb{R}} [\varphi(t,x+\gamma(t,x,z)) - \varphi(t,x) - \gamma(t,x,z)\frac{\partial \varphi}{\partial x}(t,x)]m(dz). \tag{63}$$

Let us give the definition of viscosity solution which is introduced by Crandall and Lions [4] for the first order equation, then generalized to the second order by Gimbert and Lions [8].

Definition 7.1 *Let $k = 1,...,\ell$, and $u^{(k)}$ be a continuous function.*
(i) We say that $u^{(k)}$ is a viscosity supersolution (subsolution) of (60) if

$$\min\{r\varphi(t_0,x_0) - \frac{\partial \varphi}{\partial t}(t_0,x_0) - A(t_0,x_0,\frac{\partial \varphi}{\partial x}(t_0,x_0),\frac{\partial^2 \varphi}{\partial x^2}(t_0,x_0)) - B(t_0,x_0,\frac{\partial \varphi}{\partial x}(t_0,x_0),\varphi);$$
$$\varphi(t_0,x_0) - \phi^{(k)}(t_0,x_0)\} \geq 0 \tag{64}$$

(≤ 0) *whenever $\varphi \in \mathcal{C}^{1,2}([0,T) \times \mathbb{R})$ and $u^{(k)} - \varphi$ has a strict global minimum (maximum) at $(t_0,x_0) \in [0,T) \times \mathbb{R}$.*
(ii) We say that $u^{(k)}$ is a viscosity solution of (60) if it is both super and subsolution of (60).

By Soner [17, Lemma 2.1] or Sayah [16, Proposition 2.1], we can see an equivalent formulation for viscosity solution in $\mathcal{C}_2([0,T] \times \mathbb{R})$, where

$$\mathcal{C}_2([0,T] \times \mathbb{R}) := \{\varphi \in \mathcal{C}^0([0,T] \times \mathbb{R}) / \sup_{[0,T]\times\mathbb{R}} \frac{|\varphi(t,x)|}{1+|x|^2} < +\infty\}.$$

Lemma 7.2 *Let $u^{(k)} \in \mathcal{C}_2([0,T] \times \mathbb{R})$. Then $u^{(k)}$ is a viscosity supersolution (subsolution) of (60) if and only if:*

$$\begin{aligned}\min\{ru^{(k)}(t_0,x_0) - \frac{\partial \varphi}{\partial t}(t_0,x_0) - A(t_0,x_0,D_x\varphi(t_0,x_0),D_x^2\varphi(t_0,x_0))\\ - B(t_0,x_0,D_x\varphi(t_0,x_0),u^{(k)}); u^{(k)}(t_0,x_0) - \phi^{(k)}(t_0,x_0)\} \geq 0\end{aligned} \tag{65}$$

(≤ 0) *whenever $\varphi \in \mathcal{C}^2([0,T] \times \mathbb{R})$ and $u^{(k)} - \varphi$ has a strict global minimum (maximum) at $(t_0,x_0) \in [0,T) \times \mathbb{R}$.*

Theorem 7.3 *For all $k = 1,...,\ell$, the value function $v^{(k)}$ is a viscosity solution of the HJBVI (60) on $[0,T) \times \mathbb{R}$.*

Proof. Viscosity supersolution:
Let $\varphi \in \mathcal{C}^{1,2}([0,T) \times \mathbb{R})$ and $(t_0,x_0) \in [0,T) \times \mathbb{R}$ be a strict global minimum of φ such that

$$0 = (v^{(k)} - \varphi)(t_0,x_0) = \min_{(t,x)\in[0,T)\times\mathbb{R}} (v^{(k)} - \varphi)(t,x). \tag{66}$$

From the Dynamic Programming Principle, it follows that for all $h > 0$, $\theta \in \mathcal{S}_{t_0}$

$$v^{(k)}(t_0, x_0) = \sup_{\tau \in \mathcal{S}_{t_0}} E^Q[\mathbf{1}_{\{\tau < \theta\wedge(t_0+h)\}} e^{-r(\tau - t_0)} \phi^{(k)}(\tau, X_\tau^{t_0,x_0}) + \mathbf{1}_{\{\tau \geq \theta\wedge(t_0+h)\}} e^{-r(\theta\wedge(t_0+h))} v^{(k)}(\theta \wedge (t_0+h), X_{\theta\wedge(t_0+h)}^{t_0,x_0})]$$

Let $\eta > 0$, we define the stopping time

$$\theta := \inf\{t > t_0 \quad : \quad (t, X_t^{t_0,x_0}) \notin B_\eta(t_0, x_0)\} \wedge T$$

where $B_\eta(t_0, x_0) := \{(t, x) \in [0, T] \times \mathbb{R}$ such that $|t - t_0| + |x - x_0| \leq \eta\}$. Then

$$\begin{aligned} \varphi(t_0, x_0) = v^{(k)}(t_0, x_0) &\geq E^Q\left[e^{-r(\theta\wedge(t_0+h))} v^{(k)}(\theta \wedge (t_0+h), X_{\theta\wedge(t_0+h)}^{t_0,x_0})\right] \\ &\geq E^Q\left[e^{-r(\theta\wedge(t_0+h))} \varphi(\theta \wedge (t_0+h), X_{\theta\wedge(t_0+h)}^{t_0,x_0})\right]. \end{aligned} \tag{67}$$

By applying Itô's Lemma to $e^{-rs\varphi(s,X_s^{t_0,x_0})}$ we obtain that

$$\begin{aligned} &E^Q\left[e^{-r(\theta\wedge(t_0+h))} \varphi(\theta \wedge (t_0+h), X_{\theta\wedge(t_0+h)}^{t_0,x_0})\right] - e^{-rt_0}\varphi(t_0, x_0) \\ &= E^Q\Bigg[\int_{t_0}^{\theta\wedge(t_0+h)} e^{-rs}\Bigg(- r\varphi(s, X_s^{t_0,x_0}) + \frac{\partial \varphi}{\partial s}(s, X_s^{t_0,x_0}) \\ &+ A(s, X_s^{t_0,x_0}, \frac{\partial \varphi}{\partial x}(s, X_s^{t_0,x_0}), \frac{\partial^2 \varphi}{\partial x^2}(s, X_s^{t_0,x_0})) + B(s, X_s^{t_0,x_0}, \frac{\partial \varphi}{\partial x}(s, X_s^{t_0,x_0}), \varphi)\Bigg) ds\Bigg], \end{aligned} \tag{68}$$

where A and B are defined by (62) and (63) respectively.
By using inequality (67) and dividing by h we obtain

$$\begin{aligned} 0 \geq \frac{1}{h} E^Q\Bigg[\int_{t_0}^{\theta\wedge(t_0+h)} & e^{-rs}(-r\varphi(s, X_s^{t_0,x_0}) + \frac{\partial \varphi}{\partial s}(s, X_s^{t_0,x_0}) \\ + A(s, X_s^{t_0,x_0}, \frac{\partial \varphi}{\partial x}(s, X_s^{t_0,x_0}), \frac{\partial^2 \varphi}{\partial x^2}(s, X_s^{t_0,x_0})) &+ B(s, X_s^{t_0,x_0}, \frac{\partial \varphi}{\partial x}(s, X_s^{t_0,x_0}), \varphi))ds\Bigg]. \end{aligned}$$

Sending h to 0, we deduce by the mean value theorem the a.s. convergence of the random value in the expectation. Then it follows from assumption (31) and the dominated convergence theorem that

$$r\varphi(t_0, x_0) - \frac{\partial \varphi}{\partial s}(t_0, x_0) - A\left(t_0, x_0, \frac{\partial \varphi}{\partial x}(t_0, x_0), \frac{\partial^2 \varphi}{\partial x^2}(t_0, x_0)\right) - B\left(t_0, x_0, \frac{\partial \varphi}{\partial x}(t_0, x_0), \varphi\right) \geq 0,$$

then $v^{(k)}$ is a viscosity supersolution of (60).
Viscosity subsolution:
We fix $\eta > 0$. Let $(t_0, x_0) \in [0, T) \times \mathbb{R}$ and $\varphi \in \mathcal{C}^{1,2}([0, T) \times \mathbb{R})$ be such that

$$0 = (v^{(k)} - \varphi)(t_0, x_0) > (v^{(k)} - \varphi)(t, x) \quad \text{for all } (t, x) \in [0, T) \times \mathbb{R}\backslash\{(t_0, x_0)\}. \tag{69}$$

Then there exists $\xi > 0$ such that

$$\max_{(t,x)\in\partial B_\eta(t_0,x_0)} (v^{(k)} - \varphi)(t, x) = -\xi, \tag{70}$$

where $B_\eta(t_0,x_0)=\{(t,x)\in[0,T]\times\mathbb{R}$ such that $|t-t_0|+|x-x_0|\le\eta\}$. Then,

$$\text{for all } (t,x)\in\partial B_\eta(t_0,x_0),\quad (v^{(k)}-\varphi)(t,x)\le-\xi, \tag{71}$$

where $\partial B_\eta(t_0,x_0)$ is the parabolic boundary of $B_\eta(t_0,x_0)$.
In order to prove the required result, we assume to the contrary that there exists $\varepsilon>0$ such that for all $(t,x)\in B_\eta(t_0,x_0)$

$$\min\{-\frac{\partial\varphi}{\partial t}(t,x)-\mathcal{L}\varphi(t,x);\varphi(t,x)-\phi^{(k)}(t,x)\}\ge\varepsilon, \tag{72}$$

where $\mathcal{L}\varphi(t,x):=-r\varphi(t,x)+A\left(t,x,\frac{\partial\varphi}{\partial x}(t,x),\frac{\partial^2\varphi}{\partial x^2}(t,x)\right)+B\left(t,x,\frac{\partial\varphi}{\partial x}(t,x),\varphi\right)$.
Let us define the stopping times:

$$\begin{aligned}\theta_k^1&:=\inf\{t>t_0:(t,X_t^{t_0,x_0})\in\partial B_\eta(t_0,x_0)\}\wedge T\\ \theta_k^2&:=\inf\{t>t_0:\phi^{(k)}(t,X_t^{t_0,x_0})=v^{(k)}(t,X_t^{t_0,x_0})\}\wedge T\end{aligned}$$

On the set $\{\theta_k^1<\theta_k^2\}$:
We have that $(\theta_k^1,X_{\theta_k^1}^{t_0,x_0})\in\partial B_\eta(t_0,x_0)$, so by equality (70) we obtain that $(v^{(k)}-\varphi)(\theta_k^1,X_{\theta_k^1}^{t_0,x_0})\le-\xi$ and then by applying Itô's Lemma to $e^{-r(s-t_0)}\varphi(s,X_s^{t_0,x_0})$, we obtain

$$\begin{aligned}&e^{-r(\theta_k^1-t_0)}\left(v^{(k)}(\theta_k^1,X_{\theta_k^1}^{t_0,x_0})+\xi\right)-v^{(k)}(t_0,x_0)\\ &\le e^{-r(\theta_k^1-t_0)}\varphi(\theta_k^1,X_{\theta_k^1}^{t_0,x_0})-\varphi(t_0,x_0)\\ &=\int_{t_0}^{\theta_k^1}e^{-r(s-t_0)}\left(\frac{\partial\varphi}{\partial s}(s,X_s^{t_0,x_0})+\mathcal{L}\varphi(s,X_s^{t_0,x_0})\right)ds\\ &+\int_{t_0}^{\theta_k^1}e^{-r(s-t_0)}\sigma(s,X_s^{t_0,x_0})\frac{\partial\varphi}{\partial x}(s,X_s^{t_0,x_0})dW_s\\ &+\int_{t_0}^{\theta_k^1}e^{-r(s-t_0)}\int_{\mathbb{R}}\left[\varphi(s,X_{s^-}^{t_0,x_0}+\gamma(s,X_{s^-}^{t_0,x_0},z))-\varphi(s,X_{s^-}^{t_0,x_0})\right]\tilde{v}(ds,dz)\end{aligned} \tag{73}$$

On the set $\{\theta_k^1\ge\theta_k^2\}$:
We have that $(\theta_k^2,X_{\theta_k^2}^{t_0,x_0})\in B_\eta(t_0,x_0)$, so by (72) we obtain that

$$v^{(k)}(\theta_k^2,X_{\theta_k^2}^{t_0,x_0})=\phi^{(k)}(\theta_k^2,X_{\theta_k^2}^{t_0,x_0})\le\varphi(\theta_k^2,X_{\theta_k^2}^{t_0,x_0})-\varepsilon. \tag{74}$$

Now, by applying Itô's Lemma to $e^{-r(s-t_0)}\varphi(s, X_s^{t_0,x_0})$ we obtain

$$
\begin{aligned}
e^{-r(\theta_k^2-t_0)}&\left(\varepsilon+\phi^{(k)}(\theta_k^2, X_{\theta_k^2}^{t_0,x_0})\right)-v^{(k)}(t_0,x_0)\\
&\le e^{-r(\theta_k^2-t_0)}\varphi(\theta_k^2, X_{\theta_k^2}^{t_0,x_0})-\varphi(t_0,x_0)\\
&=\int_{t_0}^{\theta_k^2} e^{-r(s-t_0)}\left(\frac{\partial\varphi}{\partial s}(s,X_s^{t_0,x_0})+\mathcal{L}\varphi(s,X_s^{t_0,x_0})\right)ds\\
&+\int_{t_0}^{\theta_k^2} e^{-r(s-t_0)}\sigma(s,X_s^{t_0,x_0})\frac{\partial\varphi}{\partial x}(s,X_s^{t_0,x_0})dW_s\\
&+\int_{t_0}^{\theta_k^2} e^{-r(s-t_0)}\int_{\mathbb{R}}[\varphi(s,X_{s^-}^{t_0,x_0}+\gamma(s,X_{s^-}^{t_0,x_0},z))\\
&-\varphi(s,X_{s^-}^{t_0,x_0})]\tilde{v}(ds,dz).
\end{aligned}
\tag{75}
$$

Let us denote by $\theta_k := \theta_k^1 \wedge \theta_k^2$. By multiplying respectively inequalities (73) and (75) by $\mathbf{1}_{\{\theta_k^1<\theta_k^2\}}$ and $\mathbf{1}_{\{\theta_k^1\ge\theta_k^2\}}$ respectively, and by getting the expectation we obtain

$$
\begin{aligned}
&E^Q\left[\left(e^{-r(\theta_k^1-t_0)}\left(v^{(k)}(\theta_k^1, X_{\theta_k^1}^{t_0,x_0})+\xi\right)-v^{(k)}(t_0,x_0)\right)\mathbf{1}_{\{\theta_k^1\le\theta_k^2\}}\right]\\
&\le E^Q\left[\left(\int_{t_0}^{\theta_k} e^{-r(s-t_0)}\left(\frac{\partial\varphi}{\partial s}(s,X_s^{t_0,x_0})+\mathcal{L}\varphi(s,X_s^{t_0,x_0})\right)ds\right)\mathbf{1}_{\{\theta_k^1\le\theta_k^2\}}\right]
\end{aligned}
\tag{76}
$$

and

$$
\begin{aligned}
&E^Q\left[e^{-r(\theta_k^2-t_0)}\left(\varepsilon+\phi^{(k)}(\theta_k^2, X_{\theta_k^2}^{t_0,x_0})-v^{(k)}(t_0,x_0)\right)\mathbf{1}_{\{\theta_k^1>\theta_k^2\}}\right]\\
&\le E^Q\left[\left(\int_{t_0}^{\theta_k} e^{-r(s-t_0)}\left(\frac{\partial\varphi}{\partial s}(s,X_s^{t_0,x_0})+\mathcal{L}\varphi(s,X_s^{t_0,x_0})\right)ds\right)\mathbf{1}_{\{\theta_k^1>\theta_k^2\}}\right].
\end{aligned}
\tag{77}
$$

By adding inequalities (76) and (77) and using inequality (72) we obtain that

$$
\begin{aligned}
&\varepsilon E^Q\left[e^{-r(\theta_k^2-t_0)}\mathbf{1}_{\{\theta_k^1>\theta_k^2\}}\right]+E^Q\left[e^{-r(\theta_k^1-t_0)}v^{(k)}(\theta_k^1, X_{\theta_k^1}^{t_0,x_0})\mathbf{1}_{\{\theta_k^1\le\theta_k^2\}}\right]\\
&-v^{(k)}(t_0,x_0)+\xi E^Q\left[e^{-r(\theta_k^1-t_0)}\mathbf{1}_{\{\theta_k^1\le\theta_k^2\}}\right]+E^Q\left[e^{-r(\theta_k^2-t_0)}\phi^{(k)}(\theta_k^2, X_{\theta_k^2}^{t_0,x_0})\mathbf{1}_{\{\theta_k^1>\theta_k^2\}}\right]\\
&\le -\varepsilon E^Q\left[\int_{t_0}^{\theta_k} e^{-r(s-t_0)}ds\right]<0.
\end{aligned}
\tag{78}
$$

Let us suppose that for all $\xi'>0$, we have that

$$
H := \xi E^Q\left[e^{-r(\theta_k^1-t_0)}\mathbf{1}_{\{\theta_k^1\le\theta_k^2\}}\right]+\varepsilon E^Q\left[e^{-r(\theta_k^2-t_0)}\mathbf{1}_{\{\theta_k^1>\theta_k^2\}}\right]\le\xi' \tag{79}
$$

then

$$
0\le \xi E^Q\left[e^{-r(\theta_k^1-t_0)}\mathbf{1}_{\{\theta_k^1\le\theta_k^2\}}\right]<\xi'.
$$

By sending ξ' to 0 we obtain that $\mathbf{1}_{\{\theta_k^1\le\theta_k^2\}}=0$ a.s., then $\theta_k^1>\theta_k^2$ a.s.. So we obtain that $H=\varepsilon E^Q\left[e^{-r(\theta_k^2-t_0)}\right]<\xi'$. By sending ξ' to 0 we obtain that $\varepsilon E^Q\left[e^{-r(\theta_k^2-t_0)}\right]\le 0$, which is in contradiction with the fact that

$\varepsilon E^Q\left[e^{-r(\theta_k^2-t_0)}\right] > 0$. We conclude then that there exists $\xi' > 0$ such that $H \geq \xi'$.
On the other hand, we have that $\{e^{-rt}v^{(k)}(t, X_t^{t_0,x_0}), t_0 \leq t \leq T\}$ is a supermartingale,then by [9, Theorem D.9, p.355], the stopped supermartingale $\{e^{-r(t\wedge\theta_k^2)}v^{(k)}(t\wedge\theta_k^2, X_{t\wedge\theta_k^2}^{t_0,x_0}), t_0 \leq t \leq T\}$ is a martingale. From the growth condition on $v^{(k)}$, the martingale $\{e^{-r(t\wedge\theta_k^2)}v^{(k)}(t\wedge\theta_k^2, X_{t\wedge\theta_k^2}^{t_0,x_0}), t_0 \leq t \leq T\}$ is bounded in $L^1(Q)$ and so uniformly integrable. By the Stopping Theorem we obtain that

$$e^{-rt_0}v^{(k)}(t_0, x_0) = E^Q\left[e^{-r(t\wedge\theta_k)}v^{(k)}(t\wedge\theta_k, X_{t\wedge\theta_k}^{t_0,x_0})\right], \quad \forall t \in \mathcal{S}_{t_0} \tag{80}$$

and so

$$\begin{aligned} v^{(k)}(t_0, x_0) = & E^Q\left[e^{-r(\theta_k^1-t_0)}v^{(k)}(\theta_k^1, X_{\theta_k^1}^{t_0,x_0})\mathbf{1}_{\{\theta_k^1\leq\theta_k^2\}}\right] \\ & + E^Q\left[e^{-r(\theta_k^2-t_0)}\phi^{(k)}(\theta_k^2, X_{\theta_k^2}^{t_0,x_0})\mathbf{1}_{\{\theta_k^1>\theta_k^2\}}\right]. \end{aligned} \tag{81}$$

From inequality (78) and the fact that $H \geq \xi'$, we deduce that $\xi' \leq 0$, which contradicts the fact that $\xi' > 0$.
We conclude then that the value function $v^{(k)}$ is a viscosity subsolution of the equation (60) on $[0, T) \times \mathbb{R}$. □
Let us now prove the uniqueness of viscosity solutions. First, we recall the notion of parabolic superjet and parabolic subjet as introduced in P.L. Lions [11].
Let $v \in \mathcal{C}^0([0, T] \times \mathbb{R})$ and $(t, x) \in [0, T) \times \mathbb{R}$, we define the parabolic superjet:

$$\begin{aligned} \mathcal{P}^{2,+}v(t,x) = \{ & (p_0, p, a) \in \mathbb{R}\times\mathbb{R}\times\mathbb{R}/v(s,y) \leq v(t,x) + p_0(s-t) + p(y-x) \\ & + \frac{1}{2}a(y-x)^2 + o(|s-t| + |y-x|^2) \text{ as } (s,y) \to (t,x)\} \end{aligned}$$

and its closure

$$\begin{aligned} \bar{\mathcal{P}}^{2,+}v(t,x) = \{ & (p_0, p, a) = \lim_{n\to+\infty}(p_{0,n}, p_n, a_n) \\ & \text{with } (p_{0,n}, p_n, a_n) \in \mathcal{P}^{2,+}v(t_n, x_n) \\ & \text{and } \lim_{n\to+\infty}(t_n, x_n, v(t_n, x_n)) = (t, x, v(t, x))\}. \end{aligned}$$

The parabolic subjet is defined by $\mathcal{P}^{2,-}v(t,x) = -\mathcal{P}^{2,+}(-v)(t,x)$.
As in Pham [14, Lemma 2.2], we have an intrinsic formulation of viscosity solutions in $\mathcal{C}_2([0, T] \times \mathbb{R})$.

Lemma 7.4 *Let $v^{(k)} \in \mathcal{C}_2([0, T] \times \mathbb{R})$ be a viscosity supersolution (resp. subsolution) of (60). Then, for all $(t, x) \in [0, T) \times \mathbb{R}$, for all $(p_0, p, a) \in \bar{\mathcal{P}}^{2,-}v^{(k)}(t, x)$ (resp. $\bar{\mathcal{P}}^{2,+}v^{(k)}(t, x)$), we have*

$$\min\{rv^{(k)}(t,x) - p_0 - A(t,x,p,a) - B(t,x,p,v^{(k)}); v^{(k)}(t,x) - \phi^{(k)}(t,x)\} \geq 0 \tag{82}$$

(resp. ≤ 0).

Theorem 7.5 ***(Comparison Theorem)***
Assume that the assumptions (31), (32), (33) and the Lipschitz continuity of ϕ hold. Let $u^{(k)}$ (resp. $v^{(k)}$), $k = 1, ..., \ell$, be a viscosity subsolution (resp. supersolution) of (60). Assume also that $u^{(k)}$ and $v^{(k)}$ are Lipschitz, have a linear growth in x and holder in t. If

$$u^{(k)}(T, x) \le v^{(k)}(T, x) \qquad \forall x \in \mathbb{R}, \tag{83}$$

then

$$u^{(k)}(t, x) \le v^{(k)}(t, x) \qquad \forall (t, x) \in [0, T] \times \mathbb{R}. \tag{84}$$

Proof. Let $k \in \{1, ..., \ell\}$. We have that $u^{(k)}$ and $v^{(k)}$ are continuous in $t = 0$, then it suffices to prove inequality (84) for all $(t, x) \in (0, T] \times \mathbb{R}$. Let β, ε, δ and $\lambda > 0$, we define the function ψ_k in $(0, T] \times \mathbb{R}$:

$$\psi_k(t, x, y) = u^{(k)}(t, x) - v^{(k)}(t, y) - \frac{\beta}{t} - \frac{1}{2\varepsilon}|x - y|^2 - \delta e^{\lambda(T-t)}(|x|^2 + |y|^2). \tag{85}$$

By the continuity and the linear growth condition of $u^{(k)}$ and $v^{(k)}$ we can see that ψ_k admits a maximum at $(\bar{t}, \bar{x}, \bar{y}) \in (0, T] \times \mathbb{R} \times \mathbb{R}$, to simplify the notation we omit the dependance on β, ε, δ and λ. We can see that $2\psi_k(\bar{t}, \bar{x}, \bar{y}) \ge \psi_k(\bar{t}, \bar{x}, \bar{x}) + \psi_k(\bar{t}, \bar{y}, \bar{y})$, so we obtain

$$\frac{1}{\varepsilon}|\bar{x} - \bar{y}|^2 \le u^{(k)}(\bar{t}, \bar{x}) - u^{(k)}(\bar{t}, \bar{y}) + v^{(k)}(\bar{t}, \bar{x}) - v^{(k)}(\bar{t}, \bar{y}).$$

By using the Lipschitz condition of $u^{(k)}$ and $v^{(k)}$ we deduce that

$$|\bar{x} - \bar{y}| \le C\varepsilon, \tag{86}$$

where C is a positive constant independent of ε.
From the inequality $\psi_k(T, 0, 0) \le \psi_k(\bar{t}, \bar{x}, \bar{y})$, we obtain that

$$\begin{aligned} \delta e^{\lambda(T-\bar{t})}(|\bar{x}|^2 + |\bar{y}|^2) &\le u^{(k)}(\bar{t}, \bar{x}) - v^{(k)}(\bar{t}, \bar{y}) + \frac{\beta}{T} - \frac{\beta}{\bar{t}} - u^{(k)}(T, 0) + v^{(k)}(T, 0) \\ &\le C(1 + |\bar{x}| + |\bar{y}|), \end{aligned} \tag{87}$$

where the last inequality is deduced from the linear growth condition in x of $u^{(k)}$ and $v^{(k)}$ and C is a positive constant which is independent of ε, we deduce then

$$\delta(|\bar{x}|^2 + |\bar{y}|^2) \le C(1 + |\bar{x}| + |\bar{y}|).$$

By using Young's inequality we obtain that there exists a positive constant C_δ such that

$$|\bar{x}|, |\bar{y}| \le C_\delta. \tag{88}$$

From (86)-(88) we deduce that there exists a subsequence of $(\bar{t}, \bar{x}, \bar{y})$ which goes to $(t_0, x_0, x_0) \in [0, T] \times \mathbb{R} \times \mathbb{R}$, as $\varepsilon \to 0^+$.
If $\bar{t} = T$ then $\psi_k(t, x, x) \le \psi_k(T, \bar{x}, \bar{y})$, which gives that

$$\begin{aligned} u^{(k)}(t, x) - v^{(k)}(t, x) - \frac{\beta}{t} - 2\delta e^{\lambda(T-t)}|x|^2 &\le u^{(k)}(T, \bar{x}) - v^{(k)}(T, \bar{x}) + v^{(k)}(T, \bar{x}) - v^{(k)}(T, \bar{y}) \\ &\le 0, \end{aligned} \tag{89}$$

where the last inequality follows from assumption (83) and the fact that when $\varepsilon \to 0^+$, $u^{(k)}(T,\bar{x}) - v^{(k)}(T,\bar{y}) \to u^{(k)}(T,x_0) - v^{(k)}(T,x_0) \leq 0$. Then the inequality (89) became

$$u^{(k)}(t,x) - v^{(k)}(t,x) - \frac{\beta}{t} - 2\delta e^{\lambda(T-t)}|x|^2 \leq 0.$$

By sending β and δ to 0^+ and using inequality (86), we obtain that

$$u^{(k)}(t,x) \leq v^{(k)}(t,x).$$

Let us assume then that $\bar{t} < T$. By applying Theorem 9 of Crandall-Ishii [3] to the function $\psi_k(t,x,y)$ at point $(\bar{t},\bar{x},\bar{y}) \in (0,T)\times\mathbb{R}\times\mathbb{R}$, we find $p_0 \in \mathbb{R}$, a and $d \in \mathbb{R}$ such that

$$\left(p_0 - \frac{\beta}{\bar{t}^2} - \lambda\delta e^{\lambda(T-\bar{t})}(|\bar{x}|^2 + |\bar{y}|^2), \frac{1}{\varepsilon}(\bar{x}-\bar{y}) + 2\delta e^{\lambda(T-\bar{t})}\bar{x}, a + 2\delta e^{\lambda(T-\bar{t})}\right) \in \bar{\mathcal{P}}^{2,+}u^{(k)}(\bar{t},\bar{x})$$

$$\left(p_0, \frac{1}{\varepsilon}(\bar{x}-\bar{y}) - 2\delta e^{\lambda(T-\bar{t})}\bar{y}, d - 2\delta e^{\lambda(T-\bar{t})}\right) \in \bar{\mathcal{P}}^{2,-}v^{(k)}(\bar{t},\bar{y})$$

and the Lipschitz assumption (32) on σ gives

$$\frac{1}{2}\sigma^2(\bar{t},\bar{x})a - \frac{1}{2}\sigma^2(\bar{t},\bar{y})d \leq \frac{C}{\varepsilon}|\bar{x}-\bar{y}|^2. \tag{90}$$

We have that $u^{(k)}$ and $v^{(k)}$ are respectively viscosity subsolution and supersolution of (60) in $\mathcal{C}_2([0,T]\times\mathbb{R})$, so by applying Lemma 7.4 we obtain the two inequalities:

$$\begin{aligned}\min\{ru^{(k)}(\bar{t},\bar{x}) - p_0 + \frac{\beta}{\bar{t}^2} + \lambda\delta e^{\lambda(T-\bar{t})}(|\bar{x}|^2 + |\bar{y}|^2) \\ - A(\bar{t},\bar{x},\frac{1}{\varepsilon}(\bar{x}-\bar{y}) + 2\delta e^{\lambda(T-\bar{t})}\bar{x}, a + 2\delta e^{\lambda(T-\bar{t})}) - B(\bar{t},\bar{x},\frac{1}{\varepsilon}(\bar{x}-\bar{y}) + 2\delta e^{\lambda(T-\bar{t})}\bar{x}, u^{(k)}); \\ u^{(k)}(\bar{t},\bar{x}) - \phi^{(k)}(\bar{t},\bar{x})\} \leq 0\end{aligned} \tag{91}$$

and

$$\begin{aligned}\min\{rv^{(k)}(\bar{t},\bar{x}) - p_0 - A(\bar{t},\bar{y},\frac{1}{\varepsilon}(\bar{x}-\bar{y}) - 2\delta e^{\lambda(T-\bar{t})}\bar{y}, d - 2\delta e^{\lambda(T-\bar{t})}) \\ - B(\bar{t},\bar{y},\frac{1}{\varepsilon}(\bar{x}-\bar{y}) - 2\delta e^{\lambda(T-\bar{t})}\bar{y}, v^{(k)}); v^{(k)}(\bar{t},\bar{y}) - \phi^{(k)}(\bar{t},\bar{y})\} \geq 0.\end{aligned} \tag{92}$$

It is easy to see that $\min(\alpha,\beta) - \min(\eta,\gamma) \leq 0$ implies either $\alpha - \eta \leq 0$ or $\beta - \gamma \leq 0$. So by subtracting inequalities (91) and (92) we obtain two cases:
(i) Case 1:

$$r[u^{(k)}(\bar{t},\bar{x}) - v^{(k)}(\bar{t},\bar{y})] + \frac{\beta}{\bar{t}^2} + \lambda\delta e^{\lambda(T-\bar{t})}(|\bar{x}|^2 + |\bar{y}|^2) \leq T_1 + T_2, \tag{93}$$

where

$$\begin{aligned}T_1 :=& A(\bar{t},\bar{x},\frac{1}{\varepsilon}(\bar{x}-\bar{y}) + 2\delta e^{\lambda(T-\bar{t})}\bar{x}, a + 2\delta e^{\lambda(T-\bar{t})}) \\ &- A(\bar{t},\bar{y},\frac{1}{\varepsilon}(\bar{x}-\bar{y}) - 2\delta e^{\lambda(T-\bar{t})}\bar{y}, d - 2\delta e^{\lambda(T-\bar{t})})\end{aligned}$$

$$T_2 := B(\bar{t}, \bar{x}, \frac{1}{\varepsilon}(\bar{x} - \bar{y}) + 2\delta e^{\lambda(T-\bar{t})}\bar{x}, u^{(k)}) - B(\bar{t}, \bar{y}, \frac{1}{\varepsilon}(\bar{x} - \bar{y}) - 2\delta e^{\lambda(T-\bar{t})}\bar{y}, v^{(k)}).$$

From inequality (90) and the linear growth condition of b and σ we obtain

$$\begin{aligned} T_1 =& \frac{1}{2}\left(\sigma^2(\bar{t}, \bar{x})(a + 2\delta e^{\lambda(T-\bar{t})}) - \sigma^2(\bar{t}, \bar{y})(d - 2\delta e^{\lambda(T-\bar{t})})\right) \\ &+ b(\bar{t}, \bar{x})\left(\frac{1}{\varepsilon}(\bar{x} - \bar{y}) + 2\delta e^{\lambda(T-\bar{t})}\bar{x}\right) - b(\bar{t}, \bar{y})\left(\frac{1}{\varepsilon}(\bar{x} - \bar{y}) - 2\delta e^{\lambda(T-\bar{t})}\bar{y}\right) \\ &\leq C\left(\frac{1}{\varepsilon}|\bar{x} - \bar{y}|^2 + \delta e^{\lambda(T-\bar{t})}(1 + |\bar{x}|^2 + |\bar{y}|^2)\right). \end{aligned} \tag{94}$$

We have that for all $p \in \mathbb{R}$ and $\varphi \in \mathcal{C}_2([0, T] \times \mathbb{R})$, the integrand of $B(\bar{t}, \bar{x}, p, \varphi)$ is bounded by $C_p(1 + |\bar{x}|^2)$, so from assumption (31) this integral term is finite. We deduce then that the two integral terms of T_2 are finite because we have that $u^{(k)}$ and $v^{(k)}$ are in $\mathcal{C}_2([0, T] \times \mathbb{R})$. Moreover, the difference of these two integrands is

$$\begin{aligned} &\left[u^{(k)}(\bar{t}, \bar{x} + \gamma(\bar{t}, \bar{x}, z)) - u^{(k)}(\bar{t}, \bar{x}) - \gamma(\bar{t}, \bar{x}, z)\left(\frac{1}{\varepsilon}(\bar{x} - \bar{y}) + 2\delta e^{\lambda(T-\bar{t})}\bar{x}\right)\right] \\ &- \left[v^{(k)}(\bar{t}, \bar{y} + \gamma(\bar{t}, \bar{y}, z)) - v^{(k)}(\bar{t}, \bar{y}) - \gamma(\bar{t}, \bar{y}, z)\left(\frac{1}{\varepsilon}(\bar{x} - \bar{y}) - 2\delta e^{\lambda(T-\bar{t})}\bar{y}\right)\right] \\ &= \psi_k(\bar{t}, \bar{x} + \gamma(\bar{t}, \bar{x}, z), \bar{y} + \gamma(\bar{t}, \bar{y}, z)) - \psi_k(\bar{t}, \bar{x}, \bar{y}) \\ &+ \frac{1}{2\varepsilon}|\gamma(\bar{t}, \bar{x}, z) - \gamma(\bar{t}, \bar{y}, z)|^2 + \frac{1}{2\varepsilon}|\bar{x} - \bar{y}|^2 \\ &+ \delta e^{\lambda(T-\bar{t})}[|\gamma(\bar{t}, \bar{x}, z)|^2 + |\gamma(\bar{t}, \bar{y}, z)|^2]. \end{aligned}$$

On the other hand by the definition of $(\bar{t}, \bar{x}, \bar{y})$ we have that

$$\psi_k(\bar{t}, \bar{x} + \gamma(\bar{t}, \bar{x}, z), \bar{y} + \gamma(\bar{t}, \bar{y}, z)) - \psi_k(\bar{t}, \bar{x}, \bar{y}) \leq 0.$$

Then from the Lipschitz and the linear growth conditions of γ and assumption (31) we deduce that

$$T_2 \leq C\left(\frac{1}{\varepsilon}|\bar{x} - \bar{y}|^2 + \delta e^{\lambda(T-\bar{t})}(1 + |\bar{x}|^2 + |\bar{y}|^2)\right). \tag{95}$$

By the definition of $(\bar{t}, \bar{x}, \bar{y})$ we have that $\psi_k(t, x, x) \leq \psi_k(\bar{t}, \bar{x}, \bar{y})$, i.e.

$$\begin{aligned} u^{(k)}(t, x) - v^{(k)}(t, x) - \frac{\beta}{t} - 2\delta e^{\lambda(T-t)}|x|^2 \leq& u^{(k)}(\bar{t}, \bar{x}) - v^{(k)}(\bar{t}, \bar{y}) - \frac{\beta}{\bar{t}} - \frac{1}{2\varepsilon}|\bar{x} - \bar{y}|^2 \\ &- \delta e^{\lambda(T-\bar{t})}(|\bar{x}|^2 + |\bar{y}|^2). \end{aligned} \tag{96}$$

By inequality (93) we have that

$$\begin{aligned} &u^{(k)}(\bar{t}, \bar{x}) - v^{(k)}(\bar{t}, \bar{y}) - \frac{\beta}{\bar{t}} - \frac{1}{2\varepsilon}|\bar{x} - \bar{y}|^2 - \delta e^{\lambda(T-\bar{t})}(|\bar{x}|^2 + |\bar{y}|^2) \\ &\leq \frac{1}{r}\left(T_1 + T_2 - \lambda\delta e^{\lambda(T-\bar{t})}(|\bar{x}|^2 + |\bar{y}|^2)\right). \end{aligned} \tag{97}$$

From the two last inequalities we deduce that

$$u^{(k)}(t, x) - v^{(k)}(t, x) - \frac{\beta}{t} - 2\delta e^{\lambda(T-t)}|x|^2 \leq \frac{1}{r}\left(T_1 + T_2 - \lambda\delta e^{\lambda(T-\bar{t})}(|\bar{x}|^2 + |\bar{y}|^2)\right). \tag{98}$$

By sending ε to 0^+ we deduce from estimates (94), (95) and the estimation (86) that

$$u^{(k)}(t,x) - v^{(k)}(t,x) - \frac{\beta}{t} - 2\delta e^{\lambda(T-t)}|x|^2 \leq \frac{2\delta}{r} e^{\lambda(T-t_0)} \left[C(1+2|x_0|^2) - \lambda |x_0|^2 \right]. \tag{99}$$

Let λ be sufficiently large, such that $\lambda \geq 2C$, so by sending β and δ to 0^+ we conclude that $u^{(k)}(t,x) \leq v^{(k)}(t,x)$.
(ii) Case 2:

$$u^{(k)}(\bar{t},\bar{x}) - v^{(k)}(\bar{t},\bar{y}) + \phi^{(k)}(\bar{t},\bar{y}) - \phi^{(k)}(\bar{t},\bar{x}) \leq 0.$$

Using the fact that $\phi^{(k)}$ is Lipschitz, Proposition 6.2, and inequality (86), we can see that $\limsup_{\varepsilon \to 0^+} (u^{(k)}(\bar{t},\bar{x}) - v^{(k)}(\bar{t},\bar{y})) \leq 0$. On the other hand we have that $\psi_k(t,x,x) \leq \psi_k(\bar{t},\bar{x},\bar{y})$, so

$$u^{(k)}(t,x) - v^{(k)}(t,x) - \frac{\beta}{t} - \delta e^{\lambda(T-t)}(|x|^2 + |y|^2) \leq u^{(k)}(\bar{t},\bar{x}) - v^{(k)}(\bar{t},\bar{y})$$

by sending β and δ to 0^+ we obtain that $u^{(k)}(t,x) - v^{(k)}(t,x) \leq u^{(k)}(\bar{t},\bar{x}) - v^{(k)}(\bar{t},\bar{y})$ and by sending ε to 0^+ we conclude that

$$u^{(k)}(t,x) \leq v^{(k)}(t,x).$$

□

8 Appendix

We recall the classical following theorem (see for example Karatzas Shreve (1998)).

Theorem 8.1 *(Essential supremum) Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let $\mathcal{X}$ be a non empty family of nonnegative random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Then there exists a random variable X^* satisfying*

1. *for all $X \in \mathcal{X}$, $X \leq X^*$ a.s. ,*
2. *if Y is a random variable satisfying $X \leq Y$ a.s. for all $X \in \mathcal{X}$, then $X^* \leq Y$ a.s..*

This random variable, which is unique a.s., is called the essential supremum of X and is denoted $\operatorname{ess\,sup}_{X \in \mathcal{X}} X$. Furthermore, if $\mathcal{X}$ is closed under pairwise maximization (that is: $X, Y \in \mathcal{X}$ implies $X \vee Y \in \mathcal{X}$), then there is a nondecreasing sequence $(Z_n)_{n \in \mathbb{N}}$ of random variable in $\mathcal{X}$ satisfying $X^* = \lim_{n \to \infty} Z_n$ almost surely.

Lemma 8.2 *For all $0 \leq t < s \leq T$, there exists a constant $C > 0$ such that*

$$E^Q[|X^{t,x}_{t+\delta} - X^{s,x}_{s+\delta}|] \leq C(1+|x|)\sqrt{s-t}.$$

Proof. Let X^1 and X^2 two processes such that

$$\begin{cases} X_t^1 &= x \\ dX_u^1 &= b(u, X_{u^-}^1)du + \sigma(u, X_{u^-}^1)dW_u + \int_{\mathbb{R}} \gamma(u, X_{u^-}^1, z)\tilde{v}(du, dz) \quad \forall u \in (t, t+\delta] \\ dX_u^1 &= 0 \qquad \forall u \in (t+\delta, s+\delta] \end{cases}$$

$$\begin{cases} dX_u^2 = 0 \qquad \forall u \in [t, s] \\ X_s^2 = x \\ dX_u^2 = b(u, X_{u^-}^2)du + \sigma(u, X_{u^-}^2)dW_u + \int_{\mathbb{R}} \gamma(u, X_{u^-}^2, z)\tilde{v}(du, dz) \quad \forall u \in (s, s+\delta] \end{cases}$$

We define $Y_u := X_u^1 - X_u^2$, then $Y_t = 0$.
First case: $s < t + \delta$

$$\begin{aligned} dY_u &= b(u, X_{u^-}^1)du + \sigma(u, X_{u^-}^1)dW_u + \int_{\mathbb{R}} \gamma(u, X_{u^-}^1, z)\tilde{v}(du, dz), \quad \forall u \in (t, s] \\ dY_u &= \left(b(u, X_{u^-}^1) - b(u, X_{u^-}^2)\right) du + \left(\sigma(u, X_{u^-}^1) - \sigma(u, X_{u^-}^2)\right) dW_u \\ &\quad + \int_{\mathbb{R}} \left(\gamma(u, X_{u^-}^1, z) - \gamma(u, X_{u^-}^2, z)\right) \tilde{v}(du, dz), \quad \forall u \in (s, t+\delta] \\ dY_u &= -b(u, X_{u^-}^2)du - \sigma(u, X_{u^-}^2)dW_u - \int_{\mathbb{R}} \gamma(u, X_{u^-}^2, z)\tilde{v}(du, dz), \quad \forall u \in (t+\delta, s+\delta]. \end{aligned}$$

We obtain then in addition to the Lipschitz continuity and the linear growth condition in x of b, σ and γ that

$$\begin{aligned} E^Q[|Y_{s+\delta}|^2] \le\; & CE^Q\left[\int_t^s \left(\left|b(u, X_u^1)\right|^2 + \left|\sigma(u, X_u^1)\right|^2 + \int_{\mathbb{R}} \left|\gamma(u, X_u^1, z)\right|^2 m(dz)\right) du\right] \\ & + CE^Q\Bigg[\int_s^{t+\delta} \Bigg(\left|b(u, X_u^1) - b(u, X_u^2)\right|^2 + \left|\sigma(u, X_u^1) - \sigma(u, X_u^2)\right|^2 \\ & + \int_{\mathbb{R}} \left|\gamma(u, X_u^1, z) - \gamma(u, X_u^2, z)\right|^2 m(dz)\Bigg) du\Bigg] \\ & + CE^Q\left[\int_{t+\delta}^{s+\delta} \left(|b(u, X_u^2)|^2 + |\sigma(u, X_u^2)|^2 + \int_{\mathbb{R}} |\gamma(u, X_u^2, z)|^2 m(dz)\right) du\right] \end{aligned}$$

From Lemma 3.1 of Pham [15] and assumption (31), we deduce that

$$\begin{aligned} E^Q[|Y_{s+\delta}|^2] \quad &\le \quad CE^Q\left[\int_t^s (1 + |X_u^1|)^2 du + \int_s^{t+\delta} |Y_u|^2 du + \int_{t+\delta}^{s+\delta} (1 + |X_u^2|)^2 du\right] \\ &\le \quad CE^Q\left[\int_t^s (1 + |Y_u|^2 + |x|^2) du + \int_s^{t+\delta} |Y_u|^2 du\right] \\ &+ \quad CE^Q\left[\int_{t+\delta}^{s+\delta} (1 + |X_{s+\delta}^1|^2 + |Y_u|^2) du\right] \\ &\le C \quad \left((s-t)(1 + |x|^2) + E^Q\left[\int_t^{s+\delta} |Y_u|^2 du\right]\right) \end{aligned}$$

Second case: $t+\delta \leq s$

$$dY_u = b(u, X^1_{u^-})du + \sigma(u, X^1_{u^-})dW_u + \int_{\mathbb{R}} \gamma(u, X^1_{u^-}, z)\tilde{v}(du, dz), \quad \forall u \in (t, t+\delta]$$

$$dY_u = 0, \quad \forall u \in (t+\delta, s]$$

$$dY_u = -b(u, X^2_{u^-})du - \sigma(u, X^2_{u^-})dW_u - \int_{\mathbb{R}} \gamma(u, X^2_{u^-}, z)\tilde{v}(du, dz), \quad \forall u \in (s, s+\delta].$$

We obtain then in addition to the linear growth conditions in x of b, σ and γ that

$$E^Q[|Y_{s+\delta}|^2] \leq CE^Q\left[\int_t^{t+\delta}\left(\left|b(u, X^1_u)\right|^2 + \left|\sigma(u, X^1_u)\right|^2 + \int_{\mathbb{R}}\left|\gamma(u, X^1_u, z)\right|^2 m(dz)\right)du\right]$$
$$+ CE^Q\left[\int_s^{s+\delta}\left(\left|b(u, X^1_u) - b(u, X^2_u)\right|^2 + \left|\sigma(u, X^1_u) - \sigma(u, X^2_u)\right|^2\right.\right.$$

From Lemma 3.1 of Pham [15] and assumption (31), we deduce that

$$\begin{aligned}
E^Q[|Y_{s+\delta}|^2] &\leq CE^Q\left[\int_t^{t+\delta}(1+|X^1_u|)^2du + \int_s^{s+\delta}(1+|X^2_u|)^2du\right] \\
&\leq CE^Q\left[\int_t^{t+\delta}(1+|Y_u|^2+|x|^2)du + \int_s^{s+\delta}(1+|X^1_{s+\delta}|^2+|Y_u|^2)du\right] \\
&\leq C\left((s-t)(1+|x|^2) + E^Q\left[\int_t^{s+\delta}|Y_u|^2du\right]\right)
\end{aligned}$$

We deduce then that in both cases we have that

$$\leq C\left((s-t)(1+|x|^2) + E^Q\left[\int_t^{s+\delta}|Y_u|^2du\right]\right).$$

Then by Fubini's theorem and by Gronwall's lemma we obtain that

$$E^Q\left[\left|X^{s,x}_{s+\delta} - X^{t,x}_{t+\delta}\right|^2\right] = E^Q[|Y_{s+\delta}|^2] \leq C(s-t)(1+|x|^2)$$

and then

$$E^Q\left[\left|X^{s,x}_{s+\delta} - X^{t,x}_{t+\delta}\right|\right] \leq C(1+|x|)\sqrt{s-t}.$$

□

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