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***Abstract*—This paper introduces Mean Field Games (MFG) as a framework to develop optimal estimators in some sense for a general class of nonlinear systems. We show that under suitable conditions the estimation error converges exponentially fast to zero. Computer simulations are performed to illustrate the method. In particular we provide an example where the proposed estimator converges whereas both extended Kalman filter and particle filter diverge.**

I. Introduction

The estimation problem has been a fundamental and a challenging problem in theory and applications of control systems [1], [2]. In this paper, we will use the mean field game (MFG) framework to address the state estimation problem for a nonlinear system described as

$$dx = f(x)dt + G(x)dw_t, \quad x(0) = x_0 \tag{1a}$$

$$y = h(x) + K(x)v_t, \tag{1b}$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $G : \mathbb{R}^n \to \mathbb{R}^{n\times n}$, $K : \mathbb{R}^n \to \mathbb{R}^{n\times p}$, $h : \mathbb{R}^n \to \mathbb{R}^p$, $x \in \mathbb{R}^n$ is the state, and $y \in \mathbb{R}^p$ denotes the measured output. We assume that G and K are invertible for all x and the input disturbance $w_t \in \mathbb{R}^n$ and the measurement noise $v_t \in \mathbb{R}^p$ are white Gaussian noises. Moreover, the initial condition x_0 is also characterized with a Gaussian distribution. For simplicity, we set $G(x) = \sigma I_n$ with $\sigma > 0$. The state estimation problem consists in computing an estimate $\hat{x}$ of the state x from the measurements y. This problem can be converted to the study of the Kushner equation, that under certain mild assumptions allows us to reduce it to the study of a simpler equation, the Zakai equation [3]. While for linear systems the Kalman filter is the optimal estimator in the sense that it minimizes the mean square error, for nonlinear systems the Extended Kalman Filter (EKF) is the most widely used method. Notice however that EKF is not optimal (in the same sense) as it is obtained by linearizing the nonlinear dynamics and the observation along the trajectory of the estimate.

Recently particle filters (PF) have become one of the most widespread tools to solve nonlinear estimation problems [4]. In general PF schemes uses a discretization of the probability density function associated with the state dynamics described in (1a). For each point of this discretization, called a *particle*, there is a *weight* associated. The PF is composed by two iterative steps, a prediction and an update. In the prediction step the particles are re-sampled from the probability density associated with the dynamics (1a) and in the update step the values of the weights are re-computed based on the probability associated with (1b) using the Bayes formula [4]. One extension of this scheme is to consider the limit when we have infinitely many particles and take its mean field, see [5]. An extension to the PF scheme is provided in [6], [7] where each particle is equipped with a control to minimize a coupling cost. An alternative, using a game theoretical approach is the H_∞-filter that can be understood as a stochastic differential game between the system and the uncertainty [8].

Our approach joins game theoretical ideas with the "mean field" approach. To formulate the proposed estimator consider a continuum of particles $z \in \mathbb{R}^n$ where each particle is identified by its initial condition $z(0) = z_0$ and evolves according to the dynamics (1a) but with an additional artificial control $u \in \mathbb{R}^n$ that is

$$dz = (f(z) + u)dt + \sigma dw_t. \tag{2}$$

In (2), $\sigma > 0$, and the control u is set to minimize the distance (in a distribution sense) from the underlying distribution p_z (that represents the probability distribution of the variable z driven by (2)) and the measurements distribution p_y, associated with (1b). Considering that the measurements over the system p_y are provided by a Gaussian distribution, we can express the minimization problem as follows

$$J(t,z) = \min_{u:[t,T]\to\mathbb{R}^n} \mathbb{E}\left[\int_t^T \frac{\|u(s)\|^2}{2} ds \right. \tag{3}$$

$$\left. + \int_t^T \|h(z(s)) - y(s)\|^2_{R^{-1}} - \delta \ln(p_z(s,z))\, ds\right], \tag{4}$$

where $R^{-1} = K(z)K(z)^T$ is assumed to be positive definite. An estimate for x could be for example the particle z^* that minimizes the cost $J(t,z)$ as it will be seen later in Section III. In other words, the cost function (4) is described by a minimization over a time window $[t,T]$ the following quantities: the distance of each particle to the measured output given by $\|h(z(s)) - y(s)\|^2_{R^{-1}}$, and an aggregative function

$-\ln(.)$ that forces the consensus among the particles. The role of δ is to weight how much we should rely on the actual state and how fast we should create incorporate the new measurements. Finally, the term $\|u\|^2$ penalizes the control cost. Note that by imposing in the cost a penalization term on u forces the solutions of the problem to not degenerate. Without this cost, the best possible solution would correspond to all agents being in the same position ξ such that $h(\xi) = y(s)$, leading the correspondent distance to zero and $p_z(\xi) \to \infty$, due to a peak distribution. Another remark is the fact that $\delta(t)$ tunes not only how fast the aggregation takes place but also how much we would like to consider the past, in the sense that accounts with the effect of $p_z(\xi)$ in the cost.

To the best of authors knowledge, this is the first time that MFG is used to develop an estimator. Moreover our approach can be understood as a mean field approach [5] over the feedback particle filter [6], [7].

At this point, it is convenient to stress that game theoretical approaches similar to (2)-(4) were already developed in different contexts which include the anonymous sequential population games [9], [10], the work of P. Caine and his associates in control of large scale systems [11], [12], in wireless networks e.g., S. Adlakha [13] and H. Tembine [14]–[16], and in economics by Weintraub [17], and by O.Gueant, Lasry and Lions [18]–[23], and in other applications, see [24].

This paper contains two main contributions: i) Using a MFG framework we propose an iterative estimation algorithm show that under suitable conditions the proposed estimator converges locally exponentially fast to zero. For the first order approximations of the state equation and output equation, we derive a closed-form (iterative) solution and prove local convergence. Using an example of a nonlinear system we illustrate the key benefits of the proposed algorithm.

The paper is organized as follows: Section II provides an informal way to derive mean field games, starting first by describing the Hamilton-Jacobi-Bellman (HJB) equation, followed by the derivation of the Fokker-Planck (FP) equation. These two equations are essential to formulate the MFG estimation problem. Section III contains the main results of the paper where we provide a solution to the estimation problem using MFG and provide conditions under which the estimation error converges to zero. In Section IV we illustrate our method using computer simulations and compare it with a PF and an EKF estimators. Conclusions together with further research are drawn in Section V.

Notation:

J_t - partial derivative of the cost function w.r.t. t,
J_z - partial derivative of the cost function w.r.t. z,
$(p_z)_t$ - partial derivative of the pdf p_z w.r.t. t,
$(p_z)_z$ - partial derivative of the pdf p_z w.r.t. z,
$x \in \mathbb{R}^n$ - system state,
$u^j(t) \in \mathbb{R}^n$ - control of the agent j at time t,
$p_z(t, .)$ - probability density function that represents the population of agents at time t,
$H_p(.)$ - derivative with respect to p
$\|v\|_A^2$ - Euclidean norm with weight matrix A, i.e, $v^T A v$,
$z^j(t)$ - is the location of the jth agent at time t,
div - divergence operator,
$\nabla = [\frac{d}{dz_1} \dots \frac{d}{dz_n}]^T$ - gradient operator,
$\Delta J = \sum_{i=1}^{n} \frac{\partial^2 J}{\partial z_i^2}$ - Laplacian operator,

II. Mean Field Games

In this section we briefly review the MFG framework [22], [23], [25], which is applied in Section III to develop a new nonlinear estimator.

A. Hamilton-Jacobi-Bellman equation

Consider a particle j, for $j = 1, ..., N$ with the following dynamics

$$dz^j = (f(z^j) + u^j)dt + \sigma dw_t, \tag{5}$$

subject to an initial condition $z^j(0) = z_0^j$. Let the control u^j be the result of the following optimization problem

$$J(t, z) = \min_{u^j:[t,T]} \mathbb{E}\left[J\left(T, z^j(T)\right) + \int_t^T \left\|u^j\right\|^2 ds \right],$$

for $t_0 \le t \le T$, where $J\left(T, z^j(T)\right)$ is the terminal cost.

Let $H : \mathbb{R}^n \to \mathbb{R}$ be the Legendre transform of the running cost given by

$$H(p) = \sup_{u^j \in \mathbb{R}^n} -(u^j)^T p - \left\|u^j\right\|^2.$$

It is well known that J is a viscosity solution of the Hamilton-Jacobi-Bellman (HJB) equation [26], [27]

$$-J_t - \frac{\sigma^2}{2}\Delta J + J_z^T f(z^j) + H(J_z) = 0, \tag{6}$$

and that the optimal control is given in feedback form as

$$u^j = -H_p\left(J_z\left(t_i, z_{t_i}^j\right)\right). \tag{7}$$

B. Fokker-Planck equation

Consider, instead of having just one agent, a very large number N of agents distributed throughout the space according to a probability density (pdf) $p_z(z)$ such that

$$\int_{\mathbb{R}^n} p_z(z) dz = 1,$$

for each time t. This distribution $p_z(z)$ describes the pdf associated with the random variable $z(t)$ in

$$dz = (f(z) + u)dt + \sigma dw_t, \tag{8}$$

where u is assumed to be given in a feedback form. Then the evolution of the pdf associated with (8) is described by the Fokker-Planck (FP) equation [28]

$$(p_z(z))_t(t, z) - \frac{\sigma^2}{2}\Delta p_z(z)(t, z) + \mathrm{div}\left(p_z(z)(u + f)\right)(t, z) = 0. \tag{9}$$

C. Mean Field Games

In Section II-A the HJB equation was introduced, assuming that each agent had a fixed cost function to minimize independently on the location of the other agents. A mean field game generalizes this setup, by allowing the cost function of each agent to also depend on the density function p_z of all the other agents. Such generalization can be achieved in several ways. The simplest one would be to add in the cost function a term that captures the influence of the collective of agents, as we did in (4). In that case the cost function (4) associated with (2) has to satisfy the HJB equation presented in (11) coupled together with the Fokker-Planck equation (12).

In (11), the boundary condition is the terminal cost $J(T,.) = 0$. In (12), the probability density p_z must satisfy the initial condition p_z^0 given by (13). The system composed by (11) and (12) is referred as a mean field game system. The backward evolution equation (HJB) represents particles' decisions based on where they would like to be in the future; and the forward evolution equation (FPK) represents where they actually end up, based on their initial density and both equations depend on each other.

III. Nonlinear Estimator using Mean Field Games

In this section we propose an estimator for (1) using the MFG given by the dynamics in (2) and the cost function (4). The optimal state estimate $\hat{x}$ at time t is defined as

$$\hat{x}(t) = \arg\min_{z\in\mathbb{R}^n} J(t,z). \tag{10}$$

The reasoning behind (10) is similar to the minimum energy estimator [29], where the goal is to minimize the expected minimum energy constrained to a consensus among all the possible evolutions given the initial conditions. As described in the previous section, the cost J has to satisfy the following Hamilton-Jacobi-Bellman equation

$$\begin{aligned} &J_t\left(t,z\right) + J_z(t,z)^T f\left(z\right) + \frac{\left\|J_z\left(t,z\right)\right\|^2}{2} \\ &\quad + \frac{\sigma^2}{2}\Delta J\left(t,z\right) - \frac{1}{2}\left\|y - h\left(z\right)\right\|^2_{R^{-1}} = -\delta \ln\left(p_z(t,z)\right), \end{aligned} \tag{11}$$

with terminal condition $J(T,.) = 0$ and p_z has to satisfy the Fokker-Planck equation,

$$(p_z)_t + \operatorname{div}\left(p_z(J_z\left(t,z\right) + f(z))\right) = \frac{\sigma^2}{2}\Delta p_z, \tag{12}$$

with initial condition $p_z^0(.) = p_z(t_0,.)$ given by

$$p_z(t_0,.) = K\exp\left\{-\frac{\|h(.) - y_0\|^2_{\Sigma^{-1}}}{2}\right\}, \tag{13}$$

where Σ^{-1} is positive definite, $y_0 = h(\hat{x}_0)$, $\hat{x}_0$ encodes a priori information about the state, and K is a normalization constant. We call the system composed by (10), (11) and (12) the *MFG estimator*. Note that the MFG-based estimator defines an infinite dimensional observer.

To rewrite (10) in a filtering-like form (similar to a Kalman filter) we follow similar steps proposed by [30], [31] by assuming that $J(t,z)$ is a smooth solution.

In this case, J is differentiable at $(t,\hat{x}(t))$, and therefore

$$0 = J_z\left(t,\hat{x}\left(t\right)\right). \tag{14}$$

If, in addition $\hat{x}$ is differentiable at t, we have

$$\begin{aligned} \frac{d}{dt}J\left(t,\hat{x}\left(t\right)\right) &= J_t\left(t,\hat{x}\left(t\right)\right) + J_z\left(t,\hat{x}\left(t\right)\right)\dot{\hat{x}}\left(t\right) \\ &= J_t\left(t,\hat{x}\left(t\right)\right). \end{aligned}$$

If we further assume that since J is C^2 in a neighborhood of $(t,\hat{x}\left(t\right))$ and $\hat{x}$ is differentiable in a neighborhood of t, differentiating (14) with respect to t yields

$$0 = J_{z_i t}\left(t,\hat{x}\left(t\right)\right) + J_{z_j z_i}\left(t,\hat{x}\left(t\right)\right)\dot{\hat{x}}_j\left(t\right), \tag{15}$$

where we are using Einstein's convention of summing on repeated indices. Differentiating now the HJB (11) with respect to z_i at $\hat{x}$,

$$\begin{aligned} 0 = &J_{tz_i}\left(t,\hat{x}\left(t\right)\right) + J_{z_j z_i}\left(t,\hat{x}\left(t\right)\right)f_j\left(\hat{x}\left(t\right)\right) \\ &+ h_{rz_i}\left(\hat{x}\left(t\right)\right)R^{-1}_{rs}\left(y_s\left(t\right) - h_r\left(\hat{x}\left(t\right)\right)\right) \\ &+ \frac{1}{2}J_{z_j z_j z_i}\left(t,\hat{x}\left(t\right)\right) - \delta\frac{\partial}{\partial z_i}\ln\left(p_z(t,\hat{x})\right). \end{aligned} \tag{16}$$

Thus, by the commutativity of mixed partials and replacing (15) into (16) we have

$$\begin{aligned} J_{z_j z_i}\left(t,\hat{x}\left(t\right)\right)\dot{\hat{x}}_j\left(t\right) = &J_{z_j z_i}\left(t,\hat{x}\left(t\right)\right)f_j\left(\hat{x}\left(t\right)\right) \\ &+ h_{rz_i}\left(\hat{x}\left(t\right)\right)R^{-1}_{rs}\left(y_s\left(t\right) - h_r\left(\hat{x}\left(t\right)\right)\right) \\ &+ \frac{1}{2}J_{z_j z_j z_i}\left(\hat{x}\left(t\right),t\right) - \delta\frac{\partial}{\partial z_i}\ln\left(p_z(t,\hat{x})\right). \end{aligned}$$

Moreover, since $J_{zz}\left(z\left(t\right),t\right)$ is invertible, we finally obtain in a filtering like form the MFG estimator

$$\begin{aligned} \dot{\hat{x}}\left(t\right) = &f\left(\hat{x}\left(t\right)\right) \\ &+ J^{-1}_{zz}\left(t,\hat{x}\left(t\right)\right)\Big(h^T_z\left(\hat{x}\left(t\right)\right)R^{-1}\left(y\left(t\right) - h\left(\hat{x}\left(t\right)\right)\right) \\ &+ \frac{1}{2}(\Delta J)_z - \delta\nabla\ln\left(p_z(t,\hat{x})\right)\Big). \end{aligned} \tag{17}$$

Note that at this point we still have the problem of computing the evolution of $J(t,z)$ (and its derivatives). Nevertheless if we take a linear approximation of f and h we obtain an approximation of the MFG estimator, written in a Kalman-like formula, given by

$$\dot{\hat{x}}(t) = f_z(\hat{x})\hat{x}(t) + Qh^T_z(\hat{x})(R^{-1} + \delta\Sigma^{-1}(t))\left(y(t) - h_z(\hat{x})\hat{x}(t)\right) \tag{18}$$

where $Q = P^{-1}$ and $J(t,x) = \|\hat{x}(t) - x(t)\|^2_P$ has to satisfy the HJB equation (11) from which follows the Riccati equation

$$\begin{aligned} \dot{Q} = &f_z(\hat{x}(t))Q(t) + Q(t)f^T_z(\hat{x}) + I \\ &- Q(t)h^T_z(\hat{x}(t))(R^{-1} + \delta\Sigma^{-1}(t))h_z(\hat{x}(t))Q(t) \end{aligned} \tag{19}$$

where Σ has to satisfy

$$\begin{aligned}\dot{\Sigma} =&\Sigma(t)f_z^T(\hat{x}(t)) + f_z(\hat{x})\Sigma(t) + R^{-1}\\ &- \Sigma(t)h_z^T(\hat{x}(t))(\sigma^2 I + Q(t))h_z(\hat{x}(t))\Sigma(t). \end{aligned}\tag{20}$$

Note in (11) that δ regulates the coupling between both Riccati equations.

The following result provides the convergence of the proposed estimator (20). As in [32] we assume that the stochastic filter can be analysed as a deterministic one, see [29] for more details. In particular, we assume that the signals w and v are unknown L_2 signals (bounded energy).

Theorem 1: Consider (1) with $v, w \in L^2$, the estimator (18) and suppose that

- the system is uniformly observable [32] and satisfies standard Lipschitz conditions,
- the second derivative of f with respect to x is bounded,
- $Q(t)$ and $\Sigma^{-1}(t)$ are positive definite and $\|\tilde{x}(0)\| = \|x(0) - \hat{x}(0)\|$ is sufficiently small.

Then $\|x(t) - \hat{x}(t)\| \to 0$ exponentially as $t \to \infty$. □

Proof: [Outline] This proof follows similar steps of the local convergence of the extended Kalman Filter in [32], which borrows ideas from [31]. It is decomposed in following four parts, respectively showing that:

1) Q is bounded from above,
2) $P = Q^{-1}$ is bounded from above,
3) Using the duality between estimation and control for the linear case construct an upper bound to $\|\tilde{x}(t)\| = \|x(t) - \hat{x}(t)\|$,
4) by continuity get an upper bound to $\|\tilde{x}(0)\|$.

We refer to the work in [33] that explores the existence of solutions for the case of the linear-quadratic-Gaussian mean field games. In particular, it implies that J is always quadratic with P positive definite and p_z a Gaussian distribution, copping with 1) and 2). Since p_z is always a Gaussian distribution with covariance Σ, Σ^{-1} is positive definite. On the other hand $\delta > 0$ and R^{-1} are positive definite thus $(R^{-1} + \delta\Sigma^{-1})$ is positive definite and (18) with (19) define a Kalman-like structure, allowing us to apply the demonstration in [32] and copping with 3) and 4). Taking the same scheme as in [32] we have the following estimate for the initial error

$$\|\tilde{x}(0)\| = \|x(0) - \hat{x}(0)\| < \frac{m_7 + \delta m_7'}{2(m_1 + \delta m_1')^{\frac{7}{2}} m_5^2 L}, \tag{21}$$

where $m_i > 0$ are constants described in [32] and $m_1', m_7' > 0$ are new constants originated in the proof. Their exact value is not relevant, since for the analysis one just need to understand that the basis of attraction is enlarged since δ is chosen. Moreover, if we take $\delta \to 0$ we obtain the same bound for the initial error as for the EKF. ■

Remark: From (21) one can conclude that the region of attraction of the initial error for local convergence is greater for the proposed estimator than the EKF, although, recall that δ must be such that (19) has a solution.

IV. Simulation Results

In order to understand the behavior of the proposed estimator, we provide an example for a first order nonlinear system, where under certain initial conditions, the EKF and PF converge to a local minimum. Due to the forward-backward and coupling nature of the equations(19) and (20) involved in the MFG, in the next section we use the stationary assumption to obtain numerical results.

More precisely, the MFG based estimator is described as

$$\dot{\hat{x}}(t) = f_z(\hat{x})\hat{x}(t) + Qh_z^T(\hat{x}(t))\left(R^{-1} + \delta\Sigma^{-1}\right)(y - h_z(\hat{x})\hat{x}(t)),$$

where $4(\sigma^2)^{-1}Q = \Sigma^{-1}$, and Q satisfies the following Riccati equation

$$\begin{aligned}0 =& Q\big(f_z(\hat{x}(t)) + \delta(\sigma^2)^{-1}I\big)^T + \big(f_z(\hat{x}(t)) + \delta(\sigma^2)^{-1}I\big)Q\\ &+ 2I - \frac{1}{2}Qh_z^T(\hat{x}(t))R^{-1}h_z(\hat{x}(t))Q,\end{aligned}$$

where Q is re-computed in each discretizated time step since it depends on the linearization around the estimate.

A. Example

Consider the following system [30]

$$\begin{aligned} dx &= x(1 - x^2)dt + 0.001 dw_t\\ y &= x^2 + 0.01x + 0.1v_t. \end{aligned}\tag{22}$$

where the initial condition is $x_0 = -0.5$ and the initial estimate $\hat{x}_0 = 1$. The system (without errors) has two stable equilibria at $x = \pm 1$ and an unstable equilibrium at $x = 0$. An schematic of the evolution of the system is presented in Figure 1.

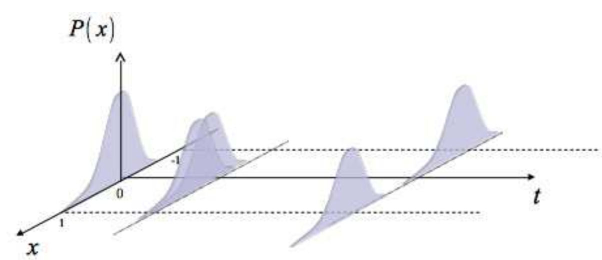

Fig. 1. Time evolution of the true state assuming initial Gaussian distribution

Under certain initial conditions, the EKF and PF fail to converge. See, Fig. 1 that illustrates the case where the initial covariance matrix in the EKF/PF/MFG was set to $P_0 = 0.1$. In the simulation, the system was discretized, using the Euler method with a discretization time step equal to 0.1. With respect to $\delta(t)$, it follows a decreasing law given by $\delta_{t_i} = \delta_{t_0}/(t_i + 1)$ where $\delta_{t_0} = 0.002$. For the particle filter, we set $N = 100$ particles with multinomial resampling.

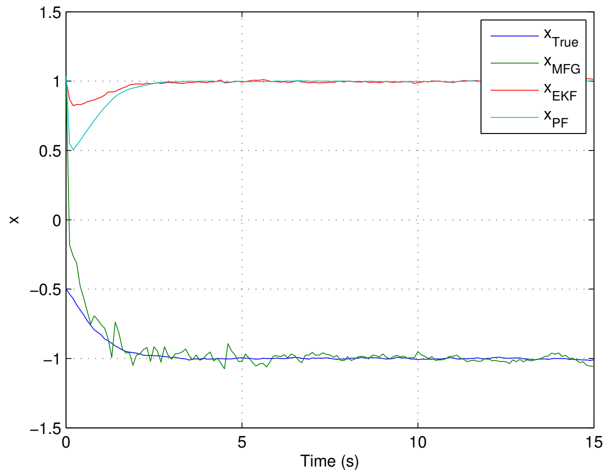

Fig. 2. Time evolution of the true state and their estimatives using MFG, PF and EKF.

B. Discussion of results

We preformed other simulations for different linear and non-linear systems and concluded that for the cases that the EKF converges we obtained compatible results for the MFG estimator. To control how fast we would like to make the population adapt some common behavior, we must tune the parameter δ. If this parameter is small, i.e, $\delta \to 0$ this means that each particle (and the estimate as well) starts to become independent of how the neighbors behave and in that case the estimate no longer depends of the population mass evolution.

V. Conclusions and Further Research

We have considered the state estimation problem of non-linear systems and proposed the MFG as a framework to develop an estimator. We derived a computationally procedure to estimate the state and showed the local convergence for the linearized case. Simulation results (using stationarity assumptions) illustrate the good performance of the proposed algorithm when compared with the EKF and PF. Future work will address the development of new techniques to compute the estimate without using the stationarity assumption and improve the computationally efficiency of the algorithm.

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