Mean-Field Games and Green Power Control

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***Abstract*—In this work, we consider a distributed wireless network where many transmitters communicate with a common receiver. Having the choice of their power control policy, transmitters are concerned with energy constraints : instantaneous energy-efficiency and long-term energy consumption. The individual optimization of the average energy-efficient utility over a finite horizon is studied by using control theory and a coupled system of Hamilton-Jacobi-Bellman-Fleming equations is obtained. Even though the existence of a solution to the corresponding stochastic differential game is proven, the game is difficult to analyze when the number of transmitters is large (in particular, the Nash equilibrium analysis becomes hard and even impossible). But when the number of transmitters is large, the stochastic differential game converges to a mean-field game which is ruled by a more tractable system of equations. A condition for the uniqueness of the equilibrium of the mean-field game is given.**

I. Introduction

Power control has always been recognized as an important problem for multiuser communications [1], [2]. With the appearance of new paradigms such as ad hoc networks [3], unlicensed band communications, and cognitive radio [4], [5], designing distributed power control policies has become especially relevant; in such networks, terminals can freely choose their power control policies and do not need to follow orders from central nodes. More recently, the need for building green communication networks has appeared to be stronger and stronger [6]. The goal is to manage energy consumption both at the mobile terminals and network infrastructure sides [7]. The work reported in this paper precisely falls into this framework that is, the design of green distributed power control policies in multiuser networks.

More precisely, we consider a network which comprises many transmitters and one receiver. Each transmitter chooses its power control policy in order to maximize its average energy-efficiency (measured in bit per Joule). Note that a similar framework was analyzed in [8]. In [8], the problem is modeled by a static game: for each transmission block, the transmitters choose their power levels strategically but independently from past blocks. In [9], it has been shown that much more efficient control policies can be obtained by exploiting long-term interactions, which is done by using the models of repeated games. But, as the channel gains associated with the different communication links vary over time, there is generally a loss of optimality when using repeated games. Indeed, in [10], [11], it is shown that the model of stochastic games is more appropriate and can lead to better policies. The problem is that, even though some special ad hoc policies can be found [11], stochastic games are not fully characterized in general; in particular, Folk theorems are only available in special cases. Additionally, the problem of characterizing the performance of distributed networks modeled by stochastic games becomes very hard and even impossible when the number of players becomes large. The same statement holds for determining individually control strategies. This is where mean-field games come into play. Mean-field games [12] represent a way of approximating a stochastic differential (or difference) game, by a much more tractable model. Typically, instead of depending of the actions and states of all the players, the mean-field utility of a player only depends on its own action and state, and depends on the others through a mean-field. It seems that the only work where mean-field games has been used for power control is [13]. Compared to the latter reference, the present work is characterized by a different utility function (no linear quadratic control assumptions is made here), the fact that the battery level of a transmitter is considered as part of a terminal state, and the existence and uniqueness analysis for of mean-field equilibria is conducted.

This article is organized as follows. In Section II, the wireless context is described and we explain how we model our problem with a stochastic differential game. In Section III, the convergence of this stochastic differential game to a mean-field game is proven and a condition for the uniqueness of the solution to this game is given. Finally, in Section IV we conclude this work.

II. Problem statement

A. Wireless context

In a wireless network depicted by Fig.1 with a set of transmitters
$\mathcal{N} = \{1, \dots, n\}$ and one receiver, the received signal is

$$y(t) = \sum_{i=1}^{n} h_i(t) a_i(t) + z(t) \tag{1}$$

with $h_i(t) \in \mathbb{C}$ the channel coefficient between transmitter i and the receiver at time t, $a_i(t)$ the symbols sequence sent by

The authors would like to thank Hamidou Tembine from École Supérieure d'Électricité, Supélec for his many constructive comments (tembine@iee.org).

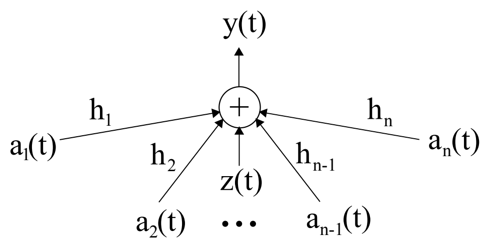

Figure 1. System model.

transmitter i during time-slot t and $z(t)$ is a Gaussian noise with variance σ^2. We denote $p_i(t) = |a_i(t)|^2$ the transmitting power of transmitter i during time-slot t.

As we consider that transmitters are concerned about their transmission rate but also about the energy they spend to reach this rate, we use energy-efficient utility as introduced by Mandayam and Goodman [8] : for each transmitter, the instantaneous utility is

$$u_i(\underline{p}(t), \underline{h}(t)) = \frac{Rf(\gamma_i(\underline{p}(t), \underline{h}(t)))}{p_i(t)} \tag{2}$$

where R in bits/s is the transmitting rate of transmitter i (as we do not use this rate as a control in our work, we consider that it is constant for all the transmitters), γ_i is the $SINR$ of transmitter i, $\underline{p}(t) = (p_1(t), \dots, p_n(t))$ and $\underline{h}(t) = (h_1(t), \dots, h_n(t))$. f is an efficiency function which represents the transmission success rate at the receiver. It takes its values in $[0, 1]$ and depends on the $SINR$ of transmitter i. More details about f can be found in [8]. The $SINR$ writes

$$\gamma_i(\underline{p}(t), \underline{h}(t)) = \frac{p_i(t)|h_i(t)|^2}{\sum_{j \neq i}^{n} p_j(t)|h_j(t)|^2 + \sigma^2}, \tag{3}$$

with $|h_i(t)|^2$ the channel gain between transmitter i and the receiver.

B. Evolution laws of the system

Depending on time, two parameters of the considered model vary. These are the energy left at transmitter and the channel coefficients between the transmitter and the receiver.

First, we consider that the manufacturer of the mobile transmitter wants its device to be able to transmit during a finite horizon which represents the desired battery life. We denote E_0 the initial available energy for each transmitter. The dynamics of $E_i(t)$, the energy left for transmitter i at time t, is directly linked to the transmitting power through the following deterministic law

$$\mathrm{d}E_i(t) = -p_i(t)\mathrm{d}t. \tag{4}$$

Secondly, we consider that for each transmitter the dynamics of its channel coefficients is a Wiener process, meaning that $\forall i \in \mathcal{N}$

$$\mathrm{d}h_i(t) = \eta \mathrm{d}\mathbb{W}_i(t) \tag{5}$$

where $\forall i \in \mathcal{N}$, $\mathbb{W}_i(t)$ are mutually independent Wiener processes of dimension 2 and η is the variance of h_i ($\eta < +\infty$). Contrary to the evolution of the energy left at the transmitter, the channel coefficients evolve according to a stochastic law.

C. A stochastic differential game

It is assumed that each transmitter wants to maximize its utility during a finite horizon $T \to T'$ while taking into account the dynamics of the system, then the problem can be written $\forall i \in \mathcal{N}$

$$\begin{aligned} &v_i(T) = \sup_{p_i(T \to T')} \mathbb{E}\left[\int_T^{T'} u_i(\underline{p}(t), \underline{h}(t))\mathrm{d}t + q(\underline{E}(T'), \underline{h}(T'))\right] \\ &\mathrm{d}h_i(t) = \eta \mathrm{d}\mathbb{W}_i(t) \\ &\mathrm{d}E_i(t) = -p_i(t)\mathrm{d}t \end{aligned} \tag{6}$$

where $v_i(T)$ is the Bellman function, the expectation of the continuous sum of utility for the optimal control path, $\underline{E}(t) = (E_1(t), \dots, E_n(t))$, and $q(\underline{E}(T'), \underline{h}(T'))$ is the instantaneous utility value for the final state.

We denote $X(t) = (\underline{E}(t), \underline{h}(t))^{\mathrm{T}}$ the state of the system at time t. Then

$$\mathrm{d}X(t) = (-\underline{p}(t)\mathrm{d}t, \eta \mathrm{d}\mathbb{W}(t))^{\mathrm{T}}, \tag{7}$$

with $\mathbb{W}(t)$ a $2n$-dimension Wiener process. (6) can be rewritten

$$\begin{aligned} &v_i(T, X) = \sup_{p_i(T \to T')} \mathbb{E}\left[\int_T^{T'} u_i(X(t), \underline{p}(t))\mathrm{d}t + q(X(T'))\right] \\ &\mathrm{d}X(t) = (-\underline{p(t)}\mathrm{d}t, \underline{\eta}\mathbb{W}_t)^{\mathrm{T}} \end{aligned} \tag{8}$$

If $p_i(t)$ is now considered as a feedback control

$$p_i(t) = p_i(t, X(t)), \tag{9}$$

we are exactly in the context of a stochastic differential game.

D. Existence of a Nash equilibrium to the stochastic differential game

Definition 1 (Nash equilibrium of the stochastic differential game)**.** *A power profile*

$$\underline{p}^*(t, X(t)) = (p_1^*(t, X(t)), \dots, p_n^*(t, X(t))$$

is a Nash equilibrium of the stochastic differential game if and only if $\forall i \in \{1, \dots, n\}$, p_i^* is the optimal feedback for the control problem

$$\sup_{p_i(T \to T')} \mathbb{E}\left[\int_T^{T'} u_i(X(t), p_i(t, X(t)), \underline{p}_{-i}^*(t, X(t)))dt + q(X(T'))\right] \tag{10}$$

where

$$\underline{p}_{-i}^*(.) = \big(p_1^*(.), \dots, p_{i-1}^*(.), p_{i+1}^*(.), \dots, p_n^*(.)\big). \tag{11}$$

According to [14], a sufficient condition for the existence of a Nash equilibrium for the stochastic differential game is the existence of a solution to the Hamilton-Jacobi-Bellman-Fleming [15] equation for each transmitter

$$0 = \sup_{p_i(T\to T')}\Big[u_i(X(t),\underline{p}(t,X(t))) - p_i(t,X(t))\partial_{E_i} v_i(t,X(t))\Big] \\ +\partial_t v_i(t,X(t)) + \frac{\eta^2}{2}\partial^2_{hh} v_i(t,X(t)). \tag{12}$$

with $\partial_y F$ and $\partial^2_{yy} F$ respectively being the first order and the second order partial derivative of function F with regard to y

$$\begin{cases} \partial_y F = \frac{\partial F}{\partial y} \\ \partial^2_{yy} F = \frac{\partial^2 F}{\partial y^2} \end{cases}$$

However there exists a solution if the function

$$H(X(t), p_{-i}(t,X(t)), \partial_{E_i} v_i(t,X(t))) = \\ \sup_{p_i(T\to T')}\Big[u_i(X(t),\underline{p}(t,X(t))) - p_i(t,X(t))\partial_{E_i} v_i(t,X(t))\Big] \tag{13}$$

is smooth (see [16] for more details). And with a similar reasoning as in [17], we can show that finding optimal power profiles

$$\underline{p}^*(T\to T') = (p_1^*(T\to T'),\ldots,p_n^*(T\to T')) \tag{14}$$

such that $\forall i \in \mathcal{N},\ p_i^*(T\to T') \in$

$$\arg\max_{p_i(T\to T')}\Big[u_i(X(t),\underline{p}(t,X(t))) - p_i(t,X(t))\partial_{E_i} v_i(t,X(t))\Big] \tag{15}$$

requires to solve $\forall i \in \mathcal{N}$, $\forall t \in [T,T']$

$$f'(\gamma_i(t))\gamma_i(t) - f(\gamma_i(t)) = \gamma_i(t)^2 \frac{\partial_{E_i} v_i(t,X(t))}{R}\left(\frac{\sigma^2 + \sum_{j\neq i}^n |h_j|^2(t) p_j^*(t)}{|h_i|^2(t)}\right)^2 \tag{16}$$

Note that we consider that $\partial_{E_i} V_i(t,X(t)) \geq 0$, otherwise the optimal power $p_i^*(t) \to \infty$. In other words, we consider that the more energy we have in our battery, the better the payoff can be.

The existence of a non-zero solution depends on the term

$$\beta_i = \frac{\partial_{E_i} v_i(t,X(t))}{R}\left(\frac{\sigma^2 + \sum_{j\neq i}^n |h_j|^2(t) p_j^*(t)}{|h_i|^2(t)}\right)^2. \tag{17}$$

Heuristically, we can see that there exists a threshold $\beta_{\max}$ such that if $\beta_i < \beta_{\max}$, there exists a unique global maximizer γ^* different from 0 and if $\beta_i \geq \beta_{\max}$, 0 is the global maximizer. It is interesting to note that β_i being proportional to the interference of transmitter i, we can state that when the interference conditions of transmitter i are favorable, transmitter i should transmit whereas when the conditions are bad, it should not waste transmitting power.

This behavior is illustrated on Fig.2, where the considered efficiency function is

$$f(\gamma_i(t)) = \begin{cases} e^{-\frac{a}{\gamma_i(t)}} \text{ if } \gamma_i(t) > 0, \\ 0 \text{ if } \gamma_i(t) = 0. \end{cases} \tag{18}$$

an efficiency function introduced by Belmega et al. in [18]. In this simulation $a = 1$.

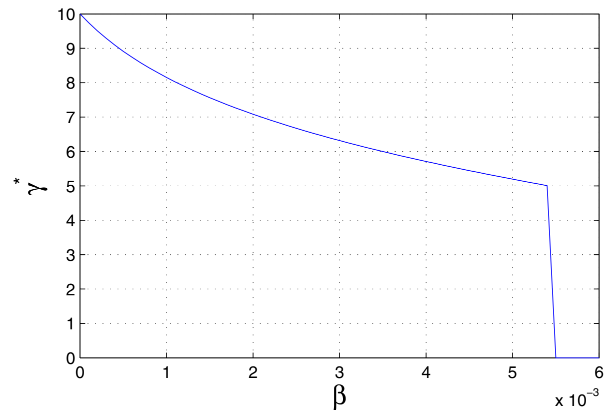

Figure 2. Instantaneous global maximizer depending on β.

In order to use the implicit function theorem, we define

$$g : [0,\beta_{\max}[\times\mathbb{R}^+ \to \mathbb{R} \\ (\beta,\gamma) \to f'(\gamma)\gamma - f(\gamma) - \beta\gamma^2. \tag{19}$$

g is C^∞, then if $g(\beta^0,\gamma^0) = 0$, there exists $\varphi : \mathbb{R} \to \mathbb{R}$ such that $\gamma^0 = \varphi(\beta^0)$. φ is C^∞ and

$$\frac{\partial\varphi}{\partial\beta}(\beta^0) = -\frac{\frac{\partial g(\beta^0,\gamma^0)}{\partial\beta}}{\frac{\partial g(\beta^0,\gamma^0)}{\partial\gamma}} \tag{20}$$

III. Convergence of the stochastic differential power control game to a mean-field game

The previous system of equations is hard to solve because for each transmitter, the associated partial differential equation depends on all the channel coefficients and all the power profiles of the other transmitters. Of course, if the number of transmitters in the system increases greatly, it is true that the complexity of the resolution increases as well. But an interesting fact is that if the number of transmitters is large enough, then for one single transmitter point of view, it becomes equivalent to consider all the other transmitters as a continuum. Therefore only the distribution of the other transmitters states is needed for one transmitter to take into account the other transmitters, which highly simplifies the formulation of the problem. This is what is developed in the present section.

Naturally, in a MAC network, the interaction between the different transmitters is expressed in the interference term. If we consider that CDMA is used in the network, then the interference seen by the receiver is

$$I_i(t) = \frac{1}{n-1}\sum_{j\neq i}^n p_j(t)|h_j(t)|^2. \tag{21}$$

We consider homogeneous admissible control in own state feedback form $\forall i \in \mathcal{N}$

$$p_i(t) = \alpha(t, s_i(t)), \tag{22}$$

with $s_i(t) = (h_i(t), E_i(t))^{\mathrm{T}}$ the own state of transmitter $i$and $\mathbb{E}[\alpha(t, s_i(t))^2] < +\infty$. Then

$$\begin{aligned} I_i(t) =& \frac{1}{n-1}\frac{n}{n}\sum_{j\neq i}^{n}\alpha(t, s_j(t))|h_j(t)|^2 \\ =& \frac{n}{n-1}\frac{1}{n}\left[\sum_{j=1}^{n}\alpha(t, s_j(t))|h_j(t)|^2 - \alpha(t, s_i(t))|h_i(t)|^2\right] \\ =& \frac{n}{n-1}\int |h|^2\alpha(t, s)M_t^n(\mathrm{d}s) \\ & -\frac{\alpha(t, s_i(t))|h_i(t)|^2}{n-1} \\ =& \frac{n}{n-1}A_t^n - \frac{1}{n-1}B_t^i. \end{aligned} \tag{23}$$

with

$$M_t^n = \frac{1}{n}\sum_{j=1}^{n}\delta_{s_j(t)}, \tag{24}$$

$$A_t^n = \int |h|^2\alpha(t, s)M_t^n(\mathrm{d}s), \tag{25}$$

$$B_t^i = \alpha(t, s_i(t))|h_i(t)|^2. \tag{26}$$

If the number of transmitters becomes very large ($n \to \infty$), we can consider that we have a continuum of transmitters. The convergence of the interference term when $n \to \infty$ needs to be proven. Using admissible control, $\mathbb{E}[B_t^i] < \infty$, then

$$\lim_{n\to\infty}\frac{B_t^i}{n-1} = 0. \tag{27}$$

As $\lim_{n\to\infty}\frac{n}{n-1} = 1$, to prove $I_i(t)$ converges weakly, it suffices to prove A_t^n converge weakly. A sufficient condition is the weak convergence of the process M_t^n. As stated by Kotelenez and Kurtz in [19], if the states and the controls are almost surely bounded and the $(s_j(0))$ are exchangeable, then there exists a distribution m_t such that

$$m_t = \lim_{n\to\infty} M_t^n. \tag{28}$$

However in our case, the evolution of the state of the transmitters does not depend on the index of the transmitter since $\forall i \in \mathcal{N}$

$$\begin{cases} \mathrm{d}E_i(t) = -\alpha(t, s_i(t))\mathrm{d}t \\ \mathrm{d}h_i(t) = \eta\mathrm{d}\mathbb{W}_t \end{cases} \tag{29}$$

Then, we set

$$\hat{I}(t, m_t) = \int |h|^2\alpha(t, s)m_t(\mathrm{d}s), \tag{30}$$

$$\widehat{\gamma}(s(t), m_t) = \frac{p(t)|h(t)|^2}{\sigma^2 + \hat{I}(t, m_t)}, \tag{31}$$

$$\hat{u}(t) = \frac{Rf(\widehat{\gamma}(s(t), m_t))}{p(t)} =: \hat{r}(s(t), p(t), m_t). \tag{32}$$

And we can formulate the mean-field response problem

$$\hat{v}_T = \sup_{p(T\to T')}\mathbb{E}\left[q(s(T')) + \int_T^{T'}\hat{r}(s(t), p(t), m_t^*)\mathrm{d}t\right] \tag{33}$$

where m_t^* is the mean-field optimal trajectory, m_0^* being assumed to be known and

$$\mathrm{d}s(t) = (-p(t)\mathrm{d}t, \eta\mathrm{d}\mathbb{W}(t))^{\mathrm{T}}, \tag{34}$$

A solution of the mean-field response problem is a solution of

$$\begin{cases} -\partial_t\hat{v}_t = \tilde{H}(s(t), \partial_E\hat{v}_t, m_t) + \frac{\eta^2}{2}\partial_{hh}^2\hat{v}_t \\ \partial_t m_t + \partial_E(m_t\partial_u\tilde{H}(s(t), \partial_E\hat{v}_t, m_t)) = \frac{\eta^2}{2}\partial_{hh}^2 m_t \end{cases} \tag{35}$$

with $\hat{v}_T = q(s(T))$ and m_0 known and

$$\tilde{H}(s, u, m) = \sup_p\{\hat{r}(s, p, m) + \langle p, -u\rangle\} \tag{36}$$

As for the stochastic differential game case, the first equation is a Hamilton-Jacobi-Bellman-Flemming equation. But it is now coupled with a Fokker-Planck-Kolmogorov equation. The former one is a backward equation whereas the latter one is a forward equation.

Similarly to section II-D, expressing $\tilde{H}(s(t), \partial_E\hat{v}_t, m_t)$ requires to solve

$$\gamma f'(\hat{\gamma}) - f(\hat{\gamma}) = \hat{\beta}\hat{\gamma}^2 \tag{37}$$

with

$$\hat{\beta} = \frac{\partial_E\hat{v}_t}{R}\frac{(\sigma^2 + \hat{I}_i(t, m_t))^2}{|h_i|^4} \tag{38}$$

If there exists a solution different from zero, we denote $\hat{\gamma}^*(s(t), \partial_E\hat{v}_t, m_t)$ this solution. With the parameters previously defined, we can derive a sufficient condition for uniqueness of the solution to the system of coupled equations (35). The proof is not given here.

Proposition 1 (Uniqueness of a solution)**.** *A sufficient condition for a solution of the solution to the mean-field game to be unique is* $\forall s, \forall\partial_E\hat{v}, \forall m$

$$\begin{aligned} &(f''(\hat{\gamma}^*(s, \partial_E\hat{v}, m) - 2\hat{\beta})f'(\hat{\gamma}^*(s, \partial_E\hat{v}, m)) \\ &+ m\hat{\gamma}^*(s, \partial_E\hat{v}, m)^2\beta^2 > 0 \end{aligned} \tag{39}$$

IV. Conclusion and Perspective

In a wireless model where the energy left for each transmitters evolves according to the power spent to transmit and the channel coefficients of these same transmitters evolve in a stochastic manner, we have shown that the problem of each transmitter maximizing the expectation of an energy-efficient utility during a given time is equivalent to a stochastic differential game. The existence of a Nash equilibrium for this game has been proven. But as the resolution of the associated system of partial differential equations is hard, especially when the number of transmitter is high, we show that the stochastic differential game converges to a mean-field game which equilibrium is the solution of a system of two partial differential equations: a backward Hamilton-Jacob-Bellman equation and a forward Fokker-Planck-Kolmogorov equation. From a game theory point of view, this model highly simplifies the complexity of the interactions between a large number of players. A condition of uniqueness for the mean-field Nash equilibrium is given.

Of course, our main objective now is to characterize the mean-field equilibrium. This can be done numerically by implementing solutions as described in the work of Achdou [20]. Then the efficiency of this solution could be compared to equilibria obtained as solutions of simpler energy-efficient power control models.

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