# Routing Games on a Circle

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***Abstract*—Rings are quite common in both road traffic networks as well as in telecommunication networks. In the road traffic context, we often find rings surrounding towns and cities. Traffic over these rings is either bidirectional or we may find two rings that surround the town carrying traffic in opposite directions (clockwise and anti-clockwise). Telecommunication networks based on rings have been often used both as local area networks as well as in metropolitan area networks and here too we find both bidirectional networks as well as networks consisting of two rings carrying traffic in opposite directions. Each decision maker (e.g. the drivers, in the case of road traffic, and perhaps Internet access providers, in the case of telecommunication) is faced with a simple routing decision: whether to go clockwise or anti-clockwise. Assuming a simple source-destination demand matrix, we analyze this problem as a non-cooperative game and derive several interesting characteristics of the equilibria.**

## I. Introduction

The analysis of selfish routing over ring networks has attracted the attention of researchers in both road traffic as well as telecommunication network engineering. Indeed, ring type roads have been deployed around big cities, allowing vehicles that do not need to enter the city to take the peripheral route, thereby suffer from and cause less congestion. Various research papers thus study the traffic assignment problem for cities that have only radial and circumferential roads (see [1], [2] and references therein). Rings are also common in telecommunication networks, with the Token Ring standard IEEE 802.5 for local area networks, and the standardized metropolitan area networks such as the FDDI and the DQDB [3]. Optimal or competitive routing over these networks has also been studied. Unlike road traffic assignment problems in which the solution concept is often taken to be the Wardrop equilibrium, in telecommunication applications, it is the Nash equilibrium that has emerged (along with its comparison to the global optimal). Papers that have studied routing in this context assume in general that flows are non-splittable [4].

In this paper we consider the framework of Nash equilibrium (splittable and non-splittable) as well as that of Wardrop equilibrium. Instead of focusing on routing choices between radial and circumference routes, as is usually done, we focus on the choice of *direction*: for a given source and destination on the peripheral network, there is a possibility to arrive at the destination going either clockwise or anti-clockwise. We consider two frameworks: (i) the network consists of a single bidirectional ring, and (ii) it consists of two unidirectional rings in opposite directions.

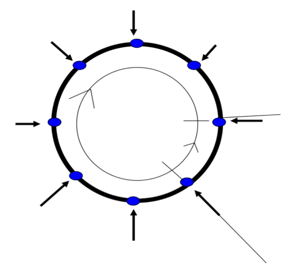

Fig. 1. Competitive routing on a bidirectional circular network.

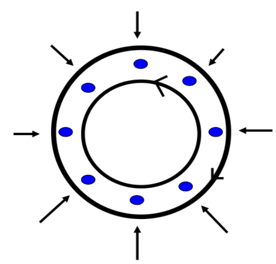

Fig. 2. Competitive routing between two unidirectional circular paths.

There are $n$ nodes on the ring, each identified with a player. We study an extreme scenario in which all traffic that arrives at a node has to be shipped to the next node. This case is extreme since it is the most unbalanced one, in the sense that the traffic that a player sends through the direct path has only one hop to go, where as the traffic that it sends through the alternative route has to go through $n-1$ hops. Even with this simple demand matrix, which is clearly biased in favor of choosing the direct path, we establish the counter-intuitive fact: *"It is possible that not all players send their traffic through the direct path at equilibrium."* Also, in the case where not all players send their traffic through the direct path, we provide interesting characterizations of the relative magnitudes of their demands.

## II. The Model

We consider $n$ nodes on a circle, indexed by $0, 1, \ldots, n-1$. Each node $k$ corresponds to one player that has to ship a demand $\phi_k (> 0)$ to the next node, i.e., to node $k+1$. It can use the direct path $(k, k+1)$ or the indirect one, $(k, k-$

$1, \ldots, 0, n-1, n-2, \ldots, k+1$). (Note that node indices are modulo $n$.) Throughout the paper, we assume that $n > 2$, since the concept of a direct/indirect path is meaningful only if $n > 2$. We consider two possibilities: (i) a single bidirectional link (Figure 1), and (ii) two unidirectional links (Figure 2). In the bidirectional case, we call the link between node $k$ and node $k+1$ simply *link* $k$. In the unidirectional case, the link from node $k$ to node $k+1$ shall be called *link* $k$, and the link from node $k+1$ to node $k$ shall be called *link* $k'$.

Let $x_k$ and $x'_k$ denote the total amount of flow on link $k$ and link $k'$, respectively. Let $x_k^k$ and $\alpha_k$ denote the *amount* and *fraction*, respectively, of class $k$ flow through the direct path, i.e., through link $k$. We call

$$\boldsymbol{\alpha} = (\alpha_0, \ldots, \alpha_{n-1})$$

the *assignment vector*. For a given *demand vector*

$$\boldsymbol{\phi} = (\phi_0, \ldots, \phi_{n-1}),$$

the assignment vector $\boldsymbol{\alpha}$ completely specifies the *flow vector*

$$\boldsymbol{x} = (x_0^0, \ldots, x_{n-1}^{n-1}) = (\alpha_0 \phi_0, \ldots, \alpha_{n-1}\phi_{n-1}).$$

When considering a symmetric demand, we denote $\phi_k$ by $\phi$. When considering a symmetric flow, we denote $x_k$ and $x_k^k$ by $x$ and $\overline{x}$, respectively. When considering a symmetric assignment, we denote $\alpha_k$ by $\alpha$.

The cost density of link $k$ (i.e., cost per unit of flow) is given by $T_k(x_k)$ when the amount of flow on link $k$ is $x_k$. Note that the link cost function $T_k(\cdot)$ of link $k$ depends only on the amount of flow on link $k$. We define the *marginal cost function* $\tau_k(x)$ of link $k$ by $\tau_k(x) := d(xT_k(x))/dx$. We assume that the costs are *additive* in that the total cost over a path is equal to the sum of costs incurred over each of the links constituting the path. When considering symmetric link costs, we denote $T_k(\cdot)$ and $\tau_k(\cdot)$ by $T(\cdot)$ and $\tau(\cdot)$, respectively.

Throughout the paper, we have the following assumptions on the link cost functions $T_k(\cdot)$.

**A 1.** The link cost functions $T_k(\cdot)$ are nonnegative, strictly increasing, continuously differentiable.

**A 2.** For all $k$, $x > 0$ implies $T_k(x) > 0$.

Throughout the paper, we will implicitly assume that **A**1 and **A**2 hold. Depending on the context, we may also assume one or more of the following.

**A 3.** The link cost functions are identical.

**A 4.** For every link $k$, the function $xT_k(x)$ is convex.

**A 5.** For all $k$, the function $T_k(\cdot)$ is subadditive.

## III. Wardrop Equilibrium and Global Optimality

Recall that a routing game is completely specified by the tuple $(G, \boldsymbol{\phi}, \boldsymbol{T})$, where $G$ denotes the network graph, $\boldsymbol{\phi}$ denotes the demand vector and $\boldsymbol{T}$ denotes the vector of link cost functions. We will need the following theorem by Beckmann et al. [5] to study global optimality (see the discussion surrounding Corollary 18.10 of [6] for an explanation). Note that, we have rephrased the original theorem by Beckmann et al. [5] to fit our model and terminology.

**Theorem 1** (Beckmann et al. [5]). *If* **A***1 and* **A***4 hold, then an assignment* $\boldsymbol{\alpha}^*$ *is an optimal assignment of the game* $(G, \boldsymbol{\phi}, \boldsymbol{T})$ *if and only if it is a Wardrop equilibrium assignment of the game* $(G, \boldsymbol{\phi}, \boldsymbol{\tau})$*, where* $\tau_k(\cdot)$ *denotes the marginal cost function corresponding to the link cost function* $T_k(\cdot)$. ■

### A. Bidirectional Link

The flow on link $k$ in this case is given by

$$x_k = x_k^k + \sum_{j \neq k} (\phi_j - x_j^j) = \alpha_k \phi_k + \sum_{j \neq k} (1 - \alpha_j)\phi_j. \quad (1)$$

Allowing the demands and the cost functions to be *asymmetric* we obtain the following theorem.

**Theorem 2.** *At Wardrop equilibrium, for all classes, except perhaps one, traffic goes through the direct path.*[1]

**Proof.** Assume that class $i$ does not send all its traffic through link $i$, i.e., $\alpha_i < 1$. Then, for all $k \neq i$, we have

$$x_k = \alpha_k \phi_k + \sum_{j \neq k} (1 - \alpha_j)\phi_j \geq (1 - \alpha_i)\phi_i > 0,$$

i.e., $x_k > 0$, implying that $T_k(x_k) > 0$. Also, by Wardrop principle, we have $\sum_{k \neq i} T_k(x_k) \leq T_i(x_i)$. Thus, we conclude that, for all $k \neq i$, $T_k(x_k) < T_i(x_i)$. The cost for player $k \neq i$ over its indirect path is at least $T_i(x_i)$, since the indirect path of a player $k \neq i$ contains link $i$, whereas its cost over its direct path equals $T_k(x_k)$. Hence, $T_k(x_k) < T_i(x_i)$ implies that, all players, other than $i$, use their direct path only. ■

The following corollaries are immediate from Theorem 2.

**Corollary 1.** *If* **A***4 holds, then the global optimum also corresponds to an assignment* $\boldsymbol{\alpha}$ *such that* $\alpha_k = 1$ *for every class* $k$*, except perhaps one class.*

**Proof.** Note that Theorem 2 still applies if we replace the costs $T_k(\cdot)$ by their marginal costs $\tau_k(\cdot)$. The corollary is then immediate from Theorems 1 and 2. ■

**Corollary 2.** *If the demands and the cost functions are such that,*

$$T_{i^*}(\phi_{i^*}) \leq \sum_{k \neq i^*} T_k(\phi_k),$$

*where* $i^* \in \arg\max_i T_i(\phi_i)$*, then, for all classes, traffic goes through the direct path at Wardrop equilibrium.*

**Proof.** Suppose that there exists a class $i$ for which not all the traffic is routed through the direct path, i.e., $\alpha_i < 1$. By Theorem 2, all the traffic of class $k \neq i$ is routed through the direct path. Thus, $x_i = \alpha_i \phi_i < \phi_i$, and, for all $k \neq i$, we have $x_k = \phi_k + (1 - \alpha_i)\phi_i > \phi_k$. Then,

$$\sum_{k \neq i} T_k(x_k) > \sum_{k \neq i} T_k(\phi_k) \geq T_i(\phi_i) > T_i(x_i),$$

[1] Recall that **A**1 and **A**2 are assumed to hold throughout the paper.

implying that, for class $i$, the cost of the indirect path is strictly larger than the cost of the direct path which contradicts $\alpha_i < 1$ at equilibrium. Thus, $\alpha_i$ must be equal to 1. ■

Next, we restrict the cost functions to be identical, and obtain the following theorem.

**Theorem 3.** *If* **A**3 *holds and not all traffic of a given class $i$ uses the direct path at Wardrop equilibrium, then the demand $\phi_i$ of this class is strictly larger than that of any other class. If, in addition,* **A**5 *also holds, then $\phi_i > \sum_{k\neq i}\phi_k$, i.e., the demand of class i is strictly larger than the sum of the demands of the remaining classes.*

**Proof.** Let $i$ be the class for which not all the traffic is routed through the direct path. By Theorem 2, all the traffic of class $k \neq i$ is routed through the direct path. Thus, for all $k \neq i$, we have $x_k > \phi_k$. Suppose that $\phi_j \geq \phi_i$ for some $j \neq i$. Then the cost of the indirect path for class $i$, satisfies

$$\sum_{k\neq i} T(x_k) > \sum_{k\neq i} T(\phi_k) > T(\phi_j) \geq T(\phi_i) > T(x_i),$$

i.e., the cost of the indirect path is strictly larger than the direct path. This contradicts equilibrium, since we assumed that a positive fraction of class $i$ traffic is routed through the indirect path. Hence, it must hold that, for all $k \neq i$, $\phi_i > \phi_k$.

For the second part of the theorem, note that, by Corollary 2, there is an index $i^*$ such that $T(\phi_{i^*}) > \sum_{k\neq i^*} T(\phi_k)$. By the first part of this theorem and strict monotonicity of the cost function, $i^* = i$. Now, by subadditivity of the cost function, we have,

$$T(\phi_i) > \sum_{k\neq i} T(\phi_k) \geq T(\sum_{k\neq i}\phi_k) \implies \phi_i > \sum_{k\neq i}\phi_k. \quad ■$$

Next, we also restrict the demands to be symmetric, and obtain the following theorem.

**Corollary 3.** *If the demands and the cost functions are symmetric, then the Wardrop equilibrium satisfy, for all k,* $\alpha_k = 1$. *In other words, only paths that are the shortest in terms of number of hops are used. If, in addition,* **A**4 *also holds, then the global optimal solution also corresponds to* $\alpha_k^* = 1$ *for all k.*

**Proof.** By Corollary 2 and by symmetry, it follows that all classes of traffic use only the direct path, and, for all $k$, $\alpha_k = 1$. By Corollary 1, under **A**4, traffic for an optimal flow goes through the direct path for all classes, except perhaps one. Suppose that $\alpha_i^* < 1$. Then, by Theorem 1, $\alpha_k^* = 1$ for all $k \neq i$. Thus, $x_i^* = \alpha_i^*\phi < \phi$ implying that $\tau(x_i^*) < \tau(\phi)$, and for all $k \neq i$, $x_k^* = \phi + (1-\alpha_i)\phi > \phi$ implying that $\tau(x_k^*) > \tau(\phi)$. Since $\alpha_i^* < 1$, Wardrop equilibrium demands that $\sum_{k\neq i}\tau(x_k^*) \leq \tau(x_i^*)$ implying that $\sum_{k\neq i}\tau(\phi) < \tau(\phi)$, which is impossible. Hence, we must have $\alpha_i^* = 1$, and the global optimum corresponds to $\alpha_k^* = 1$ for all $k$. ■

### *B. Unidirectional Links*

The amount of flow on link $k$ and $k'$ are given by

$$x_k = x_k^k = \alpha_k\phi_k \; ; \; x'_k = \sum_{j\neq k}(\phi_j - x_j^j) = \sum_{j\neq k}(1-\alpha_j)\phi_j \quad (2)$$

**Theorem 4.** *If* **A**3 *holds, then the Wardrop equilibrium consists of sending all traffic to the right, i.e., through the shortest paths in terms of number of hops if and only if*

$$\max_k T(\phi_k) \leq (n-1)T(0).$$

*If*

$$\min_k T(\phi_k) > (n-1)T(0),$$

*then there is no pure Wardrop equilibrium.*[2]

**Proof.** For each class $k$, the cost of the direct path is $T(\alpha_k\phi_k)$ and the cost of the indirect path is $\sum_{j\neq k} T(\sum_{m\neq j}(1-\alpha_m)\phi_m)$. The Wardrop equilibrium consists of sending all traffic to the right if and only if, for each class $k$, the cost of the direct path is less than or equal to the cost of the indirect path with $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$. This happens if and only if, for all $k$, $T(\phi_k) \leq (n-1)T(0)$, i.e., if and only if $\max_k T(\phi_k) \leq (n-1)T(0)$.

Suppose that there exists a pure Wardrop equilibrium with a class $i$ such that $\alpha_i = 0$. Then, for class $i$, the cost of the direct path is $T_i^D = T(0)$ and the cost of the indirect path is $T_i^I \geq (n-1)T(\phi_i)$, with equality if and only if class $i$ is the only class for which all the traffic is routed through the indirect path. Since $\alpha_i = 0$, we have $T_i^I \leq T_i^D$, which gives

$$(n-1)T(\phi_i) \leq T_i^I \leq T_i^D = T(0),$$

or

$$(n-1)T(\phi_i) \leq T(0).$$

Since

$$\min_k T(\phi_k) > (n-1)T(0) \implies T(\phi_i) > (n-1)T(0),$$

we have

$$(n-1)^2 T(0) < (n-1)T(\phi_i) \leq T(0),$$

or

$$(n-1)^2 < 1,$$

which is impossible for $n > 2$. (Recall that we assume $n > 2$.)

Hence, if $\min_k T(\phi_k) > (n-1)T(0)$, there cannot be a pure equilibrium with a class $i$ such that $\alpha_i = 0$.

The case $\alpha_1 = \alpha_2 = \cdots = \alpha_n = 1$ is ruled out by the first part of the proof. ■

The following corollary is immediate from Theorem 4.

**Corollary 4.** *If the demands and the link costs are symmetric, then the Wardrop equilibrium consists of sending all traffic to the right, i.e., through the shortest paths in terms of number of hops if and only if $T(\phi) \leq (n-1)T(0)$. If $T(\phi) > (n-1)T(0)$, then there is no pure Wardrop equilibrium.* ■

[2] An equilibrium is said to be *pure* if each class sends all its traffic on one path, i.e., for all $k$, $\alpha_k$ is either 1 or 0.

## IV. The $n$-Player Game and Nash Equilibria

### A. Non-splittable Demands

First consider the case when the demands are not splittable, i.e., when each player $k$ has to route all its traffic either through the direct path ($\alpha_k = 1$) or through the indirect path ($\alpha_k = 0$). In this case, we can have only pure Nash equilibria. The following theorem states the existence of a pure Nash equilibrium and characterizes the pure Nash equilibria.

*1) Bidirectional Link:*

**Theorem 5.** *If* **A**3 *holds, then the assignment* $\boldsymbol{\alpha} = (1, \ldots, 1)$ *is a Nash equilibrium. Moreover, there cannot be a pure Nash equilibrium with a player sending its traffic on the direct path and another player sending its traffic on the indirect path.*

**Proof.** When all players use the direct path, the cost for each player $i$ is $\phi_i T(\phi_i)$. If only player $k$ deviates and takes the indirect path, then the cost for player $k$ is $\phi_k \sum_{i \neq k} T(\phi_i + \phi_k)$ which is strictly larger than $\phi_k T(\phi_k)$. Hence, the assignment $\boldsymbol{\alpha} = (1, \ldots, 1)$ is a Nash equilibrium.

Suppose that there exist classes $i$ and $j$ such that $\alpha_i = 0$ and $\alpha_j = 1$. For all other classes $k \neq i, j$, we have $\alpha_k$ equal to 1 or 0. Then the cost for class $i$ is

$$
\begin{aligned}
J_I^i &= \phi_i \sum_{k \neq i} T(\alpha_k \phi_k + \sum_{l \neq k} (1 - \alpha_l)\phi_l) \\
&= \phi_i \sum_{k \neq i} T(\alpha_k \phi_k + \phi_i + \sum_{l \neq k, i} (1 - \alpha_l)\phi_l) \\
&= \phi_i T(\phi_j + \phi_i + \sum_{l \neq i, j} (1 - \alpha_l)\phi_l) \\
&\quad + \phi_i \sum_{k \neq i, j} T(\alpha_k \phi_k + \phi_i + \sum_{l \neq i, j, k} (1 - \alpha_l)\phi_l).
\end{aligned}
$$

If only class $i$ deviates and takes the direct path, then, taking $\alpha_i = 1$ and $\alpha_j = 1$, the cost for class $i$ would be

$$
\begin{aligned}
J_D^i &= \phi_i T(\alpha_i \phi_i + \sum_{l \neq i} (1 - \alpha_l)\phi_l) \\
&= \phi_i T(\phi_i + \sum_{l \neq i, j} (1 - \alpha_l)\phi_l).
\end{aligned}
$$

In that case, we have

$$
\begin{aligned}
J_I^i &> \phi_i T(\phi_j + \phi_i + \sum_{l \neq i, j} (1 - \alpha_l)\phi_l) \\
&> \phi_i T(\phi_i + \sum_{l \neq i, j} (1 - \alpha_l)\phi_l) = J_D^i,
\end{aligned}
$$

and player $i$ would deviate to use the direct path. ∎

The following corollary is immediate from Theorem 5.

**Corollary 5.** *If* **A**3 *holds, and*

$$T\Big(\sum_k \phi_k\Big) < \min_i \sum_{k \neq i} T\Big(\sum_{l \neq k} \phi_l\Big),$$

*then the assignment* $\boldsymbol{\alpha} = (1, \ldots, 1)$ *is the only pure Nash equilibrium.*

**Proof.** Theorem 5 rules out the possibilities of all pure Nash equilibria, except $\boldsymbol{\alpha} = (1, \ldots, 1)$ and $\boldsymbol{\alpha} = (0, \ldots, 0)$. The hypothesis of Corollary 5 rules out the possibility $\boldsymbol{\alpha} = (0, \ldots, 0)$. ∎

*2) Unidirectional Links:*

**Theorem 6.** *If* **A**3 *holds, then the assignment* $\boldsymbol{\alpha} = (1, \ldots, 1)$ *is a Nash equilibrium. Moreover, the assignment* $\boldsymbol{\alpha} = (1, \ldots, 1)$ *is the only pure Nash equilibrium.*

**Proof.** When all players use the direct path, the cost for each player $i$ is $\phi_i T(\phi_i)$. If only player $k$ deviates and routes its traffic over the indirect path, then the cost for player $k$ is $\phi_k \sum_{i \neq k} T(\phi_k) = (n-1)\phi_k T(\phi_k)$ which is strictly larger than $\phi_k T(\phi_k)$. Hence, the assignment $\boldsymbol{\alpha} = (1, \ldots, 1)$ is a Nash equilibrium.

Suppose that there exists a class $i$ such that $\alpha_i = 0$. For any other class $k$, we have $\alpha_k$ equal to 1 or 0. Then, the cost for class $i$ is

$$
\begin{aligned}
J_I^i &= \phi_i \sum_{k \neq i} T\left(\sum_{l \neq k} (1 - \alpha_l)\phi_l\right) \\
&= \phi_i \sum_{k \neq i} T\left(\phi_i + \sum_{l \neq k, i} (1 - \alpha_l)\phi_l\right) \\
&> \phi_i T(\phi_i) =: J_D^i,
\end{aligned}
$$

where $J_D^i := \phi_i T(\phi_i)$ is the cost for class $i$ if player $i$ deviates and routes all its traffic over the direct path. Hence, the assignment $\boldsymbol{\alpha} = (1, \ldots, 1)$ is the only pure Nash equilibrium. ∎

### B. Splittable Demands

For simplicity, in this subsection we assume that the demands and the cost functions are symmetric. Clearly, the equilibria are symmetric [7].

*1) Bidirectional Links:* The cost for a player $k$ is

$$J^k(\boldsymbol{x}) = x_k^k T(x_k) + (\phi - x_k^k) \sum_{j \neq k} T(x_j) \tag{3}$$

where $x_j$ is given by Equation (1). Differentiating $J^k(\boldsymbol{x})$ with respect to $x_k^k$ and equating to zero, we get

$$T(x_k) + x_k^k G(x_k) - \sum_{j \neq k} T(x_j) - (\phi - x_k^k) \sum_{j \neq k} G(x_j) = 0, \tag{4}$$

where $G(x) = dT(x)/dx$. Since the equilibrium flow is symmetric, for all $k$, $x_k^k = \overline{x}$ and

$$x_k = x = (n-1)\phi - (n-2)\overline{x}. \tag{5}$$

Applying symmetry in Equation (4), we obtain

$$\overline{x} = \frac{n-1}{n}\phi + \frac{n-2}{n}\Psi(x), \tag{6}$$

where $\Psi(x) = T(x)/G(x)$, and, for each $k$, $\overline{x}$ is the solution of the following fixed point equation

$$\overline{x} = \frac{n-1}{n}\phi + \frac{n-2}{n}\Psi((n-1)\phi - (n-2)\overline{x}).$$

Combining Equations (5) and (6), we also obtain

$$x = \frac{2(n-1)\phi}{n} - \frac{(n-2)^2}{n}\Psi(x).$$

In general, the equilibrium values of $\overline{x}$ and $x$ can be obtained by solving the above two fixed point equations.

We next present examples in which an explicit expression for the equilibrium is obtained.

*Example* 1. Assume $T(x) = \exp(\lambda x)$. Then $\Psi(x) = 1/\lambda$ and we obtain at equilibrium:

$$x = \frac{2(n-1)\phi}{n} - \frac{(n-2)^2}{n\lambda}\ ;\ \overline{x} = \frac{n-1}{n}\phi + \frac{n-2}{n\lambda}.$$

*Example* 2. Let's take $T(x) = ax^m$ for some $m > 1$. Then $\Psi(x) = x/m$, and

$$\overline{x} = \frac{(n-1)(m+n-2)\phi}{mn+(n-2)^2}\ ;\ x = \frac{2m(n-1)\phi}{mn+(n-2)^2}$$

*2) Unidirectional Links:* In this case, the cost for a player $k$ is

$$J^k(\boldsymbol{x}) = x_k^k T(x_k^k) + (\phi - x_k^k)\sum_{j\neq k} T(x'_j).$$

where $x'_k$ is given by Equation (2). Differentiating $J^k(\boldsymbol{x})$ with respect to $x_k^k$ and equating to zero, we get

$$T(x_k^k) + x_k^k G(x_k^k) - \sum_{j\neq k} T(x'_j) - (\phi_k - x_k^k)\sum_{j\neq k} G(x'_j) = 0$$

where $G(x) = dT(x)/dx$. Assuming that the equilibrium flow is symmetric, we obtain $x = (n-1)(\phi - \overline{x})$, where $\overline{x}$ can obtained by solving the following equation

$$\begin{aligned} 0 \ = \ & T(\overline{x}) + \overline{x}G(\overline{x}) - (n-1)(\phi - \overline{x})G((n-1)(\phi - \overline{x})) \\ & -(n-1)T((n-1)(\phi - \overline{x})) \end{aligned}$$

*Example* 3. Let's take $T(x) = ax^m$ for some $m > 1$. Then

$$\overline{x} = \frac{\phi}{1 + \frac{1}{n-1}\left(\frac{m+1}{m+n-1}\right)^{1/m}}\ ;\ x = \frac{\left(\frac{m+1}{m+n-1}\right)^{1/m}\phi}{1 + \frac{1}{n-1}\left(\frac{m+1}{m+n-1}\right)^{1/m}}$$

## V. Concluding Discussion

We considered routing games on a circle with a bidirectional link as well as with two unidirectional links carrying traffic in opposite directions. Even with a simple demand matrix which is the most extremely biased in favor of choosing the direct path, we showed that it is clearly not obviously true that all players send their traffic only on the direct path.

We showed that the Wardrop equilibrium for the symmetric game on the bidirectional ring is globally optimal. It also easy to show for this case that if the demands are asymmetric, then the player with the largest demand gets the benefit of being selfish by splitting its traffic over the two alternate paths, and all other players are worse off at equilibrium. For the unidirectional case, at Wardrop equilibrium, it is more likely that the players split their traffic over the two paths.

In case of Nash equilibrium with non-splittable traffic, it is more likely that all the players take the direct path. In the splittable case, we obtained the explicit solution for a few example cost functions.

In a transportation network the traffic in the two directions are dependent due to possible existence of the intersections at node locations. In the present paper, we have studied the two extreme cases, namely, *complete dependence* as in the bidirectional case *no dependence* as in the unidirectional case. As future work, we would like to study the general case of *partial dependence* between the two directions.

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