# Representation of Finite Games as Network Congestion Games

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***Abstract*—Weighted network congestion games are used for modeling interactions involving finitely many non-identical users of network resources, such as road segments or communication links. In spite of their special form, these games are not fundamentally special: *every* finite game can be represented as a weighted network congestion game. The same is true for the class of unweighted network congestion games with player-specific costs, in which the players differ in their cost functions rather than their weights. The intersection of the two classes consists of the unweighted network congestion games. These games *are* special: a finite game can be represented in this form if and only if it is an exact potential game.**

***Keywords-network games; congestion games; potential games***

## I. Introduction

In a (finite) congestion game, finitely many players share a finite set $E$ of resources but may differ in which resources they are allowed to use. Specifically, each player's set of strategies is a particular collection of nonempty subsets of $E$. The player's payoff from using a strategy is the negative of the total cost of using the resources included in the strategy. The cost of a resource depends only on its identity and on the number of users. The cost does not necessarily increase with congestion, and it may be negative, which means that using the resource contributes positively to the payoff. Rosenthal [6] showed that every congestion game admits an exact potential, which is a function $P$ on strategy profiles that exactly mirrors the players' incentives to unilaterally change their strategies. Whenever a single player moves to a different strategy, his gain or loss is equal to the corresponding change in $P$. Monderer and Shapley [5] proved the converse: essentially, *only* congestion games are exact potential games. More precisely, every finite game that admits an exact potential can be represented as a congestion game. One of this paper's findings strengths Monderer and Shapley's result by showing that an exact potential game can always be represented as a particular kind of congestion game, namely, an unweighted network congestion game.

A restriction or expansion of the meaning of 'congestion game' potentially has the same effect on the class of representable games. Examples of restriction are: congestion games with nondecreasing cost functions, in which increasing congestion never makes users better off; singleton congestion games, in which each strategy includes a single resource; and network congestion games, in which resources are represented by edges in a graph and strategies correspond to routes, which are paths in the graph that connect the player's designated origin and destination vertices. (The second restriction is a special case of the third. It corresponds to a parallel network, which is one with only two vertices.) Examples of extensions are: congestion games with player-specific costs (or payoffs [1]), where players are differently affected by congestion; and weighted congestion games, in which their contributions to it (the players' "congestion impacts") differ.

This paper shows that both weighted network congestion games and unweighted network congestion games with player-specific costs are actually capable of representing all finite games. Both representations use only nondecreasing cost functions and two-terminal, or single-commodity, networks, in which all players' routes start and terminate at the same origin and destination vertices. Thus, although the definitions of the two kinds of network congestion games involve quite specific structures, the games themselves are not in any way special. For unweighted network congestion games, which are simultaneously weighted network congestion games and unweighted network congestion games with player-specific costs, this is not so. As indicated, these games are special in that they represent (all) exact potential games.

The potential significance of these findings lies in the information that the representation as a network congestion game provides about the represented game. An example of such information is existence of a pure-strategy equilibrium [3]. For certain network topologies, the existence of equilibrium is guaranteed regardless of the exact functional form of the cost functions, the number of players and (if applicable) their weights. Thus, any finite game that can be represented as a network congestion game on such a network has at least one pure-strategy equilibrium.

## II. Preliminaries

### A. Game Theory

A finite (noncooperative) game $\Gamma$ has a finite number $n$ of players whose strategy sets $S^1, S^2, \dots, S^n$ are finite. A strategy profile $s = (s^1, s^2, \dots, s^n)$ in $\Gamma$ assigns a strategy $s^i$ to each player $i$. Two games $\Gamma$ and $\Gamma'$ with identical sets of players are *isomorphic* [5] if for each player $i$ there is a one-to-one correspondence between $i$'s strategy sets in $\Gamma$ and $\Gamma'$, such that each strategy profile $s$ in $\Gamma$ yields the same payoffs to all players as the corresponding strategy profile $s'$ in $\Gamma'$.

Essentially, isomorphic games are just alternative representations of a single game. Two games $\Gamma$ and $\Gamma'$ with identical sets of players and respective strategy sets are *similar* if for each player the difference between the payoffs in $\Gamma$ and $\Gamma'$ does not depend on the player's own strategy, and can thus be expressed as a function of the other players' strategies.

A game $\Gamma$ is an *exact potential game* [5] if it is similar to some game $\Gamma'$ in which all the players have the same payoff function $P$, which is said to be an exact potential for $\Gamma$.

### B. Graph Theory

An *undirected multigraph* consists of a finite set of vertices and a finite set of edges. Each edge $e$ joins two distinct vertices, $u$ and $v$, which are referred to as the *end vertices* of $e$. Thus, loops are not allowed but more than one edge can join two vertices. An edge $e$ and a vertex $v$ are *incident* with each other if $v$ is an end vertex of $e$. A (simple) *path* of length $m$ is an alternating sequence of vertices and edges $v_0 e_1 v_1 \cdots v_{m-1} e_m v_m$, beginning and ending with vertices, in which each edge is incident with the two vertices immediately preceding and following it and all the vertices (and necessarily all the edges) are distinct. Every path traverses each of its edges $e$ in a particular *direction*: from the end vertex that precedes $e$ in the path to the vertex that follows it.

A *two-terminal network*, or simply *network*, $G$ is an undirected multigraph together with a distinguished ordered pair of (distinct) *terminal* vertices, the *origin* $o$ and the *destination* $d$, such that each vertex and each edge belongs to at least one path in which the first vertex is $o$ and the last vertex is $d$. Any path with these first and last vertices is referred to as a *route* in $G$.

Two networks $G$ and $G'$ may be connected *in parallel* if they have the same origin and the same destination but no other common vertices or edges, and *in series* if they have only one common vertex which is the destination in $G$ and the origin in $G'$. In both cases, the set of vertices and the set of edges in the resulting network are the unions of the corresponding sets in $G$ and $G'$, and the origin and destination are those in $G$ and $G'$, respectively (as well as in $G'$ and $G$, respectively, in the case of connection in parallel). The connection of an arbitrary number of networks in parallel or in series is defined recursively.

### C. Network Congestion Games

A *weighted network congestion game* on a (two-terminal[1]) network $G$ is a finite, $n$-player game that is defined as follows. First, an allowable direction, a (possibly, empty) set of allowable users and a nondecreasing *cost function* $c_e: (0, \infty) \to (-\infty, \infty)$ are specified for each edge $e$ in $G$, such that each edge is traversed in the allowed direction by at least one route and each player $i$ has at least one *allowable route*, that is, a route in $G$ that includes only edges that $i$ is allowed to use and traverses them in the allowed direction. The *strategy set* $S^i$ of each player $i$ is the collection of his allowable routes. Second, a *weight* $w^i > 0$ is specified for each player $i$, which represents the player's congestion impact.[2] The total weight $f_e$ of the players whose chosen route includes an edge $e$ is the *flow* (or load) on $e$. The (not necessarily negative) cost of $e$ for each of its users is $c_e(f_e)$. A player's *payoff* in the game is the negative of the total cost of the edges in his route.

A weighted network congestion game is called an *unweighted* network congestion game if the players' weights are all identical and equal to $1$. The equality entails, in particular, that the cost of an edge is not affected by the identities of its users but only by their number. A generalization that allows for a dependence of the cost on the user's identity is *unweighted network congestion game with player-specific costs*. In such a game, each edge $e$ is associated with a (player-specific) nondecreasing cost function $c_e^i: (0, \infty) \to (-\infty, \infty)$ for each player $i$, and its cost for that player is $c_e^i(f_e)$, where (the flow) $f_e$ is the total number of players using $e$.

## III. Presentation Results

Every unweighted network congestion game is in particular a congestion game in the sense of belonging to the class of games presented by Rosenthal [6] and studied by Monderer and Shapley [5] (see the Introduction). In fact, as the following theorems show, the class of unweighted network congestion games essentially *coincides* with that of all congestion games. Weighted network congestion games and unweighted network congestion games with player-specific costs are generally not congestion games (in the above sense). The next theorem shows that both classes essentially coincide with that of *all* finite games.

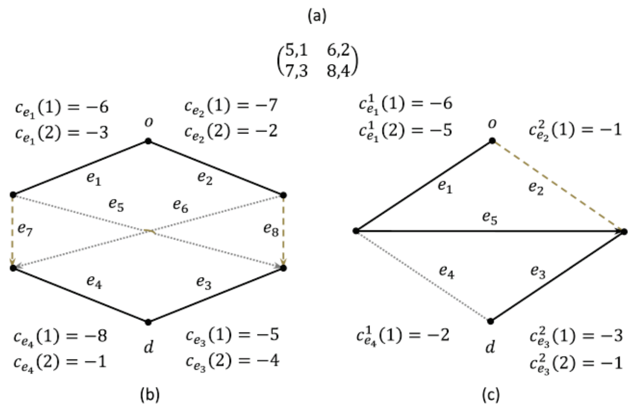


Figure 1. Representations of a $2 \times 2$ exact potential game (a) as a weighted network congestion game (b), with weights $w^1 = 1$ and $w^2 = 2$, and as an unweighted network congestion game with player-specific costs (c). Dotted, dashed and solid edges are allowable to player 1, player 2 and both players, respectively. The allowable directions are indicated where required. All relevant costs other than those specified are zero. A player's payoff is the negative of his total cost.

[1] The assumption of a single origin–destination pair may be viewed as a normalization. Any weighted network congestion game on a *multi-commodity network*, which has multiple origin–destination pairs, may also be viewed as a game with a single such pair. In that game, each terminal vertex is incident with a single allowable edge for each player, which joins it with the player's corresponding terminal vertex in the original game.

[2] In certain contexts, it may be desirable to require the following (weak) connection between the players' weights and the cardinality of their strategy sets: For all $i$ and $j$ with $w^i < w^j$, $|S^i| \geq |S^j|$. In this paper, this requirement is not needed, but adding it would not affect any of the paper's results.

*Theorem 1.* Every finite game $\Gamma$ is isomorphic both to a weighted network congestion game $\Gamma'$ and to an unweighted network congestion game with player-specific costs $\Gamma''$. $\Gamma$ is isomorphic to an unweighted network congestion game[3] if and only if it is an exact potential game.

By the first part of the theorem, every finite game can be represented as a network congestion game where the players differ in their weights and as one where they differ in their costs functions. The existence of a representation of the latter kind was first indicated by Monderer [4]. However, Monderer's definition of network congestion game is different, and significantly less restrictive, than in this paper. The second part of Theorem 1 strengths a well-known result of Monderer and Shapley [5]. Their result is the equivalence of (ii) and (iii) in the following extension of that part of the theorem.

*Theorem 2.* For every finite game $\Gamma$, the following conditions are equivalent:

(i) $\Gamma$ is isomorphic to an unweighted network congestion game.

(ii) $\Gamma$ is isomorphic to a congestion game.

(iii) $\Gamma$ is an exact potential game.

A finite game $\Gamma$ obviously has more than a single pair of representations as in Theorem 1. The "canonical" games $\Gamma'$ and $\Gamma''$ constructed in the proof of the theorem, which share the same network $G$, are just one such pair. Other representations may be preferable in that they use a simpler network. An example is shown in Figure 1(c). That representation of the $2 \times 2$ exact potential game in Figure 1(a) as an unweighted network congestion game with player-specific costs uses the Wheatstone network, which has fewer edges than the network $G$ constructed in the proof of Theorem 1 (which, for all $2 \times 2$ games, is the one in (b)). Both networks are considerably simpler than the one constructed in the proof of Theorem 2. For an example of a simple representation of a specific variety of congestion games as weighted network congestion games, see [2].

*Proof of Theorem 1.* The following proof concerns the first part of the theorem. The second part is dealt with as part of the proof of Theorem 2 below.

Suppose that the number $n$ of players in $\Gamma$ and the cardinality $m$ of the largest strategy set are both at least two (otherwise the assertion is trivial). Without loss of generality, it may be assumed that, for all $1 \le i < j \le n$, player $i$'s number of strategies is at least as great as that of $j$ (otherwise take 'player 1' below to mean the player with the largest number of strategies, 'player 2' the player with the second-largest number, and so on). It is desirable to have the stronger properly that these numbers are actually equal: *all* players have $m$ strategies. Such equality may be achieved by (temporarily) increasing the number of strategies of one or more players $i$. To this end, one of $i$'s strategies needs to be replicated, such that choosing the original strategy or any of its replicas has the same effect on $i$'s payoff and those of the other players. Each player's strategies can now be indexed from 1 to $m$, thus identifying the collection of all strategy profiles with the product set $\{1,2,\dots,m\}^n$. Order this set in the following way:

$$\begin{aligned}&(1,1,\dots,1), (2,2,\dots,2), \dots, (m,m,\dots,m), \dots,\\ &(1,2,\dots,2), (2,3,\dots,3), \dots, (m-1,m,\dots,m), (m,1,\dots,1),\end{aligned} \tag{1}$$

where the order of the $m^n - 2m$ elements represented by the second ellipsis mark is immaterial. With each strategy profile $s = (s^1, s^2, \dots, s^n)$ associate two vertices $u_s$ and $v_s$ and an edge $e_s$ that joins them. The edge is directed from $u_s$ to $v_s$ and is *public*, that is, allowable to all players. Next, for each player $i$ and $1 \le k \le m$, consider all strategy profile $s$ with $s^i = k$ and list them according to their order in (1). For each pair $s$ and $t$ of successive entries in this list, add an edge that joins $v_s$ and $u_t$, is directed from $v_s$ to $u_t$, and is *private*, that is, allowable to player $i$ only. Finally, identify all vertices of the form $u_s$, where $s$ is one of the first $m$ elements in (1), and denote this single vertex by $o$. Do the same for all vertices of the form $v_s$, where $s$ is one of the last $m$ elements in (1), and denote the result by $d$. These terminal vertices, together with the other vertices and edges specified above, constitute a network $G$, with specified direction and set of allowable users for each edge. (For $2 \times 2$ games, this is the network depicted in Figure 1(b).) For each player $i$, each allowable route $r$ in $G$ corresponds to some strategy $s^i$ in $\Gamma$. Specifically, $r$ includes all $m^{n-1}$ edges $e_t$ with $t^i = s^i$, alternating with $m^{n-1} - 1$ private edges. Different allowable routes to player $i$ have no shared edges, and no shared vertices other than the terminal ones.

The unweighted network congestion game with player-specific costs $\Gamma''$ is defined by assigning the following cost functions to each player $i$. For a public edge $e_s$, corresponding to a strategy profile $s$,

$$\begin{aligned}c_{e_s}^i(x) &= 0, \qquad x \le n-1\\ c_{e_s}^i(n) &= K^i - h^i(s),\end{aligned} \tag{2}$$

where $h^i$ is player $i$'s payoff function in $\Gamma$ and $K^i$ is any number equal to or greater than that function's maximum. For a private edge $e$ allowable only to player $i$, $c_e^i = -K^i/(m^{n-1} - 1)$. As explained above, strategy profiles in $\Gamma$ are in a one-to-one correspondence with allowable route choices in $G$. The routes that correspond to a strategy profile $s$ are such that exactly one edge, namely $e_s$, is used by all $n$ players. Therefore, for each player $i$, only $e_s$ and $m^{n-1} - 1$ private edges make a nonzero contribution to the cost. By (2), the total cost is $-h^i(s)$. Hence, player $i$'s payoff is $h^i(s)$, as in $\Gamma$.

The weighted network congestion game $\Gamma'$ is defined by, first, attaching the weight $w^i = i + n - 2$ to each player $i$ $(= 1,2,\dots,n)$. Thus, the weight uniquely identifies the player, and is less than the total weight of any two players, which is $2n - 1$ or greater. Second, the cost functions are defined as follows. For a public edge $e_s$, corresponding to a strategy profile $s = (s^1, s^2, \dots, s^n)$,

$$\begin{aligned}c_{e_s}(i+n-2) &= -\sum_{t^{-i} \in S^{-i}} \left( \prod_{j \ne i} \left( \frac{1}{m-1} - 1_{t^j = s^j} \right) \right) h^i(s^i, t^{-i})\\ &\quad -(n-i+1)K, \qquad i = 1,2,\dots,n\\ c_{e_s}(x) &= 0, \qquad x \ge 2n-1,\end{aligned}$$

[3] This condition can be expressed as the requirement that $\Gamma' = \Gamma''$.

where $h^i$ is player $i$'s payoff function in Γ and $K$ is any number large enough to make the cost functions of all public edges nondecreasing. In this definition, $S^{-i}$ is the collection of all partial strategy profiles $t^{-i} = (t^1, t^2, \dots, t^{i-1}, t^{i+1}, \dots, t^n)$, $1_{t^j = s^j}$ is defined as 1 if the indicated equality holds and 0 otherwise, and $(s^i, t^{-i}) = (t^1, t^2, \dots, t^{i-1}, s^i, t^{i+1}, \dots, t^n)$. For a private edge $e$, which is allowable only to a single player $i$, the cost is given by

$$c_e(i+n-2) = \frac{(m-1)^{n-1}}{m^{n-1}-1}(n-i+1)K.$$

These "private" costs cancel the total contribution of the second term $-(n-i+1)K$ in the definition of the "public" costs, which appears only in the $(m-1)^{n-1}$ public edges in which player $i$ is the sole user. The total contribution of the corresponding first term, which depends on the strategy profile $s$, can be computed as follows (to enhance readability, the computation is shown for $i = 1$):

$$-\sum_{\bar{s}^2 \neq s^2} \cdots \sum_{\bar{s}^n \neq s^n} \sum_{t^{-1} \in S^{-1}} \left( \prod_{j=2}^{n} \left( \frac{1}{m-1} - 1_{t^j = \bar{s}^j} \right) \right) h^1(s^1, t^{-1})$$
$$= -\sum_{t^{-1} \in S^{-1}} \left( \prod_{j=2}^{n} \sum_{\bar{s}^j \neq s^j} \left( \frac{1}{m-1} - 1_{t^j = \bar{s}^j} \right) \right) h^1(s^1, t^{-1})$$
$$= -\sum_{t^{-1} \in S^{-1}} \left( \prod_{j=2}^{n} 1_{t^j = s^j} \right) h^1(s^1, t^{-1}) = -h^1(s).$$

Again, the payoff is $h^1(s)$, as in Γ.

To complete the proof of the first part of the theorem it only remains to dispose of the spurious strategies (potentially) introduced by replication at its beginning. These strategies may be eliminated simply by deleting or disallowing the use of all the private edges that belong to the corresponding routes. ■

*Proof of Theorem 2.* The fact that every congestion game, and in particular every unweighted network congestion game, is an exact potential game is well known [5],[6]. The reverse implications are proved below, using a variant of the proof in [5].

Suppose that Γ has an exact potential $P$. As in the proof of Theorem 1, and for similar reasons, it is sufficient to consider the case in which the number $n$ of players is at least 2 and the strategy set $S^i$ of each player $i$ includes at least two elements. Strategy profiles in Γ are naturally identifiable with subsets of the disjoint union $\coprod_{i=1}^n S^i$ (where 'disjoint' means that strategies of different players are viewed as distinct elements): $s$ is identified with the set $\{s^1, s^2, \dots, s^n\}$. Similarly, a partial strategy profile $s^{-i}$, which is obtained from a strategy profile $s$ by ignoring the coordinate corresponding to a particular player $i$, is identified below with the set $\{s^1, s^2, \dots, s^n\} \cup S^i$. Player $i$ (who, because of the assumption of at least two strategies for each player, is the only player $j$ with $S^j \subseteq s^{-i}$) is said to *own* $s^{-i}$.

A congestion game isomorphic to Γ is defined as follows. Each strategy profile in Γ and each partial strategy profile are viewed as resources. (Thus, a resource is identified with a specific subset of $\coprod_{i=1}^n S^i$.) For each player $i$ and strategy $s^i$ of that player in Γ, the corresponding strategy in the congestion game is the selection of all the resources that include $s^i$. These resources are of three kinds: all partial strategy profiles owned by $i$, all partial strategy profiles owned by another player but containing $s^i$, and all strategy profiles containing $s^i$. The resources of the first kind are included also in every other strategy of player $i$ in the congestion game but those of the other two kinds are exclusive. It follows that player $i$ is the only user of (the resource identified with) a partial strategy profile if and only if (a) he owns it, i.e., it is of the form $s^{-i}$, and (b) the strategy of every other player $j$ is *not* (that corresponding to) $s^j$, player $j$'s coordinate in $s^{-i}$. The cost for a player of using a resource (which is the negative of the resource's contribution to the player's payoff) is defined as follows. For (the resource identified with) a strategy profile $s$, the cost is $-P(s)$ if all $n$ players use the resource and 0 otherwise. For a partial strategy profile $s^{-i}$, the cost is

$$\frac{1}{|S^i|} \sum_{t \in S} \left( \prod_{j \neq i} \left( \frac{1}{|S^j|-1} - 1_{t^j = s^j} \right) \right) \left( P(t) - h^j(t) \right) \tag{3}$$

if no one else uses the resource (which, as indicated above, is possible only if the player considered is $i$ himself) and 0 otherwise. In (3), $h^i$ is player $i$'s payoff function in Γ, $S$ is the collection of all strategy profiles, and $1_{t^j = s^j}$ is defined as 1 if the indicated equality holds and 0 otherwise. Note that a sufficient condition for the cost functions to be nondecreasing is that both $P$ and (3) are negative. Without loss of generality, this condition may be assumed to hold. It is not difficult to show that subtracting any constant $c$ from $P$ leaves it an exact potential function, and subtracts a certain positive multiple of $c$ from (3).

To prove that the congestion game defined above is isomorphic to Γ it has to be shown that, for every player $i$ and strategy profile $s$, the total cost of $i$'s route is $-h^i(s)$, the negative of his payoff in Γ. Two kinds of resources make a nonzero contribution to the cost: the single resource corresponding to the strategy profile $s$, which contributes $-P(s)$, and those corresponding to partial strategy profiles owned by $i$ in which the strategy $\bar{s}^j$ of every player $j \neq i$ is not $s^j$. Assuming, for readability, that $i = 1$, the total contribution of the latter is

$$\sum_{\bar{s}^2 \neq s^2} \sum_{\bar{s}^3 \neq s^3} \cdots \sum_{\bar{s}^n \neq s^n} \frac{1}{|S^1|} \sum_{t \in S} \left( \prod_{j=2}^{n} \left( \frac{1}{|S^j|-1} - 1_{t^j = \bar{s}^j} \right) \right) \big( P(t) - h^1(t) \big)$$
$$= \frac{1}{|S^1|} \sum_{t \in S} \left( \prod_{j=2}^{n} \sum_{\bar{s}^j \neq s^j} \left( \frac{1}{|S^j|-1} - 1_{t^j = \bar{s}^j} \right) \right) \big( P(t) - h^1(t) \big)$$
$$= \frac{1}{|S^1|} \sum_{t \in S} \left( \prod_{j=2}^{n} 1_{t^j = s^j} \right) \big( P(t) - h^1(t) \big) = P(s) - h^1(s).$$

The last equality uses the fact that, by definition of exact potential, for every player $i$ and strategy profile $t$ the difference $h^i(t) - P(t)$ does not depend on $i$'s strategy $t^i$. This completes the proof of the isomorphism between Γ and the congestion game, which corresponds to the implication (iii) ⇒ (ii).

The above proof also constitutes the main step in the proof of the stronger assertion (iii) ⇒ (i). To complete the proof, the congestion game defined above needs to be turned into an unweighted network congestion game. To this end, the resources in the congestion game need to be viewed as public edges, allowable to all players, and supplemented with a certain number of zero-cost edges allowable to only one player. These private edges are added and connected to the public ones in the following manner. First, a single-route network is created for every strategy of every player $i$ by connecting all the public edges that $i$ uses only in that strategy (that is, resources of the second and third kinds considered above) alternately with private edges for that player, such that the first and last edges in the route are private. (The order of the public edges is immaterial.) Second, the resulting $|S^i|$ single-route networks are connected in parallel, and the resulting network is connected in series with a similar single-route network in which all the public edges that $i$ uses in all of his strategies (that is, resources of the first kind above) alternate with private edges. After this is done for all players, the origin vertices of the $n$ networks are identified and so are their destination vertices. Clearly, for each strategy of each player $i$ in the congestion game defined above, all the resources that are included in that strategy, and only them, are also included in one of $i$'s $|S^i|$ allowable routes in the network. Hence, the unweighted network congestion game is trivially isomorphic to the congestion game, and hence to Γ. ■

Another variant of network congestion games capable of representing all finite games is weighted network congestion games (in the wide sense) *without self-effect* [3]. These games are similar to "normal" weighted network congestion games in that a single nondecreasing cost function $d_e$ is associated with each edge $e$, which together with the flow $f_e$ determines the cost of $e$ for all users. However, the cost of $e$ for a user $i$ is $d_e(f_e - w^i)$. It thus depends only on the total weight of the *other* users of $e$, and may consequentially be different for users of different weights. In this respect, weighted network congestion games without self-effect are similar to unweighted network congestion games with player-specific costs. Using arguments similar to those given in the proof of Theorem 1, it is not difficult to show that every finite game can be represented as such a game.

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