Revisiting Collusion in Routing Games: a Load Balancing Problem

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***Abstract*—Is it profitable for players to unite and merge to a single player? Obviously, the sum of utilities at an equilibrium cannot exceed the sum obtained if all players join together. But what happens if only a subset of players join together? Previous work on collusion have already shown that the society may either gain or loose from collusion of a subset of players. In this paper we show for a simple load balancing example that not only the society may loose, but also the subset of players that collude may end up with a worse performance than without collusion. In doing so, we introduce new concepts that measure the price of collusion.**

***Index Terms*—Asymmetric game, Braess paradox, collusion measures, load balancing, Nash equilibrium.**

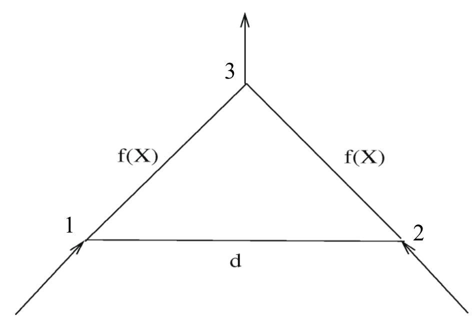

Fig. 1. Load balancing topology

I. Introduction

To evaluate the efficiency of an equilibrium, simple measures such as the price of anarchy (or the price of stability) can summarize how bad does worst (respectively, the best) possible equilibrium perform with respect to the socially optimal solution. Although this indeed gives insight into the inefficiency of not joining together, these measures are far from being sufficient for evaluating possible benefits from aggregating several players into a single one.

First, one may question why we compare the sum of utilities. In fact, quite often one is interested in maximizing other function of the utilities rather than the sum. One may be interested in the impact of the merging on the performance of the player that has the worst performance. More generally, one may prefer a measure that indicates how well does an equilibrium perform in terms of Pareto efficiency. This direction has been carried by Kameda in [2] who provides a new definition of the efficiency of an equilibrium.

Next, one may wish to assess the effect of players merging together to one or more coalitions, on the equilibrium without requesting that all players merge to a single player (which is called then a grand coalition). One can then question how beneficial is it for the society to see the set of players replaced by coalitions of these players. In [1], the authors define the price of collusion as the ratio between the social cost at equilibrium before and after the (worst possible) collusion scenario.

We argue that coalitions tend to form if there is an advantage to those involved in forming a coalition in merging together. Therefore, it seems to us fundamental to introduce a measures for price of collusion that would quantify the effect of collusion(s) on those involved in the collusion(s). We thus propose several other alternative definitions of price of collusion: one related to a single coalition, and another related to various coalitions that may occur coincidentally.

We investigate these concepts through a motivating example of a load balancing routing game. In particular we show that when a few players merge together, not only the society may loose but also the subgroup that colludes.

II. The load balancing problem

Model. Assume there is a set $\mathcal{N}_1$ of n players such that player i among them has an amount of $\theta(i)$ to ship from node 1 to node 3. Each player among a set $\mathcal{N}_2$ of m players with $m \geq n$, has an amount of $\theta(i)$ to ship from node 2 to node 3. Node 3 has a link from both node 2 as well as from node 1, each with the same link cost of f. Let g denote the derivative of f. Links 1 and 2 are connected by a delay line with a fixed cost of d. The network is depicted in figure 1.

We denote for all user i, x_1^i (resp. x_2^i) the amount of traffic that uses the link 1-3 (resp. the link 2-3). Note that for user $i \in \mathcal{N}_2$, x_1^i follows the two links 2-1 and 1-3. We set x_1 (resp x_2) the total amount of traffic on link 1-3 (resp. on link 2-3). We have: $x_1 = \sum_{j=1}^{2} \sum_{i \in \mathcal{N}_j} x_1^i$, and $x_2 = \sum_{j=1}^{2} \sum_{i \in \mathcal{N}_j} x_2^i$. For any player i, the cost of player i is given by

$$J^i(\mathbf{x}) = \begin{cases} x_1^i f(x_1) + (\theta(i) - x_1^i) f(x_2) + (\theta(i) - x_1^i) d \\ \qquad \text{for } i \in \mathcal{N}_1, \\ x_1^i f(x_1) + (\theta(i) - x_1^i) f(x_2) + x_1^i d \\ \qquad \text{for } i \in \mathcal{N}_2. \end{cases} \tag{1}$$

$$\text{Thus,}\quad \frac{\partial J^i(\mathbf{x})}{\partial x_1^i} = f(x_1) + x_1^i g(x_1) - f(x_2) - (\theta(i) - x_1^i) g(x_2) + \delta(i). \tag{2}$$

where $\delta(i) = d$ for $i \in \mathcal{N}_2$ and $\delta(i) = -d$ for $i \in \mathcal{N}_1$. Assume that at equilibrium, x_1^i and x_2^i are strictly positive for all i. Then the equilibrium flows are obtained by equating (2) to zero. This gives

$$x_1^i = \frac{f(x_2) - f(x_1) + \theta(i)g(x_2) - \delta(i)}{g(x_1) + g(x_2)}. \tag{3}$$

Define $\xi = \sum_{j=1}^{2} \sum_{i \in \mathcal{N}_j} \theta(i)$ as the total demand on the network.

Definitions of Measures for Collusion In [1], the authors define the price of collusion as the ratio between the social cost at equilibrium before and after the (worst possible) collusion scenario. We are interested in addition to quantify the effect of collusion on those involved in the collusion. We thus propose several other alternative definitions of price of collusion.

Consider a game $\Gamma = (I, \{A(i), C^i, i \in I\})$ where I is the set of players, $A(i)$ is the set of strategies and C^i the cost for player i (which is a function of the actions of all players). For a given multistrategy $\mathbf{a} = (a(i) \in A(i), i \in I)$ and $\mathbf{a}' = (a'(i) \in A(i), i \in I)$ we define the following:

- $\mathbf{a}(J)$ is given by $(a(j) \in A(j), j \in J)$.
- $\mathbf{a}(-J)$ is given by $(a(j) \in A(j), j \notin J)$.
- $[\mathbf{a}'(J), \mathbf{a}(-J)]$ is the multistrategy where player $j \notin J$ uses action $a(j)$ and player $i \in J$ plays $a'(i)$.

Let H be a partition of I. It is thus given as a set of disjoint subsets of I whose union is I. We shall call H a collusion pattern. Each $h \in H$ will be identified with a player in a new game $\Gamma(H) = (H, \{A(h), C^h, h \in H\})$. We can view a set of players that collude as a coalition. C^h is called (as in coalition games) the imputation for a coalition h.

For discrete games we define for any $h \subset I$, $C[h](\mathbf{a}) = \sum_{i \in h} C^{\mathbf{a}}$.

Definition 1: Consider a game $\Gamma = (I, \{A(i), C^i, i \in I\})$ along with a collusion pattern H. $\mathbf{a}[\mathbf{h}]$ is called a H-equilibrium if it is an equilibrium in the game $\Gamma(H)$.

Definition 2: We say that a collusion pattern H is a single collusion if there is some $J \subset I$ such that $H = H(J) := \{J, \{i\}, i \notin J\}$. In other words, there is only one coalition that is formed by all $i \in J$ merging together. Let SC denote the set of all possible single collusions.

Definition 3: Let $\mathbf{a}$ be an equilibrium in a game Γ and let $\mathbf{a}[\mathbf{H}]$ be a corresponding equilibrium in the game $\Gamma(H)$. We define

- The individual single collusion-price: we consider the impact of a single collusion, that of all players within a group h acting together as a single player. The collusion pattern is thus $H(h)$.
$$ISCP(h) = \frac{C[h](\mathbf{a})}{C^h(\mathbf{a}[\mathbf{H}])}.$$
This ratio measures the harm for a group h of players to collude together. After colluding, they get together at the new equilibrium $C^h(\mathbf{a}[\mathbf{H}])$. If they did not collude then they would get together $C[h](\mathbf{a})$ at equilibrium.
- The individual collusion-price (ICP): we now allow for several colluding groups and define the ICP as the worst degradation over all new coalitions.
$$ICP(H) = \sup_{h \in H} \frac{C[h](\mathbf{a})}{C^h(\mathbf{a}[\mathbf{H}])}.$$
- The social single collusion-price:
$$SSCP(h) = \frac{C[I](\mathbf{a})}{C[I](\mathbf{a}[\mathbf{H}])}.$$
where $H = H(h)$. This measures the impact on the whole society of a single collusion among players in $h \subset I$.
- The social collusion-price:
$$SCP(H) = \sup_{h \subset I} \frac{C[I](\mathbf{a})}{C[I](\mathbf{a}[\mathbf{H}])}.$$
This measures the impact on the whole society of a collusion pattern H.
- The single collusion externality-price:
$$SCEP(h) = \frac{C[I \setminus h](\mathbf{a})}{C[I \setminus h](\mathbf{a}[\mathbf{H}])}.$$
This measures the impact of a collusion h over the non-colluding players.
- The collusion externality-price:
$$CEP(H) = \sup_{h \subset I} \frac{C[I \setminus h](\mathbf{a})}{C[I \setminus h](\mathbf{a}[\mathbf{H}])}.$$
This measures the worst degradation over all coalitions on the non-colluding players.

We shall use the above definition in cases where the equilibrium is unique both before as well as after collusion. In case of several equilibria, one may define the worst case and the best case ratios (as in the definitions of price of anarchy and of price of stability).

III. The symmetric case, Braess paradox and Collusions

In this section, we observe similar results as in [4] in which authors study in details symmetric systems.

Assume that $n = m$. Taking the sum over i, we get

$$x_1 = \frac{2nf(x_2) - 2nf(x_1) + \xi g(x_2)}{g(x_1) + g(x_2)}, \tag{4}$$

which does not depend on d any more!

Assume that we have symmetry in the demands as well, in the sense that for every player $i_1 \in \mathcal{N}_1$ there is a player in $\mathcal{N}_2$ with the same demand. Since we know that there is a unique equilibrium, we shall derive it by showing that there is an equilibrium when restricting to symmetric ones. In a symmetric equilibrium, $x_1 = x_2 = \xi/2$. This gives by substituting in (3) $x_1^i = \frac{\theta(i)}{2} - \frac{\delta(i)}{2g(\xi/2)}$ and $x_2^i = \frac{\theta(i)}{2} + \frac{\delta(i)}{2g(\xi/2)}$. The above values of x_j^k indeed satisfies the positivity assumption provided that $d \leq \gamma$ where $\gamma := g(\xi/2)\theta(i)$. If this condition is not satisfied then there is no flow forwarded between one source node to another, i.e. on the link 1-2.

A. Braess-type paradox

It follows from (1) that at equilibrium, $J^i(x) = \theta(i) f(\xi/2) + d\left(\frac{\theta(i)}{2} - \frac{d}{2g(\xi/2)}\right)$. we conclude that $\frac{\partial J^i}{\partial d} = \frac{\theta(i)}{2} - \frac{d}{g(\xi/2)}$, which is nonnegative for $d \in (\gamma/2, \gamma)$. In this region J^i decreases as d increases, which is a Braess type paradox. This paradox was obtained in the context of a load balancing network in [3].

B. Collusions

Assume that a group K of $k \leq n$ players among $\mathcal{N}_1$ collude together to become a single player. Assume that at the same time, the set K' of players among $\mathcal{N}_2$ corresponding to K (a player in $\mathcal{N}_2$ corresponds to one in $\mathcal{N}_1$ if they both have the same demand) also collude to become one single player. We then have $2(n-k+1)$ players instead of $2n$. Then we observe the following:

- The performance of all players other than the colluding ones is unchanged.
- The sum of the performance of the colluding players is also unchanged over the region $d \leq g(\xi/2)\min_i \theta(i)$. This is the region in which we had mutual forwarding among all pairs of corresponding players in K and K'. The mutual forwarding remains after the collusion.
- We see however that after the collusion, we still have mutual forwarding in larger region of delays: $d \leq g(\xi/2)\sum_i \theta(i)$. Thus there is strict deterioration in the sum of the costs when $g(\xi/2)\min_i \theta(i) \leq d \leq g(\xi/2)\sum_i \theta(i)$.

We thus showed that a particular type of collusion in which two sets of players merged into two coalitions, resulted in a worse performance at equilibrium. It is natural to ask whether one can observe deterioration due to collusion when only one group is formed.

IV. Asymmetric Load Balancing Games

The behavior of colluding players was easy to describe in the symmetric case, since we know for symmetric routing games that the costs are also symmetric. To use this tool for the study of collusions, we needed to restrict to collusions that kept the system symmetric. This excluded the situation in which only one group of players collude. To study the latter, we therefore have to go beyond symmetric games, which we do next.

We shall obtain an implicit equation to compute the total equilibrium flow over link 1. This is used in the following section to compute the equilibrium flow in the case of linar costs. Summing over i, in (2) we get

$$\Delta := \sum_{i=1}^{n+m} \frac{\partial J^i(\mathbf{x})}{\partial x_1^i} = (n+m) f(x_1) + x_1 g(x_1)$$
$$-(n+m) f(x_2) - (\xi - x_1) g(x_2) + d(m-n).$$

Substituting the flow conservation constraint $x_2 = \xi - x_1$,

$$\Delta = (n+m) f(x_1) + x_1 g(x_1) - (n+m) f(\xi - x_1)$$
$$-(\xi - x_1) g(\xi - x_1) + d(m-n).$$

Define $R(x) = (n+m-1)\int_0^x f(s)ds + x f(x)$, and $D(x) = (m-n)xd$. Then Δ is the derivative of $\overline{J}(x)$ at $x = x_1$ where $\overline{J}(x) := R(x) + D(x) + R(\xi - x)$.

Assume that there exists an equilibrium in which all players send strictly positive amount of traffic to both paths available to them. Then the equilibrium flow x_1 satisfies $\Delta = 0$. In the next section, in order to give explicit expressions of the collusion measures defined in section II, we consider the linear cost function.

A. Linear cost function

By considering the linear cost function, we are able to determine explicitly the new collusion measures. So let us define the following function $f(x) = ax$.

Proposition 1: Considering the linear cost function, the equilibrium rates are described as follows.

- If $d > a(m+n+1)\frac{\theta(1)}{2m+1}$, then $(x_1^i)^* = \theta(1)$, $\forall i \in \mathcal{N}_1$, otherwise $(x_1^i)^* = \frac{\theta(i)}{2} + \frac{d}{2a}\frac{2m+1}{n+m+1}$.
- If $d > a(m+n+1)\frac{\theta(2)}{2n+1}$, then $(x_1^i)^* = 0$, $\forall i \in \mathcal{N}_2$, otherwise $(x_1^i)^* = \frac{\theta(i)}{2} - \frac{d}{2a}\frac{2n+1}{n+m+1}$.

Proof We get from previous results that:

$$\Delta = ax_1[2(n+m)+2] - a(n+m)\xi - a\xi + d(m-n).$$

Thus the solution of $\Delta = 0$ is $x_1^* = \frac{\xi}{2} - \frac{d(m-n)}{2a(m+n+1)}$ and hence, $x_2^* = \frac{\xi}{2} + \frac{d(m-n)}{2a(m+n+1)}$. Substituting in (3) gives:

$$(x_1^i)^* = \frac{\theta(i)}{2} - \frac{\delta(i)}{2a}\frac{2|\mathcal{N}_{-i}|+1}{n+m+1},$$

where $|\mathcal{N}_{-i}|$ is the number of players in the other side of the network, i.e. $|\mathcal{N}_{-1}| = |\mathcal{N}_2| = m$ and $|\mathcal{N}_{-2}| = |\mathcal{N}_1| = n$. We get

$$\forall i \in \mathcal{N}_1, (x_1^i)^* = \frac{\theta(i)}{2} + \frac{d}{2a}\frac{2m+1}{n+m+1},$$
$$\text{and} \quad \forall i \in \mathcal{N}_2, \quad (x_1^i)^* = \frac{\theta(i)}{2} - \frac{d}{2a}\frac{2n+1}{n+m+1}.$$

Given those traffic at equilibrium, we have the following constraints from the positivity assumption:

$$\forall i \in \mathcal{N}_1, \quad (x_1^i)^* \leq \theta(i), \quad \text{and} \quad \forall i \in \mathcal{N}_2, \quad (x_1^i)^* \geq 0.$$

As we assume through the rest of the paper that the demand for each user on the same side of the network is the same. Then, for all user $i \in \mathcal{N}_1$, (resp. $i \in \mathcal{N}_2$) the demand for the user i is $\theta(i) = \theta(1)$ (resp. $\theta(i) = \theta(2)$). Then the two previous constraints imply the following one:

$$d \leq a(m+n+1)\min\{\frac{\theta(1)}{2m+1}, \frac{\theta(2)}{2n+1}\} := d_{\max}.$$

If the condition is not satisfied, i.e. $d > d_{\max}$, then we have the two following cases.

- $d_{\max} := a(m+n+1)\frac{\theta(1)}{2m+1}$ then if $d > d_{\max}$ we have $\forall i \in \mathcal{N}_1, \quad (x_1^i)^* = \theta(1)$.

- $d_{\max} := a(m+n+1)\dfrac{\theta(2)}{2n+1}$ then if $d > d_{\max}$ we have $\forall i \in \mathcal{N}_2, \quad (x_1^i)^* = 0$.
- if $d > a(m+n+1)\max\{\dfrac{\theta(1)}{2m+1}, \dfrac{\theta(2)}{2n+1}\}$, then $\forall i \in \mathcal{N}_1, \quad (x_1^i)^* = \theta(1)$ and $\forall i \in \mathcal{N}_2, \quad (x_1^i)^* = 0$.

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Given the equilibrium rates, the cost of player $i \in \mathcal{N}_1$ is:

$$J^i(x_1^*, x_2^*) = -\frac{d^2}{2a}\left(\frac{2m+1}{n+m+1}\right)^2 + \frac{d\theta(1)}{2} + \theta(1)\frac{a\xi}{2}.$$

We observe that this cost is a concave function in d and has a unique maximum for $d = \overline{d}_1 := \dfrac{a\theta(1)}{2}\left(\dfrac{m+n+1}{2m+1}\right)^2$. Then for d higher than $\overline{d}$ the cost is decreasing, it is Braess type paradox. Moreover, we can see that $\overline{d}_1 < d_{\max}$.

We have the similar result for any player $i \in \mathcal{N}_2$. The cost function is given by

$$J^i(x_1^*, x_2^*) = -\frac{d^2}{2a}\left(\frac{2n+1}{n+m+1}\right)^2 + \frac{d\theta(2)}{2} + \theta(2)\frac{a\xi}{2},$$

and this function has a unique maximum at $d = \overline{d}_2 := \dfrac{a\theta(2)}{2}\left(\dfrac{m+n+1}{2n+1}\right)^2 < d_{\max}$. Then we observe also in this asymmetric network a Braess paradox.

B. Equilibrium and costs with collusion

We consider that k players in $\mathcal{N}_1$ decide to collude. Then we have $n-k$ users in $\mathcal{N}_1$ that do not collude, one collusion with k users from $\mathcal{N}_1$ and m users in $\mathcal{N}_2$ which do not collude as well. The total number of individuals is still $n+m$ but as we see the collusion as an individual, we are faced with a non-symmetric system with $m+n-k+1$ users. We call the collusion h. For all user $i \in \mathcal{N}_1 \setminus h$, the demand is $\theta(1)$, for all user $j \in \mathcal{N}_2$, the demand is $\theta(2)$ and for the "collusion" player h the demand is $k\theta(1)$. We denote by y the amount of traffic of the "collusion" player that uses the link 1-3.

Proposition 2: The equilibrium rates are given by:

- If $d > a\dfrac{n-k+2+m}{2m+1}$ then $(x_1^i)^* = \theta(1)$ and $y^* = k\theta(1)$, $\forall i \in \mathcal{N}_1 \setminus g$, otherwise $(x_1^i)^* = \dfrac{\theta(1)}{2} + \dfrac{d}{2a}\dfrac{2m+1}{n-k+2+m}$ and $y^* = \dfrac{k\theta(1)}{2} + \dfrac{d}{2a}\dfrac{2m+1}{n-k+2+m}$.
- If $d > a\dfrac{n-k+2+m}{2(n-k+1)+1}$ then $(x_1^j)^* = 0$, $\forall j \in \mathcal{N}_2$, otherwise, $(x_1^j)^* = \dfrac{\theta(2)}{2} - \dfrac{d}{2a}\dfrac{2(n-k+1)+1}{n-k+2+m}$.

Proof In order to compute the cost of the individuals that are colluding, we study the atomic game with $n-k+1+m$ non-symmetric players. We denote by $y \in [0, k\theta(1)]$ the amount of traffic sent by the collusion h through the link 1-3. The cost function for the "collusion" player g is:

$$J^h(x, y) = yf(x_1) + (k\theta(1) - y)f(x_2) + (k\theta(1) - y)d,$$

where $x_1 = \sum_{j=1}^2 \sum_{i \in \mathcal{N}_j} x_1^i + y$ is the total amount of traffic in the link 1-3, and $x_2 = \sum_{j=1}^2 \sum_{i \in \mathcal{N}_j} x_2^i + (k\theta(1) - y)$ is the total amount of traffic in the link 2-3. The cost functions $J^i(x,y)$ and $J^j(x,y)$, respectively, for a user $i \in \mathcal{N}_1$ and for a user $j \in \mathcal{N}_2$ are not influenced by the collusion and are given by equations 1 (for $i \in \mathcal{N}_1$ and for $i \in \mathcal{N}_2$). Then the derivatives of these functions are:

$$\frac{\partial J^i(x,y)}{\partial x_1^i} = f(x_1) + x_1^i g(x_1) - f(x_2) - (\theta(i) - x_1^i)g(x_2) + \delta(i),$$

where $\delta(i) = d$ for $i \in \mathcal{N}_2$ and $\delta(i) = -d$ for $i \in \mathcal{N}_1$. Moreover we have the derivative for user h:

$$\frac{\partial J^h(x,y)}{\partial y} = f(x_1) + yg(x_1) - f(x_2) - (k\theta(1) - y)g(x_2) - d.$$

Then the summation of all the derivatives Δ is:

$$\begin{aligned}
\Delta &= \sum_{i \in N_1} \frac{\partial J^i(x,y)}{\partial x_1^i} + \sum_{j \in N_2} \frac{\partial J^j(x,y)}{\partial x_1^i} + \frac{\partial J^h(x,y)}{\partial y} \\
&= (n-k)\frac{\partial J^i(x,y)}{\partial x_1^i} + m\frac{\partial J^j(x,y)}{\partial x_1^i} + \frac{\partial J^h(x,y)}{\partial y} \\
&= (n-k+1+m)f(x_1) + x_1 g(x_1) \\
&\quad -(n-k+1+m)f(x_2) \\
&\quad -(\xi - x_1)g(x_2) + d(m - (n-k) - 1),
\end{aligned}$$

where $\xi = (n-k)\theta(1) + k\theta(1) + m\theta(2) = n\theta(1) + m\theta(2)$ is the total amount of traffic in the network. Taking the linear cost function $f(x) = ax$, we obtain the amount of traffic x_1^* on the link 1-3 at equilibrium and on link 2-3:

$$x_1^* = \frac{\xi}{2} - \frac{d}{2a}\frac{m-n+k-1}{n-k+2+m}, \quad x_2^* = \frac{\xi}{2} + \frac{d}{2a}\frac{m-n+k-1}{n-k+2+m}.$$

Given that the derivative of the cost function for each player is equal to zero at equilibrium, we get:

$$\forall i \in \mathcal{N}_1 \setminus g, \quad (x_1^i)^* = \frac{\theta(1)}{2} + \frac{d}{2a}\frac{2m+1}{n-k+2+m},$$

$$\forall j \in \mathcal{N}_2, \quad (x_1^j)^* = \frac{\theta(2)}{2} - \frac{d}{2a}\frac{2(n-k+1)+1}{n-k+2+m},$$

$$\text{and } y^* = \frac{k\theta(1)}{2} + \frac{d}{2a}\frac{2m+1}{n-k+2+m}.$$

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Then we have the following cost for any type of player:

$$\begin{aligned}
J^i(x^*, y^*) = -\frac{d^2}{2a}\left(\frac{2m+1}{n-k+m+2}\right)^2 + \frac{d\theta(1)}{2} \\
+\theta(1)\frac{a\xi}{2}, \quad \forall i \in \mathcal{N}_1 \setminus g,
\end{aligned}$$

$$\begin{aligned}
J^i(x^*, y^*) = -\frac{d^2}{2a}\left(\frac{2(n-k+1)+1}{n-k+m+2}\right)^2 + \frac{d\theta(2)}{2} \\
+\theta(2)\frac{a\xi}{2}, \quad \forall i \in \mathcal{N}_2,
\end{aligned}$$

and $C^h(\mathbf{x}[\mathbf{H}]) = J^h(x^*, y^*) = -\dfrac{d^2}{2a}\left(\dfrac{2m+1}{n-k+m+2}\right)^2 + \dfrac{kd\theta(1)}{2}$ $+k\theta(1)\dfrac{a\xi}{2}$.

C. Measures of Collusion

The individual single collusion-price (ISCP) is defined in section II as the ratio between the summation of individual costs without collusion of the players that collude and the global cost of the collusion seen as a single player. If the ISCP is lower than 1, then it means that the collusion is not of benefit to the individuals who want to collude. The ISCP for a single collusion h is expressed by:

$$ISCP(h) = \frac{C[h](\mathbf{x})}{C^h(\mathbf{x}[\mathbf{H}])}.$$

The numerator of the ISCP is the summation of the individual cost J^k at equilibrium where there is no collusion, for all user k that will collude in h, i.e.

$$C[h](\mathbf{x}) = \sum_{k\in h} J^k(x_1^*, x_2^*)$$
$$= -\frac{kd^2}{2a}\left(\frac{2m+1}{n+m+1}\right)^2 + \frac{kd\theta(1)}{2} + k\theta(1)\frac{a\xi}{2}.$$

Then the ISCP is equal to:

$$ISCP(h) = \frac{\frac{kd^2}{2a}\left(\frac{2m+1}{n+m+1}\right)^2 - \frac{kd\theta(1)}{2} - k\theta(1)\frac{a\xi}{2}}{\frac{d^2}{2a}\left(\frac{2m+1}{n-k+m+2}\right)^2 - \frac{kd\theta(1)}{2} - k\theta(1)\frac{a\xi}{2}}.$$

Proposition 3: If the size k of the collusion h is higher than $(-1/2 + 1/2\sqrt{1+4(n+m+1)})^2$, then it is beneficial to the collusion users to collude together; otherwise it is not beneficial to them, i.e.

$$ISCP(h) > 1 \Leftrightarrow k > (-1/2 + 1/2\sqrt{1+4(n+m+1)})^2.$$

Proof $ISCP(h) > 1$ is equivalent to

$$-\frac{kd^2}{2a}\left(\frac{2m+1}{n+m+1}\right)^2 + \frac{kd\theta(1)}{2} + k\theta(1)\frac{a\xi}{2}$$
$$> -\frac{d^2}{2a}\left(\frac{2m+1}{n-k+m+2}\right)^2 + \frac{kd\theta(1)}{2} + k\theta(1)\frac{a\xi}{2},$$
$$\text{for} \quad \sqrt{k} < \frac{n+m+1}{n-k+m+2} := \frac{X}{X+1-k}.$$

Define $X := n+m+1$, then $ISCP(h) > 1$ is equivalent to $X\sqrt{k} - k\sqrt{k} + \sqrt{k} - X < 0$. Finally we define $u := \sqrt{k}$ and then the equation $ISCP(h) > 1$ is equivalent to $u^3 - u(X+1) + X > 0$. This equation has the three solutions: $\{1, -1/2 - 1/2\sqrt{1+4X}, -1/2 + 1/2\sqrt{1+4X}\}$. We have that $-1/2 + 1/2\sqrt{1+4X} > 1$ because $X = n+m+1 > 2$. We are considering the case where $u = \sqrt{k} > 1$. Then if $u = \sqrt{k} \in [1, -1/2 + 1/2\sqrt{1+4X}[$ the polynome $u^3 - u(X+1) + X$ is negative, which is equivalent to $ISCP(h) < 1$. And if $u = \sqrt{k} > -1/2 + 1/2\sqrt{1+4X}$, then the polynome $u^3 - u(X+1) + X$ is positive, which is equivalent to $ISCP(h) > 1$.

Thus, $ISCP(h) > 1$ is equivalent to $k > (-1/2 + 1/2\sqrt{1+4(n+m+1)})^2$. ■

Proposition 4: The individual single-collusion price is minimized when $k = \frac{m+n+2}{3}$.

Proof The individual single collusion price $ISCP(k)$ is defined by the following expression:

$$ISCP(k) = \frac{k(A+B)}{A(\frac{X-1}{X-k})^2 + kB},$$

$$\text{where} \quad A = \frac{-d^2}{2a}\left(\frac{2m+1}{n+m+1}\right)^2, \; B = \frac{d\theta(1)}{2} + \theta(1)\frac{a\xi}{2}$$

and $X = n+m+2$. We consider the continuous function ISCP in order to compute its derivative. Then we have the following derivative:

$$\frac{\partial ISCP(k)}{\partial k} = \frac{(A+B)A(X-1)^2(X-3k)}{(X-k)^3(A(\frac{X-1}{X-k})^2 + kB)^2}.$$

We have that $\frac{A(X-1)^2}{(X-k)^3(A(\frac{X-1}{X-k})^2 + kB)^2} < 0$ and $A+B > 0$, then, the derivative of the individual single collision price is increasing if and only if $k > \frac{X}{3}$. Thus the ISCP is minimized when $k = X/3$. ■

Given this result, we have the worst lost of cost induced by a collusion for colluding players which is given by $\max_h(1 - ISCP(h))$, that is:

$$\max_h(1 - ISCP(h)) = ISCP(\frac{m+n+2}{3})$$
$$= \left\{-\frac{\frac{m+n+2}{3}d^2}{2a}\left(\frac{2m+1}{n+m+1}\right)^2 + \frac{\frac{m+n+2}{3}d\theta(1)}{2} + \frac{m+n+2}{3}\theta(1)\frac{a\xi}{2}\right\}$$
$$\Big/\left\{-\frac{d^2}{2a}\left(\frac{2m+1}{\frac{2(m+n+2)}{3}}\right)^2 + \frac{\frac{m+n+2}{3}d\theta(1)}{2} + \frac{m+n+2}{3}\theta(1)\frac{a\xi}{2}\right\}.$$

Note that this result is an approximation as the size of the collusion is an integer and the ratio $(m+n+2)/3$ is generally a real number.

Proposition 5: The optimal collusion size in order to maximize the individual single collusion price is given by:

$$k^* = \begin{cases} 1 & \text{if} \quad n < (m+1)^2, \\ n & \text{otherwise.} \end{cases}$$

Proof We have proved in the previous proposition that the ISCP is first decreasing and increasing with the size k of the collusion. Then it is the most beneficial for a colluding player, in terms of ISCP, when all players collude together, i.e.

$k = n$. Moreover, this should be satisfied when $ISCP(n) > 1$. Otherwise the best collusion size is $k = 1$ because it yields $ISCP(1) = 1$.

We have $ISCP(n) > 1$ if and only if: $-\frac{n}{(n+m+1)^2} < \frac{1}{(m+2)^2}$, which is equivalent to $n > (m+1)^2$. ■

In other words, the worst degradation over all coalitions in N_1, measured with $ICP(N_1)$, depends only on n and m. Thus, we have the following result:

$$ICP(N_1) = \begin{cases} 1 \quad \text{if} \quad n < (m+1)^2, \\ \dfrac{-\frac{nd^2}{2a}\left(\frac{2m+1}{n+m+1}\right)^2 + \frac{nd\theta(1)}{2} + n\theta(1)\frac{a\xi}{2}}{-\frac{d^2}{2a}\left(\frac{2m+1}{m+2}\right)^2 + \frac{nd\theta(1)}{2} + n\theta(1)\frac{a\xi}{2}} \\ \quad \text{otherwise.} \end{cases}$$

Now we look at the impact of a collusion on the whole society, meaning through the social welfare. For doing this, we use the measure of the social single collusion price (SSCP) which is defined as the ratio between the social welfare without collusion and the social welfare when there is a collusion. The social welfare of the system without a collusion is given by:

$$\begin{aligned} C[I](a) &= nJ^1(x_1^*, x_2^*) + mJ^2(x_1^*, x_2^*), \\ &= \frac{-d^2(n(2m+1)^2 + m(2n+1)^2)}{2a(n+m+1)^2} \\ &\quad +(\frac{d}{2} + \frac{a\xi}{2})\xi, \end{aligned}$$

with $\xi = n\theta(1) + m\theta(2)$ is the total demand. We have the social welfare when there is a collusion h:

$$\begin{aligned} C[I](a(H)) &= \sum_{i \in \mathcal{N}_1 \setminus g} J^i(x^*, y^*) \\ &\quad + \sum_{i \in \mathcal{N}_2} J^i(x^*, y^*) + J^g(x^*, y^*), \\ &= \frac{-d^2}{2a(n-k+m+2)^2}((n-k+1)(2m+1)^2 \\ &\quad + m(2(n-k+1)+1)^2) + (\frac{d}{2} + \frac{a\xi}{2})\xi. \end{aligned}$$

Thus the social single collusion price, SSCP, is expressed by the following ratio:

$$\left\{\frac{-d^2(n(2m+1)^2 + m(2n+1)^2)}{2a(n+m+1)^2} + \Big(\frac{d}{2} + \frac{a\xi}{2}\Big)\xi\right\} \Bigg/ \left\{\frac{-d^2}{2a(n-k+m+2)^2}((n-k+1)(2m+1)^2 + m(2(n-k+1)+1)^2) + (\frac{d}{2} + \frac{a\xi}{2})\xi\right\}.$$

Finally, we look at the impact of the collusion only on the non colluding users $\mathcal{N}_1 \setminus h \bigcup \mathcal{N}_2$. We use the measure called the single collusion externality price (SCEP) defined in section II which is defined as the ratio between total cost perceived by the non colluding users when there is no collusion and when there is a collusion. The total cost perceived by those users when there is no collusion is given by:

$$\begin{aligned} C[I \backslash h](a) &= (n-k)J^1(x_1^*, x_2^*) + mJ^2(x_1^*, x_2^*), \\ &= \frac{-d^2}{2a(n+m+1)^2}((n-k)(2m+1)^2 + m(2n+1)^2) \\ &\quad +(\frac{d}{2} + \frac{a\xi}{2})((n-k)\theta(1) + m\theta(2)). \end{aligned}$$

The total cost perceived by those users when there is a collusion is given by:

$$\begin{aligned} C[I \setminus h](a[H]) &= \sum_{i \in \mathcal{N}_1 \setminus g} J^i(x^*, y^*) + \sum_{i \in \mathcal{N}_2} J^i(x^*, y^*), \\ &= \frac{-d^2((n-k)(2m+1)^2}{2a(n-k+m+2)^2} \\ &\quad + m(2(n-k+1)+1)^2) \\ &\quad +(\frac{d}{2} + \frac{a\xi}{2})((n-k)\theta(1) + m\theta(2)). \end{aligned}$$

Thus the single collusion externality price (SCEP) of a collusion h is:

$$\left\{\frac{-d^2}{2a(n+m+1)^2}((n-k)(2m+1)^2 + m(2n+1)^2) + (\frac{d}{2} + \frac{a\xi}{2})((n-k)\theta(1) + m\theta(2))\right\} \Bigg/ \left\{\frac{-d^2}{2a(n-k+m+2)^2}((n-k)(2m+1)^2 + m(2(n-k+1)+1)^2) + (\frac{d}{2} + \frac{a\xi}{2})((n-k)\theta(1) + m\theta(2))\right\}.$$

V. Concluding Remarks

Our contribution has been to define different new measures for collusion and to illustrate them through the load balancing routing game. In particular, we have shown that merging into a coalition may be harmful not only for society but also to those who collude.

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