About Fokker-Planck equation with measurable coefficients and applications to the fast diffusion equation

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Abstract. The object of this paper is the uniqueness for a d-dimensional Fokker-Planck type equation with non-homogeneous (possibly degenerated) measurable not necessarily bounded coefficients. We provide an application to the probabilistic representation of the so called Barenblatt solution of the fast diffusion equation which is the partial differential equation $\partial_t u = \partial^2_{xx} u^m$ with $m \in (0,1)$. Together with the mentioned Fokker-Planck equation, we make use of small time density estimates uniformly with respect to the initial condition.

Keywords. Fokker-Planck, fast diffusion, probabilistic representation, non-linear diffusion, stochastic particle algorithm.

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1. Introduction

The first part of the paper focuses on some uniqueness result on Fokker-Planck type equation with measurable non-negative (possibly degenerated) multidimensional unbounded coefficients. The second part of the paper develops one application to the probabilistic representation of a fast diffusion equation. In the whole paper $T > 0$ will stand for a fixed final time.

In one dimension ($x \in \mathbb{R}^d$ with $d = 1$), Fokker-Planck equation is of the type

$$\begin{cases} \partial_t u(t,x) &= \partial^2_{xx}(a(t,x)u(t,x)) - \partial_x(b(t,x)u(t,x)), \ \ t \in]0,T], \\ u(0,\cdot) &= \mu(dx), \end{cases} \tag{1.1}$$

where, $a, b : [0,T] \times \mathbb{R} \to \mathbb{R}$ are measurable locally bounded coefficients and μ is a finite measure. The study of Fokker-Planck equation for measures is a quite widely studied subject in the literature in finite and infinite dimension. Recent work in the case of time-dependent coefficients was done by [9, 16, 31] in the case $d \geq 1$ with some minimal regularity. In infinite dimension some interesting work was produced by [8].

In this paper we concentrate in the case of measurable (possibly) degenerate coefficients. Our interest is devoted to the irregularity of the diffusion coefficient, so we will set $b = 0$. A first result in that direction was produced in [7] where a was

bounded, possibly degenerated and the difference of two solutions was supposed to be in $L^2([\kappa, T]\times\mathbb{R})$, for every $\kappa > 0$ (ASSUMPTION (A)). This result was exploited for studying the probabilistic representation of a porous media type equation with irregular coefficients. We will come back later to this point. We remark that it is not possible to obtain uniqueness without ASSUMPTION (A). In particular Remark 3.11 of [7] provides two measure-valued solutions when a is time-homogeneous, continuous with $\frac{1}{a}$ integrable in a neighborhood of zero.

One natural question concerns what happens when a is not bounded and $x \in \mathbb{R}^d$. A partial answer to this question is given in Theorem 3.1 and Remark 3.5. That Theorem is probably the most important result of the paper; it is a generalization of Theorem 3.8 in [7] where the non-homogeneous function a was bounded. Theorem 3.1 handles the multidimensional case and it allows a to be unbounded.

An application of Theorem 3.1 concerns the following parabolic problem for $x \in \mathbb{R}$:

$$\begin{cases} \partial_t u(t,x) &= \partial^2_{xx}(u^m(t,x)), \ \ t \in]0,T], \\ u(0,\cdot) &= \delta_0, \end{cases} \tag{1.2}$$

where, δ_0 is the Dirac delta function at zero. u^m will denote $u|u|^{m-1}$.

It is well known that, for $m > 1$, there exists an exact solution to (1.2), the so-called *Barenblatt density*, [3]. Its explicit formula is recalled for instance in Chapter 4 of [35] and more precisely in [4]. Equation (1.2) is the *classical* porous medium equation.

In this paper, we focus on (1.2), for some $m \in (0,1)$, the *Fast Diffusion equation*. In fact, an analogous Barenblatt type solution also exists in that case, see Chapter 4 of [35] and references therein; it is given by the following expression

$$\mathcal{U}(t,x) = t^{-\alpha}\big(D + \tilde{k}|x|^2 t^{-2\alpha}\big)^{-\frac{1}{1-m}}, \tag{1.3}$$

where,

$$\alpha = \frac{1}{m+1},\ \tilde{k} = \frac{1-m}{2(m+1)m},\ D = \left(\frac{I}{\sqrt{\tilde{k}}}\right)^{\frac{2(1-m)}{m+1}},\ I = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} [\cos(x)]^{\frac{2m}{1-m}}\, dx. \tag{1.4}$$

Equation (1.2) is a particular case of the so called generalized porous media type equation

$$\begin{cases} \partial_t u(t,x) &= \partial^2_{xx}\beta\left(u(t,x)\right), \ \ t \in]0,T], \\ u(0,x) &= u_0(dx), \quad x \in \mathbb{R}, \end{cases} \tag{1.5}$$

where $\beta : \mathbb{R} \to \mathbb{R}$ is a monotone non-decreasing function such that $\beta(0) = 0$.

In the case of fast diffusion, we have $\beta(u) = u^m,\ m \in (0,1)$ and the initial condition u_0 is a finite measure which equals δ_0 in the Barenblatt case. In this application two difficulties arise: first the coefficient β is of singular type since it is not locally Lipschitz, second the initial condition is a measure. Another type of singular coefficient is $\beta(u) = H(u-u_c)u$, where H is a Heaviside function, see e.g. [2].

The problem (1.2) with $m \in (0,1)$ was studied by several authors. For a bounded integrable function as initial condition, equation in (1.2), is well-stated in the sense of distributions, as a byproduct of the classical papers [10, 6] on (1.5) with general monotonous coefficient β.

More specifically, [19], proved existence for (1.2), when the initial data is locally integrable. This result was extended to the finite Radon measures in a bounded domain by [11], and to locally finite measures in the whole space by [30]. The Barenblatt solution is an *extended continuous solution* as defined in [14, 13]; Theorem 5.2 of [13] says that (1.2) admits such a solution and it is unique. As far as we know there is no uniqueness argument in the literature as far as the initial condition is a finite measure in the general sense of distributions.

In [15], the authors investigated large time behavior of solutions to (1.2); [25] showed existence of solutions to the fast diffusion equation perturbed by a right-hand side source term, being a general finite and positive Borel measure.

The present paper provides the probabilistic representation of the (Barenblatt) solution of (1.2) and exploits this fact in order to approach it via a Monte Carlo simulation; the committed L^2 error is around 10^{-3}. We make use of the probabilistic procedure developed in Section 4 of [4] and we compare it to the exact form of the solution $\mathcal{U}$ of (1.2), which is given by the explicit formulae (1.3)-(1.4).

In the case when $\beta(u) = H(u-u_c)u$, numerical simulations based on the same procedure were obtained in [4]. In that paper we compared this with a deterministic numerical analysis recent approach developed in [12], which was very performing in that case. At this stage, the implementation of the same deterministic method for the fast diffusion equation does not give satisfying results; this constitutes a further justification for the probabilistic representation.

The probabilistic representation of $\mathcal{U}$ consists in finding a suitable stochastic process Y such that the law of Y_t has $\mathcal{U}(t,\cdot)$ as density. Y will be a (weak) solution of the non-linear SDE

$$\begin{cases} Y_t &= \int_0^t \Phi(\mathcal{U}(s,Y_s))dW_s, \\ \mathcal{U}(t,\cdot) &= \text{Law density of } Y_t, \ \ \forall\, t>0, \end{cases} \tag{1.6}$$

where, W is a Brownian motion on some suitable filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

Moreover, we define

$$\Phi(u) = \sqrt{2}|u|^{\frac{m-1}{2}}, \ \ u \in \mathbb{R}, \ \ m \in (0,1).$$

To the best of our knowledge, the first author who considered a probabilistic representation of (1.5) was McKean [27], particularly in relation with the so called *propagation of chaos*. In his case, β was smooth, but the equation also included a first order coefficient. From then on the literature steadily grew and nowadays there is a vast amount of contributions to the subject, especially when the non-linearity is in the

first order part, as e.g. in Burgers equation. We refer the reader to the excellent survey papers [34] and [18].

A probabilistic interpretation of (1.5) when $\beta(u) = u.|u|^{m-1}$, $m > 1$, was provided for instance in [5]. For the same β, though the method could be adapted to the case where β is Lipschitz, in [22] the author has studied the evolution equation (1.5) when the initial condition and the evolution takes values in the class of probability distribution functions on $\mathbb{R}$. Therefore, instead of an evolution equation in $L^1(\mathbb{R})$, he considers a state space of functions vanishing at $-\infty$ and with value 1 at $+\infty$. He studies both the probabilistic representation and the propagation of chaos. An alternative study to chaos propagation when $\beta(u) = u^2$ and $\beta(u) = u^m, m > 1$ was proposed in [29] and [17]. The probabilistic representation in the case of possibly discontinuous β was treated in [7] when β is non-degenerate and in [2] when β is degenerate; the latter case includes the case $\beta(u) = H(u - u_c)u$.

As a preamble for the probabilistic representation we make a simple even though crucial observation. Let W be a standard Brownian motion.

Proposition 1.1. *Let $\beta : \mathbb{R} \to \mathbb{R}$, such that $\beta(u) = \Phi^2(u).u$, $\Phi : \mathbb{R} \to \mathbb{R}_+$ and u_0 be a probability real measure.*

Let Y be a solution to the problem

$$\begin{cases} Y_t &= Y_0 + \int_0^t \sqrt{2}\Phi(u(s, Y_s))dW_s, \\ u(t,\cdot) &= \textit{Law density of } Y_t, \ \ \forall\, t > 0, \\ u(0,\cdot) &= u_0(dx). \end{cases} \tag{1.7}$$

Then, $u : [0,T] \times \mathbb{R} \to \mathbb{R}$ is solution to (1.5)*.*

The proof of the result above is based on the following lemma.

Lemma 1.2. *Let $a : [0,T] \times \mathbb{R} \to \mathbb{R}_+$ be measurable. Let (Y_t) be a process which solves the SDE*

$$Y_t = Y_0 + \int_0^t \sqrt{2a(s, Y_s)}dW_s, \quad t \in [0,T].$$

Consider the function $t \mapsto \rho(t,\cdot)$ from $[0,T]$ to the space of finite real measures $\mathcal{M}(\mathbb{R})$, defined as $\rho(t,\cdot)$ being the law of Y_t.

Then, ρ is a solution, in the sense of distributions (see (3.1)*), of*

$$\begin{cases} \partial_t u &= \partial^2_{xx}(au), \\ u(0,\cdot) &= \textit{Law of } Y_0. \end{cases} \tag{1.8}$$

Proof of Lemma 1.2. This is a classical result see for instance [33], Chapter 4. The proof is based on an application of Itô's formula to $\varphi(Y_t)$, $\varphi \in \mathcal{S}(\mathbb{R})$. □

Proof of Proposition 1.1. We set $a(s,y) = \Phi^2(u(s,y))$. We apply Lemma 1.2, setting $\rho(t,y) = u(t,y)dy$, $t > 0$ and $\rho(0,\cdot) = u_0$. □

When u_0 is the Dirac measure at zero and $\beta(u) = u^m$, with $m \in]\frac{3}{5}, 1[$, Theorem 5.7 states the converse of Proposition 1.1, providing a process Y being the unique (weak) solution of (1.6).

The first step consists in reducing the proof of that Theorem to the proof of Proposition 5.3, where the Dirac delta measure, as initial condition of (1.2), is replaced by the function $\mathcal{U}(\kappa,\cdot),\ 0 < \kappa \leq T$. This corresponds to the shifted Barenblatt solution through a time κ, which will be denoted by $\overline{\mathcal{U}}$. In this case, Proposition 5.3 allows even to obtain a unique strong solution of the corresponding non-linear SDE. That reduction is possible through a weak convergence argument of the solutions given by Proposition 5.3, when $\kappa \to 0$.

The idea of the proof of Proposition 5.3 is the following. Let W be a standard Brownian motion and a r.v. $\overline{Y_0}$ distributed as $\mathcal{U}(\kappa,\cdot)$; since $\Phi(\overline{\mathcal{U}})$ is Lipschitz, the SDE

$$\overline{Y_t} = \overline{Y_0} + \int_0^t \Phi(\overline{\mathcal{U}}(s,\overline{Y_s}))dW_s,$$

admits a unique strong solution. The marginal laws of $(\overline{Y_t})$ and $\overline{\mathcal{U}}$ can be shown to be both solutions to (1.8) for $a(s,y) = (\overline{\mathcal{U}}(s,y))^{m-1}$; that a will be in the sequel denoted by $\overline{a}$. The leading argument of the proof is carried by Theorem 3.1 which states uniqueness for measure valued solutions of the Fokker-Planck type PDE (1.8) under some **Hypothesis(B)**. More precisely, to conclude that the marginal laws of $(\overline{Y_t})$ and $\overline{\mathcal{U}}$ coincide via Theorem 3.1, we show that they both verify the so-called **Hypothesis(B2)**. In order to prove that for $\overline{\mathcal{U}}$, we will make use of Lemma 4.2. The verification of **Hypothesis(B2)** for the marginal laws of $\overline{Y}$ is more involved. It makes use of small time (uniformly with respect to the initial condition) upper bound for a non-homogeneous diffusion flow with linear growth (unbounded) coefficients, even though the diffusion term is non-degenerate. This is the object of Proposition 5.1, whose proof is based on an application of Malliavin calculus. In our opinion, that result has an interest by itself; we were not able to find it in the literature.

When the paper was practically finished we have discovered an interesting recent result of M. Pierre, presented in Chapter 6 of [20] obtained independently. That result holds with non-degenerate locally bounded coefficients in dimension 1, with initial condition having a first moment. In that case, they do not need an Hypothesis of type (B). In particular, it allows to establish Proposition 5.3 but not Theorem 5.7, where the coefficients are not locally bounded on $[0,T] \times \mathbb{R}$.

The paper is organized as follows. Section 2, is devoted to basic notations. Section 3 is concentrated on Theorem 3.1 which concerns uniqueness for the deterministic, time inhomogeneous Fokker-Planck type equation. Section 4, presents some properties of the Barenblatt solution $\mathcal{U}$ to (1.2). The probabilistic representation of $\mathcal{U}$, is treated in

Section 5. Proposition 5.1 is one basic tool performing small time density estimates for time-inhomogeneous diffusions, whose proof is located in the Appendix. Finally, Section 6 is devoted to numerical experiments.

2. Preliminaries

We start with some basic analytical framework. Let $d \geq 1$. If $f : \mathbb{R}^d \to \mathbb{R}^d$ is a bounded function we will denote $\|f\|_\infty = \sup_{x\in\mathbb{R}^d} |f(x)|$. By $S(\mathbb{R}^d)$ we denote the space of rapidly decreasing infinitely differentiable functions $\varphi : \mathbb{R}^d \to \mathbb{R}$, by $S'(\mathbb{R}^d)$ its dual (the space of tempered distributions). We denote by $\mathcal{M}(\mathbb{R}^d)$ the set of finite Borel measures on $\mathbb{R}^d$. If $x \in \mathbb{R}^d$, $|x|$ will denote the usual Euclidean norm.

For $\varepsilon > 0$, let K_ε be the Green function of $\varepsilon - \Delta$, that is the kernel of the operator $(\varepsilon - \Delta)^{-1} : L^2(\mathbb{R}^d) \to H^2(\mathbb{R}^d) \subset L^2(\mathbb{R}^d)$. In particular, for all $\varphi \in L^2(\mathbb{R}^d)$, we have

$$B_\varepsilon \varphi := (\varepsilon - \Delta)^{-1}\varphi(x) = \int_{\mathbb{R}} K_\varepsilon (x - y)\, \varphi(y) dy. \tag{2.1}$$

For more information about the corresponding analysis, the reader can consult [32]. If $\varphi \in C^2(\mathbb{R}^d) \bigcap S'(\mathbb{R}^d)$, then $(\varepsilon - \Delta)\varphi$ coincides with the classical associated PDE operator.

Definition 2.1. We will say that a function $\psi : [0, T] \times \mathbb{R} \to \mathbb{R}$ is *non-degenerate* if there is a constant $c_0 > 0$, such that $\psi \geq c_0$.

Definition 2.2. We will say that a function $\psi : [0, T] \times \mathbb{R} \to \mathbb{R}$ has linear growth (with respect to the second variable), if there is a constant C, such that $|\psi(\cdot, x)| \leq C(1+|x|)$.

3. Uniqueness for the Fokker-Planck equation

Now, we state the main result of the paper which concerns uniqueness for Fokker-Planck type equation with measurable, time-dependent, (possibly degenerated and unbounded) coefficients. It generalizes Theorem 3.8 of [7] where the coefficients were bounded and one-dimensional.

The theorem below holds with two classes of hypotheses: **(B1)**, operating in the multidimensional case and **(B2)** more specifically in the one-dimensional case.

Theorem 3.1. *Let $d \geq 1$ and a be a Borel nonnegative function on $[0, T] \times \mathbb{R}^d$.*

Let $z_i : [0, T] \to \mathcal{M}(\mathbb{R}^d)$, $i = 1, 2$, be continuous with respect to the weak topology on finite measures on $\mathcal{M}(\mathbb{R}^d)$.

Let z^0 be an element of $\mathcal{M}(\mathbb{R}^d)$. Suppose that both z_1 and z_2 solve the problem $\partial_t z = \Delta(az)$ in the sense of distributions with initial condition $z(0, \cdot) = z^0$, i.e.

$$\int_{\mathbb{R}^d} \phi(x) z(t, dx) = \int_{\mathbb{R}^d} \phi(x) z^0(dx) + \int_0^t ds \int_{\mathbb{R}^d} \Delta\phi(x) a(s, x) z(s, dx) \tag{3.1}$$

for every $t \in [0,T]$ and any $\phi \in C_0^\infty(\mathbb{R}^d)$.

Then $z := (z_1 - z_2)(t,\cdot)$ is identically zero for every t, under the following requirement.

Hypothesis (B).

There is $\tilde{z} \in L^1_{loc}([0,T]\times\mathbb{R}^d)$, such that $z(t,\cdot)$ admits $\tilde{z}(t,\cdot)$ as density for almost all $t \in [0,T]$. $\tilde{z}$ will still be denoted again by z. Moreover, either **(B1)** or **(B2)** below is fulfilled.

(B1)

(i) $\int\limits_{[0,T]\times\mathbb{R}^d} |z(t,x)|^2 dtdx < +\infty,$

(ii) $\int\limits_{[0,T]\times\mathbb{R}^d} |az|^2(t,x) dtdx < +\infty.$

(B2) We suppose $d = 1$. For every $t_0 > 0$, we have

(i) $\int\limits_{[t_0,T]\times\mathbb{R}} |z(t,x)|^2 dtdx < +\infty,$

(ii) $\int\limits_{[0,T]\times\mathbb{R}} |az|(t,x) dtdx < +\infty,$

(iii) $\int\limits_{[t_0,T]\times\mathbb{R}} |az|^2(t,x) dtdx < +\infty.$

Remark 3.2.

The weak continuity of $z(t,\cdot)$ and Remark 3.10 of [7], imply that

$$\sup_{t\in[0,T]} \|z(t,\cdot)\|_{\text{var}} < +\infty.$$

In particular, $\sup\limits_{0<t\leq T} \int_{\mathbb{R}^d} |z(t,x)|dx < +\infty$.

Remark 3.3.

1) If a is bounded, then the first item of Hypothesis(B1) implies the second one.

2) If a is non-degenerated, assumption (ii) of Hypothesis(B1) implies assumption (i).

Remark 3.4. Let $d = 1$.

1) If a is non-degenerate, the third assumption of Hypothesis(B2) implies the first one.

2) If $z(t,x) \in L^\infty([t_0,T]\times\mathbb{R})$, then the first item of Hypothesis(B2) is always verified.

3) If a is bounded, then assumption (ii) of Hypothesis(B2) is always verified by Remark 3.2; the first item of Hypothesis(B2) implies the third one. So, Theorem 3.1, is a strict generalization of Theorem 3.8 in [7].

4) Let $(z(t,\cdot), t \in [0,T])$ be the law of a stochastic process solving

$$Y_t = Y_0 + \int_0^t \sqrt{2a(s,Y_s)}dW_s,$$

with Y_0 distributed as z^0, such that $\int_{\mathbb{R}} |x|^2 z^0(dx) < +\infty$.

If $\sqrt{a}$ has linear growth, it is well known that

$$\sup_{t\leq T} \mathbb{E}(|Y_t|^2) < +\infty; \text{ so}$$

$$\int_{[0,T]\times\mathbb{R}} |a(s,x)z(s,x)|dsdx = \mathbb{E}\left[\int_0^T a(s,Y_s)ds\right] < +\infty.$$

Therefore, assumption (ii) in Hypothesis(B2) is always fulfilled.

Proof of Theorem 3.1.

Let z_1, z_2 be two solutions of (3.1), we set $z := z_1 - z_2$. We evaluate for every $t \in [0,T]$, the quantity

$$g_\varepsilon(t) = \|z(t,\cdot)\|^2_{-1,\varepsilon},$$

where, $\|f\|_{-1,\varepsilon} = \|(\varepsilon - \Delta)^{-\frac{1}{2}} f\|_{L^2}$.

Similarly to the first part of the proof of Theorem 3.8 in [7], assuming we can show that,

$$\lim_{\varepsilon\to 0} g_\varepsilon(t) = 0, \quad \forall t \in [0,T], \tag{3.2}$$

we are able to prove that $z(t) \equiv 0$, for all $t \in]0,T]$. We explain this fact.

Let $t \in]0,T]$. We recall the notation $B_\varepsilon f = (\varepsilon - \Delta)^{-1} f$, if $f \in L^2(\mathbb{R}^d)$. Since $z(t,\cdot) \in L^2(\mathbb{R}^d)$, then $B_\varepsilon z(t,\cdot) \in H^2(\mathbb{R}^d)$ and so $\nabla B_\varepsilon z(t,\cdot) \in H^1(\mathbb{R}^d)^d \subset L^2(\mathbb{R}^d)^d$.

This gives,

$$\begin{aligned} g_\varepsilon(t) &= \int_{\mathbb{R}^d} B_\varepsilon z(t,x) z(t,x) dx, \\ &= \varepsilon \int_{\mathbb{R}^d} (B_\varepsilon z(t,x))^2 dx - \int_{\mathbb{R}^d} B_\varepsilon z(t,x) \Delta B_\varepsilon z(t,x) dx, \\ &= \varepsilon \int_{\mathbb{R}^d} (B_\varepsilon z(t,x))^2 dx + \int_{\mathbb{R}^d} |\nabla B_\varepsilon z(t,x)|^2 dx. \end{aligned}$$

Since the two terms of previous sum are non-negative, if (3.2) holds then,

$\sqrt{\varepsilon} B_\varepsilon z(t,\cdot) \to 0$, (resp. $|\nabla B_\varepsilon z(t,\cdot)| \to 0$) in $L^2(\mathbb{R}^d)$ (resp. in $L^2(\mathbb{R}^d)^d$).

So, for all $t \in]0,T]$,

$$z(t,\cdot) = \varepsilon B_\varepsilon z(t,\cdot) - \Delta B_\varepsilon z(t,\cdot) \to 0,$$

in the sense of distributions, as ε goes to zero. Therefore, $z \equiv 0$.

We proceed now with the proof of (3.2). We have the following identities, in the sense of distributions:

$$z(t,\cdot) = \int_0^t \Delta(az)(s,\cdot)ds = \int_0^t (\Delta - \varepsilon)(az)(s,\cdot)ds + \varepsilon \int_0^t (az)(s,\cdot)ds, \quad (3.3)$$

which implies

$$B_\varepsilon z(t,\cdot) \;=\; -\int_0^t (az)(s,\cdot)ds + \varepsilon \int_0^t B_\varepsilon(az)(s,\cdot)ds. \quad (3.4)$$

Let $\delta > 0$ and (ϕ_δ) a sequence of mollifiers converging to the Dirac delta function at zero. We set

$$z_\delta(t,x) = \int_{\mathbb{R}^d} z(t,y)\phi_\delta(x-y)dy.$$

Note that $z_\delta \in (L^1 \bigcap L^\infty)([0,T]\times\mathbb{R}^d)$. Moreover, (3.3), gives

$$z_\delta(t,\cdot) = \int_0^t \Delta(az)_\delta(s,\cdot)ds.$$

We suppose now Hypothesis(B1) (resp. (B2)). Let $t_0 = 0$ (resp. $t_0 > 0$). By assumption (B1)(ii) (resp. (B2)(iii)), we have $\Delta(az)_\delta \in L^2([t_0,T]\times\mathbb{R}^d)$. Thus, z_δ can be seen as a function belonging to $C([t_0,T];L^2(\mathbb{R}^d))$.

Besides, identities (3.3) and (3.4) lead to

$$z_\delta(t,\cdot) \;=\; z_\delta(t_0,\cdot) + \int_{t_0}^t (\Delta-\varepsilon)(az)_\delta(s,\cdot)ds + \varepsilon\int_{t_0}^t (az)_\delta(s,\cdot)ds, \quad (3.5)$$

$$B_\varepsilon z_\delta(t,\cdot) \;=\; B_\varepsilon z_\delta(t_0,\cdot) - \int_{t_0}^t (az)_\delta(s,\cdot)ds + \varepsilon\int_{t_0}^t B_\varepsilon(az)_\delta(s,\cdot)ds. \quad (3.6)$$

Now, proceeding through integration by parts with values in $L^2(\mathbb{R}^d)$, we get

$$\begin{aligned}\|z_\delta(t,\cdot)\|^2_{-1,\varepsilon} \;-\; \|z_\delta(t_0,\cdot)\|^2_{-1,\varepsilon} &= -2\int_{t_0}^t ds < z_\delta(s,\cdot),(az)_\delta(s,\cdot) >_{L^2} \\ &+\; 2\varepsilon\int_{t_0}^t ds < (az)_\delta(s,\cdot), B_\varepsilon z_\delta(s,\cdot) >_{L^2}. \end{aligned} \quad (3.7)$$

Then, letting δ go to zero, using assumptions (B1)(i)-(ii) (resp. (B2)(i) and (B2)(iii)) and Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}\|z(t,\cdot)\|^2_{-1,\varepsilon} \;-\; \|z(t_0,\cdot)\|^2_{-1,\varepsilon} &= -2\int_{t_0}^t ds \int_{\mathbb{R}^d} a(s,x)|z|^2(s,x)dx \\ &+\; 2\varepsilon\int_{t_0}^t ds < (az)(s,\cdot), B_\varepsilon z(s,\cdot) >_{L^2}. \end{aligned} \quad (3.8)$$

At this stage of the proof, we assume that **Hypothesis(B1)** is in force. Since $t_0 = 0$, we have $z(t_0, \cdot) = 0$. Using the inequality $c_1 c_2 \leq \frac{c_1^2 + c_2^2}{2}$, c_1, $c_2 \in \mathbb{R}$, and Cauchy-Schwarz, (3.8) implies

$$\begin{aligned} \|z(t,\cdot)\|^2_{-1,\varepsilon} &\leq -2\int_0^t ds \int_{\mathbb{R}^d} a(s,x)|z|^2(s,x)dx \\ &+ \varepsilon \int_0^t ds \|az(s,\cdot)\|^2_{L^2} + \varepsilon \int_0^t ds \|B_\varepsilon z(s,\cdot)\|^2_{L^2}, \\ &\leq \varepsilon \int_0^t ds \|az(s,\cdot)\|^2_{L^2} + \int_0^t ds \|z(s,\cdot)\|^2_{-1,\varepsilon}, \end{aligned} \tag{3.9}$$

because for $f = z(s,\cdot)$, we have

$$\begin{aligned} \varepsilon \|B_\varepsilon f\|^2_{L^2} &= \varepsilon \int_{\mathbb{R}^d} \frac{(\mathcal{F}(f))^2(\xi)}{(\varepsilon + |\xi|^2)^2} d\xi \\ &\leq \int_{\mathbb{R}^d} \frac{(\mathcal{F}(f))^2(\xi)}{\varepsilon + |\xi|^2} d\xi = \|f\|^2_{-1,\varepsilon}. \end{aligned}$$

We observe that the first integral of the right-hand side of (3.9) is finite by assumption (B1)(ii). Gronwall lemma applied to (3.9), gives

$$\|z(t,\cdot)\|^2_{-1,\varepsilon} \leq \varepsilon e^T \int_0^T ds \|az(s,\cdot)\|^2_{L^2}.$$

Letting $\varepsilon \to 0$, it follows that $\|z(t,\cdot)\|^2_{-1,\varepsilon} = 0$, $\forall t \in [0,T]$. This concludes the first part of the proof.

We suppose now that **Hypothesis(B2)** is in force, in particular $d = 1$. By Lemma 2.2 of [7], we have

$$\sup_x 2\varepsilon |B_\varepsilon z(s,x)| \leq \sqrt{\varepsilon} \|z(s,\cdot)\|_{\text{var}}.$$

Consequently, (3.8) gives

$$\|z(t,\cdot)\|^2_{-1,\varepsilon} - \|z(t_0,\cdot)\|^2_{-1,\varepsilon} \leq \sqrt{\varepsilon} \sup_{t \leq T} \|z(t,\cdot)\|_{\text{var}} \int\limits_{[t_0,T]\times\mathbb{R}} |az|(s,x) ds dx. \tag{3.10}$$

Besides, arguing like in the proof of Theorem 3.8 of [7], we obtain that

$$\lim_{t_0 \to 0} \|z(t_0,\cdot)\|^2_{-1,\varepsilon} = 0.$$

We first let $t_0 \to 0$ in (3.10), which implies

$$\|z(t,\cdot)\|^2_{-1,\varepsilon} \leq \sqrt{\varepsilon} \sup_{t \leq T} \|z(t,\cdot)\|_{\text{var}} \int\limits_{[0,T]\times\mathbb{R}} |az|(s,x) ds; \tag{3.11}$$

we remark that the right-hand side of (3.11) is finite by assumption (B2)(ii). Letting ε go to zero, the proof of (3.2) is finally established.

□

Remark 3.5. The validity of Theorem 3.1 holds with slight different assumptions. For instance, assumption (B1)(ii) can be replaced by

Assumption (B1)(ii)': For every $t \in]0,T]$,

$$(y_1,y_2) \mapsto \int_0^t z(s,y_1)(az)(s,y_2)ds \quad \text{is continuous on } D=\{(y_1,y_2)|y_1=y_2\}.$$

In the proof of Theorem 3.1, we exploit items (i) and (ii) of Hypothesis(B1). Under assumption (B1)(ii)', we proceed as follows.

We choose $\phi_\delta(y) = \frac{1}{\sqrt{2\pi}\delta^{\frac{d}{2}}} \exp(\frac{-y^2}{2\delta})$ as a sequence of mollifiers. For every $t_0 \geq 0$, we can prove that

$$\int_{t_0}^t ds < z_\delta(s,\cdot),(az)_\delta(s,\cdot) >_{L^2} \longrightarrow \int_{t_0}^t ds \int_{\mathbb{R}^d} (az^2)(s,x)dx, \tag{3.12}$$

when δ goes to zero. In fact, since $(az)_\delta \in L^2([0,T]\times\mathbb{R}^d)$, so the left-hand side of (3.12), gives

$$\begin{aligned}
&\int_{t_0}^t ds \int_{\mathbb{R}^d} z_\delta(s,x)(az)_\delta(s,x)dx,\\
&= \int_{t_0}^t ds \int_{\mathbb{R}^d} dx \int_{\mathbb{R}^d} z(s,y_1)\phi_\delta(x-y_1)dy_1 \int_{\mathbb{R}^d} (az)(s,y_2)\phi_\delta(x-y_2)dy_2,\\
&= \int_{t_0}^t ds \int_{\mathbb{R}^{2d}} dy_1dy_2 z(s,y_1)(az)(s,y_2) \int_{\mathbb{R}^d} \phi_\delta(x-y_1)\phi_\delta(x-y_2)dx,\\
&= \int_{t_0}^t ds \int_{\mathbb{R}^{2d}} dy_1dy_2 z(s,y_1)(az)(s,y_2)\phi_{2\delta}(y_1-y_2).
\end{aligned}$$

Since $dy_1dy_2\phi_\delta(y_1-y_2)$ weakly converges to $dy_1\delta_{y_2}(y_1-dy_2)$, previous expression converges to the right-hand side of (3.12), as $\delta \to 0$. Similar considerations can be formulated in order to replace item (iii) of Hypothesis(B2), when $t_0 > 0$.

4. Basic facts on the fast diffusion equation

We go on providing some properties of the Barenblatt solution $\mathcal{U}$ to (1.2) when $m \in (0,1)$ and given by (1.3)-(1.4).

Proposition 4.1.

(i) *$\mathcal{U}$ is a solution in the sense of distribution to* (1.2). *In particular, for every $\varphi \in \mathcal{S}(\mathbb{R})$, we have*

$$\int_{\mathbb{R}} \varphi(x)\mathcal{U}(t,x)dx = \varphi(0) + \int_0^t ds \int_{\mathbb{R}} \mathcal{U}^m(s,x)\varphi''(x)dx. \tag{4.1}$$

(ii)

$$\int_{\mathbb{R}} \mathcal{U}(t,x)dx = 1, \quad \forall t > 0. \tag{4.2}$$

In particular, for any $t > 0$, $\mathcal{U}(t,\cdot)$ is a probability density.

(iii) *The Dirac measure δ_0, is the initial trace of $\mathcal{U}$, in the sense that*

$$\int_{\mathbb{R}} \gamma(x)\mathcal{U}(t,x)dx \to \gamma(0), \text{ as } t \to 0, \tag{4.3}$$

for every $\gamma : \mathbb{R} \to \mathbb{R}$, continuous and bounded.

Proof of Proposition 4.1. See Appendix 7.2. □

Note that (4.2) allows to determine the explicit expression of the constant D.

Lemma 4.2.

(i) *Suppose that, $\frac{1}{3} < m < 1$. Then there is $p \geq 2$ and a constant $\mathcal{C}_p$ such that, for $0 \leq s < \ell \leq T$*

$$\int_{]s,\ell]\times\mathbb{R}} dtdx \, (\mathcal{U}(t,x))^{\frac{p(m-1)}{2}+1} \leq \mathcal{C}_p(\ell - s). \tag{4.4}$$

(ii) *In particular, if $p = 2$, we have*

$$\int_{]0,T]\times\mathbb{R}} dtdx \, (\mathcal{U}(t,x))^m < +\infty, \tag{4.5}$$

again when m belongs to $]\frac{1}{3}, 1[$.

(iii) If $\frac{1}{5} < m < 1$,

$$\int\limits_{]0,T]\times\mathbb{R}} dtdx\,(\mathcal{U}(t,x))^{2m} < +\infty. \tag{4.6}$$

(iv) If m *belongs to* $]\frac{3}{5}, 1[$ *, then*

$$\forall \kappa > 0,\ \int\limits_{\mathbb{R}} |x|^4 \mathcal{U}(\kappa, x)dx < +\infty. \tag{4.7}$$

Proof of Lemma 4.2.

(i) Using (1.3), we have

$$\int\limits_{]s,\ell]\times\mathbb{R}} (\mathcal{U}(t,x))^{\frac{p(m-1)}{2}+1} dtdx = \int\limits_{]s,\ell]\times\mathbb{R}} t^{\frac{-\alpha p(m-1)}{2}-\alpha}\big(D + \tilde{k}|x|^2 t^{-2\alpha}\big)^{\frac{p}{2}-\frac{1}{1-m}} dtdx.$$

Then, setting $y = t^{-\alpha}x\sqrt{\frac{\tilde{k}}{D}}$, we get

$$\begin{aligned}\int\limits_{]s,\ell]\times\mathbb{R}} (\mathcal{U}(t,x))^{\frac{p(m-1)}{2}+1} dtdx &= \frac{D^{\frac{p+1}{2}-\frac{1}{1-m}}}{\sqrt{\tilde{k}}} \int\limits_s^l t^{\frac{p}{2}\alpha(1-m)} dt \int_{\mathbb{R}} (1+y^2)^{\frac{p}{2}-\frac{1}{1-m}} dy \\ &\le \frac{D^{\frac{p+1}{2}-\frac{1}{1-m}}}{\sqrt{\tilde{k}}} T^{\frac{p}{2}\alpha(1-m)}(l-s) \int_{\mathbb{R}} (1+y^2)^{\frac{p}{2}-\frac{1}{1-m}} dy.\end{aligned}$$

Previous integral is finite if $(p+1)(1-m) < 2$. This implies (4.4).

(ii) is a particular case of (i) and (iii) follows by similar arguments as for the proof of (i).

(iv) Now, we assume that $m \in]\frac{3}{5}, 1[$. For $\kappa > 0$, we have

$$\int\limits_{\mathbb{R}} |x|^4 \mathcal{U}(\kappa, x)dx = \frac{D^{\frac{3-5m}{2(1-m)}}}{\tilde{k}^{5/2}} \kappa^{4\alpha} \int\limits_{\mathbb{R}} |y|^4 (1+y^2)^{-\frac{1}{1-m}} dy, \tag{4.8}$$

where, the previous equality was obtained setting $y = \kappa^{-\alpha}x\sqrt{\frac{\tilde{k}}{D}}$.

Clearly, since $m \in]\frac{3}{5}, 1[$, the integral in the right-hand side of (4.8) is finite . Therefore, (4.7), is fulfilled. □

Let $\kappa \in]0,T]$. Given $u : [0,T] \times \mathbb{R} \to \mathbb{R}$, we associate

$$\overline{u}(t,x) = u(t+\kappa, x), \quad (t,x) \in [0,T] \times \mathbb{R}. \tag{4.9}$$

In particular, we have

$$\overline{\mathcal{U}}(t,x) = \mathcal{U}(t+\kappa, x). \tag{4.10}$$

Moreover, we denote, for every $x \in \mathbb{R}$,

$$u_{0,\kappa}(x) = \mathcal{U}(\kappa, x). \tag{4.11}$$

Remark 4.3.

The function $\overline{\mathcal{U}}$ solves the problem,

$$\begin{cases} \partial_t u &= \partial^2_{xx}(u^m), \\ u(0,\cdot) &= u_{0,\kappa}, \end{cases} \tag{4.12}$$

5. The probabilistic representation of the fast diffusion equation

We are now interested in a non-linear stochastic differential equation rendering the probabilistic representation related to (1.2) and given by (1.6). Suppose for a moment that Y_0 is a random variable distributed according to δ_0, so, $Y_0 = 0$ a.s.

We recall that if there exists a process Y being a solution in law of (1.6), then Proposition 1.1 implies that u solves (1.2) in the sense of distributions.

In this subsection, we shall prove existence and uniqueness of solutions in law, for (1.6). In this respect, we first state a tool, given by Proposition 5.1 below, concerning the existence of an upper bound for the density law of some process Y, being the solution of a non-homogeneous SDE with unbounded coefficients. This result has an independent interest.

Proposition 5.1.

Let, $\sigma,\ b : [0,T] \times \mathbb{R} \to \mathbb{R}$, continuous (not necessarily bounded) such that $\sigma(t,\cdot)$, $b(t,\cdot)$ are smooth with bounded derivatives of orders greater or equal to one. σ is supposed to be non-degenerate.

Let $x_0 \in \mathbb{R}$. Let $Y_t = (Y_t^{x_0})_{t\in[0,T]}$, be the solution of

$$Y_t = x_0 + \int_0^t \sigma(r, Y_r) dW_r + \int_0^t b(r, Y_r) dr. \tag{5.1}$$

Then, for every $s > 0$, the law of Y_s admits a density denoted $p_s(x_0,\cdot)$.

Moreover, we have

$$\sup_{(s,x)\in]0,T]\times\mathbb{R}} p_s(x_0,x) \leq \frac{K}{\sqrt{s}} \left(1 + |x_0|^4\right), \tag{5.2}$$

where, K is a constant which depends on $\|\sigma'\|_\infty$, $\|b'\|_\infty$ and T but does not depend on x_0.

Remark 5.2.

(i) The proof of Proposition 5.1 above is given in Appendix 7.1.

(ii) If σ and b were bounded, the classical Aronson estimates imply that (5.2) holds even without the $|x_0|^4$ multiplicative term. If σ and b are unbounded, [1] provide an adaptation of Aronson's estimates; unfortunately first they considered time-homogeneous coefficients, and second their result does not imply (5.2).

Let Y_κ be a random variable distributed according to $u_{0,\kappa}$. Now, we are interested in the following result.

Proposition 5.3.

Assume that $m \in]\frac{3}{5}, 1[$. Let B be a classical Brownian motion independent of Y_κ. Then, there exists a unique (strong) solution $\overline{Y} = \overline{Y}_{t\in[0,T]}$, of

$$\begin{cases} \overline{Y_t} &= Y_\kappa + \int_0^t \Phi(\overline{\mathcal{U}}(s,\overline{Y_s}))dB_s, \\ \overline{\mathcal{U}}(t,\cdot) &= \textit{Law density of } \overline{Y_t}, \ \ \forall\, t \geq 0, \\ \overline{\mathcal{U}}(0,\cdot) &= u_{0,\kappa}, \end{cases} \tag{5.3}$$

In particular, pathwise uniqueness holds.

Corollary 5.4.

Let W be a classical Brownian motion independent of Y_κ. Therefore, there is a unique (strong) solution $Y^\kappa = Y^\kappa_{t\in[\kappa,T]}$, of

$$\begin{cases} Y^\kappa_t &= Y_\kappa + \int_\kappa^t \Phi(\mathcal{U}(s,Y^\kappa_s))dW_s, \\ \mathcal{U}(t,\cdot) &= \textit{Law density of } Y^\kappa_t, \ \ \forall\, t \geq \kappa, \\ \mathcal{U}(\kappa,\cdot) &= u_{0,\kappa}, \end{cases} \tag{5.4}$$

Proof of Corollary 5.4.

We start with the proof of uniqueness. Let $\kappa > 0$. We consider two solutions $Y^{\kappa,1}$ and $Y^{\kappa,2}$ of (5.4). Then , we set $\overline{Y^i_t} = Y^{\kappa,i}_{t+\kappa}$, $\forall t \geq 0$, $i = 1, 2$ and $B_t = W_{t+k} - W_t$, $\forall t \geq 0$.

Clearly, we get that $\overline{Y^1_t}$ and $\overline{Y^2_t}$ solve (5.3). Therefore, using Proposition 5.3, we deduce uniqueness for problem (5.4). Existence follows by similar arguments. □

Proof of Proposition 5.3.

Let W be a classical Brownian motion on some filtered probability space. Given the function $\overline{\mathcal{U}}$, defined in (4.10), we construct below a unique process $\overline{Y}$ being the strong solution of

$$\overline{Y_t} = \overline{Y_0} + \int_0^t \Phi(\overline{\mathcal{U}}(s,\overline{Y_s}))dW_s. \tag{5.5}$$

From (4.10), for every $(s,y) \in [0,T] \times \mathbb{R}$, we have

$$\Phi(\overline{\mathcal{U}}(s,y)) = \sqrt{2\overline{a}(s,y)},$$

where,

$$\overline{a}(s,y) = (s+\kappa)^{\alpha(1-m)}(D + \tilde{k}|y|^2(s+\kappa)^{-2\alpha}). \tag{5.6}$$

In fact, $\Phi(\overline{\mathcal{U}})$ is continuous with all its space derivatives of order greater or equal to one being bounded; in particular $\Phi(\overline{\mathcal{U}})$ is Lipschitz and it has linear growth. Therefore, (5.5) admits a strong solution.

By Lemma 1.2, the function $t \mapsto \rho(t,\cdot)$ from $[0,T]$ to $\mathcal{M}_+(\mathbb{R})$, where $\rho(t,\cdot)$ is the law of $\overline{Y_t}$, is a solution to

$$\begin{cases} \partial_t \rho &= \partial^2_{xx}(\overline{a}\rho), \\ \rho(0,\cdot) &= u_{0,\kappa}. \end{cases} \tag{5.7}$$

To conclude it remains to prove that $\overline{\mathcal{U}}(t,y)dy$ is the law of $\overline{Y_t}$, $\forall t \in [0,T]$; in particular the law of the r.v $\overline{Y_t}$ admits a density. For this we will apply Theorem 3.1, for which we need to check the validity of Hypothesis(B2) when $a = \overline{a}$ and for $z := z_1 - z_2$, where $z_1 := \rho$ and $z_2 := \overline{\mathcal{U}}$. By additivity, this will be of course fulfilled if we prove it separately for $z := \rho$ and $z := \overline{\mathcal{U}}$, which are both solutions to (5.7).

Since $\overline{a}$ is non-degenerate, by Remark 3.4(1), we only need to check items (ii) and (iii) of the mentioned Hypothesis(B2).

On one hand, since $\overline{a}(s,y) = \overline{\mathcal{U}}^{m-1}(s,y)$, $z := \overline{\mathcal{U}}$ verifies Hypothesis(B2) because of items (ii) and (iii) of Lemma 4.2.

On the other hand, since $\sqrt{\overline{a}}$ has linear growth, by Remark 3.4.(4), ρ fulfills item (ii) of Hypothesis(B2). Moreover, by Lemma 5.5 below, ρ verifies also item (iii) of Hypothesis(B2). Finally, Theorem 3.1 implies that $\overline{\mathcal{U}} \equiv \rho$.

□

Lemma 5.5.

Let $\psi : [0,T] \times \mathbb{R} \to \mathbb{R}_+$, continuous (not necessarily bounded) such that $\psi(t,\cdot)$, is smooth with bounded derivatives of orders greater or equal to one. ψ is supposed to be non-degenerate.

We consider a stochastic process $X = (X_t)_{t\in[0,T]}$, being a strong solution of the SDE

$$X_t = X_0 + \int_0^t \psi(s, X_s) dW_s, \tag{5.8}$$

where, X_0 is a random variable distributed according to $u_{0,\kappa}$, defined in (4.11) *with $m \in]\frac{3}{5}, 1[$.*

For $t \in]0,T]$, the law of X_t has a density $\nu(t,\cdot)$ such that, $(\psi^2\nu)(t,x)$ belongs to $L^2([t_0,T]\times\mathbb{R})$, for every $t_0 > 0$.

Proof of Lemma 5.5.

If $X_0 = x_0$, where x_0 is a real number; then Proposition 5.1, implies that for every $t \in]0,T]$, the law of X_t admits a density $p_t(x_0,\cdot)$. Consequently, the unique strong solution of (5.8) has a density for each $t>0$, $\nu(t,\cdot)$, given by

$$\nu(t,x) = \int_{\mathbb{R}} u_{0,\kappa}(x_0) p_t(x_0,x) dx_0.$$

By (5.2) in Proposition 5.1, it follows

$$\sup_{(t,x)\in[t_0,T]\times\mathbb{R}} p_t(x_0,x) \leq K_0(1+|x_0|^4), \quad \text{where, } K_0 = \frac{K}{\sqrt{t_0}}. \tag{5.9}$$

In this proof we will use constants $K_0,\dots,K_5$ which only depend on t_0, T and ψ.

Using (5.9), we get

$$\sup_{(t,x)\in[t_0,T]\times\mathbb{R}} |\nu(t,x)| \leq K_0 \int_{\mathbb{R}} (1+|x_0|^4)\mathcal{U}(\kappa,x_0)dx_0 \leq K_1; \tag{5.10}$$

the latter inequality is valid because of (4.7) in Lemma 4.2.

Furthermore,

$$\int_{[t_0,T]\times\mathbb{R}} ((\psi^2\nu)(t,x))^2\, dtdx \leq \sup_{(t,x)\in[t_0,T]\times\mathbb{R}} |\nu(t,x)| \mathbb{E}\left[\int_0^T \psi^4(t,X_t)dt\right].$$

Since ψ has linear growth, previous expression is bounded by

$$\begin{aligned} & K_2 \sup_{(t,x)\in[t_0,T]\times\mathbb{R}} |\nu(t,x)| \left(T + \mathbb{E}\left[\int_0^T |X_t|^4 dt\right]\right) \\ \leq\ & K_1K_2T\left(1 + \int_0^T \mathbb{E}\left[\sup_{t\in[0,T]}|X_t|^4\right] dt\right). \end{aligned} \tag{5.11}$$

(5.11), follows because of (5.10).

Besides, by Burkholder-Davis-Gundy's and Jensen's inequalities, it follows that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|X_t|^4\right] \leq K_3\left(\mathbb{E}\left[|X_0|^4\right] + \int_0^T \mathbb{E}\left[\psi^4(s,X_s)\right] ds\right).$$

Using again the linear growth of ψ, we get

$$\mathbb{E}\left[\sup_{t\in[0,T]}|X_t|^4\right] \leq K_4\left(\mathbb{E}\left[|X_0|^4\right] + \int_0^T \mathbb{E}\left[\sup_{s\in[0,T]}|X_s|^4\right] ds + T\right).$$

Then, by Gronwall's lemma, there is a further constant K_5, such that

$$\mathbb{E}\left[\sup_{t\in[0,T]}|X_t|^4\right] \leq K_5\left(1+\int_{\mathbb{R}}|x_0|^4\mathcal{U}(\kappa,x_0)dx_0\right). \tag{5.12}$$

Finally, (5.11), (5.12) and (5.10), allow to conclude the proof. □

We are now ready to provide the probabilistic representation related to the solution $\mathcal{U}$. For this we only have a solution in law of (1.6).

Definition 5.6. We say that (1.6) admits a weak (in law) solution if there is a probability space $(\Omega,\mathcal{F},\mathbb{P})$, a Brownian motion $(W_t)_{t\geq 0}$ and a process $(Y_t)_{t\geq 0}$ such that, the system (1.6) holds. (1.6), admits uniqueness in law if given (W^1,Y^1), (W^2,Y^2) solving (1.6) on some related probability space, it follows that Y^1 and Y^2 have the same law.

Theorem 5.7.

Assume that $m\in]\frac{3}{5},1[$. Then, there is a unique weak solution (in law) , Y, of the problem (1.6).

Remark 5.8. Indeed the assumption on $m\in]\frac{3}{5},1[$ is only required for the application of Theorem 3.1. The arguments under the present proof only use $m>\frac{1}{3}$.

Proof of Theorem 5.7.

First we start with the existence of a weak solution for (1.6). Let $\mathcal{U}$ be again the solution of (1.2). We consider the solution $(Y^\kappa_t)_{t\in[\kappa,T]}$ provided by Corollary 5.4 extended to $[0,\kappa]$, setting $Y^\kappa_t=Y_\kappa$, $t\in[0,\kappa]$.

We prove that the laws of the processes Y^κ are tight. For this, we implement the classical Kolmogorov's criterion, see Problem 4.11, Section 2.4 of [24], for instance. We will show the existence of $p>2$ such that

$$\mathbb{E}\left[|Y^\kappa_t-Y^\kappa_s|^p\right]\leq \mathcal{C}_p|t-s|^{\frac{p}{2}},\quad \forall s,t\in[0,T], \tag{5.13}$$

where $\mathcal{C}_p$ will stand for a constant (not always the same), depending on p but not on κ.

Let $s,t\in]0,T]$. Let $p>2$. By Burkholder-Davis-Gundy's inequality, we obtain

$$\mathbb{E}\left[|Y^\kappa_t-Y^\kappa_s|^p\right]\leq \mathcal{C}_p\mathbb{E}\left[\left|\int_s^t\Phi^2(\mathcal{U}(r,Y^\kappa_r))dr\right|^{\frac{p}{2}}\right].$$

Then, using Jensen's inequality and the fact that $\mathcal{U}(r,\cdot)$ is the law density of Y^κ_r, $r\geq\kappa$, we get

$$\mathbb{E}\left[|Y^\kappa_t-Y^\kappa_s|^p\right]\leq \mathcal{C}_p|t-s|^{\frac{p}{2}-1}\int_s^t dr\int_{\mathbb{R}}\Phi^p(\mathcal{U}(r,y))\mathcal{U}(r,y)dy. \tag{5.14}$$

We have,

$$\int_s^t dr \int_{\mathbb{R}} \Phi^p(\mathcal{U}(r,y))\mathcal{U}(r,y)dy = \int_s^t dr \int_{\mathbb{R}} dy\, (\mathcal{U}(r,y))^{\frac{p(m-1)}{2}+1},$$

and by Lemma 4.2 (i), the result follows.

Consequently, there is a subsequence $Y^n := Y^{\kappa_n}$, converging in law (as $C([0,T])-$valued random elements) to some process Y. Let P^n the corresponding laws on the canonical space $\mathbf{\Omega} = C([0,T])$ equipped with the Borel σ-field. Y will denote the canonical process $Y_t(\omega) = \omega(t)$. Let P the weak limit of (P^n).

1) We first observe, that the marginal laws of Y under P^n converge to the marginal law of Y under P. Let $t \in]0,T]$. If the sequence (κ_n) is lower than t, the law of Y_t under P^n equals the constant law $\mathcal{U}(t,x)dx$. Consequently, for every $t \in]0,T]$, the law of Y_t under P is $\mathcal{U}(t,x)dx$.

2) We need now to prove that Y is a (weak) solution of (1.6), under P. By similar arguments as for the classical stochastic differential equations, see [33], Chapter 6, it is enough to prove that Y (under P) fulfills the martingale problem i.e., for every $f \in C_b^2(\mathbb{R})$, the process

$$\textbf{(MP)} \quad f(Y_t) - f(0) - \frac{1}{2}\int_0^t f''(Y_s)\Phi^2(\mathcal{U}(s,Y_s))ds,$$

is an $(\mathcal{F}_s)$-martingale, where $(\mathcal{F}_s)$ is the canonical filtration associated with Y.

Let $\mathbb{E}$ (resp. $\mathbb{E}^n$) the expectation operator with respect to P (resp. P^n). Let $s,t \in [0,T]$, with $s<t$ and $R = R(Y_r, r \le s)$ an $\mathcal{F}_s-$measurable, bounded and continuous random variable. In order to show the martingale property **(MP)** of Y, we have to prove that

$$\mathbb{E}\left[\left(f(Y_t) - f(Y_s) - \frac{1}{2}\int_s^t f''(Y_r)\Phi^2(\mathcal{U}(r,Y_r))dr\right)R\right] = 0, \quad f \in C_b^2(\mathbb{R}). \tag{5.15}$$

We first consider the case when $s > 0$. There is $n \ge n_0$, such that $\kappa_n < s$. Let $f \in C_b^2(\mathbb{R})$, since $(Y_s)_{s\ge\kappa_n}$ under P^n are still martingales, we have

$$\mathbb{E}^n\left[\left(f(Y_t) - f(Y_s) - \frac{1}{2}\int_s^t f''(Y_r)\Phi^2(\mathcal{U}(r,Y_r))dr\right)R\right] = 0. \tag{5.16}$$

We are able to prove that (5.15) follows from (5.16). Let $\varepsilon > 0$ and $N > 0$ such that

$$\int_s^t dr \int_{\{|y|>\frac{N}{C}-1\}} \mathcal{U}^m(r,y)dy \le \varepsilon, \tag{5.17}$$

where, C is the linear growth constant of $\Phi^2 \circ \mathcal{U}$ in the sense of Definition 2.2.

In order to conclude passing to the limit in (5.16), we will only have to show that

$$\lim_{n\to+\infty} \mathbb{E}^n\left[F(Y)\right] - \mathbb{E}\left[F(Y)\right] = 0, \tag{5.18}$$

where,

$$F(\ell) = \int_s^t dr \Phi^2(\mathcal{U}(r,\ell(r))R(\ell(r), r \le s).$$

$F : C([0,T]) \to \mathbb{R}$ is continuous but not bounded.

The left-hand side of (5.18) equals

$$\mathbb{E}^n\left[F(Y) - F^N(Y)\right] + \mathbb{E}^n\left[F^N(Y)\right] - \mathbb{E}\left[F^N(Y)\right] + \mathbb{E}\left[F^N(Y) - F(Y)\right], \tag{5.19}$$

$$:= \mathcal{E}_1(n,N) + \mathcal{E}_2(n,N) + \mathcal{E}_3(n,N),$$

where,

$$F^N(\ell) = \int_s^t dr \left(\Phi^2(\mathcal{U}(r,\ell(r)) \wedge N\right) R(\ell(r), r \le s).$$

Since $\kappa_n < s$, for N large enough, we get

$$\begin{aligned}
|\mathcal{E}_1(n,N)| &\le \|R\|_\infty \int_s^t dr \int_{\{\Phi^2(\mathcal{U}(r,y))\ge N\}} \left(\Phi^2(\mathcal{U}(r,y) - N\right)\mathcal{U}(r,y)dy \\
&\le \|R\|_\infty \int_s^t dr \int_{\{|y|>\frac{N}{C}-1\}} \mathcal{U}^m(r,y)dy \\
&\le \varepsilon\|R\|_\infty,
\end{aligned} \tag{5.20}$$

taking into account (5.17) and the second item of Lemma 4.2.

For fixed N chosen in (5.17), we have

$$\lim_{n\to+\infty} \mathcal{E}_2(n,N) = 0,$$

since F^N is bounded and continuous.

Again, since the law density of Y_t, $t \ge s$ under P is $\mathcal{U}(t,\cdot)$, similarly as for (5.20), we obtain

$$|\mathcal{E}_3(n,N)| \le \varepsilon\|R\|_\infty.$$

Finally, coming back to (5.19),

$$\limsup_{n\to+\infty} |\mathbb{E}^n [F(Y)] - \mathbb{E} [F(Y)]| \leq 2\varepsilon \|R\|_\infty;$$

since $\varepsilon > 0$ is arbitrary, (5.18) is established. So (5.15) is verified for $s > 0$.

3) Now, we consider the case when $s = 0$. We first prove that

$$\mathbb{E}\left[\int_0^T \Phi^2(\mathcal{U}(r, Y_r))dr\right] < +\infty. \tag{5.21}$$

By item 1) of this proof, the law of Y_r, $r > 0$ admits $\mathcal{U}(r, \cdot)$ as density.

Consequently, the left-hand side of (5.21) gives,

$$\int_{]0,T]} dr \int_{\mathbb{R}} \Phi^2(\mathcal{U}(r, y))\mathcal{U}(r, y)dy = \int_0^T dr \int_{\mathbb{R}} \mathcal{U}^m(r, y)dy,$$

which is finite, by the second item of Lemma 4.2.

Coming back to (5.15), we can now let s go to zero. Since, Y is continuous and f is bounded we clearly have

$$\lim_{s\to 0} \mathbb{E} [f(Y_s)R] = \mathbb{E} [f(Y_0)R] .$$

Moreover,

$$\lim_{s\to 0} \mathbb{E}\left[\left(\int_s^t f''(Y_r)\Phi^2(\mathcal{U}(r, Y_r))dr\right) R\right] = \mathbb{E}\left[\left(\int_0^t f''(Y_r)\Phi^2(\mathcal{U}(r, Y_r))dr\right) R\right],$$

using Lebesgue's dominated convergence theorem and (5.21).

Consequently, we obtain

$$\mathbb{E}\left[\left(f(Y_t) - f(Y_0) - \frac{1}{2}\int_0^t f''(Y_r)\Phi^2(\mathcal{U}(r, Y_r))dr\right) R\right] = 0. \tag{5.22}$$

It remains to show that $Y_0 = 0$ a.s. This follows because $Y_t \to Y_0$ a.s., and also in law to δ_0 by the third item of Proposition 4.1.

Finally, we have shown that the limiting process Y verifies **(MP)**, which proves existence of solutions to (1.6).

4) We prove now uniqueness. Since $\mathcal{U}$ is fixed, our problem reduces to show uniqueness for the first line equation of (1.6).

Let $(Y_t^i)_{t\in[0,T]}$, $i=1,2$, be two solutions. In order to show that the laws of Y^1 and Y^2 are identical, according to Lemma 2.5 in [21], we will verify that their finite marginal distributions are the same.

For this, we consider $0=t_0<t_1<\ldots<t_N=T$. Let $0<\kappa<t_1$. Obviously, we have $Y_{t_0}^i=0$, a.s., in the corresponding probability space, $\forall i\in\{1,2\}$.

Let $\kappa>0$. Both restrictions $Y^1|_{[\kappa,T]}$ and $Y^2|_{[\kappa,T]}$ verify (5.4). Since that equation admits pathwise uniqueness, it also admits uniqueness in law by Yamada-Watanabe theorem. Consequently, $Y^1|_{[\kappa,T]}$ and $Y^2|_{[\kappa,T]}$ have the same law and in conclusion the law of $(Y_{t_1}^1,\ldots,Y_{t_N}^1)$ coincides with the law of $(Y_{t_1}^2,\ldots,Y_{t_N}^2)$. □

6. Numerical experiments

In order to avoid singularity problems due to the initial condition being a Dirac delta function, we will consider a time translation of $\mathcal{U}$, denoted v and defined by

$$v(t,\cdot)=\mathcal{U}(t+1,\cdot),\ \ \forall t\geq 0.$$

Obviously, v still solves equation (1.2), for $m\in(0,1)$, but now with a smooth initial data given by

$$v_0(x)=\mathcal{U}(1,x),\quad \forall x\in\mathbb{R}. \tag{6.1}$$

Indeed, we have the following formula

$$v(t,x)=(t+1)^{-\alpha}\big(D+\tilde{k}|x|^2(t+1)^{-2\alpha}\big)^{-\frac{1}{1-m}}, \tag{6.2}$$

where, α, $\tilde{k}$ and D are still given by (1.4).

We wish now to compare the exact solution of problem (1.2), to a numerical probabilistic solution. In fact, in order to perform such approximated solutions, we use the algorithm described in Sections 4 of [4]. We recall that, this method was implemented in Matlab. We focus on the case $m=\frac{1}{2}$. In that case, the exact solution v of the PDE (1.2), with initial condition v_0 is given by the following explicit formula: ,

$$v(t,x)=(t+1)^{-\frac{2}{3}}\left(D+\frac{x^2}{3(t+1)^{\frac{4}{3}}}\right)^{-2},\ \ (t,x)\in[0,T]\times\mathbb{R}, \tag{6.3}$$

where $D=\left(\frac{\pi\sqrt{3}}{2}\right)^{\frac{2}{3}}$.

The first step of the simulation concerns the initial condition. In order to perform numerical experiments, we need to simulate random variables according to the density function v_0 defined below,

$$v_0(x)=\left(D+\frac{|x|^2}{3}\right)^{-2}, \tag{6.4}$$

where, D is given in (6.3). For this, we apply a rejection sampling method based on the fact that

$$\forall x \in \mathbb{R}, \;\; v_0(x) \leq kq(x), \;\; \text{where, } k = \pi\sqrt{3}D^{-3/2}, \tag{6.5}$$

and q is the density probability function of a Cauchy law with parameter $\sqrt{3D}$, denoted $\mathcal{C}(\sqrt{3D})$, for which it is well-known that we can generate observations using the inverse transform sampling technique. In fact we have,

$$X = \sqrt{3D}\tan(\pi(U - 1/2)) \sim \mathcal{C}(\sqrt{3D}), \tag{6.6}$$

where, U is a Uniform random variable on $[0, 1]$. For more information about those simulation techniques, one can consult [23], for instance.

Simulation experiments: we compute the numerical solution over the time-space grid $[0, 1.5] \times [-15, 15]$. We use $n = 50000$ particles and a time step $\Delta t = 2 \times 10^{-4}$. Figures 1.(a)-(b)-(c)-(d), display the exact and the numerical solutions at times $t = 0$, $t = 0.5$, $t = 1$ and $t = T = 1.5$, respectively. The exact solution for the fast diffusion equation (1.2), given in (6.3), is depicted by solid lines. Besides, Figure 1.(e) describes the time evolution of the discrete L^2 error on the time interval $[0, 1.5]$.

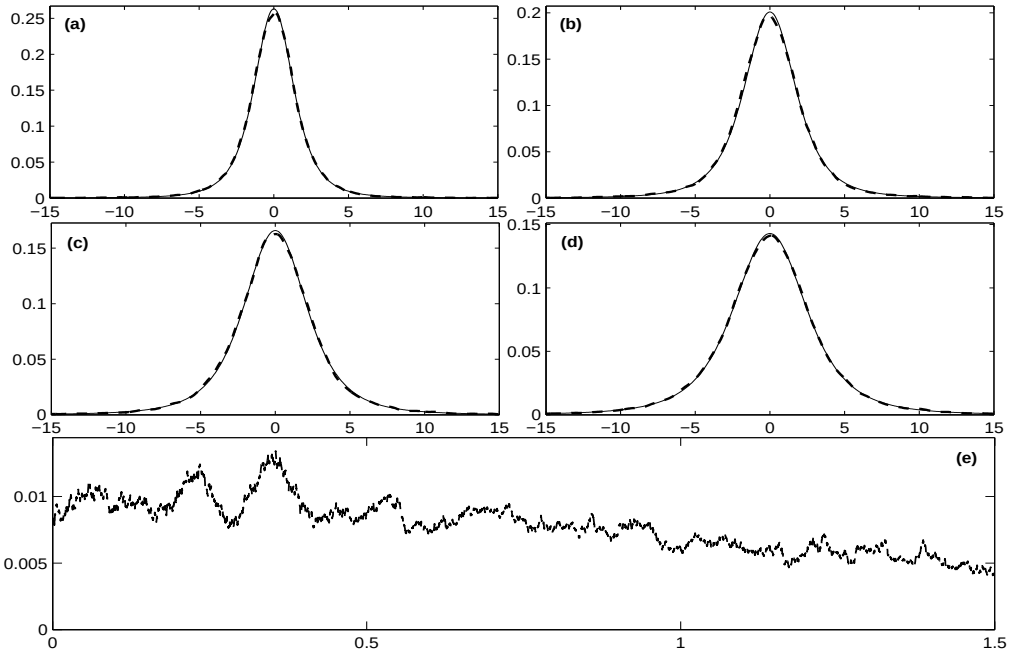

Figure 1. Numerical (dashed line) and exact solutions (solid line) values at t=0 (a), t=0.5 (b), t=1 (c) and t=1.5 (d). The evolution of the L^2 error over the time interval $[0, 1.5]$ (e).

7. Appendix

7.1. Proof of Proposition 5.1

We start with some notations for Malliavin calculus. The set $\mathbb{D}^{\infty}$, represents the classical Sobolev-Malliavin space of smooth test random variables. $\mathbb{D}^{1,2}$ is defined in the lines after Lemma 1.2.2 of [28] and $\mathbb{L}^{1,2}$ is introduced in Definition 1.3.2 of [28]. See also [26] for a complete monograph on Malliavin calculus.

Now we state a preliminary result.

Proposition 7.1.

Let N be a non-negative random variable. Suppose the existence of the constants $C(p)$ and $\epsilon_0(p)$, for every $p \geq 1$, such that

$$\mathbb{P}(N \leq \epsilon) \leq C(p).\epsilon^{p+1}, \quad \forall \epsilon \in]0, \epsilon_0(p)]. \tag{7.1}$$

Then, for every $p \geq 1$,

$$\mathbb{E}(N^{-p}) \leq \epsilon_0(p).C(p+1) + \epsilon_0(p)^{-p}.\mathbb{P}(N > \epsilon_0(p)). \tag{7.2}$$

Proof of Proposition 7.1.

Let $p \geq 1$ and $\epsilon_0(p) > 0$. Setting, $F(x) = \mathbb{P}(N \leq x)$ for every $x \in \mathbb{R}_+$, we have

$$\mathbb{E}(N^{-p}) = I_1 + I_2, \tag{7.3}$$

where,

$$I_1 = \int_0^{\epsilon_0(p)} x^{-p} dF(x), \text{ and } I_2 = \int_{\epsilon_0(p)}^{+\infty} x^{-p} dF(x).$$

(7.1) implies that I_1 and I_2 are well-defined. Indeed on one hand, applying integration by parts on I_1, we get

$$I_1 = \left[x^{-p} F(x)\right]_0^{\epsilon_0(p)} + p \int_0^{\epsilon_0(p)} x^{-p-1} F(x) dx;$$

moreover, there is a constant $C(p)$ such that

$$I_1 \leq (p+1)\epsilon_0(p) C(p). \tag{7.4}$$

On the other hand, again (7.1) says that

$$I_2 \leq \epsilon_0(p)^{-p}(1 - F(\epsilon_0(p))). \tag{7.5}$$

Consequently, using (7.4) and (7.5) and coming back to (7.3), (7.2) is established. □

Proof of Proposition 5.1.

In this proof σ' (resp. b') stands for $\partial_x\sigma$ (resp. $\partial_x b$). Let $Y = (Y_t^{x_0})_{t\in[0,T]}$, be the solution of (5.1). According to Theorem 2.2.2 of [28], we have $Y_s \in \mathbb{D}^\infty$, $\forall s \in [0,T]$. Let $s > 0$. Since σ is non-degenerate, by Theorem 2.3.1 of [28], the law of Y_s admits a density that we denote by $p_s(x_0,\cdot)$.

The second step consists in a re-scaling, transforming the time s into a noise multiplicative parameter λ; we set $\lambda = \sqrt{s}$. Indeed, (Y_t) is distributed as $(Y^\lambda_{\frac{t}{\lambda^2}})$, where

$$Y_t^\lambda = x_0 + \lambda \int_0^t \sigma(r\lambda^2, Y_r^\lambda) dW_r + \lambda^2 \int_0^t b(r\lambda^2, Y_r^\lambda) dr.$$

In particular, $Y_s \sim Y_1^\lambda$. By previous arguments, $Y_t^\lambda \in \mathbb{D}^\infty$, for every $t > 0$; it admits a density denoted $p_t^\lambda(x_0,\cdot)$.

Our aim consists in showing the existence of a constant K, such that

$$\sup_{y\in\mathbb{R},\lambda\in]0,\sqrt{T}]} p_1^\lambda(x_0,y) \le \frac{K}{\lambda}(1+|x_0|^4), \tag{7.6}$$

where, K is a constant which does not depend on x_0 and λ.

In fact, we will prove that, for every $\lambda \in]0,\sqrt{T}]$

$$\sup_{y\in\mathbb{R},t\in]0,1]} p_t^\lambda(x_0,y) \le \frac{K}{\lambda}(1+|x_0|^4). \tag{7.7}$$

We set

$$Z_t^\lambda = \frac{Y_t^\lambda - x_0}{\lambda}, \;\; t \in [0,1],$$

so that, the density q_t^λ of Z_t^λ fulfills

$$q_t^\lambda(z) = \lambda p_t^\lambda(x_0, \lambda z + x_0), \quad (t,z) \in [0,1]\times\mathbb{R}.$$

In fact, we will have attained (7.7), if we show

$$\sup_{z\in\mathbb{R},\lambda\in]0,\sqrt{T}]} q_t^\lambda(z) \le K(1+|x_0|^4), \; t\in]0,1]. \tag{7.8}$$

We express the equation fulfilled by Z; it yields

$$Z_t^\lambda = \int_0^t \sigma^\lambda(r, Z_r^\lambda) dW_r + \int_0^t b^\lambda(r, Z_r^\lambda) dr, \tag{7.9}$$

where, for every $(r,z) \in [0,1]\times\mathbb{R}$, we set

$$\sigma^\lambda(r,z) = \sigma(r\lambda^2, \lambda z + x_0), \;\; \text{and } b^\lambda(r,z) = \lambda b(r\lambda^2, \lambda z + x_0).$$

At this stage, we state the following lemma.

Lemma 7.2.
For $\lambda \in]0,1]$, we set $Z_t = (Z_t^\lambda)_{t\in[0,T]}$, as the solution of (7.9)*. Then, for every $\gamma \geq 1$, we have*

$$\sup_{\lambda\in]0,\sqrt{T}]} \mathbb{E}\left[\sup_{t\in[0,1]} |Z_t|^\gamma\right] \leq C(1+|x_0|^\gamma), \tag{7.10}$$

where, C is a constant depending on $\|\sigma'\|_\infty$, $\|b'\|_\infty$ and T, but which does not depend on x_0.

Remark 7.3.

(i) For simplicity in the whole proof of Proposition 5.1, we will set $T = 1$.

(ii) Since there is no more ambiguity, we will use again the letter s in the considered integrals.

Proof of Lemma 7.2.
Let $\lambda \in]0,1]$ and $\gamma \geq 1$. In the proof C_1 is a constant depending on T, and C_2, C_3 depend on T, $\|\sigma'\|_\infty$ and $\|b'\|_\infty$. Using Burkholder-Davis-Gundy and Jensen inequalities, we get

$$\mathbb{E}\left[\sup_{\rho\in[0,t]} |Z_\rho|^\gamma\right] \leq C_1\left(\int_0^t \mathbb{E}\left[|\sigma(s\lambda^2, \lambda Z_s + x_0)|^\gamma\right] ds + \int_0^t \mathbb{E}\left[|\lambda b(s\lambda^2, \lambda Z_s + x_0)|^\gamma\right] ds\right).$$

Since σ' and b' are bounded, σ and b have linear growth. Therefore, previous expression is bounded by

$$C_2(1+\lambda^\gamma)\left(1 + |x_0|^\gamma + \lambda^\gamma \int_0^t \mathbb{E}\left[\sup_{\rho\in[0,s]} |Z_\rho|^\gamma\right] ds\right).$$

Since $\lambda \in]0,1]$, we obtain

$$\mathbb{E}\left[\sup_{\rho\in[0,t]} |Z_\rho|^\gamma\right] \leq C_3\left(1 + |x_0|^\gamma + \int_0^t \mathbb{E}\left[\sup_{\rho\in[0,s]} |Z_\rho|^\gamma\right] ds\right),$$

Consequently, using Gronwall's lemma, the result follows. □

Now, in order to perform (7.8), we make use of Malliavin calculus deriving in the sense of Malliavin the expression (7.9). Omitting λ in the notation Z_t^λ, we get

$$D_r Z_t = \sigma(r\lambda^2, \lambda Z_r + x_0)\mathbb{1}_{[r,1]}(t) + \lambda \int_r^t \sigma'(s\lambda^2, \lambda Z_s + x_0) D_r Z_s dW_s + \lambda^2 \int_r^t b'(s\lambda^2, \lambda Z_s + x_0) D_r Z_s ds.$$

Consequently,

$$D_r Z_t = \sigma(r\lambda^2, \lambda Z_r + x_0)\mathcal{E}\left(\lambda \int_r^t \sigma'(s\lambda^2, \lambda Z_s + x_0)dW_s + \lambda^2 \int_r^t b'(s\lambda^2, \lambda Z_s + x_0)ds\right),\ r < t,$$

where, $\mathcal{E}(S)$ denotes the Doléans exponential of the continuous semi-martingale,

$$S_t = \lambda \int_r^t \sigma'(s\lambda^2, \lambda Z_s + x_0)dW_s + \lambda^2 \int_r^t b'(s\lambda^2, \lambda Z_s + x_0)ds,\ t \in [0,1].$$

We recall that, $\mathcal{E}(S_t) = \exp(S_t - \frac{1}{2}[S]_t)$. Consequently, for fixed $t \in]0,1]$, we have

$$< DZ_t, DZ_t >= \int_0^t \sigma^2(r\lambda^2, \lambda Z_r + x_0)\mathcal{E}^2(\lambda; r, t)dr, \tag{7.11}$$

where,

$$\mathcal{E}(\lambda; r, t) = \exp\left(\lambda \int_r^t \sigma'(s\lambda^2, \lambda Z_s + x_0)dW_s + \lambda^2 \int_r^t (b' - \frac{(\sigma')^2}{2})(s\lambda^2, \lambda Z_s + x_0)ds\right). \tag{7.12}$$

We set, for every $s \leq t$,

$$G(\lambda, s) = \frac{D_s Z_t}{< DZ_t, DZ_t >}.$$

In view of the application of Proposition 2.1.1 in [28], which implies the useful expression (7.26) for the density of Z_t, we will need to show that G belongs to the domain of the divergence operator δ, denoted by, $Dom\ \delta$. It will be the case if $G \in \mathbb{L}^{1,2}(H)$, with $H = L^2([0,T])$. In fact, by the lines after Definition 1.3.2 in [28], we know that $\mathbb{L}^{1,2} \subset Dom\ \delta$.

Since $Z_t \in \mathbb{D}^\infty$, we can deduce that $\frac{1}{< DZ_t, DZ_t >}$ belongs to $\mathbb{D}^\infty$, provided that we prove, that

$$\frac{1}{< DZ_t, DZ_t >} \in L^p(\mathbf{\Omega}),\ \ \forall p \geq 1, \tag{7.13}$$

see Lemma 2.1.6 of [28]. Since $\mathbb{D}^\infty$ is an algebra; $G(\lambda, s) \in \mathbb{D}^\infty$ for $s \in]0,T]$ and so $G(\lambda, s) \in \mathbb{D}^{1,2}$. (7.13), will be the object of Proposition 7.5. According to Definition 1.3.2 of [28], to affirm that $G(\lambda, \cdot)$ belongs to $\mathbb{L}^{1,2}$ it remains to show the existence of a measurable version of $D_{s_1}G(\lambda, s)$, $(s_1, s) \in ([0,t])^2$, such that

$$\mathbb{E}\left[\int_{[0,t]^2} (D_{s_1}G(\lambda, s))^2 ds_1 ds\right] < +\infty. \tag{7.14}$$

We first state the following Lemma.

Lemma 7.4.
For every $q > 1$, there exists a constant $C_0(q)$, such that

$$\sup_{0<r\leq t\leq 1,\lambda\in]0,1]} \mathbb{E}\left[(\mathcal{E}(\lambda;r,t))^q\right] \leq C_0(q). \tag{7.15}$$

Proof of Lemma 7.4.
Let $\lambda \in]0,1]$ and $q > 1$. For fixed, $0 < r \leq t \leq 1$, (7.12), gives

$$\mathcal{E}^q(\lambda;r,t) = M(\lambda;r,t,q).\exp\left(\frac{\lambda^2(q^2-q)}{2}\int_r^t (\sigma')^2(\rho\lambda^2,\lambda Z_\rho + x_0)d\rho + q\lambda^2\int_r^t b'(\rho\lambda^2,\lambda Z_\rho + x_0)d\rho\right),$$

where,

$$M(\lambda;r,t,q) = \exp\left(\int_r^t \lambda q\sigma'(\rho\lambda^2,\lambda Z_\rho + x_0)dW_\rho - \frac{1}{2}\int_r^t (q\lambda\sigma')^2(\rho\lambda^2,\lambda Z_\rho + x_0)d\rho\right). \tag{7.16}$$

In fact, since σ' is bounded, the stochastic exponential $M(\lambda;r,t,q)$ verifies Novikov's condition; therefore it is a martingale. So, $\mathbb{E}(M(\lambda;r,t,q)) = 1$.

In addition, since b' is also bounded and $\lambda \in]0,1]$, we get

$$\mathbb{E}\left[(\mathcal{E}(\lambda;r,t))^q\right] \leq C_0(q),$$

where, $C_0(q) = \exp\left(2(q^2-q)\|\sigma'\|_\infty^2 + 2q\|b'\|_\infty\right)$. Consequently, (7.15) is established. □

Proposition 7.5.
There is a constant $\mathcal{C}$ (not depending on x_0), such that

$$\sup_{(t,\lambda)\in(]0,1])^2} \mathbb{E}[(< DZ_t, DZ_t >)^{-p}] \leq \mathcal{C}, \ \ \forall p \geq 1.$$

Proof of Proposition 7.5.
Let $t \in]0,1]$ fixed, $\epsilon_0 = \frac{c_0 t}{8}$, where c_0 is a non-degeneracy constant of σ^2, in the sense of Definition 2.1. Consider $\epsilon \in]0,\epsilon_0[$, we set $N := N^\lambda =< DZ_t^\lambda, DZ_t^\lambda >$, where we recall that $< DZ_t, DZ_t >$ appears in (7.11) and (7.12). According to Proposition 7.1, we have to evaluate $\mathbb{P}(N \leq \epsilon)$.

Since σ is non-degenerate, we have

$$\begin{aligned}\mathbb{P}(N \le \epsilon) &\le \mathbb{P}\left(\int_0^t dr\mathcal{E}^2(\lambda;r,t) \le \frac{\epsilon}{c_0}\right), &(7.17)\\ &\le \mathbb{P}\left(\left(\int_{t-\frac{4\epsilon}{c_0}}^t dr\mathcal{E}^2(\lambda;r,t)\right)^{\frac{1}{2}} \le \sqrt{\frac{\epsilon}{c_0}}\right), &(7.18)\\ &\le \mathbb{P}\left(\left(\int_{t-\frac{4\epsilon}{c_0}}^t dr\right)^{\frac{1}{2}} - \left(\int_{t-\frac{4\epsilon}{c_0}}^t \mathcal{E}^2(\lambda;r,t)dr\right)^{\frac{1}{2}} \ge \sqrt{\frac{\epsilon}{c_0}}\right).\end{aligned}$$

By the inverse triangle inequality of the $L^2([t-\frac{4\epsilon}{c_0},t])$-norm, we get

$$\mathbb{P}(N \le \epsilon) \le \mathbb{P}\left(\int_{t-\frac{4\epsilon}{c_0}}^t (1-\mathcal{E}(\lambda;r,t))^2 dr \ge \frac{\epsilon}{c_0}\right).$$

Let $p \ge 1$. By Chebyshev's inequality, this is lower than

$$\left(\frac{c_0}{\epsilon}\right)^{p+1}\mathbb{E}\left[\left(\int_{t-\frac{4\epsilon}{c_0}}^t (1-\mathcal{E}(\lambda;r,t))^2 dr\right)^{p+1}\right].$$

Then, using Jensen's inequality, we get

$$\mathbb{P}(N \le \epsilon) \le 4^p\left(\frac{\epsilon}{c_0}\right)^{-1}\int_{t-\frac{4\epsilon}{c_0}}^t \mathbb{E}\left[(1-\mathcal{E}(\lambda;r,t))^{2(p+1)}\right]dr. \tag{7.19}$$

Furthermore, (7.12) implies that $\mathcal{E}(\lambda;r,t)$ solves

$$\mathcal{E}(\lambda;r,t) = 1+\lambda\int_r^t \mathcal{E}(\lambda;r,s)\sigma'(s\lambda^2,\lambda Z_s+x_0)dW_s+\lambda^2\int_r^t \mathcal{E}(\lambda;r,s)b'(s\lambda^2,\lambda Z_s+x_0)ds.$$

Thus,

$$\begin{aligned}\mathbb{E}\left[(\mathcal{E}(\lambda;r,t)-1)^{2(p+1)}\right] \le{}& 2^{2(p+1)}\mathbb{E}\left[\left|\lambda\int_r^t \mathcal{E}(\lambda;r,s)\sigma'(s\lambda^2,\lambda Z_s+x_0)dW_s\right|^{2(p+1)}\right]\\ &+2^{2(p+1)}\mathbb{E}\left[\left|\lambda^2\int_r^t \mathcal{E}(\lambda;r,s)b'(s\lambda^2,\lambda Z_s+x_0)ds\right|^{2(p+1)}\right]\end{aligned} \tag{7.20}$$

On one hand, using Jensen's inequality and $\lambda \in]0,1]$, we obtain

$$\mathbb{E}\left[\left|\lambda^2 \int_r^t \mathcal{E}(\lambda;r,s)b'(s\lambda^2,\lambda Z_s+x_0)ds\right|^{2(p+1)}\right] \leq \|b'\|_\infty |t-r|^{2p+1} \int_r^t \mathbb{E}\left[(\mathcal{E}(\lambda;r,s))^{2(p+1)}\right] ds. \tag{7.21}$$

On the other hand, by Burkholder-Davis-Gundy's inequality, we get

$$\mathbb{E}\left[\left|\lambda \int_r^t \mathcal{E}(\lambda;r,s)\sigma'(s\lambda^2,\lambda Z_s+x_0)dWs\right|^{2(p+1)}\right] \leq \mathbb{E}\left[\left|\lambda^2 \int_r^t \mathcal{E}^2(\lambda;r,s)(\sigma')^2(s\lambda^2,\lambda Z_s+x_0)ds\right|^{\mathrm{p+1}}\right].$$

Applying again Jensen's inequality gives

$$\mathbb{E}\left[\left|\lambda \int_r^t \mathcal{E}(\lambda;r,s)\sigma'(s\lambda^2,\lambda Z_s+x_0)dWs\right|^{2(p+1)}\right] \leq \|\sigma'\|_\infty |t-r|^{p} \int_r^t \mathbb{E}\left[(\mathcal{E}(\lambda;r,s))^{2(p+1)}\right] ds. \tag{7.22}$$

Therefore, (7.21), (7.22) and (7.20), lead to

$$\mathbb{E}\left[(\mathcal{E}(\lambda;r,t)-1)^{2(p+1)}\right] \leq C(T,\|\sigma'\|_\infty,\|b'\|_\infty)|t-r|^p \int_r^t \mathbb{E}\left[(\mathcal{E}(\lambda;r,s))^{2(p+1)}\right] ds. \tag{7.23}$$

By Lemma 7.4 and (7.23), there is a constant $C_0(2(p+1))$, such that

$$\mathbb{E}\left[(\mathcal{E}(\lambda;r,t)-1)^{2(p+1)}\right] \leq C_1(T,\|\sigma'\|_\infty,\|b'\|_\infty)C_0(2(p+1)). \tag{7.24}$$

Then, coming back to (7.19) and using (7.24), we obtain

$$\forall \epsilon \in]0,\epsilon_0] \quad \mathbb{P}(N \leq \epsilon) \leq C(p)\epsilon^{p+1}, \tag{7.25}$$

where, $C(p) = \dfrac{4^{2(p+1)}C_0(2(p+1))C_1(T,\|\sigma'\|_\infty,\|b'\|_\infty)}{p+1}$. Finally, using Proposition 7.1, the result follows. □

We go on with the proof of Proposition 5.1 taking into account the considerations before Lemma 7.4. In fact, Proposition 2.1.1 of [28], allows to express, for fixed $t \in]0,1]$

$$q_t^\lambda(z) = \mathbb{E}\left[\mathbb{1}_{\{Z_t>z\}}\delta(G)\right]; \tag{7.26}$$

using Cauchy-Schwarz inequality, it implies that

$$q_t^\lambda(z) \leq \sqrt{\mathbb{E}\left[|\delta(G)|^2\right]}. \tag{7.27}$$

According to (1.48) in [28], (7.27) implies

$$q_t^{\lambda}(z) \leq \left(\mathbb{E}\left[\int_0^t G^2(\lambda, s) ds \right] + \mathbb{E}\left[\int_{[0,t]^2} (D_{s_1} G(\lambda, s))^2 ds_1 ds \right] \right)^{\frac{1}{2}}. \tag{7.28}$$

Now, we state a result that provides an estimation of the two terms in the right-hand side of (7.28). Indeed, we have

Proposition 7.6. *For every $\lambda \in]0,1]$, $G(\lambda, \cdot) \in \mathbb{L}^{1,2}$. Moreover, the following statements hold*

(i) $\mathbb{E}\left[\int_0^t G^2(\lambda, s) ds \right] \leq \mathcal{C}_1 \left(1 + |x_0|^2\right)$,

(ii) $\mathbb{E}\left[\int_{[0,t]^2} (D_{s_1} G(\lambda, s))^2 ds_1 ds \right] \leq \mathcal{C}_2 \left(1 + |x_0|^8\right)$,

where, $\mathcal{C}_1$ and $\mathcal{C}_2$ depend on T, $\|\sigma'\|_\infty$ and $\|b'\|_\infty$, but do not depend on x_0.

Proof of Proposition 7.6.

(i) First, we set $I_1 = \mathbb{E}\left[\int_0^t G^2(\lambda, s) ds \right]$.

Moreover, we recall that

$$G(\lambda, s) = \frac{\sigma(s\lambda^2, \lambda Z_s + x_0) \mathcal{E}(\lambda; s, t)}{G_{\text{den}}}, \text{ where, } G_{\text{den}} = < DZ_t, DZ_t > .$$

By Cauchy-Schwarz inequality, we have

$$I_1 \leq \left(\mathbb{E}\left[\int_0^t \sigma^4(s\lambda^2, \lambda Z_s + x_0) ds \right] \mathbb{E}\left[\int_0^t \frac{\mathcal{E}^4(\lambda; s, t)}{G_{\text{den}}^4} ds \right] \right)^{\frac{1}{2}}. \tag{7.29}$$

Since σ has linear growth, by Lemma 7.2 and using again Cauchy-Schwarz inequality, there is a constant C_1 such that

$$I_1 \leq C_1 \left(1 + |x_0|^2\right) \left(\mathbb{E}\left[{G_{den}}^{-8} \right] \int_0^t \mathbb{E}\left[\mathcal{E}^8(\lambda; s, t) \right] ds \right)^{\frac{1}{4}}, \tag{7.30}$$

Consequently, using Proposition 7.5 and Lemma 7.4, the first item of Proposition 7.6 is established.

(ii) We set $I_2 = \mathbb{E}\left[\int_{[0,t]^2} (D_{s_1} G(\lambda, s))^2 ds_1 ds \right]$.

On one hand, using integration by parts, we obtain

$$\begin{aligned} D_{s_1}G(\lambda,s) &= \lambda\sigma'(s\lambda^2,\lambda Z_s+x_0)\sigma(s_1\lambda^2,\lambda Z_{s_1}+x_0)\mathcal{E}(\lambda;s_1,s)\frac{\mathcal{E}(\lambda;s,t)}{G_{\text{den}}}\mathbb{1}_{[0,s]}(s_1) \\ &+ \sigma(s\lambda^2,\lambda Z_s+x_0)\frac{D_{s_1}\mathcal{E}(\lambda;s,t)}{G_{\text{den}}} \\ &- \sigma(s\lambda^2,\lambda Z_s+x_0)\mathcal{E}(\lambda;s,t)\frac{D_{s_1}G_{\text{den}}}{G_{\text{den}}^2}. \end{aligned} \tag{7.31}$$

The right-hand side being measurable with respect to $\Omega\times[0,T]^2$, $G(\lambda,\cdot)$ will belong to $\mathbb{L}^{1,2}$ if (ii) is established.

Now, we need to evaluate $D_{s_1}\mathcal{E}(\lambda;s,t)$. From now on, for the sake of simplicity, we will only expose the calculations in the case when $b\equiv 0$. In fact, we have

$$\begin{aligned} D_{s_1}\mathcal{E}(\lambda;s,t) &= \mathcal{E}(\lambda;s,t)D_{s_1}\left(\lambda\int_s^t\sigma'(\ell\lambda^2,\lambda Z_\ell+x_0)dW_\ell-\frac{\lambda^2}{2}\int_s^t(\sigma')^2(\ell\lambda^2,\lambda Z_\ell+x_0)d\ell\right) \\ &= \mathcal{E}(\lambda;s,t)\Bigg(\lambda\sigma'(s_1\lambda^2,\lambda Z_{s_1}+x_0)\mathbb{1}_{[s,t]}(s_1) \\ &+\lambda^2\sigma(s_1\lambda^2,\lambda Z_{s_1}+x_0)\int_s^t\mathbb{1}_{[s_1,t]}(\ell)\sigma''(\ell\lambda^2,\lambda Z_\ell+x_0)\mathcal{E}(\lambda;s_1,\ell)dW_\ell \\ &-\lambda^3\sigma(s_1\lambda^2,\lambda Z_{s_1}+x_0)\int_s^t\mathbb{1}_{[s_1,t]}(\ell)(\sigma'\sigma'')(\ell\lambda^2,\lambda Z_\ell+x_0)\mathcal{E}(\lambda;s_1,\ell)d\ell\Bigg). \end{aligned} \tag{7.32}$$

On the other hand, we get

$$\begin{aligned} D_{s_1}G_{\text{den}} &= 2\lambda\int_0^t\sigma\sigma'(\xi\lambda^2,\lambda Z_\xi+x_0)\mathbb{1}_{[s_1,t]}(\xi)\sigma(s_1\lambda^2,\lambda Z_{s_1}+x_0)\mathcal{E}(\lambda;s_1,\xi)\mathcal{E}^2(\lambda;\xi,t)d\xi \\ &+2\int_0^t\sigma^2(\xi\lambda^2,\lambda Z_\xi+x_0)\mathcal{E}(\lambda;\xi,t)D_{s_1}\mathcal{E}(\lambda;\xi,t)d\xi. \end{aligned} \tag{7.33}$$

Therefore, coming back to (7.31) and using (7.32), we obtain that

$$I_2\le 4\left[J_1+J_2+J_3\right], \tag{7.34}$$

where,

$$J_1 = \mathbb{E}\left[\int_0^t ds \int_0^s ds_1 \left|\sigma'(s\lambda^2, \lambda Z_s + x_0)\sigma(s_1\lambda^2, \lambda Z_{s_1} + x_0)\mathcal{E}(\lambda; s_1, s)\frac{\mathcal{E}(\lambda; s, t)}{G_{\text{den}}}\right|^2\right],$$

$$J_2 = \mathbb{E}\left[\int_{[0,t]^2} ds_1 ds \sigma^2(s\lambda^2, \lambda Z_s + x_0)\frac{\mathcal{E}^2(\lambda; s, t)}{G_{\text{den}}^2}\left(\mathbb{1}_{[s,t]}(s_1)\sigma'(s_1\lambda^2, \lambda Z_{s_1} + x_0)\right.\right.$$

$$\left.\left. + \sigma(s_1\lambda^2, \lambda Z_{s_1} + x_0)\int_s^t \mathbb{1}_{[s_1,t]}(\ell)\sigma''(\ell\lambda^2, \lambda Z_\ell + x_0)\mathcal{E}(\lambda; s_1, \ell)\left[dW_\ell - \sigma'\sigma''(\ell\lambda^2, \lambda Z_\ell + x_0)d\ell\right]\right)^2\right],$$

$$J_3 = \mathbb{E}\left[\int_{[0,t]^2} ds_1 ds \frac{\sigma^2(s\lambda^2, \lambda Z_s + x_0)}{G_{\text{den}}^4}\mathcal{E}^2(\lambda; s, t)\left(D_{s_1}G_{\text{den}}\right)^2\right],$$

with, $D_{s_1}G_{\text{den}}$ calculated in (7.33).

In the sequel, we will enumerate constants K_1 to K_{20}; all those will not depend on x_0 or t, but eventually on T, σ and b. We start estimating J_1.

Since σ' is bounded, by Cauchy-Schwarz inequality we have

$$J_1 \leq K_1\left(\mathbb{E}\left[\int_0^t ds \int_0^s ds_1 \sigma^4(s_1\lambda^2, \lambda Z_{s_1} + x_0)\right]\mathbb{E}\left[\int_0^t ds \int_0^s ds_1 \mathcal{E}^4(\lambda; s, t)\frac{\mathcal{E}^4(\lambda; s_1, s)}{G_{\text{den}}^4}\right]\right)^{\frac{1}{2}}.$$

Since σ has linear growth, Lemma 7.2 and a further use of Cauchy-Schwarz inequality imply that J_1 is bounded by

$$K_2\left(1 + |x_0|^2\right)\left(\mathbb{E}\left[G_{\text{den}}^{-8}\right]\right)^{\frac{1}{4}}\left(\mathbb{E}\left[\int_0^t ds \int_0^s ds_1 \mathcal{E}^{16}(\lambda; s, t)\right]\mathbb{E}\left[\int_0^t ds \int_0^s ds_1 \mathcal{E}^{16}(\lambda; s_1, s)\right]\right)^{\frac{1}{8}}.$$

Therefore, by Proposition 7.5 and Lemma 7.4, we obtain that

$$J_1 \leq K_3\left(1 + |x_0|^2\right), \tag{7.35}$$

Now, we go on with the analysis of J_2. Since σ', σ'' are bounded, we have

$$J_2 \leq K_4\mathbb{E}\left[\int_{[0,t]^2} ds ds_1 \sigma^2(s\lambda^2, \lambda Z_s + x_0)\frac{\mathcal{E}^2(\lambda; s, t)}{G_{\text{den}}^2}\left(1 + \sigma^2(s_1\lambda^2, \lambda Z_{s_1} + x_0)M^2(s, s_1; t)\right)\right],$$

where, $M(s,s_1;t) = \int_{s\vee s_1}^{t} \sigma''(\ell\lambda^2, \lambda Z_\ell + x_0)\mathcal{E}(\lambda;s_1,\ell)dW_\ell$, $_{t\geq s\vee s_1}$, is a martingale with all moments, by Lemma 7.4.

Since σ has linear growth, using Cauchy-Schwarz inequality and Lemma 7.2, we get

$$J_2 \leq K_5(1+|x_0|^4)\left(\mathbb{E}\left[\int_0^t ds\mathcal{E}^{16}(\lambda;s,t)\right]\mathbb{E}\left[G_{\text{den}}^{-16}\right]\right)^{\frac{1}{8}}\left(\int_{[0,t]^2} dsds_1 E[M^8(s,s_1;t)]\right)^{\frac{1}{4}}.$$

Then, Burkholder-Davis-Gundy's inequality, Lemma 7.4 and Proposition 7.5 imply

$$J_2 \leq K_6(1+|x_0|^4). \tag{7.36}$$

Finally, we treat J_3. Applying Cauchy-Schwarz inequality, we have

$$J_3 \leq \left(\mathbb{E}\left[\int_{[0,t]^2} dsds_1 \frac{\sigma^4(s\lambda^2, \lambda Z_s + x_0)}{G_{\text{den}}^8}\mathcal{E}^4(\lambda;s,t)\right]\mathbb{E}\left[\int_{[0,t]^2} dsds_1 (D_{s_1}G_{\text{den}})^4\right]\right)^{\frac{1}{2}}.$$

Since σ has linear growth, again by Cauchy-Schwarz inequality and Lemma 7.2, we get

$$J_3 \leq K_7(1+|x_0|^2)\left(\mathbb{E}\left[G_{\text{den}}^{-32}\right]\int_0^t ds\mathbb{E}\left[\mathcal{E}^{16}(\lambda;s,t)\right]\right)^{\frac{1}{8}}\left(\mathbb{E}\left[\int_0^t ds_1 (D_{s_1}G_{\text{den}})^4\right]\right)^{\frac{1}{2}},$$

then, by Lemma 7.4 and Proposition 7.5, it follows

$$J_3 \leq K_8(1+|x_0|^2)\left(\mathbb{E}\left[\int_0^t ds_1 (D_{s_1}G_{\text{den}})^4\right]\right)^{\frac{1}{2}}. \tag{7.37}$$

Since σ' is bounded, (7.33) and Jensen's inequality give

$$\mathbb{E}\left[\int_0^t ds_1 (D_{s_1}G_{den})^4\right] \leq K_9\left(A_1 + A_2\right), \tag{7.38}$$

where,

$$A_1 = \mathbb{E}\left[\int_0^t ds_1 \int_{s_1}^t d\xi\sigma^4(s_1\lambda^2, \lambda Z_{s_1} + x_0)\sigma^4(\xi\lambda^2, \lambda Z_\xi + x_0)\mathcal{E}^4(\lambda;s_1,\xi)\mathcal{E}^8(\lambda;\xi,t)\right],$$

$$A_2 = \mathbb{E}\left[\int_0^t ds_1 \int_0^t d\xi\sigma^8(\xi\lambda^2, \lambda Z_\xi + x_0)\mathcal{E}^4(\lambda;\xi,t)\left(D_{s_1}\mathcal{E}(\lambda;\xi,t)\right)^4\right].$$

Since σ has linear growth, Cauchy-Schwarz inequality and Lemma 7.2 imply that A_1 is bounded by

$$K_{10}\left(1+|x_0|^8\right)\left(\mathbb{E}\left[\int_0^t ds_1\int_{s_1}^t d\xi\mathcal{E}^8(\lambda;s_1,\xi)\mathcal{E}^{16}(\lambda;\xi,t)\right]\right)^{\frac{1}{2}}.$$

Again, by Cauchy-Schwarz inequality and Lemma 7.4, we obtain

$$A_1 \leq K_{11}\left(1+|x_0|^8\right). \tag{7.39}$$

Now, we go on with the estimation of A_2. Using Cauchy-Schwarz inequality, A_2 is bounded by

$$K_{12}\left(\mathbb{E}\left[\int_{[0,t]^2} ds_1 d\xi\sigma^{16}(\xi\lambda^2,\lambda Z_\xi+x_0)\right]\mathbb{E}\left[\int_{[0,t]^2} ds_1 d\xi\mathcal{E}^8(\lambda;\xi,t)\left(D_{s_1}\mathcal{E}(\lambda;\xi,t)\right)^8\right]\right)^{\frac{1}{2}}.$$

Since σ has linear growth, Cauchy-Schwarz inequality and Lemma 7.2 lead to

$$A_2 \leq K_{13}\left(1+|x_0|^8\right)\left(\mathbb{E}\left[\int_{[0,t]^2} ds_1 d\xi\mathcal{E}^{16}(\lambda;\xi,t)\right]\mathbb{E}\left[\int_{[0,t]^2} ds_1 d\xi\left(D_{s_1}\mathcal{E}(\lambda;\xi,t)\right)^{16}\right]\right)^{\frac{1}{4}};$$

Lemma 7.4 implies

$$A_2 \leq K_{14}\left(1+|x_0|^8\right)\left(\mathbb{E}\left[\int_{[0,t]^2} ds_1 d\xi\left(D_{s_1}\mathcal{E}(\lambda;\xi,t)\right)^{16}\right]\right)^{\frac{1}{4}}. \tag{7.40}$$

Since σ' and σ'' are bounded, using (7.32) and Jensen's inequality, it follows that

$$\mathbb{E}\left[\int_{[0,t]^2} ds_1 d\xi\left(D_{s_1}\mathcal{E}(\lambda;\xi,t)\right)^{16}\right] \leq K_{15}\left(R_1+R_2\right), \tag{7.41}$$

where,

$$R_1 = \mathbb{E}\left[\int_{[0,t]^2} ds_1 d\xi\mathcal{E}^{16}(\lambda;\xi,t)\right],$$

$$R_2 = \mathbb{E}\left[\int_{[0,t]^2} ds_1 d\xi\sigma^{16}(s_1\lambda^2,\lambda Z_{s_1}+x_0)\mathcal{E}^{16}(\lambda;\xi,t)\left(M^{16}(s_1,\xi;t)+\int_{\xi\vee s_1}^t \mathcal{E}^{16}(\lambda;s_1,\rho)d\rho\right)\right],$$

where, $M(s_1,\xi;t)=\int\limits_{\xi\vee s_1}^{t}\sigma''(\rho\lambda^2,\lambda Z_\rho+x_0)\mathcal{E}(\lambda;s_1,\rho)dW_\rho$ is a martingale, by Lemma 7.4.

Clearly, by Lemma 7.4, R_1 is uniformly bounded in t and x_0. On the other hand, using Cauchy-Schwarz and Jensen inequalities R_2 is bounded by

$$K_{16}\left(\mathbb{E}\left[\int\limits_{[0,t]^2}ds_1d\xi\sigma^{32}(s_1\lambda^2,\lambda Z_{s_1}+x_0)\mathcal{E}^{32}(\lambda;\xi,t)\right]\mathbb{E}\left[\int\limits_{[0,t]^2}ds_1d\xi\left(M^{32}(s_1,\xi;t)+\int\limits_{\xi\vee s_1}^{t}\mathcal{E}^{32}(\lambda;s_1,\rho)d\rho\right)\right]\right)^{\frac{1}{2}}.$$

Since σ has linear growth, again by Cauchy-Schwarz inequality, Lemma 7.2 and Lemma 7.4, we get

$$R_2\leq K_{17}\left(1+|x_0|^{16}\right)\mathbb{E}\left[\int\limits_{[0,t]^2}ds_1d\xi M^{32}(s_1,\xi;t)\right].$$

Then, Burkholder-Davis-Gundy's inequality gives

$$R_2\leq K_{18}\left(1+|x_0|^{16}\right). \tag{7.42}$$

Coming back to (7.40), using (7.42) and (7.41), we obtain

$$A_2\leq K_{19}\left(1+|x_0|^{12}\right); \tag{7.43}$$

thus, replacing (7.39) and (7.43) in (7.38) and coming back to (7.37), imply

$$J_3\leq K_{20}\left(1+|x_0|^{8}\right). \tag{7.44}$$

Consequently, substituting (7.35), (7.36) and (7.44) in (7.34), item (ii) of Proposition 7.6 is established. □

Now, coming back to the proof of Proposition 5.1 and substituting in (7.28), the first and the second item of Proposition 7.6, expression (5.2) is verified. Finally, this concludes the proof of Proposition 5.1. □

7.2. Proof of Proposition 4.1

(i) This is a well-known fact which can be established by inspection.

(ii) For $M\geq 1$, we consider a sequence of smooth functions (φ^M), such that

- $\varphi^M(x)\begin{cases}=0, & \text{if } |x|\geq M+1;\\ \leq 1, & \text{if } |x|\in[M,M+1];\\ =1, & \text{if } |x|\leq M.\end{cases}$

- $(\varphi^M)''(x) = 0$, if $|x| \notin [M, M+1]$.
- $(\varphi^M)''$ is uniformly bounded in M by some constant C.

Clearly, $\varphi^M \to 1$ and $(\varphi^M)'' \to 0$, pointwise, as $M \to +\infty$.

By (4.1), we have

$$\int_{\mathbb{R}} \varphi^M(x)\mathcal{U}(t,x)dx = 1 + \int_0^t ds \int_{\mathbb{R}} \mathcal{U}^m(s,x)(\varphi^M)''(x)dx. \tag{7.45}$$

Letting $M \to +\infty$, by Lebesgue's dominated convergence theorem, the left-hand side of (7.45) converges to $\int_{\mathbb{R}} \mathcal{U}(t,x)dx$. The integral on the right-hand side of (7.45) is bounded by

$$\begin{aligned} C\int_0^t ds \int_M^{M+1} \mathcal{U}^m(s,x)dx &\leq C\int_0^t s^{-\alpha m}\left(D + \tilde{k}M^2 s^{-2\alpha}\right)^{\frac{-m}{1-m}} ds, \\ &\leq \frac{C}{(\tilde{k}M^2)^{\frac{m}{1-m}}}\int_0^T s^{\frac{m}{1-m}} ds. \end{aligned}$$

Previous integral is finite since $\frac{m}{1-m} > 0$, for every $m \in (0,1)$.

Therefore, the integral in the right-hand side of (7.45) goes to zero as $M \to +\infty$. This concludes the proof of the second item of Proposition 4.1.

(iii) Let γ be a continuous and bounded function on $\mathbb{R}$. For every $t > 0$, we have

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{U}(t,x)\gamma(x)dx &= \int_{\mathbb{R}} t^{-\alpha}\left(D + \tilde{k}|x|^2 t^{-2\alpha}\right)^{-\frac{1}{1-m}}\gamma(x)dx, \\ &= t^{-\alpha}D^{-\frac{1}{1-m}}\int_{\mathbb{R}} dx\gamma(x)\left(1 + \frac{\tilde{k}|x|^2 t^{-2\alpha}}{D}\right)^{-\frac{1}{1-m}}, \\ &= \frac{D^{-\frac{1+m}{2(1-m)}}}{\sqrt{\tilde{k}}}\int_{\mathbb{R}} dy\gamma\left(\sqrt{\frac{D}{\tilde{k}}}yt^{\alpha}\right)(1+y^2)^{-\frac{1}{1-m}}, \end{aligned}$$

where, for the last equality we set $y = \sqrt{\frac{\tilde{k}}{D}}xt^{-\alpha}$. Then, using the change of variable $y = \tan(z)$, we get

$$\int_{\mathbb{R}} \mathcal{U}(t,x)\gamma(x)dx = \frac{D^{-\frac{1+m}{2(1-m)}}}{\sqrt{\tilde{k}}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \gamma\left(\sqrt{\frac{D}{\tilde{k}}}\tan(z)t^{\alpha}\right)[\cos(z)]^{\frac{2m}{1-m}}dz.$$

Then, by Lebesgue's dominated convergence theorem, letting t go to zero, we obtain

$$\lim_{t\to 0}\int_{\mathbb{R}}\mathcal{U}(t,x)\gamma(x)dx=\gamma(0)\frac{D^{-\frac{1+m}{2(1-m)}}}{\sqrt{\tilde{k}}}I,$$

where I is defined in (1.4).

Finally, replacing D by its expression in (1.4), the result follows.

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