COOPERATION AND COMPETITION IN MULTIDISCIPLINARY OPTIMIZATION Application to the aero-structural aircraft wing shape optimization

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Abstract This article aims to contribute to numerical strategies for PDE-constrained multiobjective optimization, with a particular emphasis on CPU-demanding computational applications in which the different criteria to be minimized (or reduced) originate from different physical disciplines that share the same set of design variables. Merits and shortcuts of the most-commonly used algorithms to identify, or approximate, the Pareto set are reviewed, prior to focusing on the approach by Nash games. A strategy is proposed for the treatment of two-discipline optimization problems in which one discipline, the primary discipline, is preponderant, or fragile. Then, it is recommended to identify, in a first step, the optimum of this discipline alone using the whole set of design variables. Then, an orthogonal basis is constructed based on the evaluation at convergence of the Hessian matrix of the primary criterion and constraint gradients. This basis is used to split the working design space into two supplementary subspaces to be assigned, in a second step, to two virtual players in competition in an adapted Nash game, devised to reduce a secondary criterion while causing the least degradation to the first. The formulation is proved to potentially provide a set of Nash equilibrium solutions originating from the original single-discipline optimum point by smooth continuation, thus introducing competition gradually. This approach is demonstrated over a testcase of aero-structural aircraft wing shape optimization, in which the eigen-split-based optimization reveals clearly superior. Thereafter, a result of convex analysis is established for a general unconstrained multiobjective problem in which all the gradients are assumed to be known. This results provides a descent direction common to all criteria, and adapting the classical steepest-descent algorithm by using this direction, a new algorithm is defined referred to as the multiple-gradient descent algorithm (MGDA). The MGDA realizes a phase of cooperative optimization yielding to a point on the Pareto set, at which a competitive optimization phase can possibly be launched on the basis of the local eigenstructure of the different Hessian matrices.

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1 Multidisciplinary competition in complex design optimization

In the engineering office, the optimization problems that are raised by designers of complex systems are by nature *multiobjective*. For instance, in aerodynamic shape optimization for the design of commercial airplanes, one focus is the lift maximization in the critical phase of take-off or landing, another is drag minimization in the cruise regime since it directly determines kerosene consumption or range, but other criteria are also important : those related to stability or maneuverability that are linked to aerodynamic moments, or those imposed by manufacturing constraints, etc. Evidently, the resulting multiobjective optimization problems are inevitably also *multipoint*, since they are associated with different flight regimes (different Mach and Reynolds numbers, and angles of attack) and configurations (e.g. possible deployment of special high-lift devices). Consequently, the accurate evaluation of such criteria by means of high-fidelity models requires the efficient simulation of several flowfields by the numerical approximation of the gasdynamics equations, typically by finite volumes. In addition, different couplings of aerodynamics with other physical phenomena are also of critical importance in the performance evaluation of a design : structural deformation, stress and fatigue, dynamic fluid-structure interaction, acoustics, thermal load analysis, etc. These aspects can be treated in various ways with advanced numerical procedures. For example, in her doctoral thesis [17], M. Marcelet, in preparation of an aerodynamic aircraft wing shape optimization, has considered a model in which the compressible Reynolds-Averaged Navier-Stokes (RANS) equations have been used to compute the three-dimensional flow about the wing, whereas the structure has been modeled as a beam subject to bending and torsion under the aerodynamic forces, and thus established the expression for the discrete gradient of aerodynamic coefficients accounting for this coupling. In this area, where functional gradients of complex coupled discrete systems are calculated, *Automatic Differentiation* as it is more and more routinely developed in tools such as TAPENADE (cf. http://www-sop.inria.fr/tropics), is expected to become increasingly useful. Considering more generally, the application of gradient-based methods to aerodynamic and structural wing design, the article by Leoviriyakit and Jameson [16] reflects the potentials of state-of-the-art computational methods.

In a different perspective, in the literature, the expression "multidisciplinary optimization" (MDO) most often refers to methodologies for analyzing, and locally optimizing single-discipline subsystems, and integrating them in a larger coupled system for purpose of design. In particular, the design of aeronautical complex systems has stimulated many basic developments. A commonly-used approach is the Bi-level Integrated System Synthesis (BLISS) of Sobieszczanski-Sobieski and co-authors in which the integration is organized after a distinction is made among the design variables between the global (or public) variables common ta all disciplines, and the local (or private) variables associated with separate subsystems [25] [26]. A formal presentation and a comparison of collaborative optimization approaches was made by Alexandrov in [3]. The DIVE approach of [23] has been proposed recently as a variant of the BLISS in which the coupling between subsystems is reinforced by the solution of an additional nonlinear equation. From the original developments, MDO concepts have matured and we refer to the textbook by Keane and Nair [15] for a general presentation, and to [29] for a recent review.

In our perspective, MDO processes are viewed as game strategies [5] [21] of particular types, and our developments are linked to MDO in this light.

From the standpoint of numerical analysis, how should the public variables be optimized concurrently to account for antagonistic criteria originating from different disciplines? This article focuses on this question sometimes referred to as "*concurrent engineering*". In optimum-shape design, often the different physical phenomena are accurately modeled by partial-differential equations to be solved in domains that are identical or distinct but share a common geometrical boundary at which appropriate conditions are enforced and whose shape is to be optimized. Besides the case of the aero-structural design of an aircraft wing cited above, in the design of a stealth airplane, one would optimize the wing-shape w.r.t. an appropriate aerodynamic criterion, or several such criteria, concurrently with an electromagnetic criterion, such as cross-radar section (RCS) reduction. In the latter case, both distributed P.D.E. systems are formulated in the domain exterior to the aircraft, but have very different computational characteristics in particular concerning mesh requirements.

In the area of pure numerical simulation of multidisciplinary coupled systems, the computational cost to evaluate a configuration may be very high. *A fortiori*, in multidisciplinary optimization, one is led to evaluate a number of different configurations to iterate on the design parameters. This observation motivates the search for the most innovative and computationally efficient approaches in all the sectors of the computational chain : at the level of the solvers (using a hierarchy of physical models), the meshes and geometrical parametrizations for shape, or shape deformation, the implementation (on a sequential or parallel architecture; grid computing), and the optimizers (deterministic or semi-stochastic, or hybrid; synchronous, or asynchronous).

In the present approach, we concentrate on situations typically involving a small number of disciplines assumed to be strongly antagonistic, and a relatively moderate number of related objective functions. However, our objective functions are functionals, that is, PDE-constrained, and thus costly to evaluate. The aerodynamic and structural optimization of an aircraft configuration is a prototype of such a context, when these disciplines have been reduced to a few major objectives. This is the case when, implicitly, many subsystems are taken into account by local optimizations.

Our developments are focused on the question of approximating the Pareto set in cases of strongly-conflicting disciplines. For this purpose, a general computational technique is proposed, guided by a form of sensitivity analysis, with the additional objective to be more economical than standard evolutionary approaches.

Classically, the simplest way to account for several criteria simultaneously consists in agglomerating them all in a single performance index weighting each criterion with an appropriate coefficient, or weight. For example, with two criteria J_A and J_B, consider :

$$J = \alpha \frac{J_A}{J_A^0} + \beta \frac{J_B}{J_B^0}$$

where J_A^0 and J_B^0 are reference values, for example, those associated with an initial design. Here, α and β are positive weights to be chosen somehow. This approach is very commonly-used, particularly when one disposes of an initial design that is close to be satisfactory, that is, only a better, or slightly different optimum is to be sought. However, the construction of the agglomerated criterion involves a large amount of arbitrariness, in particular (but not only) w.r.t. the weights α and β that can strongly influence the result and require to be calibrated by an experienced practitioner. Thus, this approach is poorly general, physically or mathematically relevant.

An alternative to the unique criterion by agglomeration of several objective functions, consists of a two-step process in which each criterion is first optimized alone, possibly under constraints; for the above two-objective problem, one thus gets J_A^* and J_B^* as the solutions to two independent single-objective optimizations. Then, in the second step, one solves the following single-objective constrained problem :

$$\min p$$

subject to the following inequality constraints :

$$J_A \leq J_A^* + \alpha p \quad \text{and} \quad J_B \leq J_B^* + \beta p$$

In this alternative, assuming all the cited single-objective problems make sense separately, without physical coupling, the difficulty is here to treat a problem with functional inequality constraints of physically-different nature. Additionally, the same arbitrariness resides in the calibration of the weights α and β.

A real alternative to the unique agglomerated objective approach, is to establish the front of *Pareto-optimal solutions.* To introduce this, we first recall the notion of *dominance* and *non-dominance* :

Definition : When considering the minimization of several criteria concurrently (J_A, J_B, etc), a design point $D^{(1)}$ in the parameter space is said to dominate the design $D^{(2)}$ in efficiency, which we denote as follows :

$$D^{(1)} \succ D^{(2)},$$

iff, for all the criteria J to be minimized, the following holds :

$$J\left[D^{(1)}\right] \leq J\left[D^{(2)}\right],$$

and if, for at least one criterion, the inequality is strict. Inversely, if instead :

$$D^{(1)} \not\succ D^{(2)}, \text{ and } D^{(2)} \not\succ D^{(1)},$$

the two design-points $D^{(1)}$ and $D^{(2)}$ are said to be non-dominated.

This notion can be used to sort a collection, or population of design-points evaluated w.r.t. the various criteria J_A, J_B, etc, according to the so-called *Pareto fronts*. The first front is made of all the design-points dominated by no other; the second, the front of those dominated by no other in the remaining set; etc. The result of this sorting process is sketched at Fig. 1.

Relying on this sorting process, Srinivas and Deb [27] have proposed the genetic algorithm *NSGA (Non-dominated Sorting Genetic Algorithm)* which utilizes essentially the front index as the *fitness function*, the engine of the GA. Goldberg [12] improved the method by introducing a *niching* technique in order to prevent the accumulation of non-dominated design-points on a given front. To illustrate the *NSGA*, we present an experiment made by Marco *et al* [18] in which an airfoil shape was optimized to reduce drag (in transonic flow conditions) and maximize lift (in subsonic flow conditions) concurrently. The *NSGA* was implemented in two independent experiments corresponding to finite-volume simulations of the compressible Euler equations using different meshes, a coarse and a fine. The totality

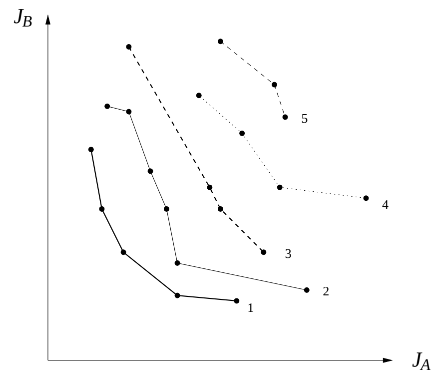


Fig. 1 Sketch of a population of design-points sorted in Pareto fronts

of the design-points accumulated during the successive generations in the two experiments indistinctly, are represented on Fig. 2 a). In each experiment, the set of design-points does not cover the entire quarter plane : not all pairs (J_A, J_B) can be achieved by the system. The boundary of the domain of realizable pairs is made of Pareto-optimal solutions. The corresponding two (discrete) fronts and the associated shapes (for the fine-mesh experiment only) are depicted on Fig. 2 b) and c).

This experiment allows us to point out the principal merits and weaknesses of this approach. The method provides the designer with a rich and unbiased information on the behavior of the criteria when the parameters vary, but one can also regret the lack of hierarchy between the Pareto-optimal solutions, among which a definite operating design-point requires to be elected on the basis of some other criterion still to be introduced. Other experiments in the literature have shown that the method is very general since it has been applied to cases where the Pareto-equilibrium front was either non-convex or discontinuous. On the other hand, the computational cost of a standard application of the *NSGA* is fairly high since a large number of configurations ought to be evaluated, if an accurate identification of the front is sought. In our example, this was achieved by instantiations of a two-dimensional Eulerian flow code for purpose of demonstration; however today, realistic flow simulations about aircraft wings are based on three-dimensional turbulent Navier-Stokes equations. The cost-efficiency issue can be somewhat alleviated by the usage of parallel computing, which is possible at several levels : the parallelization of the analysis code by domain decomposition, the natural parallelization of its independent instantiations, as well as the parallelization the crossover operator in the GA [19]. Various evolutionary algorithms other than the *NSGA* have been proposed for multiobjective optimization on the basis of similar principles (e.g. *NPGA* [14], *MOGA* [11], *SPEA* [31]).

When the front of Pareto-optimal solutions is convex and smooth, it may be possible to identify it point-wise, by treating all but one criterion as equality constraints, as depicted on Fig. 3. However this approach is much less general since, as mentioned before, functional constraints are difficult to implement; additionally, the identification is usually logically complex in cases of more than two objectives.

An alternate treatment of multiobjective problems that circumvents the usually very arbitrary question of adjusting penalty constants in the agglomerated-criterion approach, and that is much more economical than an *NSGA*-type method to establish the Pareto-equilibrium front, consists in simulating a dynamic game in which the design variables are *a) Realizable design-point accumulation by application of the NSGA (independent Eulerian flow simulations on coarse and fine grids)*

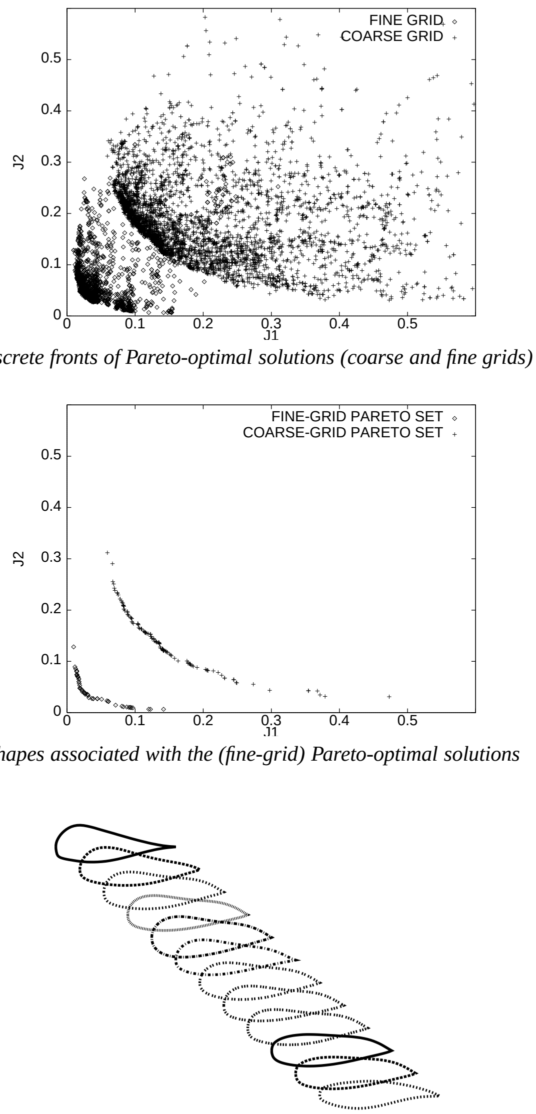


Fig. 2 An illustration of the NSGA in which an airfoil shape is optimized to reduce drag and maximize lift concurrently; in c) the upper-left airfoil has the highest lift, and the lower-right the lowest drag.

first split in complementary subsets and distributed to virtual players as individual strategies. Symmetrical as well as unsymmetrical (or hierarchical) games can be considered [21], [5]. In a symmetrical Nash game [21], each player accommodates its own strategy to the other players strategies to optimize only one criterion. If an equilibrium point is reached, a trade-off between the various criteria is achieved.

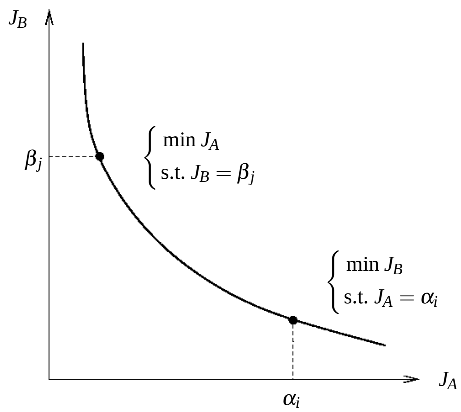


Fig. 3 Schematic of a Pareto front point-wise identification by the treatment of certain criteria as equality constraints

In his doctoral thesis, B. Abou El Majd [1] has realized a number of aero-structural shape-optimization exercises related to a generic business-jet wing using either Nash or Stackelberg games, some of which are reported here for illustration, some of which have also been reported in [2].

Here, we focus on the symmetrical formulation of Nash games involving two players A and B controlling the subvectors Y_A and Y_B composing the complete vector of design variables :

$$Y = (Y_A, Y_B)$$

In this case, the vector $\overline{Y} = \left(\overline{Y}_A, \overline{Y}_B\right)$ is said to realize a Nash equilibrium of the criteria J_A and J_B, iff :

$$\overline{Y}_A = \mathsf{Argmin}_{Y_A} J_A \left(Y_A, \overline{Y}_B\right)$$

and symmetrically :

$$\overline{Y}_B = \mathsf{Argmin}_{Y_B} J_B \left(\overline{Y}_A, Y_B\right)$$

This formulation is inspired by the negotiation mechanism of which economics and social sciences provide numerous examples.

The Nash equilibrium-point can be achieved by the following parallel algorithm [28] :

Step 1: Initialize both subvectors :

$$Y_A := Y_A^{(0)} \qquad Y_B := Y_B^{(0)}$$

Step 2: Perform in parallel optimization iterations of both subsystems (by independent and generally different analysis and optimization methods) :

Player A:

- Retrieve and maintain fixed

$$Y_B = Y_B^{(0)}$$

- Perform K_A minimization steps of $J_A\left(Y_A, Y_B^{(0)}\right)$ by iterating on Y_A alone and get $Y_A^{(K_A)}$.

<u>*Player B:*</u>

- Retrieve and maintain fixed
$$Y_A = Y_A^{(0)}$$
- Perform K_B minimization steps of $J_B\left(Y_A^{(0)}, Y_B\right)$ by iterating on Y_B alone and get $Y_B^{(K_B)}$.

Step 3: Update both subvectors in preparation of the information exchange :

$$Y_A^{(0)} := Y_A^{(K_A)} \qquad Y_B^{(0)} := Y_B^{(K_B)}$$

and go back to Step 2 or stop (at equilibrium).

Note that in practice, under-relaxation is very often essential to convergence. This point is particularly critical when the two criteria J_A and J_B originate from different physical disciplines associated with different dependencies and scales, as it is the case for optimum design w.r.t. aerodynamics and structural mechanics, or electro-magnetics. However, certain rather general mathematical stabilization techniques exist; see for example [4].

Important remark : invariance of the Nash equilibrium solution to units and scales. Assume that $\overline{Y} = \left(\overline{Y}_A, \overline{Y}_B\right)$ realizes a Nash equilibrium of the criteria J_A and J_B, and let Φ and Ψ be some arbitrary but smooth and strictly-monotone increasing functions; then, evidently, $\overline{Y}$ also realizes a Nash equilibrium of the criteria $\Phi\left[J_A\right]$ and $\Psi\left[J_B\right]$. In other words, the notion of Nash equilibrium is not only independent of the physical units used for the criteria, but also of possible changes in scales applied to them : for example, replacing J by J^{α} or $exp(J)$ has no effects other than a different conditioning of the numerical system. By this invariance property, the Nash game formulation contrasts outstandingly from the agglomerated criterion approach in which dimensioning the penalty constants has a strong, and usually unknown influence on the solution. The equilibrium solution, unique or not, is only determined by the split of the design vector, which is here referred to as the *split of territory* by which each virtual player is allocated a subspace of action, or territory.

This approach has been tested successfully over a number of cases related to optimum design in aeronautics, in particular within the framework of the Jacques-Louis Lions Laboratory common to the University of Paris 6 and Dassault Aviation. One of the earliest contributions has been Wang's doctoral thesis [30] in which multicriterion optimization problems in aerodynamics have been treated by Nash games by taking the best advantage of a distributed environment. Nevertheless, note that in some cases of multipoint drag minimization, the lift constraint was introduced by the penalty approach; thus, somewhat artificially, all the criteria were unconstrained and this results in a simplification, because it allows the Nash equilibrium to be sought from an initial point where the functional gradient is equal to zero, and the dynamic game develops in a region in which the functional is not very sensitive to parameter changes.

For purpose of illustration, we reproduce here partially the results of a two-point airfoil shape aerodynamic optimization taken from [28]. The targets are to maximize the lift in a subsonic regime representative of take-off and landing ($M_\infty = 0.2\,,\ \alpha = 10.8^o$) defining the first point, and concurrently minimize the drag in a transonic flow representative of cruise ($M_\infty = 0.77\,,\ \alpha = 1^o$) defining the second point. For both points, the airfoil is assumed to be immersed in a compressible Eulerian flow. Here, both optimizations are treated as inverse problems. A first airfoil shape is associated with the subsonic point; this airfoil is considered satisfactory w.r.t. lift in this regime, and the corresponding pressure distribution along the airfoil is denoted p_{sub}. This airfoil may be the result of a single-point optimization. However, this airfoil should be improved w.r.t. drag at the transonic point. A second airfoil has the opposite characteristics. It is satisfactory w.r.t. drag at the transonic point, and the corresponding pressure distribution along the airfoil is denoted p_{trans}, but not w.r.t. lift at the subsonic point. Then one seeks an airfoil shape that produces in each point a pressure distribution as close as possible to the relevant target profile, p_{sub} or p_{trans}.

For this, the airfoil boundary Γ_c is split into two complementary territories Γ_1 and Γ_2, corresponding approximately to the fore and aft regions of the airfoil (see Fig. 4, top). The airfoil is parametrized classically by means of the Hicks-Henne basis functions, and the associated weights are the design variables of the experiment. One such design variable is allocated to either territory depending on the location of the maximum of the corresponding bell-shaped function. In this way, Γ_1 and Γ_2 are associated with specific distinct subsets of the design variables. A trade-off between the two target airfoils is then sought by realizing a Nash equilibrium associated with the following formulation :

$$\min_{\Gamma_1} I_1 = \int_{\Gamma_c} \left(p - p_{\text{sub}}\right)^2 \tag{1}$$

(in which it is implicit that the field is calculated in the subsonic conditions that define the first point), and

$$\min_{\Gamma_2} I_2 = \int_{\Gamma_c} \left(p - p_{\text{trans}}\right)^2 \tag{2}$$

(in which it is implicit that the field is calculated in the transonic conditions that define the second point).

Starting with some appropriate initial airfoil, a virtual player performs 5 design cycles to reduce criterion I_1 by acting only on the subset of the design variables associated with Γ_1, and maintaining the other variables fixed. The optimizer is a steepest-descent-type method based on a functional gradient resulting from discretizing a continuous adjoint equation. In parallel, another virtual player performs 10 design cycles to reduce criterion I_2 by acting only on the subset of the design variables associated with Γ_2, and maintaining the other variables fixed. Then, both players exchange their best respective subvectors of design variables, and so on until an equilibrium is reached. The iterative convergence of this process is indicated at Fig. 4 (bottom) : both criteria approach a stable asymptote.

Fig. 5 illustrates how the trade-off airfoil shape corresponding to the Nash equilibrium solution compares with the initial and target airfoils, and Fig. 6 provides the pressure distributions over this optimized geometry in the two calculation points.

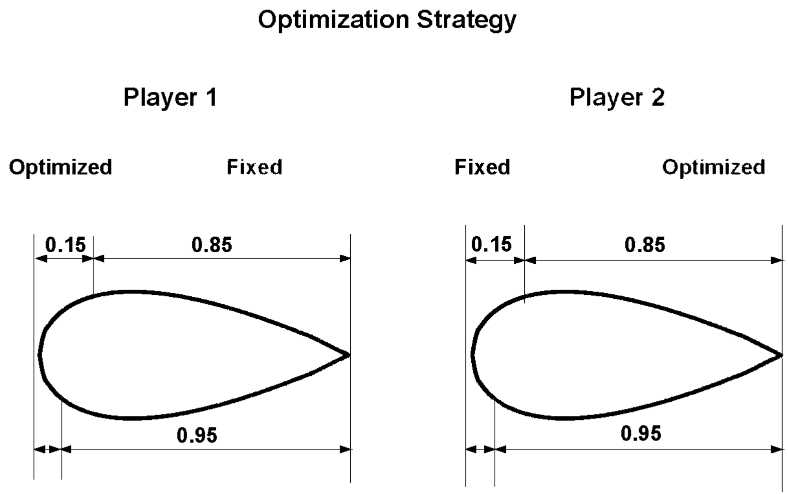
Another example of application of a Nash formulation to the treatment of a complex geometrical optimization problem is given by [13], in which two disciplines, elasticity and thermal analysis, have been considered as governing models in the competition between the structural and the cooling material topologies.

Hence, in summary and referring also to [28], in PDE-constrained optimization, the Nash game formulation has the following most important merits :

1. the iteration applies to a set of design variables, and not to a population of such vectors;
2. it permits straightforwardly to couple physical disciplines represented by independent codes through the exchange of design variables;
3. parallel computing can be exploited readily;
4. the multiobjective solution satisfies the above property of invariance to units and scales.

Keeping the above example in mind, we return now to our general discussion on multiobjective, or multidiscipline optimization. In optimum-shape design in aerodynamics, we are facing two major difficulties.

a) Split of geometrical parameters



b) Convergence of the two criteria

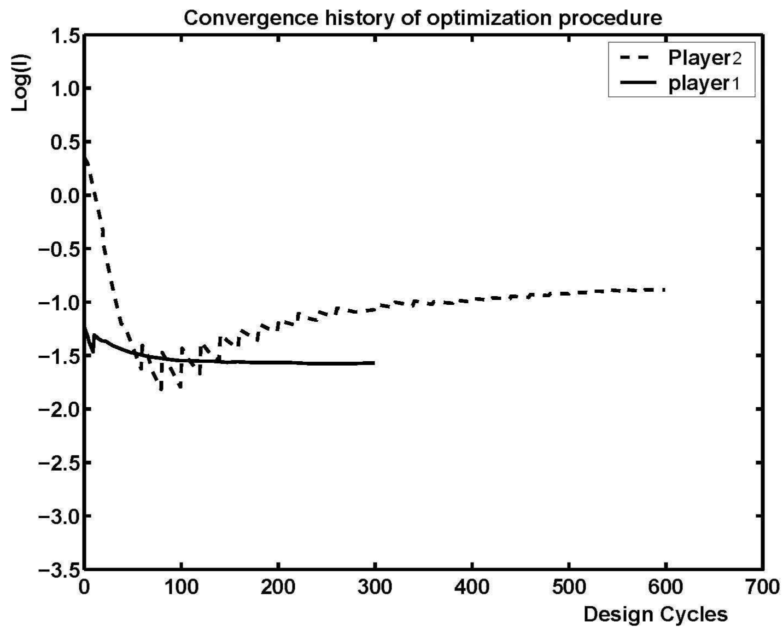


Fig. 4 Split of territory and optimization strategy; information exchange every 5 || 10 parallel optimization iterations (top); asymptotic convergence of the two criteria towards a Nash equilibrium (bottom); from [28]

The first difficulty is related to the fact that only the simulation of a complex flow by a high-fidelity model (e.g. by the RANS equations) can provide a reliable evaluation of the aerodynamic coefficients. For instance, the solution of the three-dimensional compressible Euler equations, not so long ago considered as an accomplishment, only provides the wave

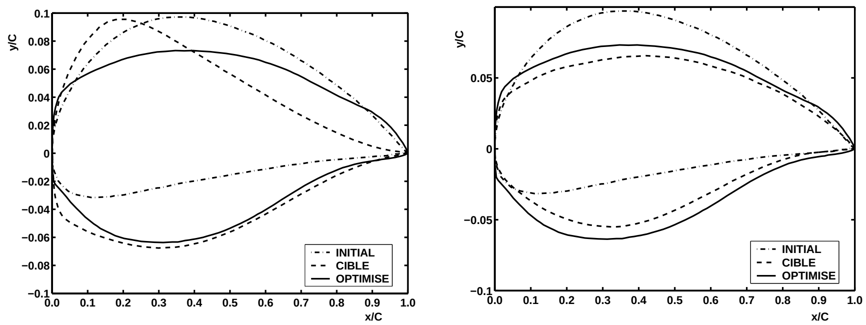


Fig. 5 Comparison between the optimized airfoil (solid line) with the initial airfoil and the target airfoil associated with the subsonic conditions of the first point (left), and the target airfoil associated with the transonic conditions of the second point (right); from [28]

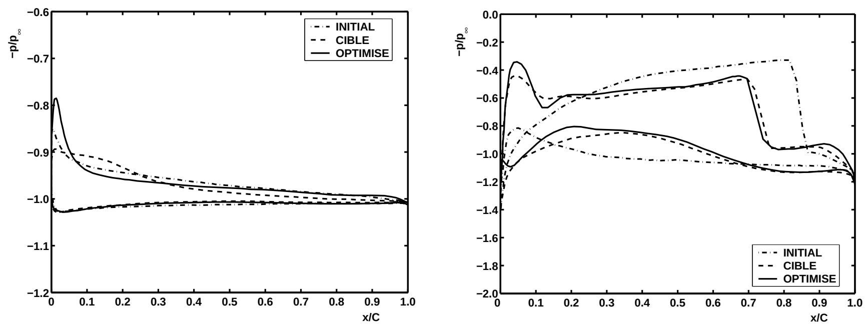


Fig. 6 Pressure distributions over the initial, target and optimized (solid line) shapes in the subsonic conditions of the first point (left), and in the transonic conditions of the second point (right); from [28]

drag and friction forces are neglected, as well as turbulence effects. The computational cost of an accurate evaluation of the aerodynamic functionals is thus very high.

Secondly, by nature, transonic flows are only weak solutions to the partial-differential equations of gasdynamics. As such, they are very sensitive to variations in boundary conditions, such as shape variations. The aerodynamic performance is therefore very fragile, in particular drag, and tolerance margins are small. By coupling aerodynamics with one or more other disciplines in a multidisciplinary optimization, it is imperative to maintain the aerodynamic performance near the optimal level.

This observation has led us to introduce the notion of *primary functional* w.r.t. which sub-optimality should be maintained, and *secondary functional* to be reduced under possible constraints.

In our notations, the dimension of the full design space is N. A first optimization step is completed in which the sole principal criterion J_A is minimized w.r.t. the totality of the N design variables, yielding a vector Y_A^* that realizes, by hypothesis, a local or global minimum of this criterion. It is also assumed that at this point, K ($K < N$) scalar constraints ($g_k = 0$, $k = 1, 2, ..., K$, or more compactly $g = 0$) are active. Then, one wishes to conduct a second optimization step, multiobjective and competitive in nature, by establishing a Nash equilibrium between the criteria J_A and J_B. To extend the formulation of the previous experiment, the following more general *split of territory* is introduced :

$$Y = Y(U,V) = Y_A^* + S \begin{pmatrix} U \\ V \end{pmatrix} \tag{3}$$

where :

$$U = \begin{pmatrix} u_1 \\ \vdots \\ u_{N-p} \end{pmatrix}, \quad V = \begin{pmatrix} v_p \\ \vdots \\ v_1 \end{pmatrix} \tag{4}$$

in which S is an adjustable matrix of dimension $N \times N$, referred to as the *splitting matrix*, and to utilize the subvectors U ($U \in \mathbb{R}^{N-p}$) and V ($V \in \mathbb{R}^p$) as strategies, or territories of two virtual players A and B in charge of the minimization of J_A and J_B respectively.

The Nash equilibrium point, if it exists, is denoted $\overline{Y} = Y\left(\overline{U}, \overline{V}\right)$, and it is associated with the following coupled optimization formulation :

$$\begin{cases} \min\limits_{U \in \mathbb{R}^{N-p}} J_A\left[Y\left(U, \overline{V}\right)\right] \\ \text{Subject to : } g\left[Y\left(U, \overline{V}\right)\right] = 0 \end{cases} \tag{5}$$

and :

$$\begin{cases} \min\limits_{V \in \mathbb{R}^{p}} J_B\left[Y\left(\overline{U}, V\right)\right] \\ \text{Subject to : no constraints} \end{cases} \tag{6}$$

The dimension p of subvector V which controls the subspace of action of player B is adjustable ($p \geq 1$); however, the dimension $N-p$ of subvector U must be at least equal to 1, and at least equal to the number K ($K \geq 0$) of active constraints; this gives the following bounds on p :

$$1 \leq p \leq N - \max(K,1) \tag{7}$$

In the limiting case ($N - p = K$), in the above Nash game formulation, the minimization of J_A under constraints reduces to the adjustment of the K components of subvector U to satisfy the K scalar constraints. This case has been examined in [7]. Hereafter, unless mentioned otherwise, a strict inequality is assumed instead.

In the examples cited above, [30] and [28], the split is a partition of the *primitive variables*, that is, the original components of the design vector Y. Our new and more general formulation encompasses this particular case obtained when the splitting matrix is a permutation matrix.

In a parametric shape optimization, the primitive variables are geometrical control parameters, such as the weights put on the different Hicks-Henne basis functions, or the coordinates of control points in a Bézier or B-spline parametrization. Thus, typically, these variables are associated with specific locations of the optimized geometry. Hence, when the splitting is a permutation, the permutation reflects our intuitive understanding of the dependency of the physical functionals on the geometry, or regions of it. For instance, in the example of Fig. 4, the split was guided by the knowledge that in a transonic flow, the wave drag is the result of the shock intensity and it depends mostly on the delicate design of the geometry on the upper surface near the shock, whereas, in a subsonic flow, the lift is essentially proportional to the airfoil thickness. In his doctoral thesis, Wang [30] demonstrated that iterations based on choices for the splitting opposite to this physical sense, unsurprisingly, diverge.

These considerations lead us to raise the following question : how should the split be defined in a general and systematic manner to respect the physical sense? In particular, if the Nash game is initiated from a viable, physically-relevant solution corresponding to an optimum of the primary criterion J_A, can near-optimality of this criterion be maintained at equilibrium?

With the formulation of (3), the subspace spanned by the first $N-p$ column vectors of the splitting matrix S can be viewed as the territory assigned to player A in charge of minimizing the primary criterion J_A, and the subspace spanned by the last p column vectors as the territory assigned to player B in charge of minimizing the secondary criterion J_B. Thus the above open questions are those of the adequacy of the split of territory. The option which is adopted here consists in making this choice statically (and not adaptively in the course of the dynamic game), at completion of the first step of the procedure in which the primary criterion is minimized alone (possibly under constraints) in full dimension N, yielding the optimal design vector Y_A^*, and before any competitive strategy is initiated. Thus the choice is made, here once for all, on the basis of the analysis of the sensitivity of this criterion only. We specifically enforce the following condition : the second step of the optimization procedure, the competitive step, should be such that infinitesimal perturbations of the parameters about Y_A^* that lie in the subspace identified as the territory of the secondary criterion should cause the least possible degradation of the primary criterion (w.r.t. the minimum achieved at completion of the first step). As a basis for the identification of the optimal splitting, one considers the formal Taylor's expansion of the primary functional to second order about Y_A^* in the direction of a unit vector $\omega \in \mathbb{R}^N$:

$$J_A\,(Y_A^* + \varepsilon\omega) = J_A\,(Y_A^*) + \varepsilon\,\nabla J_A^*\,.\,\omega + \frac{\varepsilon^2}{2}\,\omega\,.\,H_A^*\,\omega + O(\varepsilon^3) \tag{8}$$

Our goal is to propose a sensible splitting associated with the definition of a vector basis $\{\,\omega^k\,\}$ $(k = 1,\ldots,N)$. To fix the ideas, let us assume that the first few elements, $\{\,\omega^k\,\}$ $(k = 1,2,\ldots)$, of the basis are dedicated to player A in charge of reducing the primary criterion J_A, and inversely, the tail elements, $\{\,\omega^k\,\}$ $(k = N, N-1\ldots)$, to player B in charge of reducing the secondary criterion J_B. Note that the direction of maximum sensitivity of the primary criterion J_A, or steepest-descent direction, is given by the gradient, ∇J_A^* at $Y = Y_A^*$. Thus, the following two conditions should be satisfied by the basis :

1. the first few elements should span the gradient, ∇J_A^*;
2. inversely, the difference $|J_A\,(Y_A^* + \varepsilon\omega) - J_A\,(Y_A^*)|$, when ε is small and fixed, should be as small as possible when ω is a tail element of the basis.

At $Y = Y_A^*$, the optimality conditions imply that the gradient ∇J_A^* is a linear combination of the K active constraint gradients, the coefficients being the Lagrange multipliers. Thus a way to achieve the first condition is to enforce that the first K elements of the basis have the same span as the gradients of the K active constraints. For this, one requires that $\{\,\omega^k\,\}$ $(k = 1,2,\ldots,K)$ be the result of applying the Gram-Schmidt orthogonalization process to the constraint gradients $\{\,\nabla g_k^*\,\}$ $(k = 1,2,\ldots,K)$. Then, let P be the following projection matrix :

$$P = I - \sum_{k=1}^{K} \left[\omega^k\right]\,\left[\omega^k\right]^T \tag{9}$$

where $\left[\omega^k\right]$ denotes the column-vector matrix made of the components of vector ω^k, and consider the following real-symmetric matrix :

$$H_A' = P\,H_A^*\,P \tag{10}$$

We claim that the eigenvectors of the matrix H'_A, ordered appropriately, constitute the best choice.

First, these eigenvectors contain the null space of the projection matrix P, that is, $\{\omega^k\}$ $(k = 1, 2, \ldots, K)$. Thus the first condition is satisfied simply if the ordering is such that these vectors appear first.

Second, the basis is orthogonal; hence the tail elements are orthogonal to the first K, and to ∇J_A^* as a consequence of the first condition. Thus, for $\omega = \omega^k$ $(k \geq K+1)$, the principal term in the expansion of the difference, $|J_A(Y_A^* + \varepsilon\omega) - J_A(Y_A^*)|$ is the quadratic term. This term, including the absolute value, reduces to the Rayleigh quotient associated with the matrix H'_A (assuming positive-definiteness), and the classical characterization of eigenvectors, here by decreasing eigenvalue, holds. □

Starting from the above observations, the following theorem, taken from [7], exploits this basic principle and draws certain additional consequences related to the Nash game. It is assumed that the two criteria J_A and J_B are strictly positive and such that :

$$J_A^* = J_A(Y_A^*) > 0\,, \quad J_B^* = J_B(Y_A^*) > 0 \tag{11}$$

If necessary the problem can easily be reformulated to meet these requirements.

Theorem 1
Let N, p and K be positive integers such that :

$$1 \leq p \leq N - \max(K, 1) \tag{12}$$

Let J_A, J_B and, if $K \geq 1$, $\{g_k\}$ $(1 \leq k \leq K)$, be $K+2$ smooth real-valued functions of the vector $Y \in \mathbb{R}^N$. Assume that J_A and J_B are positive, and consider the following primary optimization problem,

$$\min_{Y \in \mathbb{R}^N} J_A(Y) \tag{13}$$

that is either unconstrained ($K = 0$), or subject to the following K equality constraints :

$$g(Y) = (g_1, g_2, \ldots, g_K)^T = 0 \tag{14}$$

Assume that the above minimization problem admits a local or global solution at a point $Y_A^* \in \mathbb{R}^N$ at which $J_A^* = J_A(Y_A^*) > 0$ and $J_B^* = J_B(Y_A^*) > 0$, and let H_A^* denote the Hessian matrix of the criterion J_A at $Y = Y_A^*$.

If $K = 0$, let $P = I$ and $H'_A = H_A^*$; otherwise, assume that the constraint gradients, $\{\nabla g_k^*\}$ $(1 \leq k \leq K)$, are linearly independent and apply the Gram-Schmidt orthogonalization process to the constraint gradients, and let $\{\omega^k\}$ $(1 \leq k \leq K)$ be the resulting orthonormal vectors. Let P be the matrix associated with the projection operator onto the K-dimensional subspace tangent to the hyper-surfaces $g_k = 0$ $(1 \leq k \leq K)$ at $Y = Y_A^*$,

$$P = I - \sum_{k=1}^{K} \left[\omega^k\right] \left[\omega^k\right]^T \tag{15}$$

where $\left[\omega^k\right]$ denotes the column-vector matrix made of the components of vector ω^k, and consider the following real-symmetric matrix :

$$H'_A = P\,H_A^*\,P \tag{16}$$

Let Ω be an orthogonal matrix whose column-vectors are normalized eigenvectors of the matrix H'_A organized in such a way that the first K are precisely $\{\omega^k\}$ ($1 \le k \le K$), and the subsequent $N-K$ are arranged by decreasing order of the eigenvalue

$$h'_k = \omega^k . H'_A \, \omega^k = \omega^k . H^*_A \, \omega^k \quad (K+1 \le k \le N) \tag{17}$$

Consider the splitting of parameters defined by :

$$Y = Y^*_A + \Omega \begin{pmatrix} U \\ V \end{pmatrix}, \quad U = \begin{pmatrix} u_1 \\ \vdots \\ u_{N-p} \end{pmatrix}, \quad V = \begin{pmatrix} v_p \\ \vdots \\ v_1 \end{pmatrix} \tag{18}$$

Let ε be a small positive parameter ($0 \le \varepsilon \le 1$), and let $\overline{Y}_\varepsilon$ denote the Nash equilibrium point associated with the concurrent optimization problem :

$$\begin{cases} \min\limits_{U \in \mathbb{R}^{N-p}} J_A \\ \textit{Subject to : } g = 0 \end{cases} \quad \textit{and} \quad \begin{cases} \min\limits_{V \in \mathbb{R}^{p}} J_{AB} \\ \textit{Subject to : no constraints} \end{cases} \tag{19}$$

in which again the constraint $g = 0$ is not considered when $K = 0$, and

$$J_{AB} := \frac{J_A}{J^*_A} + \varepsilon \left(\theta \, \frac{J_B}{J^*_B} - \frac{J_A}{J^*_A} \right) \tag{20}$$

where θ is a strictly-positive relaxation parameter ($\theta < 1$: under-relaxation; $\theta > 1$: over-relaxation).

Then :

- *[Optimality of orthogonal decomposition] If the matrix H'_A is positive semi-definite, which is the case in particular if the primary problem is unconstrained ($K = 0$), or if it is subject to linear equality constraints, its eigenvalues have the following structure :*

$$h'_1 = h'_2 = \ldots = h'_K = 0 \quad h'_{K+1} \ge h'_{K+2} \ge \ldots \ge h'_N \ge 0 \tag{21}$$

  and the tail associated eigenvectors $\{\omega^k\}$ ($K+1 \le k \le N$) have the following variational characterization :

$$\begin{aligned} \omega^N &= \mathrm{Argmin}_\omega \, |\omega . H^*_A \, \omega| \ \textit{s.t.} \ \|\omega\| = 1 \textit{ and } \omega \perp \{\omega^1, \omega^2, \ldots, \omega^K\} \\ \omega^{N-1} &= \mathrm{Argmin}_\omega \, |\omega . H^*_A \, \omega| \ \textit{s.t.} \ \|\omega\| = 1 \textit{ and } \omega \perp \{\omega^1, \omega^2, \ldots, \omega^K, \omega^N\} \\ \omega^{N-2} &= \mathrm{Argmin}_\omega \, |\omega . H^*_A \, \omega| \ \textit{s.t.} \ \|\omega\| = 1 \textit{ and } \omega \perp \{\omega^1, \omega^2, \ldots, \omega^K, \omega^N, \omega^{N-1}\} \\ &\vdots \end{aligned} \tag{22}$$

- *[Preservation of optimum point as a Nash equilibrium] For $\varepsilon = 0$, a Nash equilibrium point exists and it is :*

$$\overline{Y}_0 = Y^*_A \tag{23}$$

– *[Robustness of original design] If the Nash equilibrium point $\overline{Y}_\varepsilon$ exists for $\varepsilon > 0$ and sufficiently small, and if it depends smoothly on this parameter, the functions :*

$$j_A(\varepsilon) = J_A\left(\overline{Y}_\varepsilon\right), \quad j_{AB}(\varepsilon) = J_{AB}\left(\overline{Y}_\varepsilon\right) \tag{24}$$

are such that :

$$j_A'(0) = 0 \tag{25}$$
$$j_{AB}'(0) = \theta - 1 \leq 0 \tag{26}$$

and

$$j_A(\varepsilon) = J_A^* + O(\varepsilon^2) \tag{27}$$
$$j_{AB}(\varepsilon) = 1 + (\theta - 1)\,\varepsilon + O(\varepsilon^2) \tag{28}$$

– *In case of linear equality constraints, the Nash equilibrium point satisfies identically :*

$$u_k(\varepsilon) = 0 \quad (1 \leq k \leq K) \tag{29}$$
$$\overline{Y}_\varepsilon = Y_A^* + \sum_{k=K+1}^{N-p} u_k(\varepsilon)\,\omega^k + \sum_{j=1}^{p} v_j(\varepsilon)\,\omega^{N+1-j} \tag{30}$$

– *For $K = 1$ and $p = N - 1$, the Nash equilibrium point* $\overline{Y}_\varepsilon$ is Pareto optimal.

We have seen already why the proposed basis of eigenvectors is optimal for the problem raised by the case of a preponderant or fragile discipline, in relation with the performance of the Nash equilibrium solution; shortly speaking, the spliiting is such that a minimal degradation of J_A is caused by the reduction of J_B. Another aspect is the existence itself of this equilibrium. With respect to this, and without entering all the details of the full proof, given in [7], let us examine the mechanism by which the present choice of territory splitting also permits to guarantee the preservation of initial optimum point of discipline A alone, Y_A^*, as a Nash equilibrium of the above formulation for $\varepsilon = 0$, as stated in (23).

For $\varepsilon = 0$, let the criterion $J_A = J$ for notational simplicity. The criteria J_{AB} and J are functionally proportional, and so are their gradients. We wish to establish that $U = V = 0$ indeed corresponds to a Nash equilibrium.

On one side, for fixed $V = 0$, the subvector $U = 0$ indeed realizes the minimum of $J_A = J$ subject to the constraint $g = 0$, because the optimization of U is equivalent to the minimization of J_A in a subset that contains the solution Y_A^* of the minimization in the full design space.

On the other side, for fixed $U = 0$, the (unconstrained) derivative of J_{AB} w.r.t. V is proportional to :

$$\frac{\partial J}{\partial V} = \nabla J \,.\, \frac{\partial Y}{\partial V} = \nabla J_A^* \,.\, \frac{\partial Y}{\partial V} = 0$$

because, by construction of the split, the vector $\frac{\partial Y}{\partial V}$ is a linear combination of the tail elements of the eigenvector basis, and these are orthogonal to the first K elements, and those span a subspace containing ∇J_A^*. Hence, for fixed $U = 0$, the unconstrained criterion $J_{AB} \sim J$ is also stationary w.r.t. subvector V at $V = 0$. □

In summary, this theorem establishes two main achievements related to the Nash equilibrium solution :

- A potential performance result : it permits to identify abstractly an orthogonal decomposition of the parameter-space that is such that for given dimension p ($p \leq N - \max(K,1)$), the tail p vectors of the basis correspond to the directions of least variation of the primary functional J_A from its minimum value under possible equality constraints; in this sense, these eigenvectors span the subspace of dimension p in which the primary functional is the most insensitive to the small variations in the design vector that will be made, in a second phase of optimization, to reduce a secondary functional, J_B;
- An existence result : a procedure involving a continuation parameter ε ($0 \leq \varepsilon \leq 1$) has been set up permitting to introduce gradually and smoothly the secondary functional J_B in competition with the primary functional J_A in a Nash game; for $\varepsilon = 0$, it is established that the original optimal solution Y_A^* is a Nash equilibrium point of the initially-trivial game formulation; consequently, by continuity, the Nash equilibrium solution exists, at least for ε sufficiently small. Another parameter θ appears in the formulation; it allows under or over-relaxation of the process; if $\theta < 1$, the auxiliary criterion J_{AB} at the Nash equilibrium point $\overline{Y}_\varepsilon$ decreases when ε increases, but remains sufficiently small; since $\overline{Y}_0 = Y_A^*$, the locus of $\overline{Y}_\varepsilon$ as ε varies is viewed as a continuation of the original optimum point of the primary functional alone.

The construction of the orthogonal basis is made at full convergence of the minimization of the primary functional by diagonalization of the Hessian matrix restricted to the subspace tangent to the hypersurfaces representing the active constraints. To identify this tangent subspace, a Gram-Schmidt orthogonalization process is applied to the constraint gradients. In practice, the Hessian can be calculated exactly either formally or by automatic differentiation; otherwise, an approximation can be made by differentiating a *meta-model* for the primary functional and constraints valid in a neighborhood of the optimal solution Y_A^*. This meta-model can be, for example, an artificial neural network or a Kriging model (see for instance [6] [10]).

We close this section by emphasizing again the merit of our formulation, when equality constraints are active, to remain consistent with the single-criterion minimization of the primary functional alone at the initial point $\varepsilon = 0$ of the continuation procedure ($\overline{Y}_0 = Y_A^*$). This nontrivial property usually does not hold when the split is made over the primitive variables as formerly proposed in [30] [28], unless the constraints are treated by the penalty approach. The variations in the primary functional are initially second-order in ε; thus the new formulation permits to identify smoothly the locus of Nash equilibrium solutions as ε varies, by an algorithm whose iterative convergence is facilitated by this robustness property, since the potential antagonism between the two criteria can be introduced as smoothly as necessary by small enough steps in the continuation parameter ε.

2 Application of territory splitting to the aero-structural shape optimization of a business jet wing

In order to illustrate the influence of the split of territory on the result of a practical two-discipline optimization, the main results achieved by B. Abou El Majd in his doctoral thesis [1] concerning a case of aero-structural shape optimization of a business jet wing, also in [2], are reproduced here. In his thesis, a number of algorithmic variants, including some whose formulations rely on a hierarchical Stackelberg game (instead of a symmetrical Nash game), have been described in details, tested and analyzed systematically.

Aerodynamics is treated as the preponderant discipline; it will also reveal to be a fragile discipline. The flow about the wing is computed by a finite-volume simulation of the three-dimensional Euler equations. The method handles unstructured grids by the construction of a dual finite-volume mesh, whose generic cell is around a node and its boundary is made of portions of medians of the elements. The approximation scheme relies on a Roe-type upwind solver. The computation yields the wave drag coefficient, C_D, as well as other aerodynamic coefficients, such as lift, C_L. The simulation point is transonic ($M_\infty = 0.83$, $\alpha = 2^o$). The primary objective is to minimize the drag coefficient augmented by a penalty term which is active when a minimal lift coefficient constraint is violated. Thus, the primary criterion admits the following expression :

$$J_A = \frac{C_D}{C_{D_0}} + 10^4 \max\left(0, 1 - \frac{C_L}{C_{L_0}}\right) \tag{31}$$

in which the reference quantities, indicated by the subscript $_0$ correspond to an initial geometry defined by an initial three-dimensional unstructured grid about the wing.

Throughout the optimization process, the geometry is iteratively modified according to the so-called *Free-Form Deformation (FFD)* method which originates from computer vision, and was proposed in the context of an aero-structural design loop by Samareh [24]. In this approach, a formula is given *a priori*, in a closed form involving adjustable parameters, to a three-dimensional *deformation field*, formally and independently of the discrete or continuous representation of the geometry itself, here an unstructured volume mesh. By construction, the deformation field is made to be smooth and equal to zero outside of a support, which is usually a bounding box of simple shape whose boundaries are not made in general of meshpoints. At a given optimization iteration, the deformation field is redefined and applied to the meshpoints lying inside the support, thus permitting an update of the surface meshpoints, but also of meshpoints in the computed volume in the vicinity of the optimized surface. In this way, an initial unstructured volume mesh evolves according to a deformation defined explicitly in terms of the *FFD* parameters. These parameters are taken to be the design variables of the optimization loop and they are updated here according to the Nelder-Mead [22] simplex method to reduce the above criterion J_A.

This procedure results in a simple and fairly robust iterative algorithm. In our experience, this procedure is less subject to mesh overlapping than a volume mesh reconstruction from the displacement of the boundary meshpoints by a pseudo-elasticity equation, such as the spring method.

In our experiments, a system of generalized coordinates (ξ, η, ζ) is defined and corresponds to longitudinal, vertical and span-wise directions.When the bounding box is a parallelepiped, the transfinite interpolation of the Cartesian coordinates suffices to define these transformed coordinates throughout the box. Then, the deformation field is defined as a linear combination of products of three Bernstein polynomials of these coordinates. Precisely, an arbitrary point q is given the following displacement Δq :

$$\Delta q = \sum_{i=0}^{n_i} \sum_{j=0}^{n_j} \sum_{k=0}^{n_k} B_{n_i}^i(\xi_q)\, B_{n_j}^j(\eta_q)\, B_{n_k}^k(\zeta_q)\, \Delta P_{ijk} \tag{32}$$

in which, for the kth Bernstein polynomial of degree n,

$$B_n^k(t) = \frac{n!}{k!\,(n-k)!}\, t^k\, (1-t)^{n-k} \tag{33}$$

The degrees of the parametrization in the three physical directions, (n_i, n_j, n_k), are fixed, and the vector-valued weighting coefficients $\{\Delta P_{ijk}\}$ ($0 \le i \le n_i$, $0 \le j \le n_j$, $0 \le k \le n_k$) are the design variables of the optimization. Such a geometrical parametrization generalizes the Bézier curve formula, and combined with the classical degree-elevation process, it facilitates the construction of multilevel optimization algorithms inspired by multigrid methods. More details on this method, and more examples of application can be found in [9] [2].

The deformation field was chosen to be linear span-wise from root to tip ($n_k = 1$). Additionally, the leading and trailing edges, and the eight vertices of the bounding box were fixed throughout the optimization. Finally, only vertical displacements were considered for simplicity.

In a first experiment (see Fig. 7), 6 control points at the root and at the tip were considered, for a total of 12 degrees of freedom.

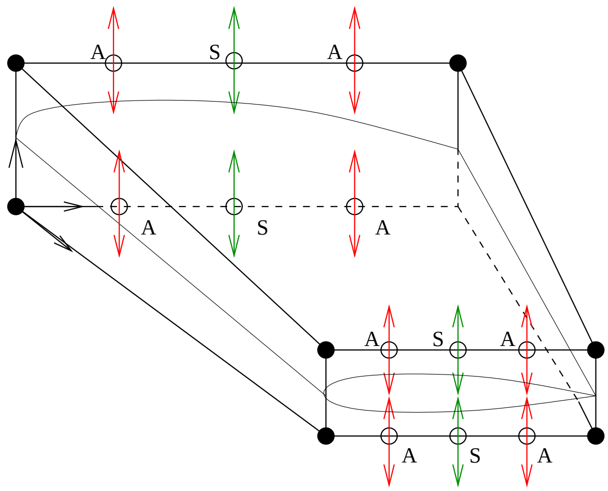


Fig. 7 Aero-structural shape optimization of a business jet wing; first split of territory, according to the primitive variables : parameters marked A are associated to aerodynamics, and those marked S to structural design.

In order to define an exercise in which the wing shape is optimized w.r.t. two disciplines, aerodynamics and structural design, that share a common set of design variables, the wing structure was treated as a thin shell which deforms under the load of aerodynamic forces. The distribution of stresses over the shell has been calculated by linear-elasticity, using a code of the public domain, ASTER developed by *Electricité de France (EDF)*.

The four degrees of freedom located at mid-chord (at root and tip, over the upper and lower surfaces), marked S on Fig. 7, were assigned to a player B (or S) in charge of minimizing the following secondary criterion :

$$J_B = J_S = \iint_S \|\sigma.n\| \, dS + K_1 \max\left(0, 1 - \frac{V}{V_A}\right) + K_2 \max\left(0, \frac{S}{S_A} - 1\right) \qquad (34)$$

in which σ is the stress tensor, S_A and V_A are the wing outer surface and volume at convergence of the purely-aerodynamic optimization, and K_1 and K_2 and penalty constants. By the reduction of this criterion, one expects a more uniform distribution of the load, and thus a more robust structure.

The remaining 8 degrees of freedom, marked A on Fig. 7, were assigned to a player A in charge of minimizing the primary criterion, J_A.

It was possible to achieve a Nash equilibrium solution associated with the above split of the *primitive variables*, as indicated on Fig. 8 which displays the convergence history of the aerodynamic and structural criteria. The sudden and occasional peaks correspond to iterations at which the constraint on lift is violated. The simplex method accommodates to this situation by discarding the point. Evidently, a stable Nash equilibrium is reached eventually.

Regrettably, this Nash-equilibrium configuration is totally unacceptable from a physical standpoint. The drag coefficient has doubled. The wing shape presents oscillations and the flow has been profoundly disrupted as indicated by the Mach number field (see Fig. 9).

Besides, the number of iterations in this experiment may be found excessive. It should be pointed out that drag reduction problems are well known to be multimodal. They exhibit a *very large number of local minimums*. Gradient-based methods are very cost efficient and useful in the final stage of convergence. But, if they converge in tens of iterations, in practice, they notably fail to provide a good estimate of the global optimum, unless the initial point is itself very close to it. Inversely, semi-stochastic methods, such as Genetic Algorithms, or Particle-Swarm optimizers, are far more robust, but often prohibitively expensive in aerodynamic optimum-shape design, due to the large number of flows required to be computed. For these reasons, for problems of intermediate difficulty, an acceptable compromise is often realized by the simplex method, which is deterministic, but fairly robust. With this optimizer, the number of iterations, or computed flows governed by the compressible Euler equations in three dimensions, can be substantial, to achieve a satisfactory convergence on a nontrivial mesh, say, in hundreds, as in subsequent experiments (Figures 11 and 13). The even slower convergence in Figure 8 precisely reveals an inappropriate coupling. Nevertheless, by exploiting the resources of a parallel architecture, this experiment could be realized in a day.

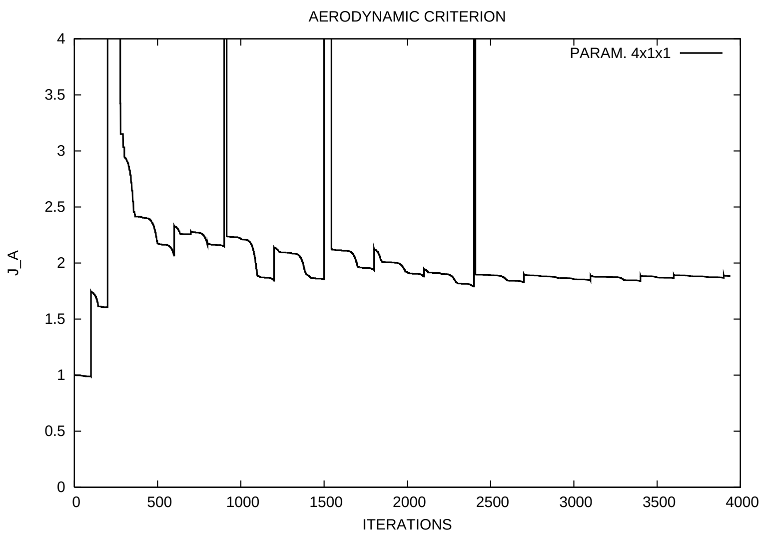
By this first experiment, we emphasize that even in case of convergence to a Nash equilibrium, the achieved configuration makes sense only if the split of variables is physically relevant.

In a second experiment, the number of design variables was reduced to 8 by considering a deformation field, only vertical and associated with the polynomial degrees $(3,1,1)$ along the longitudinal, vertical and span-wise directions. After a number of unsuccessful trials, a certain split of the primitive variables yielded acceptable results. The split corresponds to assign the 4 degrees of freedom at the root to player S $(=B)$ in charge of reducing the structural criterion, and the other 4, at the tip, to player A in charge of reducing the aerodynamic criterion (see Fig. 10).

The convergence history of the two criteria in the dynamic game corresponding to this new split of design variables is indicated at Fig. 11. The aerodynamic criterion is subject to numerous jumps due to the violation of the constraint on lift, but, as mentioned above, the simplex method accommodates to this. This phase of optimization is interrupted, somewhat arbitrarily after some 380 structural design steps; strictly speaking, convergence is not achieved, but the solution satisfactory since it realizes a visible improvement of the structural criterion of about 5 %, while the aerodynamic criterion has been increased of about the same percentage (only).

The cross sections at root, mid-span and wing tip corresponding to the initial and optimized shapes are represented on Fig. 12. It appears that the structural control parameters tend to round out very slightly the root cross section for a better load distribution. This trend augments the drag, but here in proportions still acceptable, because the process was inter-

a) Drag (J_A)



b) Stress integral (J_S)

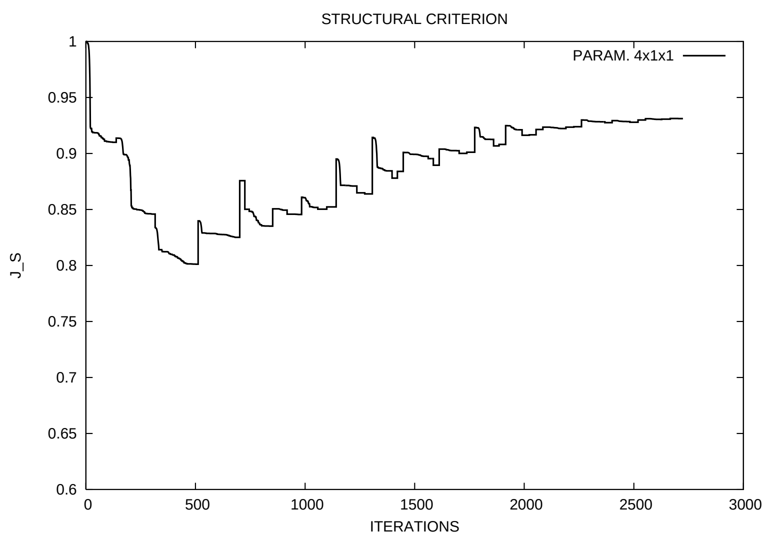
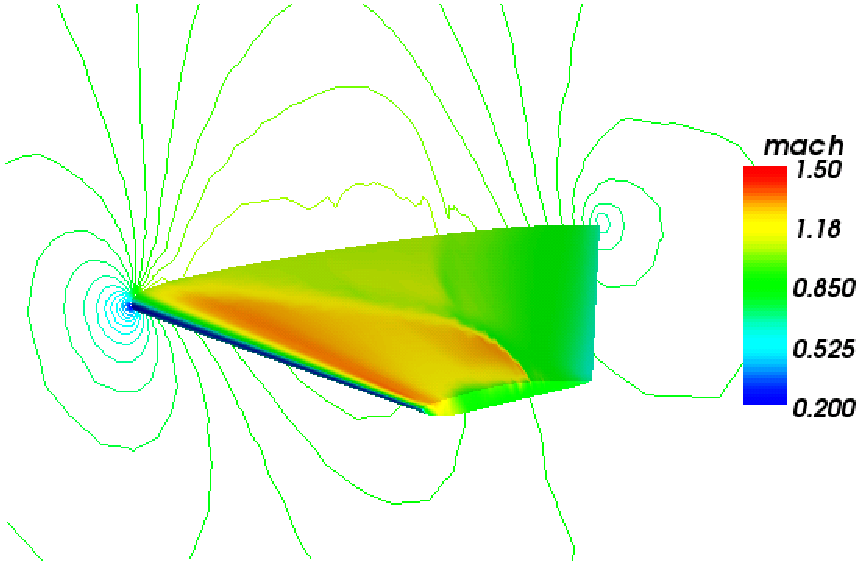


Fig. 8 Aero-structural shape optimization of a business jet wing; first split of territory, according to the primitive variables : convergence history of the aerodynamic and structural criteria.

rupted after a variation of 5 % of each criterion. In fact, at this level of only partial convergence, the shape variations are still very small in amplitude because the coupling mechanism realized by the dynamic game is very stringent. Additionally, our *a priori* knowledge of the flow led us to locate the aerodynamic control parameters near the wing tip in the vicinity of the most sensitive region of the shock wave. Thus, this experiment does not reflect a blind *a) Purely aerodynamic optimization*



b) Nash equilibrium (unacceptable)

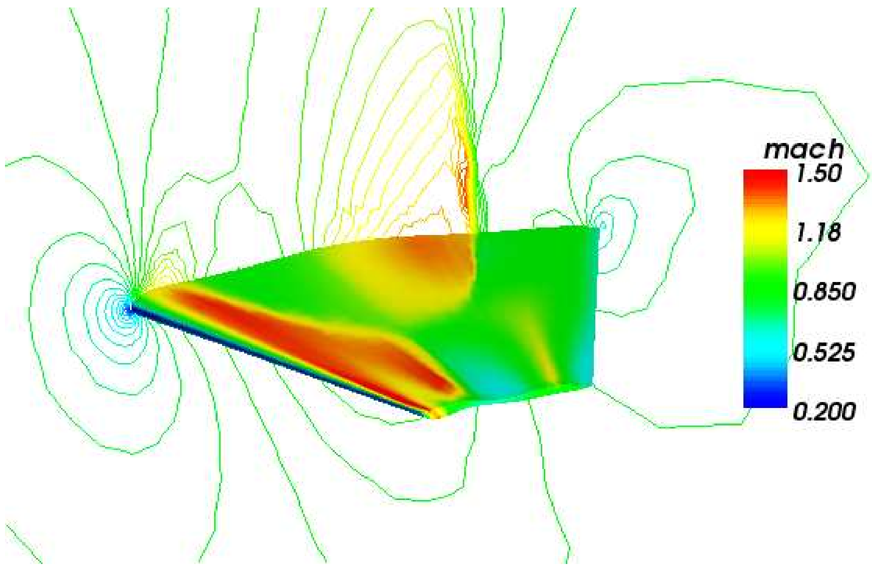


Fig. 9 Aero-structural shape optimization of a business jet wing; first split of territory, according to the primitive variables : shape and Mach number field : a) purely aerodynamic optimization, and b) Nash equilibrium.

split of variables, but instead one that was anticipated to be physically sound; and this was confirmed.

In the third experiment, the split of variables based on the proposed orthogonal decomposition of the restricted Hessian was implemented. Once the optimum of aerodynamics alone has been found at $Y = Y_A^*$, a number of independent simulations corresponding to de-

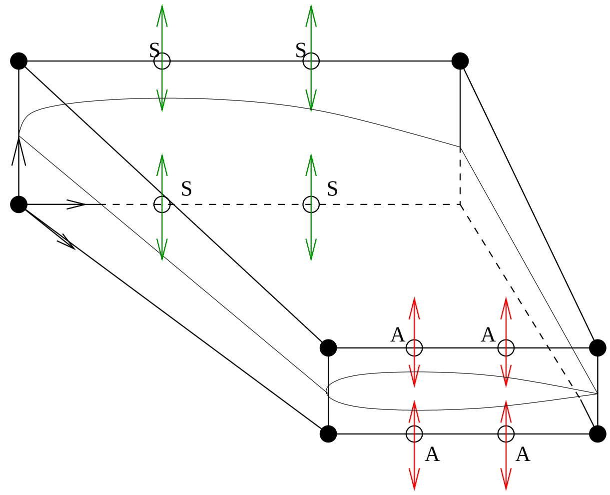


Fig. 10 Aero-structural shape optimization of a business jet wing; second split of territory, according to the primitive variables : parameters marked A are associated to aerodynamics, and those marked S to structural design.

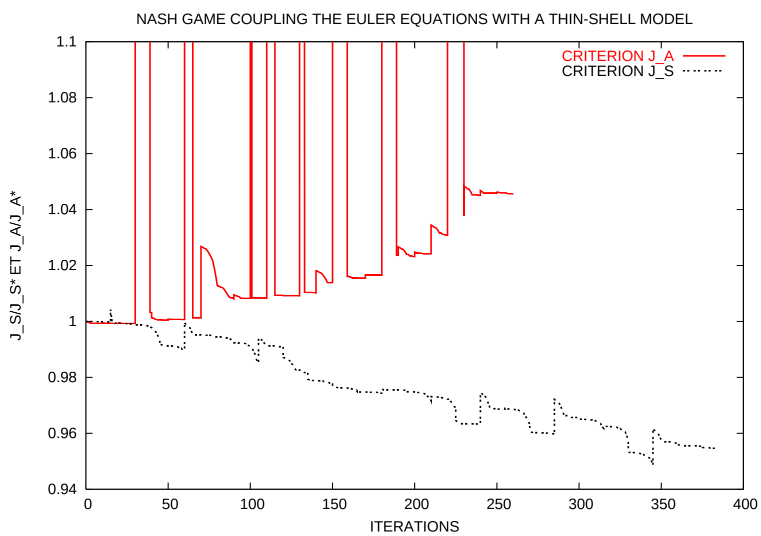


Fig. 11 Aero-structural shape optimization of a business jet wing; second split of territory, according to the primitive variables : convergence history of the two criteria

sign vectors close to Y_A^* have been made to set up a database to model the behavior of the primary criterion J_A in terms of Y by an RBF neural network [6] [10]. This meta-model was then used to approximate the gradient of C_D, the primary criterion to be minimized, the gradient of C_L, the constrained quantity, and the Hessian of C_D to form the restricted Hessian matrix. After diagonalization, the corresponding eigenvectors have been sorted by decreasing order of the associated eigenvalue, and split evenly in two subsets of four. Those associated with the four largest eigenvalues have been assigned to player A in charge of

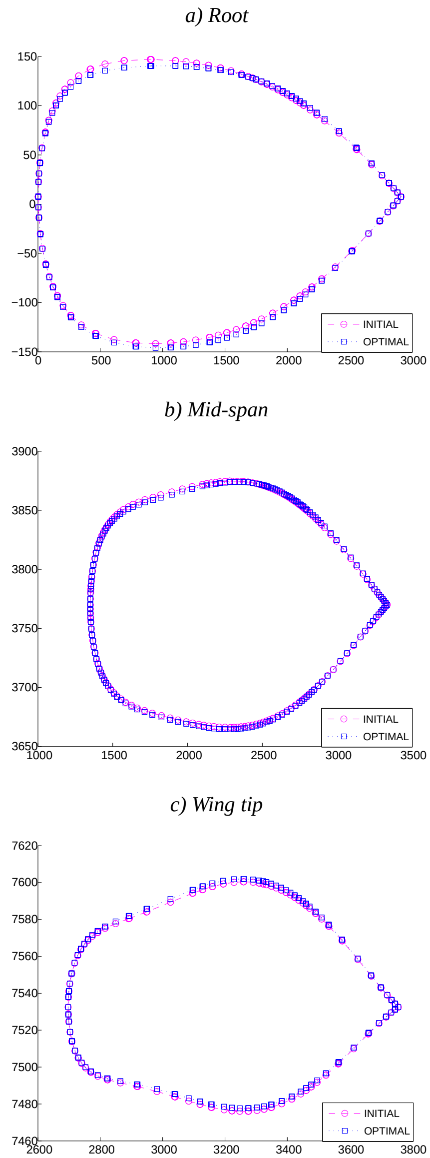


Fig. 12 Aero-structural shape optimization of a business jet wing; second split of territory, according to the primitive variables : cross-section variations at a) root, b) mid-span, and c) wing tip.

aerodynamics, and the remaining four to player S (=B) in charge of reducing the criterion of structural design.

The proposed eigensplit led to a new dynamic Nash game, whose convergence history is indicated on Fig. 13. The process was continued to a stage of convergence similar to previously in terms of coupling iterations. However, a notably superior performance was achieved : while the aerodynamic criterion was here only degraded of 3 %, the structural criterion was reduced of 8 %; equivalently, at equal stage of drag degradation, the improvement on the structural criterion is nearly three times larger. Note how the envelopes of the two curves are apparently initially tangent to the horizontal axis, a hint that in this formulation, the initial point is a robust design.

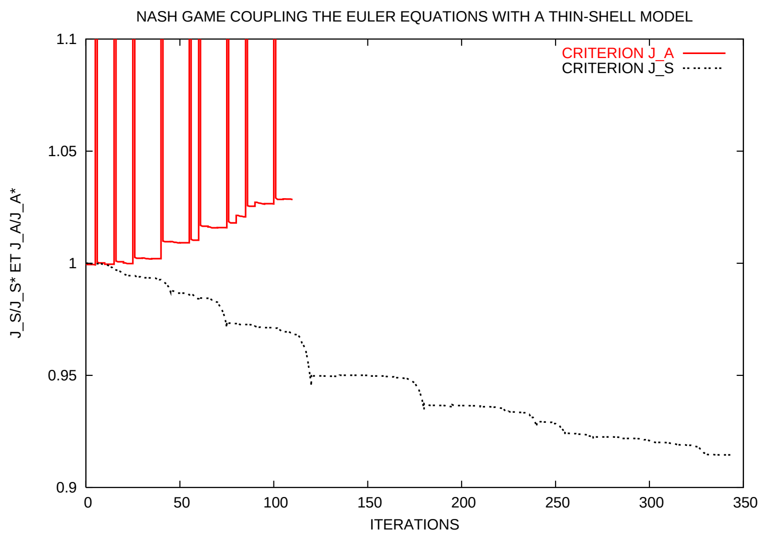


Fig. 13 Aero-structural shape optimization of a business jet wing; split of variables according to the orthogonal decomposition; convergence history of the two criteria (after 50 couplings).

Fig. 14 indicates the evolution of cross-sections at root, mid-chord and wing tip. It clearly appears from this figure that the shape variations are of larger amplitude in this experiment than before, in the previous two experiments, but more distinctly located, as for example, on the lower surface of the wing at the root. Thus a wider operational territory for the secondary criterion is identified to cause a small and acceptable degradation only of the first criterion.

The split based on the orthogonal decomposition has permitted us to identify by a blind and automatic procedure, a set of structural parameters for which variations of larger amplitude, mostly visible on the lower surface of the wing, are possible without excessively affecting the shape in the critical region of the shock wave. Consequently, the principal characteristics of the flow are preserved, as indicated on Fig. 15 which shows that the Mach number field has not been much altered from that obtained by pure aerodynamic optimization.

Thus, in conclusion, a significant reduction of 8 % of the structural criterion was realized while maintaining the flowfield configuration close to optimality (drag increase $< 3\,\%$), by an automatic procedure of orthogonal decomposition of the parameter space.

3 Cooperative multiobjective optimization

In the previous sections, we have seen how competition between two disciplines could be organized when starting from an initial design that is optimal w.r.t. one discipline, considered

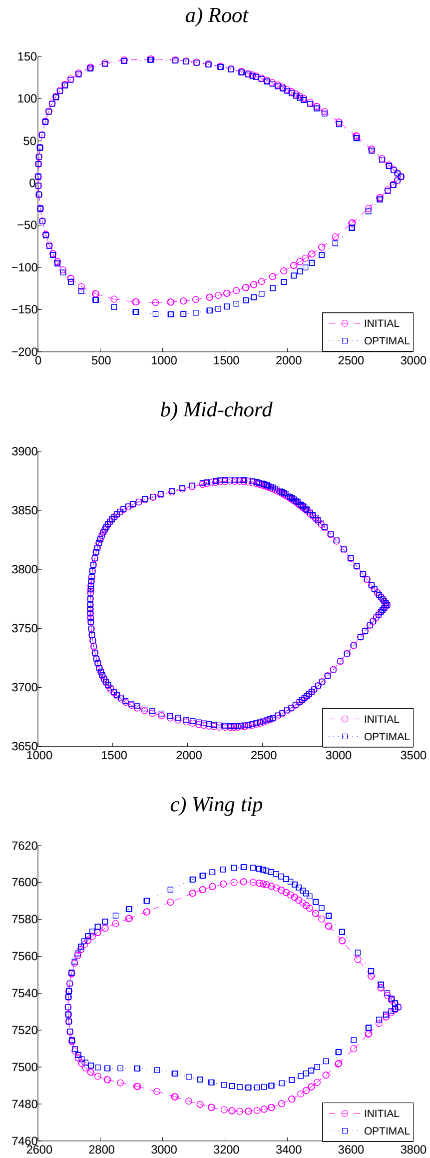
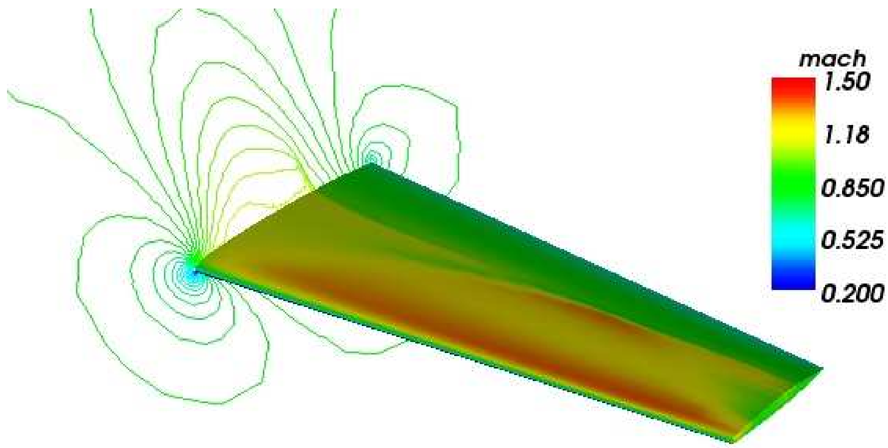


Fig. 14 Aero-structural shape optimization of a business jet wing; split of variables according to the orthogonal decomposition; cross-section variations at a) root, b) mid-chord, and c) wing tip.

to be preponderant or fragile. However, in more general situations, the initial design solution may be far from Pareto optimality w.r.t. the criteria under consideration. Then, the possibility exists to firstly improve all the criteria prior to organizing a competition between them. Additionally, we would like to provide some recommendation for cases of more than two criteria.

a) Initial aerodynamic optimum solution



b) Aero-structural Nash game solution using the orthogonal decomposition

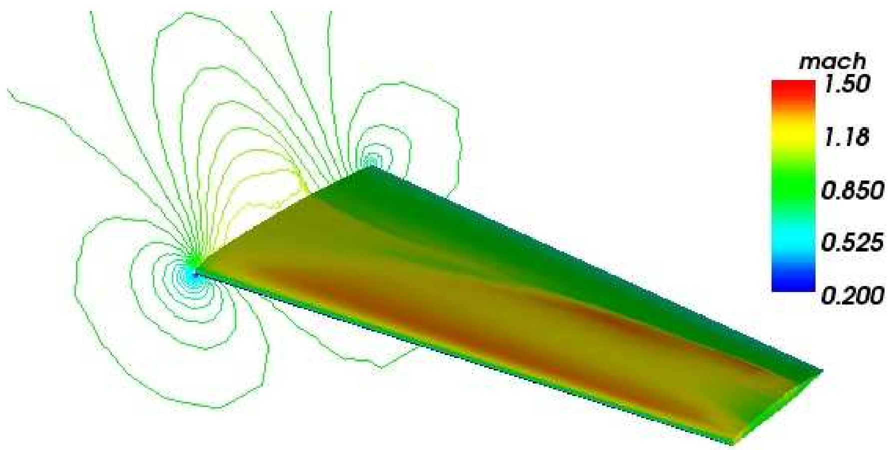


Fig. 15 Geometrical configuration and Mach number field : a) initial aerodynamic optimum solution, and b) aero-structural Nash game solution using the orthogonal decomposition.

In this section, we examine more general situations in which the number of disciplines, n, and the initial design solution are both arbitrary. In such cases, we define a preliminary optimization phase, cooperative in nature, throughout which all disciplines improve, to be followed by a competitive two-criterion optimization phase.

We first refer to the textbook by K. Miettinen [20] for a detailed review of fundamentals in nonlinear multiobjective optimization, and much more. Here, we simply formulate a number of theoretical results that are basic, but essential to our subsequent algorithmic construction.

Thus consider n smooth criteria $J_i(Y)$ (Y : design vector; $Y \in \mathscr{H}$; $\mathscr{H}$: working space, a Hilbert space usually equal to $\mathbb{R}^N$, but possibly a subspace of L^2 also). In practice, these functions or functionals are assumed to be of class C^2 in some working open ball of the design space $\mathscr{H}$. Throughout this report, unless specified otherwise, the symbol N denotes the dimension of the design-space $\mathscr{H}$ when it is finite, in which case it is assumed that $n \leq N$, and the symbol ∞ otherwise. Then the following holds :

Lemma 1
Let Y^0 be a Pareto-optimal point of the smooth criteria $J_i(Y)$ ($1 \leq i \leq n \leq N$), and define the gradient-vectors $u_i^0 = \nabla J_i\left(Y^0\right)$ in which ∇ denotes the gradient operator. There exists a convex combination of the gradient-vectors that is equal to zero:

$$\sum_{i=1}^{n} \alpha_i u_i^0 = 0\,, \qquad \alpha_i \geq 0\ (\forall i)\,, \qquad \sum_{i=1}^{n} \alpha_i = 1\,. \tag{35}$$

Proof : The proof can be made by examining different cases according to the rank of the family made of the gradient-vectors. We refer to e.g. [8] for details. □

This result has led us to introduce and use throughout the following very natural definition, although the terminology does not seem to be standard :

Definition 1 (Pareto-stationarity)
The smooth criteria $J_i(Y)$ ($1 \leq i \leq n \leq N$) are said to be Pareto-stationary at the design-point Y^0 iff there exists a convex combination of the gradient-vectors, $u_i^0 = \nabla J_i(Y^0)$, that is equal to zero:

$$\sum_{i=1}^{n} \alpha_i u_i^0 = 0\,, \qquad \alpha_i \geq 0\ (\forall i)\,, \qquad \sum_{i=1}^{n} \alpha_i = 1\,. \tag{36}$$

Thus, in general, for smooth unconstrained criteria, Pareto-stationarity is a necessary condition for Pareto-optimality. Inversely, if the smooth criteria $J_i(Y)$ ($1 \leq i \leq n$) are not Pareto-stationary at a given design-point Y^0, descent directions common to all criteria exist. We now examine how such a direction can be identified. We have the following :

Lemma 2
Let $\mathscr{H}$ be a Hilbert space of finite or infinite dimension N, and $\{u_i\}$ ($1 \leq i \leq n \leq N$) a family of n vectors in $\mathscr{H}$. Let $\mathscr{U}$ be the set of strict convex combinations of these vectors :

$$\mathscr{U} = \left\{ w \in \mathscr{H} \ / \ w = \sum_{i=1}^{n} \alpha_i u_i\,;\ \alpha_i > 0\ (\forall i)\,;\ \sum_{i=1}^{n} \alpha_i = 1 \right\} \tag{37}$$

and $\overline{\mathscr{U}}$ its closure (the convex hull of the family). Then, there exists a unique element $\omega \in \overline{\mathscr{U}}$ of minimum norm, and :

$$\forall \bar{u} \in \overline{\mathscr{U}} \ :\ (\bar{u}, \omega) \geq (\omega, \omega) = \|\omega\|^2 := C_\omega \tag{38}$$

Proof : the convex hull $\overline{\mathscr{U}}$ is a closed and convex set, and this implies existence and uniqueness of the element ω of minimum norm in $\overline{\mathscr{U}}$.

Then, let $\bar{u}$ be an arbitrary element of $\overline{\mathscr{U}}$; set $r = \bar{u} - \omega$ so that $\bar{u} = \omega + r$. Since the convex hull $\overline{\mathscr{U}}$ is convex,

$$\forall \varepsilon \in [0,1]\,,\ \omega + \varepsilon r \in \overline{\mathscr{U}} \tag{39}$$

Since ω is the element of $\overline{\mathscr{U}}$ of minimum norm, $\|\omega+\varepsilon r\| \geq \|\omega\|$, which writes :

$$\|\omega+\varepsilon r\|^2 - \|\omega\|^2 = (\omega+\varepsilon r, \omega+\varepsilon r) - (\omega,\omega) = 2\varepsilon\,(r,\omega) + \varepsilon^2\,(r,r) \geq 0 \tag{40}$$

and since ε can be arbitrarily small, this requires that :

$$(r,\omega) = (\overline{u}-\omega,\omega) \geq 0 \tag{41}$$

from which the result follows directly. □

Combining Lemma 2 with Definition 1 yields the following :

Theorem 2

Let $\mathscr{H}$ be a Hilbert space of finite or infinite dimension N. Let $J_i(Y)$ ($1 \leq i \leq n \leq N$) be n smooth functions of the vector $Y \in \mathscr{H}$, and Y^0 a particular admissible design-point, at which the gradient-vectors are denoted $u_i^0 = \nabla J_i(Y^0)$, and

$$\mathscr{U} = \left\{ w \in \mathscr{H} \;/\; w = \sum_{i=1}^{n} \alpha_i\, u_i^0\,;\; \alpha_i > 0\ (\forall i)\,;\; \sum_{i=1}^{n} \alpha_i = 1 \right\} \tag{42}$$

Let ω be the minimal-norm element of the convex hull $\overline{\mathscr{U}}$, closure of $\mathscr{U}$. Then :

1. *either $\omega = 0$, and the criteria $J_i(Y)$ ($1 \leq i \leq n$) are Pareto-stationary at $Y = Y^0$;*
2. *or $\omega \neq 0$ and $-\omega$ is a descent direction common to all the criteria; additionally, if $\omega \in \mathscr{U}$, the inner product $(\overline{u},\omega)$ is equal to $\|\omega\|^2$ for all $\overline{u} \in \overline{\mathscr{U}}$.*

Proof : all the elements of this theorem are reformulations of previous results, except for the statement concerning the inner product $(\overline{u},\omega)$ in the second case when additionally $\omega \in \mathscr{U}$ (and not simply $\overline{\mathscr{U}}$). To establish this last point, observe that under these assumptions, the element ω is the solution to the following minimization problem :

$$\omega = u = \sum_{i=1}^{n} \alpha_i u_i^0\,,\ \alpha = \mathsf{Argmin}\,\mathsf{j}(u)\,,\ \mathsf{j}(u) = (u,u)\,,\ \sum_{i=1}^{n} \alpha_i = 1 \tag{43}$$

since by hypothesis, none of the inequality constraints, $\alpha_i > 0$, is saturated. Consequently, using the vector $\alpha \in \mathbb{R}^n$ as the finite-dimensional variable, the Lagrangian writes :

$$\mathsf{L}(\alpha,\lambda) = \mathsf{j} + \lambda \left(\sum_{i=1}^{n} \alpha_i - 1 \right) \tag{44}$$

and the optimality conditions satisfied by the vector α are the following :

$$\frac{\partial \mathsf{L}}{\partial \alpha_i} = 0\ (\forall i)\,,\ \frac{\partial \mathsf{L}}{\partial \lambda} = 0 \tag{45}$$

These equations imply that for all indices i :

$$\frac{\partial \mathsf{j}}{\partial \alpha_i} + \lambda = 0 \tag{46}$$

But, $\mathsf{j}(u) = (u,u)$ and for $u = \omega = \sum_{i=1}^{n} \alpha_i u_i$, one has :

$$\frac{\partial \mathsf{j}}{\partial \alpha_i} = 2(\frac{\partial u}{\partial \alpha_i}, u) = 2(u_i^0,\omega) = -\lambda \Longrightarrow (u_i^0,\omega) = -\lambda/2 \tag{47}$$

independently of i. Now consider an arbitrary element $\bar{u} \in \overline{\mathscr{U}}$:

$$\bar{u} = \sum_{i=1}^{n} \mu_i \, u_i^0 \tag{48}$$

where $\mu_i \geq 0 \; (\forall i)$ and $\sum_{i=1}^{n} \mu_i = 1$. Then :

$$(\bar{u}, \omega) = \sum_{i=1}^{n} \mu_i \, (u_i^0, \omega) = -\lambda/2 \text{ (a constant)} = \|\omega\|^2 \tag{49}$$

where the constant has been evaluated by letting $\bar{u} = \omega$. □

In summary, one is led to identify the vector

$$\omega = \sum_{i=1}^{n} \alpha_i \, u_i^0 \tag{50}$$

by solving the following quadratic-form constrained minimization problem in $\mathbb{R}^n$:

$$\min_{\alpha \in \mathbb{R}^n} \left\| \sum_{i=1}^{n} \alpha_i \, u_i^0 \right\|^2 \tag{51}$$

subject to :

$$\alpha_i \geq 0 \; (\forall i) \, , \; \sum_{i=1}^{n} \alpha_i = 1 \tag{52}$$

Note that in a finite-dimensional setting, and in a functional-space setting as well, the above problem can be solved in $\mathbb{R}^n$, so long as the gradients $\{u_i^0\}$ $(1 \leq i \leq n)$ and their inner products $\{u_{ij}^0 := (u_i^0, u_j^0)\}$ are known. Then, a call to a library procedure should be sufficient.

4 Combining cooperation with competition in a strategy for multiobjective optimization

In this section, we collect the results of the previous two sections to develop a global strategy for multiobjective optimization. The criteria under consideration, denoted $J_i(Y)$ $(1 \leq i \leq n)$, where $n \geq 2$, are again smooth functions of the design vector Y. At the initial design-point Y^0, the condition of Pareto-stationarity is not satisfied. Then, we propose to develop the optimization process in several stages described in the following subsections.

4.1 Optional preliminary reformulation of criteria

In numerical experiments, it is preferable that the various criteria all be positive, and scaled in a somewhat unified way. To achieve this, we propose to modify the definitions of the criteria without altering the sense of the associated minimization problems.

For this purpose, let :

$$\mathscr{B}_R = \mathscr{B}\left(Y^0, R\right) \tag{53}$$

be a working ball in the design space about the initial design-point Y^0.

In a first step, we propose to replace each criterion $J_i(Y)$ by the following :

$$\tilde{J}_i(Y) = \exp\left(\alpha_i \frac{\|H_i^0\|}{\|\nabla J_i^0\|^2} \left(J_i(Y) - J_i^0\right) \right) \tag{54}$$

where :

– the superscript ^{0} indicates an evaluation at $Y = Y^0$;
– ∇J_i^0 and H_i^0 denote the gradient-vector and the Hessian matrix, and $\left\|H_i^0\right\|$ can be computed economically as : $\sqrt{\text{trace}\left[\left(H_i^0\right)^2\right]}$;
– α_i is a dimensionless constant.

In this way, the new criteria are dimensionless, they vary in the same sense as the original ones, and :

$$\forall i\,,\ \tilde{J}_i(Y^0) = 1\,,\ \nabla J_i(Y^0) = \frac{\gamma}{R} \tag{55}$$

provided the constants α_i's are chosen to satisfy :

$$\alpha_i \frac{\left\|H_i^0\right\|}{\left\|\nabla J_i^0\right\|} = \frac{\gamma}{R} \sim 1 \tag{56}$$

In the above, the dimensionless constant γ is given a value equal or close to the possibly-dimensional measure of R in the utilized system of units.

For a reason that will appear later, without altering the regularity of the criteria, we would like them to be infinite when $\|Y\|$ is infinite. For this, define the following function :

$$\phi(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ x \exp\left(-\dfrac{1}{x^2}\right) & \text{if } x > 0 \end{cases} \tag{57}$$

This function is C^∞ including at 0, and $\phi(x) \sim x$ as $x \to +\infty$. The new criterion

$$\tilde{\tilde{J}}_i(Y) = \tilde{J}_i(Y) + \varepsilon_0\, \phi\left(\frac{\left\|Y - Y^0\right\|^2}{R^2} - 1\right) \tag{58}$$

in which ε_0 is some strictly-positive constant, is identical to the former one, $\tilde{J}$, inside the working ball $\mathscr{B}_R$, and grows at least like $\|Y\|^2$ outside. The match of $\tilde{\tilde{J}}_i(Y)$ with $\tilde{J}_i(Y)$ and $J_i(Y)$ at the boundary of the working ball is infinitely smooth. Additionally :

$$\lim_{\|Y\| \to \infty} \tilde{\tilde{J}}_i(Y) = \infty \tag{59}$$

In what follows, it is implicit that the original criteria have been replaced by $\{\tilde{\tilde{J}}_i(Y)\}$ $(1 \leq i \leq n)$ and the double superscript $\tilde{\tilde{}}$ is omitted.

4.2 Cooperative-optimization phase : the Multiple-Gradient Descent Algorithm (*MGDA*)

The *MGDA* relies on the results of Theorem 2. The *MGDA* consists in iterating the following sequence :

1. Compute the gradient-vectors $u_i^0 = \nabla J_i(Y^0)$, and determine the minimum-norm element ω in the convex hull $\overline{\mathscr{U}}$. If $\omega = 0$, stop.
2. Otherwise, determine the step-size h which is, presumably optimally, the largest strictly-positive real number for which all the functions $j_i(t) = J_i(Y^0 - t\omega)$ $(1 \leq i \leq n)$ are monotone-decreasing over the interval $[0, h]$.
3. Reset Y^0 to $Y^0 - h\omega$, and return to 1.

In practice, the test $\omega = 0$ will be made with a tolerance ($\|\omega\| < tol$). In addition, note that the determination of the step-size h can be realized by the adaptation of nearly all standard one-dimensional search methods. This algorithm can be repeated a finite number of iterations if these iterations yield a design-point at which the Pareto-stationarity condition is satisfied, or indefinitely, if this never occurs.

Since at each iteration of the *MGDA*, all the criteria diminish, we refer to this process as a *cooperative-optimization* phase.

4.3 Convergence of the *MGDA*

The above *MGDA* can stop after a finite number of iterations if a Pareto-stationary design-point is reached. Otherwise, we have the following :

Theorem 3
If the sequence of iterates $\{Y^r\}$ of the MGDA is infinite, it admits a weakly convergent subsequence. (Here, the working Hilbert space $\mathscr{H}$ is assumed to be reflexive.)

Proof : the following elements hold, in part by virtue of the reformulation of the criteria :

- Since the sequence of values of any considered criterion, say $\{J_1(Y^r)\}$, is positive and monotone-decreasing, it is bounded.
- Since $J_1(Y)$ is infinite whenever $\|Y\|$ is infinite, the sequence of design-vectors $\{Y^r\}$ is itself bounded, and this implies the statement. □

Let Y^* be the limit. We conjecture that the design-point Y^* is Pareto-stationary. In what follows, Y^0 is then reset to Y^*.

4.4 Competitive-optimization phase : strategy of the greatest payoff

From the initial cooperative optimization phase, one inherits a Pareto-stationary design-point Y^0, at which (36) holds for some coefficients $\{\alpha_i\}$ $(1 \leq i \leq n)$. The whole optimization process can then be interrupted if the performance of the design-point is already considered satisfactory. Otherwise, the process can be continued by a *competitive optimization* phase. The competitive-optimization phase can be accomplished by a Nash game based on an appropriate split of variables.

In the case of two disciplines, the split may be guided by the spectral properties of local Hessian matrices (see Appendix A).

For cases of more than two disciplines the strategy is more delicate. We propose to define two criteria, J_A and J_B and to apply the strategy of Section 1. In order to maintain, at best possible, the Pareto-stationarity condition, one can let :

$$J_A = \sum_{i=1}^{n} \alpha_i J_i \qquad J_B = J_k \tag{60}$$

so that $\nabla J_A(Y^0) = 0$, and choose the index k appropriately. Of course, the choice of split may be directed by the designer's bias to improve one criterion particularly. Otherwise, we propose to choose the index k to maximize the orthogonal projection of the gradient $\nabla J_k\left(Y^0\right)$ onto the subspace assigned to the virtual player B to reduce J_B. We have seen that this subspace is entirely defined by the diagonalization of the reduced Hessian of J_A. Technically, for each k, one lets

$$u_k^0 = \nabla J_k \left(Y^0\right) = \sum_{i=1}^{N} \beta_i^0 \omega^i \tag{61}$$

where $\beta_i^0 = (u_k^0, \omega^i)$ by orthogonality. Then one chooses k to maximize $\sqrt{\sum_{N-p+1}^{N} (\beta_i^0)^2}$.

In proceeding in this way, we approximately maintain the Pareto optimality of the solution, while maximizing the potential payoff to be achieved on J_k in the subsequent competitive optimization phase.

5 Conclusions

The multiobjective optimization of an aerodynamic criterion concurrently with one or more criteria originating from other disciplines raises delicate problems to solve since the flowfields are very sensitive to parameter changes, such as perturbations in shape parameters, particularly when the flow is transonic or supersonic and contains shocks.

A theoretical formulation has been proposed for situations of this type, permitting to identify a suboptimal solution as a Nash-equilibrium solution between virtual players in charge of reducing two independent criteria. An orthogonal decomposition of the design space is made to assign the player in charge of the secondary criterion a subspace of action, or territory, in which the primary criterion has little sensitivity.

The method has been tested over a simplified testcase of aero-structural shape optimization of a business jet wing combining drag reduction under lift constraint in a transonic cruise configuration with the reduction of an integral of the stress over the structure. In this example, after a first phase of pure aerodynamic optimization, the primary criterion (drag) was modeled at convergence by an RBF neural network in order to approximate gradients and Hessians necessary to the construction of the orthogonal basis. This basis was then used as the support of a dynamic Nash game in a novel formulation. The numerical experiments, taken from B. Abou El Majd's doctoral thesis, have clearly demonstrated the superiority of concurrent optimizations realized using the orthogonal decomposition as a support, in terms of asymptotic convergence stability, and achieved performance as well.

In more general situations, we propose to conduct the multidisciplinary optimization in two phases:

1. A preliminary "cooperative-optimization" phase at each iteration of which all criteria improve until the Pareto set is reached by application of the proposed Multiple-Gradient Descent Algorithm (MGDA);
2. A subsequent "competitive-optimization" phase, in which a Nash equilibrium is sought after virtual players have been assigned supplementary subspaces as strategies; the split should be defined according to a local eigenstructure analysis of Hessians and constraint gradients in order to define the equilibrium by a smooth continuation process.

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A The two-discipline case revisited : Nash game from a Pareto-stationary design-point ($n = 2$)

In this section, we propose recommendations to construct a Nash game to carry out the competitive-optimization phase, after completion of the cooperative-optimization phase, in the case of two disciplines ($n = 2$).

In the report [7], a split of territory was defined from the knowledge of a stationary point of one discipline, the preponderant discipline. Since critical points of one discipline are particular Pareto-stationary points, this subsection is meant to generalize the results of the former report.

For simplicity, we consider the case of two disciplines only ($n = 2$). An initial Pareto-stationary design-point Y^0 is known, and should be used to define a split of variables based on local eigensystems, and a Nash equilibrium-point determined subsequently.

Here, the two criteria are denoted J_A and J_B, and at $Y = Y^0$, the following holds :

$$\alpha_A \nabla J_A^0 + \alpha_B \nabla J_B^0 = 0 \qquad \alpha_A + \alpha_B = 1 \tag{62}$$

for some $\alpha_A \in [0,1]$. Therefore, three cases are possible :

1. Pareto-stationarity of type I : $\nabla J_A^0 = \nabla J_B^0 = 0$;
2. Pareto-stationarity of type II : $\nabla J_A^0 = 0$ and $\nabla J_B^0 \neq 0$ (or vice versa);
3. Pareto-stationarity of type III : $\nabla J_A^0 + \lambda \nabla J_B^0 = 0$ for $\lambda = \frac{1-\alpha_A}{\alpha_A} > 0$ since $0 < \alpha_a < 1$.

The question is what to do next to reaching a design-point Y^0 of Pareto-stationarity of the criteria $(J_A\,,\, J_B)$? To better understand the question, let us examine first the above three cases assuming both criteria are locally convex.

Convex case :

1. Pareto-stationarity of type I : then, both criteria have simultaneously achieved at $Y = Y^0$ local minimums of their own. In general the optimization process is terminated.
2. Pareto-stationarity of type II : e.g. $\nabla J_A^0 = 0$ and $\nabla J_B^0 \neq 0$. Then, J_A has achieved a local minimum, whereas J_B is still reducible. The decision can be to interrupt the process if the achieved design is acceptable, or to continue it using the formulation of the former theory [7] : a Nash equilibrium is sought based on a hierarchical split of variables in the orthogonal basis made of the eigenvectors of matrix H_A^0.
3. Pareto-stationarity of type III : $\nabla J_A^0 + \lambda\, \nabla J_B^0 = 0$ ($\lambda > 0$). Here, Pareto-optimality has been achieved and in the absence of an additional criterion, the optimization process is terminated.

   We now turn to the general case in which the criteria are not assumed to be locally convex at $Y = Y^0$.

Non convex case : In what follows, we discuss the different cases according to various assumptions that can be made on the eigenvalues of the Hessian matrices H_A^0 and H_B^0 of the two criteria at $Y = Y^0$.

1. Pareto-stationarity of type I :
   Since both gradients are equal to zero, the principal term in the expansion of the variations of the two criteria caused by a perturbation δY of the design vector Y about Y^0 are the quadratic terms associated with the respective Hessian matrices, one of which, at least, is not positive-definite by assumption, and perhaps both.
   If H_A^0 is positive-definite and H_B^0 alone has some negative eigenvalues, J_A has achieved a minimum whereas J_B is still reducible. Then we propose to terminate the optimization process, or to continue it using the formulation of the former theory [7] : a Nash equilibrium is sought with a hierarchical split of variables based on the eigensystem of matrix H_A^0.
   If both Hessian matrices H_A^0 and H_B^0 have some negative eigenvalues, define the following families of linearly independent eigenvectors associated with these eigenvalues :

$$\mathscr{F}_A = \{\, u_1\,,\, u_2\,,\, \dots,\, u_p\,\} \qquad \mathscr{F}_B = \{\, v_1\,,\, v_2\,,\, \dots,\, v_q\,\} \tag{63}$$

   Then :
   - If the family $\mathscr{F}_A \cup \mathscr{F}_B$ is linearly dependent, say

$$\sum_{i=1}^{p} \alpha_i\, u_i - \sum_{j=1}^{q} \beta_j\, v_j = 0 \tag{64}$$

in which $\{\alpha_i\}_{i=1,\dots,p} \cup \{\beta_j\}_{j=1,\dots,q} \neq \{0\}$, the vector

$$w^r = \sum_{i=1}^{p} \alpha_i u_i = \sum_{j=1}^{q} \beta_j v_j \tag{65}$$

is not equal to zero (by linear independence of the families $\mathscr{F}_A$ and $\mathscr{F}_B$ separately), and it is a descent direction for both criteria. We then propose to make a step in that direction.

– Otherwise, $Sp\mathscr{F}_A \cap Sp\mathscr{F}_B = \{0\}$: then we propose to stop, or to determine the Nash equilibrium point using $\mathscr{F}_A$ (resp. $\mathscr{F}_B$) as the strategy of A (resp. B).

2. Pareto-stationarity of type II : say $\nabla J_A^0 = 0$ and $\nabla J_B^0 \neq 0$.
If the Hessian matrix H_A^0 is positive-definite, the criterion J_A has achieved a local minimum and this setting has been analyzed in [7] : a Nash equilibrium is sought with a hierarchical split based on the structure of the eigenvectors of H_A^0.
If instead the matrix H_A^0 has some negative eigenvalues, let

$$\mathscr{F}_A = \{u_1\,,\,u_2\,,\,\dots,\,u_p\} \tag{66}$$

be a family of associated eigenvectors. Then :

– if ∇J_B^0 is not orthogonal to $Sp\mathscr{F}_A$: a descent direction common to J_A and J_B exists in $Sp\mathscr{F}_A$: use it to reduce both criteria.
– otherwise, $\nabla J_B^0 \perp Sp\mathscr{F}_A$: we propose to identify the Nash equilibrium using $\mathscr{F}_A$ as the strategy of player A and the remaining eigenvectors of H_A^0 as the strategy of player B.

3. Pareto-stationarity of type III : $\nabla J_A^0 + \lambda \nabla J_B^0 = 0$ $(\lambda > 0)$.
Consider the direction defined by the vector :

$$\mathsf{u}_{AB} = \frac{\nabla J_A^0}{\|\nabla J_A^0\|} = -\frac{\nabla J_B^0}{\|\nabla J_B^0\|} \tag{67}$$

Along this direction, the two criteria vary in opposite ways and no rational decision can be made in the absence of other criteria. Thus consider instead possible move in the hyperplane orthogonal to u_{AB}. For this, consider reduced Hessian matrices :

$$H_A^{0'} = P_{AB} H_A^0 P_{AB} \qquad H_B^{0'} = P_{AB} H_B^0 P_{AB} \tag{68}$$

where :

$$P_{AB} = I - [\mathsf{u}_{AB}]\,[\mathsf{u}_{AB}]^t\,. \tag{69}$$

In this hyperplane, by orthogonality to the gradient-vectors, the analysis is that of Pareto-stationary point of type I in a subspace of dimension $N-1$.

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