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ON GENERALIZED KUMMER OF RANK 3 VECTOR BUNDLES OVER A GENUS 2 CURVE

ALESSANDRA BERNARDI - DAMIANO FULGHESU

1. Introduction.

Let X be a smooth projective complex curve and let $U_X(r, d)$ be the moduli space of semi-stable vector bundles of rank r and degree d on X (see [8]). It contains an open Zariski subset $U_X(r, d)^s$ which is the coarse moduli space of stable bundles, i.e. vector bundles satisfying inequality

$$\frac{d_F}{r_F} < \frac{d_E}{r_E}.$$

The complement $U_X(r, d) \setminus U_X(r, d)^s$ parametrizes certain equivalence classes of strictly semi-stable vector bundles which satisfy the equality

$$\frac{d_F}{r_F} = \frac{d_E}{r_E}.$$

Each equivalence class contains a unique representative isomorphic to the direct sum of stable bundles. Furthermore one considers subvarieties $SU_X(r, L) \subset U_X(r, d)$ of vector bundle of rank r with determinant isomorphic to a fixed line bundle L of degree d . In this work we study the variety of strictly semi-stable bundles in $SU_X(3, \mathcal{O}_X)$, where X is a genus 2 curve. We call this variety the generalized Kummer variety of X and denote it by $\text{Kum}_3(X)$. Recall that

the classical Kummer variety of X is defined as the quotient of the Jacobian variety $\text{Jac}(X) = U_X(1, 0)$ by the involution $L \mapsto L^{-1}$. It turns out that our $\text{Kum}_3(X)$ has a similar description as a quotient of $\text{Jac}(X) \times \text{Jac}(X)$ which justifies the name. We will see that the first definition allows one to define a natural embedding of $\text{Kum}_3(X)$ in a projective space (see section 4). The second approach is useful in order to give local description of $\text{Kum}_3(X)$ by following the theory developed in [1] (section 3).

We point out the use of [4] for local computations.

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2. Generalized Kummer variety.

Let A be an s -dimensional abelian variety, A^r the r -Cartesian product of A , and $A^{(r)} := A^r / \Sigma_r$ be the r -symmetric power of A . We can define the usual map $a_r : A^{(r)} \rightarrow A$ such that $a_r(\{x_1, \dots, x_r\}) = x_1 + \dots + x_r$ ¹. This surjective map is just a morphism of varieties since there is no group structure on $A^{(r)}$. However, all fibers of a_r are naturally isomorphic.

Definition 2.1. *The generalized Kummer $_r$ variety associated to an abelian variety A is*

$$\text{Kum}_r(A) := a_r^{-1}(0).$$

It is easy to see that

$$\dim(\text{Kum}_r(A)) = s(r - 1).$$

When $\dim A > 1$, $A^{(r)}$ is singular. If $\dim A = 2$, $A^{(r)}$ admits a natural desingularization isomorphic to the Hilbert scheme $A^{[r]} := \text{Hilb}(A)^{[r]}$ of 0-dimensional subschemes of A of length r (see [5]). Let $pr : A^{[r]} \rightarrow A^{(r)}$ be the usual projection. It is known that the restriction of pr over $\text{Kum}_r(A)$ is a resolution of singularities. Also $\widetilde{\text{Kum}}_r(A)$ admits a structure a holomorphic symplectic manifold (see [1]).

2.1 The Kummer variety of Jacobians.

Let X be a smooth connected projective curve of genus g and $\text{SU}_X(r, L)$ be the set of semi-stable vector bundles on X of rank r and determinant which is

¹ here $\{x_1, \dots, x_r\}$ mean an unordered set of r elements.

isomorphic to a fixed line bundle L . Let $\text{Jac}(X)$ be the Jacobian variety of X which parametrizes isomorphism classes of line bundles on X of degree 0, or, equivalently the divisor classes of degree 0. We have a natural embedding:

$$\text{Kum}_r(\text{Jac}(X)) \hookrightarrow \text{SU}_X(r, \mathcal{O}_X)$$

$$\{a_1, \dots, a_r\} \mapsto (L_{a_1} \oplus \dots \oplus L_{a_r})$$

where $L_{a_i} := \mathcal{O}_X(a_i)$. Obviously, the condition $a_1 + \dots + a_r = 0$ means that $\det(L_{a_1} \oplus \dots \oplus L_{a_r}) = 0$ and $\deg(L_{a_i}) = 0$ for all $i = 1, \dots, r$. Consequently the Kummer variety $\text{Kum}_r(\text{Jac}(X))$ describes exactly the completely decomposable bundles in $\text{SU}_X(r)$ (from now on we'll write only $\text{SU}_X(r)$ instead of $\text{SU}_X(r, \mathcal{O}_X)$).

In this paper we restrict ourselves with the case $g = 2$ and rank $r = 3$. In this case $\text{Kum}_3(\text{Jac}(X))$ is a 4-fold.

3. Singular locus of $\text{Kum}_3(\text{Jac}(X))$.

From now we let A denote $\text{Jac}(X)$. Let us define the following map:

$$\begin{aligned} \pi : A^{(2)} &\rightarrow \text{Kum}_3(A) \\ \{a, b\} &\mapsto L_a \oplus L_b \oplus L_{-a-b}. \end{aligned}$$

This map is well defined and it is a $(3 : 1)$ -covering of $\text{Kum}_3(A)$. Let now $\rho : A^2 \rightarrow A^{(2)}$ be the $(2 : 1)$ -map which sends $(x, y) \in A^2$ to $\{x, y\} \in A^{(2)}$. If we consider the map:

$$(1) \quad p := (\pi \circ \rho) : A^2 \rightarrow A^{(2)} \rightarrow \text{Kum}_3(A) \subset A^{(3)}$$

we get a $(6 : 1)$ -covering of $\text{Kum}_3(A)$.

Notations: Let X and Y be two varieties and $f : X \rightarrow Y$ be a finite morphism. We let $\text{Sing}(X)$ denote the singular locus of X , $B_f \subseteq Y$ the branch locus of f and $R_f \subseteq X$ the ramification locus of f .

Observation: $B_\pi = \pi(B_\rho)$.

Proof. Since $B_\rho = \{\{x, y\} \in A^{(2)} \mid x = y\}$ and $\pi(\{x, x\}) = \{x, x, -2x\} \in B_\pi$ we obviously get that $\pi(B_\rho) \subset B_\pi$.

Conversely, for any point $\{x, y, z\}$ of B_π , at least two of the three elements x, y, z are equal to some t . Therefore $\pi(\{t, t\}) = \{x, y, z\}$, and hence $B_\pi \subset \pi(B_\rho)$. \square

Since A^2 is smooth, we have $\text{Sing}(A^{(2)}) \subset B_\rho$. Obviously $B_\rho \subset R_\pi$, hence $\text{Sing}(\text{Kum}_3(A)) \subset B_\pi$. As a consequence we obtain that $\text{Sing}(\text{Kum}_3(A)) \subseteq B_\pi$. Therefore we have to study the $(3 : 1)$ -covering $\pi : A^{(2)} \rightarrow \text{Kum}_3(A)$.

Since π is not a Galois covering, in order to give the local description at every point $Q \in \text{Kum}_3(A)$, we have to consider the following three cases separately:

1. $Q \in \text{Kum}_3(A)$ s.t. $\pi^{-1}(Q)$ is just a point;
2. $Q \in \text{Kum}_3(A)$ s.t. $\pi^{-1}(Q)$ is a set of two different points;
3. $Q \in \text{Kum}_3(A)$ s.t. $\pi^{-1}(Q)$ is a set of exactly three points.

Let us begin studying these cases.

Case 3. When $Q \in \text{Kum}_3(A)$ s.t. $\sharp(\pi^{-1}(Q)) = 3$ we have that $Q \notin B_\pi$. Since $\pi(B_\rho) = B_\pi$ any point of $\pi^{-1}(Q)$ is smooth in $A^{(2)}$. Then Q is a smooth point of the Kummer variety.

Case 2. When $Q \in \text{Kum}_3(A)$ s.t. $\sharp(\pi^{-1}(Q)) = 2$ we fix the two points $P_1, P_2 \in A^{(2)}$ s.t. $\pi(P_1) = \pi(P_2) = Q$. In this case $Q = \{x, x, -2x\}$ with $x \neq -2x$; let us fix $P_1 = \{x, x\}$, $P_2 = \{x, -2x\}$. Let $U \subset \text{Kum}_3(A)$ be a sufficiently small analytic neighborhood of Q such that $\pi^{-1}(U) = U_1 \sqcup U_2$ where U_1 and U_2 are respectively analytic neighborhoods of P_1 and P_2 and also $U_1 \cap U_2 = \emptyset$. Let \tilde{Q} a generic point of U , so $\tilde{Q} = \{x + \epsilon, x + \delta, -2x - \epsilon - \delta\}$; the preimage of \tilde{Q} by π is $\pi^{-1}(\tilde{Q}) = \{\{x + \epsilon, x + \delta\}, \{x + \epsilon, -2x - \epsilon - \delta\}, \{x + \delta, -2x - \epsilon - \delta\}\}$, but $\{x + \epsilon, x + \delta\} \in U_1$ and $\{x + \epsilon, -2x - \epsilon - \delta\}, \{x + \delta, -2x - \epsilon - \delta\} \in U_2$, it means that P_1 has ramification order equal to 1 and P_2 has ramification order equal to 2. Therefore there is an analytic neighborhood of P_1 which is isomorphic by π to an analytic neighborhood of Q . This allows us to study a generic point of B_ρ instead of a generic point of B_π .

Case 1. When $Q \in \text{Kum}_3(A)$ s.t. $\sharp(\pi^{-1}(Q)) = 3$ we consider a point $P \in A^{(2)}$ s.t. $\pi^{-1}(Q) = P \Rightarrow Q = \{x, x, x\}$ s.t. $3x = 0 \Rightarrow x$ is a 3-torsion point of A . Now our abelian variety is a complex torus of dimension 2, so we have exactly $3^{2g} = 3^4 = 81$ such points.

Proposition 3.1. *The singular locus of $\text{Kum}_3(A)$ is a surface which coincides with the branch locus B_π of the projection $\pi : A^{(2)} \rightarrow \text{Kum}_3(A)$ and it is locally isomorphic at a generic point to $(\mathbb{C}^2 \times Q, \mathbb{C} \times o)$ where Q is a cone over a rational normal curve and o is the vertex of such a cone (see [1]).*

Moreover there are exactly 81 points of $Sing(Kum_3(X))$ whose local tangent cone is isomorphic to the spectrum of:

$$\frac{\mathbb{C}[[u_1, \dots, u_7]]}{I}$$

where I is the ideal generated by the following polynomials :

$$\begin{aligned} &u_5^2 - u_4u_6 \\ &u_4u_7 - u_5u_6 \\ &u_6^2 - u_5u_7 \\ &u_3u_4 + u_2u_5 + u_1u_6 \\ &u_3u_5 + u_2u_6 + u_1u_7. \end{aligned}$$

Proof. According to what we saw in Case 2, an analytic neighborhood of $Q \in Kum_3(A)$ such that $\sharp(\pi^{-1}(Q)) = 2$ is isomorphic to a generic element of B_ρ . We have to study the $(2 : 1)$ -covering $A^2 \rightarrow A^{(2)}$.

Since $A = Jac(X)$, A is a smooth abelian variety, this means that A is a complex torus $(\mathbb{C}^g/\mathbb{Z}^{2g})$ where g is the genus of X ; in our case X is a genus 2 curve, $A \simeq (\mathbb{C}^2/\mathbb{Z}^4)$. Thus, in local coordinates at $P \in A$, $\widehat{\mathcal{O}}_P \simeq \mathbb{C}[[z_1, z_2]]$, so we consider U_P (a neighborhood of $P \in A$) isomorphic to \mathbb{C}^2 . Therefore we obtain that locally at $Q \in A^2$, $\widehat{\mathcal{O}}_Q \simeq \widehat{\mathcal{O}}_P \otimes \widehat{\mathcal{O}}_P \simeq \mathbb{C}[[z_1, z_2; z_3, z_4]]$.

We fix a coordinate system $(z_1, z_2; z_3, z_4)$ in A^2 such that $A^2 \supset U_P \ni P = (0, 0; 0, 0)$. Let Q be a point in U_P , in the fixed coordinate system $Q = (z_1, z_2; z_3, z_4)$. Since P is such that $\rho(P) \in B_\rho$, by definition of ρ , we have: $A^{(2)} = A^2 / \langle i \rangle$, where i is the following involution of U_P :

$$(2) \quad \begin{aligned} &i : U_P \rightarrow U_P \\ &i : (z_1, z_2; z_3, z_4) \mapsto (z_3, z_4; z_1, z_2). \end{aligned}$$

The involution i is obviously linear and its associated matrix is $M = e_{1,3} + e_{3,1} + e_{2,4} + e_{4,2}$ (where $e_{i,j}$ is the matrix with 1 in the i, j position and 0 elsewhere).

Its eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$ have both multiplicity 2, so its diagonal form is:

$$\tilde{M} = (1, 1, -1, -1)$$

which in a new coordinate system:

$$\begin{cases} x_1 = \frac{z_1 + z_3}{2} \\ x_2 = \frac{z_2 + z_4}{2} \\ x_3 = \frac{z_1 - z_3}{2} \\ x_4 = \frac{z_2 - z_4}{2} \end{cases}.$$

corresponds to the linear transformation:

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, -x_3, -x_4).$$

The algebra of invariant polynomials with respect to this actions is generated by the homogeneous forms $(x_1, x_2, x_3^2, x_4^2, x_3x_4)$. Let us now consider these forms as local coordinates $(s_1, s_2, s_3, s_4, s_5)$ around $\rho(P)$, here we have that the completion of the local ring is isomorphic to the following one:

$$\left(\frac{\mathbb{C}[[s_1, \dots, s_5]]}{(s_1^2 - s_2s_3)} \right).$$

Therefore B_ρ at a generic point is locally isomorphic to $(\mathbb{C}^2 \times Q, \mathbb{C} \times o)$ where Q is a cone over a rational normal curve (we can see this rational normal curve as the image of \mathbb{P}^1 in \mathbb{P}^3 by the Veronese map $v_2 : (\mathbb{P}^1)^* \rightarrow (\mathbb{P}^3)^*$, $v_2(L) = L^2$) and o the vertex of this cone. (What we have just proved in our particular case of $\text{Kum}_3(A)$ can be found in a more general form in [1].) Therefore we have the same local description of singularity of $\text{Kum}_3(A)$ out of the correspondent points of the 81 three-torsion points of A .

Now we have to study what happens at those 3-torsion. Let Q_0 be one of them, we already know that $p^{-1}(Q_0) = (x, x) := P_0$ is such that $3x = 0$. Let us fix $(z_1, z_2; z_3, z_4) \in \mathbb{C}^2 \times \mathbb{C}^2$ a local coordinate system around P_0 in order to describe locally the $(6 : 1)$ -covering $p : A^2 \rightarrow \text{Kum}_3(A)$. We observe that for a generic P in that neighborhood, the pre-image of $p(P)$ is the set of the following 6 points:

$$P_1 := (z_1, z_2; z_3, z_4),$$

$$P_2 := (z_3, z_4; z_1, z_2),$$

$$P_3 := (z_3, z_4; (-z_1 - z_3), (-z_2 - z_4)),$$

$$P_4 := ((-z_1 - z_3), (-z_2 - z_4); z_3, z_4),$$

$$P_5 := ((-z_1 - z_3), (-z_2 - z_4); z_1, z_2),$$

$$P_6 := (z_1, z_2; (-z_1 - z_3), (-z_2 - z_4)).$$

Observe that $i(P_1) = P_2$, $i(P_3) = P_4$, $i(P_5) = P_6$ where i is the involution defined in (2). We now define a trivolution τ of $\mathbb{C}^2 \times \mathbb{C}^2$ as follows:

$$(3) \quad \tau : \mathbb{C}^2 \times \mathbb{C}^2 \rightarrow \mathbb{C}^2 \times \mathbb{C}^2$$

$$(z_1, z_2; z_3, z_4) \mapsto (z_3, z_4; (-z_1 - z_3), (-z_2 - z_4)).$$

It is easy to see that:

$$\begin{aligned} P_1 &\xrightarrow{\tau} P_3 \xrightarrow{\tau} P_5 \xrightarrow{\tau} P_1, \\ P_2 &\xrightarrow{\tau} P_6 \xrightarrow{\tau} P_4 \xrightarrow{\tau} P_2 \end{aligned}$$

The matrices that represent i and τ are respectively:

$$i = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \tau = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}$$

furthermore $\langle \tau, i \rangle \simeq \Sigma_3$, then the local description of $\text{Kum}_3(X)$ around Q_0 is isomorphic to A^2/Σ_3 .

In what follows we have used [4] program in order to do computations. First we recall Noether’s theorem ([3] pag. 331)

Theorem 3.2. *Let $G \subset GL(n, \mathbb{C})$ be a given finite matrix group, we have:*

$$\mathbb{C}[z_1, \dots, z_n]^G = \mathbb{C}[R_G(z^\beta) : |\beta| \leq |G|].$$

where R_G is the Reynolds operator.

In other words, the algebra of invariant polynomials with respect to the action of G is generated by the invariant polynomials whose degree is at most the order of the group. In our case the order of G is 6, so it is not hard to compute $\mathbb{C}[z_1, z_2, z_3, z_4]^G$. Then, after reducing the generators, we obtain that $\mathbb{C}[z_1, z_2, z_3, z_4]^G$ is generated by:

$$\begin{aligned} f_1 &:= z_2^2 + z_2z_4 + z_4^2, & f_2 &:= 2z_1z_2 + z_2z_3 + z_1z_4 + 2z_3z_4, \\ f_3 &:= z_1^2 + z_1z_3 + z_3^2, & f_4 &:= -3z_2^2z_4 - 3z_2z_4^2, \\ f_5 &:= z_2^2z_3 + 2z_1z_2z_4 + 2z_2z_3z_4 + z_1z_4^2, \\ f_6 &:= -2z_1z_2z_3 - z_2z_3^2 - z_1^2z_4 - 2z_1z_3z_4, & f_7 &:= 3z_1^2z_3 + 3z_1z_3^2. \end{aligned}$$

Let us now write $\mathbb{C}[z_1, \dots, z_4]^G = \mathbb{C}[f_1, \dots, f_7]$ as:

$$\mathbb{C}[u_1, \dots, u_7]/I_G,$$

where I_G is the syzygy ideal. It is easy to obtain that I_G is generated by the following polynomials:

$$\begin{aligned} &u_1(u_2^2 - 4u_1u_3) + 3(u_5^2 - u_4u_6) \\ &u_2(u_2^2 - 4u_1u_3) + 3(u_4u_7 - u_5u_6) \\ &u_3(u_2^2 - 4u_1u_3) + 3(u_6^2 - u_5u_7) \\ &u_3u_4 + u_2u_5 + u_1u_6 \\ &u_3u_5 + u_2u_6 + u_1u_7 \end{aligned}$$

and so we have the completion of the local ring at P :

$$\widehat{\mathcal{O}}_P \simeq \frac{\mathbb{C}[[u_1, \dots, u_7]]}{I_G}.$$

Let now calculate the tangent cone in Q_0 in order to understand which kind of singularity occurs in Q_0 . With [4] aid we find that this local cone is:

$$\text{Spec}\left(\frac{\mathbb{C}[[u_1, \dots, u_7]]}{I}\right)$$

where I is the ideal generated by the following polynomials:

$$u_5^2 - u_4u_6$$

$$u_4u_7 - u_5u_6$$

$$u_6^2 - u_5u_7$$

$$u_3u_4 + u_2u_5 + u_1u_6$$

$$u_3u_5 + u_2u_6 + u_1u_7.$$

The degree of the variety $V(I) \subset \mathbb{P}^6$ is 5, this means that Q_0 is a singular point of multiplicity 5.

What we want to do now is to describe the singular locus of the local description. Let us start to calculate the Jacobian of $V(I_G)$, what we find is the following 5×7 matrix:

$$J_G := \begin{pmatrix} u_2^2 - 8u_1u_3 & 2u_1u_2 & -4u_1^2 & -3u_6 & 6u_5 & -3u_4 & 0 \\ -4u_2u_3 & 3u_2^2 - 4u_1u_3 & -4u_1u_2 & 3u_7 & -3u_6 & -3u_5 & 3u_4 \\ -4u_3^2 & 2u_2u_3 & u_2^2 - 8u_1u_3 & 0 & -3u_7 & 6u_6 & -3u_5 \\ u_6 & u_5 & u_4 & u_3 & u_2 & u_1 & 0 \\ u_7 & u_6 & u_5 & 0 & u_3 & u_2 & u_1 \end{pmatrix}$$

Local equations define a fourfold, so we have to find the locus where the dimension of $\text{Ker}(J_G)$ is at least 5. In order to do it we calculate the minimal system of generators of all 3×3 minors of J_G , we intersect the corresponding variety with $V(I_G)$, we find a minimal base of generators of the ideal corresponding to this intersection and we compute its radical; the polynomials we find define, after suitable change of coordinates, the (reduced) variety of singular locus

$V(I_S)$, where $I_S = (u_6^2 - u_5u_7, u_5u_6 - u_4u_7, u_5^2 - u_4u_6, u_3u_6 - u_2u_7, u_3u_5 - u_1u_7, u_2u_6 - u_1u_7, u_3u_4 - u_1u_6, u_2u_5 - u_1u_6, u_2u_4 - u_1u_5, u_2^2 - u_1u_3, u_3^3 - u_7^2, u_2u_3^2 - u_6u_7, u_1u_3^2 - u_5u_7, u_1u_2u_3 - u_4u_7, u_1^2u_3 - u_4u_6, u_1^2u_2 - u_4u_5, u_1^3 - u_4^2)$. We verified that the only one singular point of $V(I_S)$ is the origin. Now, let us consider the map from \mathbb{C}^2 to \mathbb{C}^7 such that:

$$(4) \quad (t, s) \mapsto (t^2, ts, s^2, t^3, t^2s, ts^2, s^3).$$

This is the parametrization of $V(I_S)$; as we have already done we can find relations between these polynomials and verify that the ideal we get is equal to I_S . Now we can consider the following smooth parametrization from \mathbb{C}^2 to \mathbb{C}^9 :

$$(t, s) \mapsto (t, s, t^2, ts, s^2, t^3, t^2s, ts^2, s^3)$$

(which is nothing but the graph of (4)) whose projective closure is the Veronese surface $v_3(\mathbb{P}^2) = V_{2,3}$ where $v_3 : (\mathbb{P}^2)^* \rightarrow (\mathbb{P}^9)^*$, $v_3(L) = L^3$.

What we want to find now is the tangent cone in Q_0 seen inside the singular locus. Using [4] we find that its corresponding ideal \tilde{I}_C is generated by following polynomials:

$$\begin{matrix} u_7^2 & u_6u_7 & u_5u_7 & u_4u_7 & u_4u_6 \\ u_4u_5 & u_4^2 & u_6^2 & u_5u_6 & u_5^2 \\ u_3u_6 - u_2u_7 & u_3u_5 - u_1u_7 & u_2u_6 - u_1u_7 & u_3u_4 - u_1u_6 & u_2u_5 - u_1u_6 \\ u_2u_4 - u_1u_5 & u_2^2 - u_1u_3. & & & \end{matrix}$$

The ideal \tilde{I}_C has multiplicity 4 (the corresponding variety has degree four) and its radical is the following ideal:

$$I_C = (u_2^2 - u_3u_1, u_4, u_5, u_6, u_7).$$

Then $V(I_C)$ is a cone and $V(\tilde{I}_C)$ is a double cone.

This gives the description of the singularity at one of the 81 3-torsion points. \square

4. Degree of $\text{Kum}_3(A)$.

To find the degree of $\text{Kum}_3(A)$, we have to recall some general facts about theta divisors.

4.1 The Riemann theta divisor.

Let X be a curve of genus g and $\Theta_{\text{Jac}(X)}$ is the *Riemann theta divisor*. It is known that it is an ample divisor and

$$\dim |r\Theta_{\text{Jac}(X)}| = r^g - 1$$

(see [6] Theorem p. 317). Recall that for any fixed point $q_0 \in X$ there exists an isomorphism:

$$\psi_{g-1,0} : \text{Pic}^{g-1}(X) \rightarrow \text{Jac}(X) = \text{Pic}^0(X).$$

The set W_{g-1} of effective line bundles of degree $g-1$ is a divisor in $\text{Pic}^{g-1}(X)$ denoted by $\Theta_{\text{Pic}^{g-1}(X)}$. By Riemann's Theorem there exists a divisor k of degree 0 such that:

$$\psi_{g-1,0}(\Theta_{\text{Pic}^{g-1}(X)}) = \Theta_{\text{Jac}(X)} - k.$$

In a similar way we can define the *generalized theta divisor* as follows:

$$\Theta_{\text{SU}_X(r,L)}^{\text{gen}} = \{E \in \text{Pic}^{g-1}(X) : h^0(E \otimes L) > 0\}.$$

It is known that

$$\text{Pic}(\text{SU}_X(r, L)) = \mathbb{Z}\Theta_{\text{SU}_X(r,L)}^{\text{gen}},$$

and there exists a canonical isomorphism:

$$|r\Theta_{\text{Pic}^{g-1}(X)}| \simeq |\Theta_{\text{SU}_X(r)}^{\text{gen}}|^*$$

(see [2]).

4.2 Degree of $\text{Kum}_3(A)$

Let us consider the $(2 : 1)$ -map

$$\phi_3 : \text{SU}_3(X) \longrightarrow |3\Theta_{\text{Pic}^1(X)}| \simeq |\Theta_{\text{SU}_X(3)}^{\text{gen}}|^*$$

$$E \longmapsto D_E = \{L \in \text{Pic}^1(X) : h^0(E \otimes L) > 0\}.$$

Definition 4.1. $\Theta_\eta := \{E \in \text{SU}_X(3) : h^0(E \otimes \eta) > 0\} \subset \text{SU}_X(3)$ where η is a fixed divisor in $\text{Pic}^1(X)$.

Observation: $\phi_3(\Theta_\eta) = H_\eta \subset |3\Theta_{\text{Pic}^1(X)}|$ and H_η is a hyperplane. Since $\phi_3|_{\text{Kum}_3(A)} : \text{Kum}_3(A) \rightarrow \phi_3(\text{Kum}_3(A))$ is a $(1 : 1)$ -map (it is a well known fact but we will see it in the next section), we have that $\Theta_\eta \cap \text{Kum}_3(X) \simeq H_\eta \cap \phi_3(\text{Kum}_3(X))$. In order to study the degree of $\text{Kum}_3(A)$ we have to take four generic divisors $\eta_1, \dots, \eta_4 \in \text{Pic}^1(X)$ and consider the respective $\Theta_{\eta_1}, \dots, \Theta_{\eta_4} \subset \text{SU}_X(3)$. The intersection $\Theta_{\eta_i} \cap \text{Kum}_3(A)$ is equal to $\{L_a \oplus L_b \oplus L_{-a-b} \in \text{Kum}_3(X) : h^0(L_a \oplus L_b \oplus L_{-a-b} \otimes \eta_i) > 0\} = \{L_a \oplus L_b \oplus L_{-a-b} \in \text{Kum}_3(A) : h^0(L_a \otimes \eta_i) > 0\} \cup \{L_a \oplus L_b \oplus L_{-a-b} \in \text{Kum}_3(A) : h^0(L_b \otimes \eta_i) > 0\} \cup \{L_a \oplus L_b \oplus L_{-a-b} \in \text{Kum}_3(A) : h^0(L_{-a-b} \otimes \eta_i) > 0\}$ for all $i = 1, \dots, 4$. If $L_a \oplus L_b \oplus L_{-a-b}$ is a generic element of $\text{Kum}_3(A)$ and p is the $(6 : 1)$ -covering of $\text{Kum}_3(A)$ defined as in (1), then $p^{-1}(L_a \oplus L_b \oplus L_{-a-b}) \subset A^2$ is a set of 6 points. It's easy to see that $p((a, b)) \in \Theta_{\eta_i} \cap \text{Kum}_3(X)$ if and only if or $h^0(L_a \otimes \eta_i) > 0$ or $h^0(L_b \otimes \eta_i) > 0$ or $h^0(L_{-a-b} \otimes \eta_i) > 0$ where $(a, b) \in A^2$ and $L_a, L_b, L_{-a-b} \in \text{Pic}^0(X)$ are three line bundles respectively associated to $a, b, -a - b \in A$.

Let us recall Jacobi's Theorem ([6] page: 235):

Jacobi's Theorem: *Let X be a curve of genus g , $q_0 \in X$ and $\omega_1, \dots, \omega_g$ a basis for $H^0(X, \Omega^1)$. For any $\lambda \in \text{Jac}(X)$ there exist g points $p_1, \dots, p_g \in X$ such that*

$$\mu\left(\sum_{i=1}^g (p_i - q_0)\right) = \lambda,$$

where

$$\mu : \text{Div}^0(X) \rightarrow \text{Jac}(X)$$

$$\sum_i (p_i - q_i) \mapsto \left(\sum_i \int_{q_i}^{p_i} \omega_1, \dots, \sum_i \int_{q_i}^{p_i} \omega_g \right).$$

Since $\text{Jac}(X)$ is isomorphic to $\text{Pic}^0(X)$, this theorem has the following two corollaries:

1. if q_0 is a fixed point of C , then for all $L_a \in \text{Pic}^0(X)$, there are two points P_1, P_2 in X such that $L_a \simeq \mathcal{O}_X(P_1 + P_2 - 2q_0)$;
2. Consider the isomorphism

$$\psi_{1,0} : \text{Pic}^1(X) \xrightarrow{\sim} \text{Pic}^0(X)$$

$$\eta \mapsto \eta \otimes \mathcal{O}_X(-q_0).$$

For every $i = 1, \dots, 4$ there are $q_{i_1}, q_{i_2} \in C$ such that $\eta_i \simeq \mathcal{O}_X(q_{i_1} + q_{i_2} - q_0)$.

Now these two facts imply that $h^0(L_a \otimes \eta_i) > 0$ if and only if $h^0(\mathcal{O}_X(P_1 + P_2 - 2q_0) \otimes \mathcal{O}_X(q_{i,1} + q_{i,2} - q_0)) > 0$, and this happens if and only if $h^0(\mathcal{O}_X(P_1 + P_2 + q_{i,1} + q_{i,2} - 3q_0)) > 0$.

Notations: Θ_{-k} is a translate of theta divisor by $k \in \text{Pic}^0(X)$.

By Riemann's Singularity Theorem (see [6], p. 348) the dimension $h^0(\mathcal{O}_X(P_1 + P_2 + q_{i,1} + q_{i,2} - 3q_0))$ is equal to the multiplicity of $\psi_{1,0}(P_1 + P_2 + q_{i,1} + q_{i,2} - 3q_0)$ in Θ_{-k} (by a suitable $k \in \text{Pic}^0(X)$), i.e. it is equal to the multiplicity of $(P_1 + P_2 + q_{i,1} + q_{i,2} - 4q_0)$ in Θ_{-k} . It follows from this fact that $h^0(\mathcal{O}_X(P_1 + P_2 + q_{i,1} + q_{i,2} - 3q_0))$ is greater than zero if and only if $(P_1 + P_2 + q_{i,1} + q_{i,2} - 4q_0) \in \Theta_{-k}$.

Notations:

$$\begin{aligned} \Theta_i &:= \Theta_{-k-\eta_i+q_0}; \\ R_i &:= \{(a, b) \in A^2 : (a + b) \in \{-\Theta_i\}\}; \\ \Xi_i &:= (\Theta_i \times A) \cup (A \times \Theta_i) \cup R_i. \end{aligned}$$

Now $(P_1 + P_2 + q_{i,1} + q_{i,2} - 4q_0) \in \Theta_{-k}$ iff $P_1 + P_2 - 2q_0 \in \Theta_i$ which is equivalent to say that L_a belongs to Θ_i , but this implies that $p((a, b)) \in \Theta_{\eta_i} \cap \text{Kum}_3(A)$ if and only if $L_a \in \Theta_i$ or $L_b \in \Theta_i$ or $L_{-a-b} \in \Theta_i$ (or equivalently L_{a+b} belongs to $\{-\Theta_i\}$), i.e. $(a, b) \in \Xi_i$.

Therefore we can conclude:

$(a, b) \in A^2$ is such that $p((a, b)) \in \text{Kum}_3(A) \cap \Theta_{\eta_i}$, $i = 1, \dots, 4$ if and only if $(a, b) \in \Xi_i$.

The last conclusion together with the observation that $\sharp(pr^{-1}(L_a \oplus L_b \oplus L_{-a-b})) = 6$ gives the following proposition:

Proposition 4.2. $\text{deg}(\text{Kum}_3(A)) = \frac{1}{6}(\sharp(\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4))$.

Proof. $\sharp(\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4) = 6 \cdot \sharp(\text{Kum}_3(A) \cap \Theta_{\eta_1} \cap \Theta_{\eta_2} \cap \Theta_{\eta_3} \cap \Theta_{\eta_4}) = 6 \cdot \text{deg}(\text{Kum}_3(A))$. \square

Notations:

$$\begin{aligned} R_j^{a,i} &= \{(a, b) \in A^2 : a \in \Theta_i \text{ and } (a + b) \in \{-\Theta_j\}\}, \\ R_j^{b,i} &= \{(a, b) \in A^2 : b \in \Theta_i \text{ and } (a + b) \in \{-\Theta_j\}\} \text{ and} \\ R_{1,2} &= \{(a, b) \in A^2 : (a + b) \in \{-\Theta_1\} \cap \{-\Theta_2\}\}. \end{aligned}$$

Instead of computing directly $\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4$, we will compute $(\Xi_1 \cap \Xi_2) \cap$

$(\Xi_3 \cap \Xi_4)$:

$$\begin{aligned} \Xi_1 \cap \Xi_2 &= ((\Theta_1 \cap \Theta_2) \times A) \cup (A \times (\Theta_1 \cap \Theta_2)) \cup (\Theta_1 \times \Theta_2) \cup \\ &\quad (\Theta_2 \times \Theta_1) \cup (R_b^{a,1}) \cup (R_2^{b,1}) \cup (R_1^{a,2}) \cup (R_1^{b,2}) \cup (R_{1,2}). \\ \Xi_3 \cap \Xi_4 &= ((\Theta_3 \cap \Theta_4) \times A) \cup (A \times (\Theta_3 \cap \Theta_4)) \cup (\Theta_3 \times \Theta_4) \cup \\ &\quad (\Theta_4 \times \Theta_3) \cup (R_b^{a,3}) \cup (R_4^{b,3}) \cup (R_3^{a,4}) \cup (R_3^{b,4}) \cup (R_{3,4}). \end{aligned}$$

At the end we will obtain that $\sharp(\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4) = 216$ (see also tables 1. and 2.) and so:

Proposition 4.3. $\deg(\text{Kum}_3(A)) = 36$.

Proof. In the following two tables we write at place (i, j) the cardinality of intersection of the subset of $\Xi_1 \cap \Xi_2$ which we write at the place $(0, j)$, with the subset of $\Xi_3 \cap \Xi_4$ which we write at the place $(i, 0)$.

\cap	$(\Theta_1 \cap \Theta_2) \times A$	$A \times (\Theta_1 \cap \Theta_2)$	$\Theta_1 \times \Theta_2$	$\Theta_2 \times \Theta_1$
$(\Theta_3 \cap \Theta_4) \times A$	0	4	0	0
$A \times (\Theta_3 \cap \Theta_4)$	4	0	0	0
$\Theta_3 \times \Theta_4$	0	0	4	4
$\Theta_4 \times \Theta_3$	0	0	4	4
$R_4^{a,3}$	0	4	4	4
$R_3^{a,4}$	0	4	4	4
$R_4^{b,3}$	4	0	4	4
$R_3^{b,4}$	4	0	4	4
$R_{3,4}$	4	4	4	4

Table 1.

In order to be more clear we show some cases:

$\mathbf{R}_2^{a,1} \cap \mathbf{R}_4^{b,3}$: $R_2^{a,1} \cap R_4^{b,3} = \{(a, b) \in A^2 : a \in \Theta_1 \text{ and } b \in \Theta_3 \text{ and } (a + b) \in \{-\Theta_2\} \cap \{-\Theta_4\}\}$. Recall that $\Theta_i \cdot \Theta_j = 2$. So $(a + b) \in \{k_1, k_2\}$ where $\{k_1, k_2\} = \{-\Theta_2\} \cap \{-\Theta_4\}$. Fix for a moment $(a+b) = k_1$. If we translate Θ_1 and Θ_3 by $-k_1$ we get that $a \in (\Theta_1)_{-k_1}$, $b \in (\Theta_3)_{-k_1}$ and $a + b = 0$, then b must be equal to $-a$ and $a \in ((\Theta_1)_{-k_1}) \cap ((-\Theta_3)_{+k_1})$. Then for

\cap	$R_2^{a,1}$	$R_1^{a,2}$	$R_2^{b,1}$	$R_1^{b,2}$	$R_{1,2}$
$(\Theta_3 \cap \Theta_4) \times A$	0	0	4	4	4
$A \times (\Theta_3 \cap \Theta_4)$	4	4	0	0	4
$\Theta_3 \times \Theta_4$	4	4	4	4	4
$\Theta_4 \times \Theta_3$	4	4	4	4	4
$R_4^{a,3}$	4	4	4	4	0
$R_3^{a,4}$	4	4	4	4	0
$R_4^{b,3}$	4	4	4	4	0
$R_3^{b,4}$	4	4	4	4	0
$R_3^{b,4}$	4	4	4	4	0
$R_{3,4}$	0	0	0	0	0

Table 2.

fixed $a+b$ the couple (a, b) has to belong to $\{(h_1, -h_1), (h_2, -h_2)\}$ where $((\Theta_1)_{+k_1}) \cap ((-\Theta_3)_{-k_1}) = \{h_1, h_2\}$. Therefore $\sharp(R_2^{a,1} \cap R_4^{b,3}) = 2 \cdot 2 = 4$.

$(\Theta_1 \times \Theta_2) \cap \mathbf{R}_{3,4}$: $(\Theta_1 \times \Theta_2) \cap R_{3,4} = \{(a, b) \in A^2 : a \in \Theta_1, b \in \Theta_2 \text{ and } (a+b) \in \{-\Theta_3\} \cap \{-\Theta_4\}\}$. Then, as in the previous case, we have $\sharp((\Theta_1 \times \Theta_2) \cap R_{3,4}) = 4$.

$\mathbf{R}_2^{a,1} \cap ((\Theta_3 \cap \Theta_4) \times A)$: $R_2^{a,1} \cap ((\Theta_3 \cap \Theta_4) \times A) = \{(a, b) \in A^2 : a \in \Theta_1 \cap \Theta_3 \cap \Theta_4, (a+b) \in \{-\Theta_2\}\}$, but since Θ_i are generic curves on a surface, their intersection two by two is the empty set, then $\sharp(R_2^{a,1}) \cap ((\Theta_3 \cap \Theta_4) \times A) = 0$. \square

4.3 The degree of $\text{Sing}(\text{Kum}_3(A))$

As we have already seen, the singular locus of $\text{Kum}_3(A)$ is a surface. What we want to do now is to compute its degree. We use the notation from the previous section.

Let us fix two divisors Ξ_1 and Ξ_2 in A^2 . We denote by Δ the diagonal of $A \times A$.

Proposition 4.4. $\deg(\text{Sing}(\text{Kum}_3(A))) = \sharp(\Xi_1 \cap \Xi_2 \cap \Delta)$.

Proof. It is sufficient to consider the restriction to Δ of the map p defined as in (1) and get out the $(1 : 1)$ -map $p|_{\Delta} : \Delta \rightarrow \text{Sing}(\text{Kum}_3(A))$. \square

Proposition 4.5. $\deg(\text{Sing}(\text{Kum}_3(A))) = 42$.

Proof. The following table is used in the same way as we used Table 1 and Table 2 in the previous section:

\cap	Δ
$(\Theta_1 \cap \Theta_2) \times A$	2
$A \times (\Theta_1 \cap \Theta_2)$	/
$\Theta_1 \times \Theta_2$	/
$\Theta_2 \times \Theta_1$	/
$R_2^{a,1}$	4
$R_1^{a,2}$	4
$R_2^{b,1}$	/
$R_1^{b,2}$	/
$R_{1,2}$	32

Table 3.

The following list describes Table 3:

- $\Delta \cap A \times (\Theta_1 \cap \Theta_2)$: we have not considered the intersection points between Δ and $A \times (\Theta_1 \cap \Theta_2)$, $\Theta_1 \times \Theta_2$, $\Theta_2 \times \Theta_1$ because we have already counted them in $((\Theta_1 \cap \Theta_2) \times A) \cap \Delta$.
- $\Delta \cap R_2^{b,1}$: the previous argument can be used for $\Delta \cap R_2^{b,1}$ and $\Delta \cap R_1^{b,2}$: we have already counted these intersection points respectively in $R_2^{a,1}$ and in $R_1^{a,2}$.
- $R_2^{a,1} \cap \Delta$: we have now to show that $\sharp(R_2^{a,1} \cap \Delta) = 4$. The set $R_2^{a,1} \cap \Delta$ is $\{(a, a) \in A \times A \mid a \in \Theta_1, 2a \in (-\Theta_2)\}$ which is equal to $\{(a, a) \in A \times A : 2a \in ((-\Theta_2) \cap (2 \cdot \Theta_1)) \text{ and } a \in \Theta_1\}$. Let now L_1 be the line bundle on A associated to Θ_1 . The line bundle L_1^2 is associated to $(2 \cdot \Theta_1)$ and its divisor is linearly equivalent to $2\Theta_1$. As a consequence of this fact we have that $2a \in (2\Theta_1 \cap (-\Theta_2))$ then $\sharp\{2\Theta_1 \cap (-\Theta_2)\} = 4$. Now, since the map from Θ_1 to $(2 \cdot \Theta_1)$ is $1 : 1$ we get the conclusion.
- $R_{1,2} \cap \Delta$: finally we have that $(R_{1,2} \cap \Delta)$ is equivalent to the set $\{a \in A \mid 2a \in ((-\Theta_1) \cap (-\Theta_2))\}$ whose cardinality is 32. \square

5. On action of the hyperelliptic involution and $\text{Kum}_3(A)$.

Let X be a curve of genus 2. Consider the degree 2 map:

$$\phi_3 : \text{SU}_X(3) \xrightarrow{2:1} \mathbb{P}^8 = |3\Theta_{\text{Pic}^1(X)}|$$

$$E \longmapsto D_E = \{L \in \text{Pic}^1(X) / h^0(E \otimes L) > 0\}$$

(see [7]). Let τ' be the involution on $\text{SU}_X(3)$ acting by the duality:

$$\tau'(E) = E^*$$

and τ the hyperelliptic involution on $\text{Pic}^1(X)$:

$$\tau(L) = \omega_X \otimes L^{-1}.$$

We will use the following well known relation:

$$\tau \circ \phi_3(E) = \phi_3 \circ \tau'(E).$$

On $\text{SU}_X(3)$ there is also the hyperelliptic involution h^* :

$$E \mapsto h^*(E)$$

induced by the hyperelliptic involution h of the curve X . We define $\sigma := \tau' \circ h^*$. It is the involution which gives the double covering of $\text{SU}_X(3)$ on \mathbb{P}^8 . The fixed locus of σ is obviously contained in $\text{SU}_X(3)$ and we recall:

$$(5) \quad \phi_3(\text{Fix}(\sigma)) = \text{Coble sextic hypersurface}$$

(see [7]). By definition, the strictly semi-stable locus $\text{SU}_X(3)^{ss}$ of $\text{SU}_X(3)$ consists of isomorphism classes of split rank 3 semi-stable vector bundles of determinant \mathcal{O}_X . Its points can be represented by the vector bundles of the form $F \oplus L$ or $L_a \oplus L_b \oplus L_c$ with trivial determinant where L, L_a, L_b, L_c are line bundles and F is a rank 2 vector bundle. We want to consider the elements of the form $L_a \oplus L_b \oplus L_c$ (those belonging to $\text{Kum}_3(A)$) and actions of previous involutions on them:

- $\tau'(L_a \oplus L_b \oplus L_c) = (L_a \oplus L_b \oplus L_c)^* = L_{-a} \oplus L_{-b} \oplus L_{-c}$;
- $\tau'(h^*(L_a \oplus L_b \oplus L_c)) = L_a \oplus L_b \oplus L_c$.

This implies that $\sigma(\text{Kum}_3(A)) = \text{Kum}_3(A) \subset \text{SU}_X(3)$ which means that $\text{Kum}_3(A) \subset \text{Fix}(\sigma)$ and then $\phi_3(\text{Kum}_3(X)) \subset \text{Coble sextic}$ (see 5).

Let us now consider rank 2 semistable vector bundles of trivial determinant: $\text{SU}_X(2)$. If we take its symmetric square, we obtain a semisable rank three vector bundle with trivial determinant:

$$\text{SU}_X(2) \rightarrow \text{SU}_X(3); \quad E \mapsto \text{Sym}^2(E).$$

We want to study the action of involutions defined on the beginning of this paragraph on $\text{Sym}^2(E)$ with $E \in \text{SU}_X(2)$. Since $\text{Sym}^2(E)^* = \text{Sym}^2(E) = h^*(\text{Sym}^2(E))$, then $\sigma(\text{Sym}^2(E)) = \text{Sym}^2(E) \subset \text{SU}_X(3)$, so $\text{Sym}^2(\text{SU}_X(2)) \subset \text{Fix}(\sigma)$, and, again by (5), $\phi_3(\text{Sym}^2(\text{SU}_X(2))) \subset \text{Coble sextic}$.

Now we want to see the action of τ on $|3\Theta_{\text{Pic}^1(X)}|$. It is known that $\text{Fix}(\tau) = \mathbb{P}^4 \sqcup \mathbb{P}^3$.

Notations: We denote by \mathbb{P}_τ^3 and \mathbb{P}_τ^4 , respectively, the \mathbb{P}^3 and the \mathbb{P}^4 which are fixed by action of τ .

Since the image of $\text{Sym}^2(\text{SU}_X(2))$ by ϕ_3 in \mathbb{P}^8 has dimension 3 and also $\phi_3(\text{Sym}^2(\text{SU}_X(2))) \subset \text{Fix}(\tau)$, we obtain

$$\phi_3(\text{Sym}^2(\text{SU}_X(2))) \subset \mathbb{P}_\tau^4.$$

Let $L_a \oplus L_{-a}$ be an element of $\text{Kum}_2(X) \subset \text{SU}_X(2)$, then $\text{Sym}^2(L_a \oplus L_{-a}) = L_{2a} \oplus L_{-2a} \oplus \mathcal{O} \in \text{Kum}_3(A) \subset \text{SU}_X(3)$. It means that $\text{Sym}^2(\text{Kum}_2(A)) \subset \text{Kum}_3(A)$.

Observation: Since $\{L_{2a} \oplus L_{-2a} \oplus \mathcal{O} \in \text{SU}_X(3)\}$ is isomorphic to $S^2(\{L_a \oplus L_{-a}\})$, we can view $\{L_{2a} \oplus L_{-2a} \oplus \mathcal{O} \in \text{SU}_X(3)\}$ as the image of $\text{Kum}_2(A)$ inside $\text{SU}_X(3)$ under the symmetric square map. Moreover it follows from the surjectivity of the multiplication by 2 map $[2] : A \rightarrow A$ that the image of $\text{Kum}_2(A)$ in $\text{SU}_X(3)$ is isomorphic to $\text{Kum}_2(A)$.

We have already observed that $\phi_3|_{\text{Kum}_3(A)}$ is a $(1 : 1)$ -map on the image; this fact allows us to view $\phi_3(\text{Kum}_3(A))$ as the $\text{Kum}_3(A)$ in $|3\Theta_{\text{Pic}^1(X)}|$. For the same reason we can view $\phi_3(\text{Sym}^2(\text{SU}_X(2)))$ as $\text{Kum}_2(A) \subset |3\Theta_{\text{Pic}^1(X)}|$. Using this language we can say that $\text{Kum}_2(A)$ is left fixed by the action of τ in $\text{Kum}_3(A) \subset |3\Theta_{\text{Pic}^1(X)}|$ because $|3\Theta_{\text{Pic}^1(X)}| \supset \phi_3(\text{Kum}_3(A)) \supset \phi_3(\text{Sym}^2(\text{SU}_X(2))) = \text{Kum}_2(A) \subset \mathbb{P}^4 \subset \text{Fix}(\tau) \subset |3\Theta_{\text{Pic}^1(X)}^1|$.

Proposition 5.1. $\text{Fix}(\tau) \cap \phi_3(\text{Kum}_3(A)) = \phi_3(\text{Sym}^2(\text{Kum}_2(A)))$.

Proof. By definition $\tau(L_a \oplus L_b \oplus L_c) = L_{-a} \oplus L_{-b} \oplus L_{-c}$ then $L_a \oplus L_b \oplus L_c$ belongs to $\text{Fix}(\tau)$ if and only if $\{a, b, c\} = \{-a, -b, -c\}$. Let P belong to $\{-a, -b, -c\}$ and $a = P$.

- If P is different from $-a$, suppose that $P = -c$, then $\{-a, -b, -c\} = \{-a, -b, a\}$; moreover $a + b + c = 0$ because $L_a \oplus L_b \oplus L_c \in \text{Kum}_3(A)$, then $b = 0$.
- Now, if $P = -a$ or, equivalently $a = -a$, then $a = 0$ and $b = -c$.

In both cases $L_a \oplus L_b \oplus L_c \in \text{Kum}_3(A)$ such that $\tau(L_a \oplus L_b \oplus L_c) = L_a \oplus L_b \oplus L_c$ are of the form $L_a \oplus L_{-a} \oplus L_0$. This means that they belong to $\text{Kum}_2(A) \subset |3\Theta_{\text{Pic}^1(X)}|$. \square

The previous proposition tells us also that $\mathbb{P}_\tau^3 \cap \text{Kum}_3 A = \emptyset$. So the projection of $\text{Kum}_3(A) \subset |3\Theta_{\text{Pic}^1(X)}|$ from \mathbb{P}_τ^3 to \mathbb{P}_τ^4 is a morphism. It would be interesting to find its degree.

Our final observation is the following.

Proposition 5.2. $\text{Sing}(\text{Kum}_3(A)) \cap \text{Kum}_2(A) = \text{Sing}(\text{Kum}_2(A))$

Proof. Points of $\text{Kum}_2(A) \subset \text{Kum}_3(A)$ are of the form $(P, -P, 0)$. Singular points of $\text{Kum}_3(A)$ are those which have at least two equal components, then $\text{Sing}(\text{Kum}_3(A)) \cap \text{Kum}_2(A) = \{(P, -P, 0)\}$ where $2P = 0$ that are exactly the 15 points of 2-torsion and one more point $(\mathcal{O}_X, \mathcal{O}_X, \mathcal{O}_X)$ which are singularities of the usual $\text{Kum}_2(A)$. This implies that $\sharp(\text{Sing}(\text{Kum}_3(A)) \cap \text{Kum}_2(A)) = 16$ and $\text{Sing}(\text{Kum}_3(A)) \cap \text{Kum}_2(A) = \text{Sing}(\text{Kum}_2(A))$. \square

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