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## ON GENERALIZED KUMMER OF RANK 3 VECTOR BUNDLES OVER A GENUS 2 CURVE

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### 1. Introduction.

Let  $X$  be a smooth projective complex curve and let  $U_X(r, d)$  be the moduli space of semi-stable vector bundles of rank  $r$  and degree  $d$  on  $X$  (see [8]). It contains an open Zariski subset  $U_X(r, d)^s$  which is the coarse moduli space of stable bundles, i.e. vector bundles satisfying inequality

$$\frac{d_F}{r_F} < \frac{d_E}{r_E}.$$

The complement  $U_X(r, d) \setminus U_X(r, d)^s$  parametrizes certain equivalence classes of strictly semi-stable vector bundles which satisfy the equality

$$\frac{d_F}{r_F} = \frac{d_E}{r_E}.$$

Each equivalence class contains a unique representative isomorphic to the direct sum of stable bundles. Furthermore one considers subvarieties  $SU_X(r, L) \subset U_X(r, d)$  of vector bundle of rank  $r$  with determinant isomorphic to a fixed line bundle  $L$  of degree  $d$ . In this work we study the variety of strictly semi-stable bundles in  $SU_X(3, \mathcal{O}_X)$ , where  $X$  is a genus 2 curve. We call this variety the generalized Kummer variety of  $X$  and denote it by  $\text{Kum}_3(X)$ . Recall that

the classical Kummer variety of  $X$  is defined as the quotient of the Jacobian variety  $\text{Jac}(X) = U_X(1, 0)$  by the involution  $L \mapsto L^{-1}$ . It turns out that our  $\text{Kum}_3(X)$  has a similar description as a quotient of  $\text{Jac}(X) \times \text{Jac}(X)$  which justifies the name. We will see that the first definition allows one to define a natural embedding of  $\text{Kum}_3(X)$  in a projective space (see section 4). The second approach is useful in order to give local description of  $\text{Kum}_3(X)$  by following the theory developed in [1] (section 3).

We point out the use of [4] for local computations.

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## 2. Generalized Kummer variety.

Let  $A$  be an  $s$ -dimensional abelian variety,  $A^r$  the  $r$ -Cartesian product of  $A$ , and  $A^{(r)} := A^r / \Sigma_r$  be the  $r$ -symmetric power of  $A$ . We can define the usual map  $a_r : A^{(r)} \rightarrow A$  such that  $a_r(\{x_1, \dots, x_r\}) = x_1 + \dots + x_r$ <sup>1</sup>. This surjective map is just a morphism of varieties since there is no group structure on  $A^{(r)}$ . However, all fibers of  $a_r$  are naturally isomorphic.

**Definition 2.1.** *The generalized Kummer $_r$  variety associated to an abelian variety  $A$  is*

$$\text{Kum}_r(A) := a_r^{-1}(0).$$

It is easy to see that

$$\dim(\text{Kum}_r(A)) = s(r - 1).$$

When  $\dim A > 1$ ,  $A^{(r)}$  is singular. If  $\dim A = 2$ ,  $A^{(r)}$  admits a natural desingularization isomorphic to the Hilbert scheme  $A^{[r]} := \text{Hilb}(A)^{[r]}$  of 0-dimensional subschemes of  $A$  of length  $r$  (see [5]). Let  $pr : A^{[r]} \rightarrow A^{(r)}$  be the usual projection. It is known that the restriction of  $pr$  over  $\text{Kum}_r(A)$  is a resolution of singularities. Also  $\widetilde{\text{Kum}}_r(A)$  admits a structure a holomorphic symplectic manifold (see [1]).

### 2.1 The Kummer variety of Jacobians.

Let  $X$  be a smooth connected projective curve of genus  $g$  and  $\text{SU}_X(r, L)$  be the set of semi-stable vector bundles on  $X$  of rank  $r$  and determinant which is

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<sup>1</sup> here  $\{x_1, \dots, x_r\}$  mean an unordered set of  $r$  elements.

isomorphic to a fixed line bundle  $L$ . Let  $\text{Jac}(X)$  be the Jacobian variety of  $X$  which parametrizes isomorphism classes of line bundles on  $X$  of degree 0, or, equivalently the divisor classes of degree 0. We have a natural embedding:

$$\text{Kum}_r(\text{Jac}(X)) \hookrightarrow \text{SU}_X(r, \mathcal{O}_X)$$

$$\{a_1, \dots, a_r\} \mapsto (L_{a_1} \oplus \dots \oplus L_{a_r})$$

where  $L_{a_i} := \mathcal{O}_X(a_i)$ . Obviously, the condition  $a_1 + \dots + a_r = 0$  means that  $\det(L_{a_1} \oplus \dots \oplus L_{a_r}) = 0$  and  $\deg(L_{a_i}) = 0$  for all  $i = 1, \dots, r$ . Consequently the Kummer variety  $\text{Kum}_r(\text{Jac}(X))$  describes exactly the completely decomposable bundles in  $\text{SU}_X(r)$  (from now on we'll write only  $\text{SU}_X(r)$  instead of  $\text{SU}_X(r, \mathcal{O}_X)$ ).

In this paper we restrict ourselves with the case  $g = 2$  and rank  $r = 3$ . In this case  $\text{Kum}_3(\text{Jac}(X))$  is a 4-fold.

### 3. Singular locus of $\text{Kum}_3(\text{Jac}(X))$ .

From now we let  $A$  denote  $\text{Jac}(X)$ . Let us define the following map:

$$\begin{aligned} \pi : A^{(2)} &\rightarrow \text{Kum}_3(A) \\ \{a, b\} &\mapsto L_a \oplus L_b \oplus L_{-a-b}. \end{aligned}$$

This map is well defined and it is a  $(3 : 1)$ -covering of  $\text{Kum}_3(A)$ . Let now  $\rho : A^2 \rightarrow A^{(2)}$  be the  $(2 : 1)$ -map which sends  $(x, y) \in A^2$  to  $\{x, y\} \in A^{(2)}$ . If we consider the map:

$$(1) \quad p := (\pi \circ \rho) : A^2 \rightarrow A^{(2)} \rightarrow \text{Kum}_3(A) \subset A^{(3)}$$

we get a  $(6 : 1)$ -covering of  $\text{Kum}_3(A)$ .

**Notations:** Let  $X$  and  $Y$  be two varieties and  $f : X \rightarrow Y$  be a finite morphism. We let  $\text{Sing}(X)$  denote the singular locus of  $X$ ,  $B_f \subseteq Y$  the branch locus of  $f$  and  $R_f \subseteq X$  the ramification locus of  $f$ .

**Observation:**  $B_\pi = \pi(B_\rho)$ .

*Proof.* Since  $B_\rho = \{\{x, y\} \in A^{(2)} \mid x = y\}$  and  $\pi(\{x, x\}) = \{x, x, -2x\} \in B_\pi$  we obviously get that  $\pi(B_\rho) \subset B_\pi$ .

Conversely, for any point  $\{x, y, z\}$  of  $B_\pi$ , at least two of the three elements  $x, y, z$  are equal to some  $t$ . Therefore  $\pi(\{t, t\}) = \{x, y, z\}$ , and hence  $B_\pi \subset \pi(B_\rho)$ .  $\square$

Since  $A^2$  is smooth, we have  $\text{Sing}(A^{(2)}) \subset B_\rho$ . Obviously  $B_\rho \subset R_\pi$ , hence  $\text{Sing}(\text{Kum}_3(A)) \subset B_\pi$ . As a consequence we obtain that  $\text{Sing}(\text{Kum}_3(A)) \subseteq B_\pi$ . Therefore we have to study the  $(3 : 1)$ -covering  $\pi : A^{(2)} \rightarrow \text{Kum}_3(A)$ .

Since  $\pi$  is not a Galois covering, in order to give the local description at every point  $Q \in \text{Kum}_3(A)$ , we have to consider the following three cases separately:

1.  $Q \in \text{Kum}_3(A)$  s.t.  $\pi^{-1}(Q)$  is just a point;
2.  $Q \in \text{Kum}_3(A)$  s.t.  $\pi^{-1}(Q)$  is a set of two different points;
3.  $Q \in \text{Kum}_3(A)$  s.t.  $\pi^{-1}(Q)$  is a set of exactly three points.

Let us begin studying these cases.

**Case 3.** When  $Q \in \text{Kum}_3(A)$  s.t.  $\sharp(\pi^{-1}(Q)) = 3$  we have that  $Q \notin B_\pi$ . Since  $\pi(B_\rho) = B_\pi$  any point of  $\pi^{-1}(Q)$  is smooth in  $A^{(2)}$ . Then  $Q$  is a smooth point of the Kummer variety.

**Case 2.** When  $Q \in \text{Kum}_3(A)$  s.t.  $\sharp(\pi^{-1}(Q)) = 2$  we fix the two points  $P_1, P_2 \in A^{(2)}$  s.t.  $\pi(P_1) = \pi(P_2) = Q$ . In this case  $Q = \{x, x, -2x\}$  with  $x \neq -2x$ ; let us fix  $P_1 = \{x, x\}$ ,  $P_2 = \{x, -2x\}$ . Let  $U \subset \text{Kum}_3(A)$  be a sufficiently small analytic neighborhood of  $Q$  such that  $\pi^{-1}(U) = U_1 \sqcup U_2$  where  $U_1$  and  $U_2$  are respectively analytic neighborhoods of  $P_1$  and  $P_2$  and also  $U_1 \cap U_2 = \emptyset$ . Let  $\tilde{Q}$  a generic point of  $U$ , so  $\tilde{Q} = \{x + \epsilon, x + \delta, -2x - \epsilon - \delta\}$ ; the preimage of  $\tilde{Q}$  by  $\pi$  is  $\pi^{-1}(\tilde{Q}) = \{\{x + \epsilon, x + \delta\}, \{x + \epsilon, -2x - \epsilon - \delta\}, \{x + \delta, -2x - \epsilon - \delta\}\}$ , but  $\{x + \epsilon, x + \delta\} \in U_1$  and  $\{x + \epsilon, -2x - \epsilon - \delta\}, \{x + \delta, -2x - \epsilon - \delta\} \in U_2$ , it means that  $P_1$  has ramification order equal to 1 and  $P_2$  has ramification order equal to 2. Therefore there is an analytic neighborhood of  $P_1$  which is isomorphic by  $\pi$  to an analytic neighborhood of  $Q$ . This allows us to study a generic point of  $B_\rho$  instead of a generic point of  $B_\pi$ .

**Case 1.** When  $Q \in \text{Kum}_3(A)$  s.t.  $\sharp(\pi^{-1}(Q)) = 3$  we consider a point  $P \in A^{(2)}$  s.t.  $\pi^{-1}(Q) = P \Rightarrow Q = \{x, x, x\}$  s.t.  $3x = 0 \Rightarrow x$  is a 3-torsion point of  $A$ . Now our abelian variety is a complex torus of dimension 2, so we have exactly  $3^{2g} = 3^4 = 81$  such points.

**Proposition 3.1.** *The singular locus of  $\text{Kum}_3(A)$  is a surface which coincides with the branch locus  $B_\pi$  of the projection  $\pi : A^{(2)} \rightarrow \text{Kum}_3(A)$  and it is locally isomorphic at a generic point to  $(\mathbb{C}^2 \times Q, \mathbb{C} \times o)$  where  $Q$  is a cone over a rational normal curve and  $o$  is the vertex of such a cone (see [1]).*

Moreover there are exactly 81 points of  $Sing(Kum_3(X))$  whose local tangent cone is isomorphic to the spectrum of:

$$\frac{\mathbb{C}[[u_1, \dots, u_7]]}{I}$$

where  $I$  is the ideal generated by the following polynomials :

$$\begin{aligned} &u_5^2 - u_4u_6 \\ &u_4u_7 - u_5u_6 \\ &u_6^2 - u_5u_7 \\ &u_3u_4 + u_2u_5 + u_1u_6 \\ &u_3u_5 + u_2u_6 + u_1u_7. \end{aligned}$$

*Proof.* According to what we saw in Case 2, an analytic neighborhood of  $Q \in Kum_3(A)$  such that  $\sharp(\pi^{-1}(Q)) = 2$  is isomorphic to a generic element of  $B_\rho$ . We have to study the  $(2 : 1)$ -covering  $A^2 \rightarrow A^{(2)}$ .

Since  $A = Jac(X)$ ,  $A$  is a smooth abelian variety, this means that  $A$  is a complex torus  $(\mathbb{C}^g/\mathbb{Z}^{2g})$  where  $g$  is the genus of  $X$ ; in our case  $X$  is a genus 2 curve,  $A \simeq (\mathbb{C}^2/\mathbb{Z}^4)$ . Thus, in local coordinates at  $P \in A$ ,  $\widehat{\mathcal{O}}_P \simeq \mathbb{C}[[z_1, z_2]]$ , so we consider  $U_P$  (a neighborhood of  $P \in A$ ) isomorphic to  $\mathbb{C}^2$ . Therefore we obtain that locally at  $Q \in A^2$ ,  $\widehat{\mathcal{O}}_Q \simeq \widehat{\mathcal{O}}_P \otimes \widehat{\mathcal{O}}_P \simeq \mathbb{C}[[z_1, z_2; z_3, z_4]]$ .

We fix a coordinate system  $(z_1, z_2; z_3, z_4)$  in  $A^2$  such that  $A^2 \supset U_P \ni P = (0, 0; 0, 0)$ . Let  $Q$  be a point in  $U_P$ , in the fixed coordinate system  $Q = (z_1, z_2; z_3, z_4)$ . Since  $P$  is such that  $\rho(P) \in B_\rho$ , by definition of  $\rho$ , we have:  $A^{(2)} = A^2 / \langle i \rangle$ , where  $i$  is the following involution of  $U_P$ :

$$(2) \quad \begin{aligned} &i : U_P \rightarrow U_P \\ &i : (z_1, z_2; z_3, z_4) \mapsto (z_3, z_4; z_1, z_2). \end{aligned}$$

The involution  $i$  is obviously linear and its associated matrix is  $M = e_{1,3} + e_{3,1} + e_{2,4} + e_{4,2}$  (where  $e_{i,j}$  is the matrix with 1 in the  $i, j$  position and 0 elsewhere).

Its eigenvalues  $\lambda_1 = -1$  and  $\lambda_2 = 1$  have both multiplicity 2, so its diagonal form is:

$$\tilde{M} = (1, 1, -1, -1)$$

which in a new coordinate system:

$$\begin{cases} x_1 = \frac{z_1 + z_3}{2} \\ x_2 = \frac{z_2 + z_4}{2} \\ x_3 = \frac{z_1 - z_3}{2} \\ x_4 = \frac{z_2 - z_4}{2} \end{cases}.$$

corresponds to the linear transformation:

$$(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, -x_3, -x_4).$$

The algebra of invariant polynomials with respect to this actions is generated by the homogeneous forms  $(x_1, x_2, x_3^2, x_4^2, x_3x_4)$ . Let us now consider these forms as local coordinates  $(s_1, s_2, s_3, s_4, s_5)$  around  $\rho(P)$ , here we have that the completion of the local ring is isomorphic to the following one:

$$\left( \frac{\mathbb{C}[[s_1, \dots, s_5]]}{(s_1^2 - s_2s_3)} \right).$$

Therefore  $B_\rho$  at a generic point is locally isomorphic to  $(\mathbb{C}^2 \times Q, \mathbb{C} \times o)$  where  $Q$  is a cone over a rational normal curve (we can see this rational normal curve as the image of  $\mathbb{P}^1$  in  $\mathbb{P}^3$  by the Veronese map  $v_2 : (\mathbb{P}^1)^* \rightarrow (\mathbb{P}^3)^*$ ,  $v_2(L) = L^2$ ) and  $o$  the vertex of this cone. (What we have just proved in our particular case of  $\text{Kum}_3(A)$  can be found in a more general form in [1].) Therefore we have the same local description of singularity of  $\text{Kum}_3(A)$  out of the correspondent points of the 81 three-torsion points of  $A$ .

Now we have to study what happens at those 3-torsion. Let  $Q_0$  be one of them, we already know that  $p^{-1}(Q_0) = (x, x) := P_0$  is such that  $3x = 0$ . Let us fix  $(z_1, z_2; z_3, z_4) \in \mathbb{C}^2 \times \mathbb{C}^2$  a local coordinate system around  $P_0$  in order to describe locally the  $(6 : 1)$ -covering  $p : A^2 \rightarrow \text{Kum}_3(A)$ . We observe that for a generic  $P$  in that neighborhood, the pre-image of  $p(P)$  is the set of the following 6 points:

$$P_1 := (z_1, z_2; z_3, z_4),$$

$$P_2 := (z_3, z_4; z_1, z_2),$$

$$P_3 := (z_3, z_4; (-z_1 - z_3), (-z_2 - z_4)),$$

$$P_4 := ((-z_1 - z_3), (-z_2 - z_4); z_3, z_4),$$

$$P_5 := ((-z_1 - z_3), (-z_2 - z_4); z_1, z_2),$$

$$P_6 := (z_1, z_2; (-z_1 - z_3), (-z_2 - z_4)).$$

Observe that  $i(P_1) = P_2$ ,  $i(P_3) = P_4$ ,  $i(P_5) = P_6$  where  $i$  is the involution defined in (2). We now define a trivolution  $\tau$  of  $\mathbb{C}^2 \times \mathbb{C}^2$  as follows:

$$(3) \quad \tau : \mathbb{C}^2 \times \mathbb{C}^2 \rightarrow \mathbb{C}^2 \times \mathbb{C}^2$$

$$(z_1, z_2; z_3, z_4) \mapsto (z_3, z_4; (-z_1 - z_3), (-z_2 - z_4)).$$

It is easy to see that:

$$P_1 \xrightarrow{\tau} P_3 \xrightarrow{\tau} P_5 \xrightarrow{\tau} P_1,$$

$$P_2 \xrightarrow{\tau} P_6 \xrightarrow{\tau} P_4 \xrightarrow{\tau} P_2$$

The matrices that represent  $i$  and  $\tau$  are respectively:

$$i = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad \tau = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & 0 \\ 0 & -1 & 0 & -1 \end{pmatrix}$$

furthermore  $\langle \tau, i \rangle \simeq \Sigma_3$ , then the local description of  $\text{Kum}_3(X)$  around  $Q_0$  is isomorphic to  $A^2/\Sigma_3$ .

In what follows we have used [4] program in order to do computations. First we recall Noether’s theorem ([3] pag. 331)

**Theorem 3.2.** *Let  $G \subset GL(n, \mathbb{C})$  be a given finite matrix group, we have:*

$$\mathbb{C}[z_1, \dots, z_n]^G = \mathbb{C}[R_G(z^\beta) : |\beta| \leq |G|].$$

where  $R_G$  is the Reynolds operator.

In other words, the algebra of invariant polynomials with respect to the action of  $G$  is generated by the invariant polynomials whose degree is at most the order of the group. In our case the order of  $G$  is 6, so it is not hard to compute  $\mathbb{C}[z_1, z_2, z_3, z_4]^G$ . Then, after reducing the generators, we obtain that  $\mathbb{C}[z_1, z_2, z_3, z_4]^G$  is generated by:

$$f_1 := z_2^2 + z_2z_4 + z_4^2, \quad f_2 := 2z_1z_2 + z_2z_3 + z_1z_4 + 2z_3z_4,$$

$$f_3 := z_1^2 + z_1z_3 + z_3^2, \quad f_4 := -3z_2^2z_4 - 3z_2z_4^2,$$

$$f_5 := z_2^2z_3 + 2z_1z_2z_4 + 2z_2z_3z_4 + z_1z_4^2,$$

$$f_6 := -2z_1z_2z_3 - z_2z_3^2 - z_1^2z_4 - 2z_1z_3z_4, \quad f_7 := 3z_1^2z_3 + 3z_1z_3^2.$$

Let us now write  $\mathbb{C}[z_1, \dots, z_4]^G = \mathbb{C}[f_1, \dots, f_7]$  as:

$$\mathbb{C}[u_1, \dots, u_7]/I_G,$$

where  $I_G$  is the syzygy ideal. It is easy to obtain that  $I_G$  is generated by the following polynomials:

$$u_1(u_2^2 - 4u_1u_3) + 3(u_5^2 - u_4u_6)$$

$$u_2(u_2^2 - 4u_1u_3) + 3(u_4u_7 - u_5u_6)$$

$$u_3(u_2^2 - 4u_1u_3) + 3(u_6^2 - u_5u_7)$$

$$u_3u_4 + u_2u_5 + u_1u_6$$

$$u_3u_5 + u_2u_6 + u_1u_7$$

and so we have the completion of the local ring at  $P$ :

$$\widehat{\mathcal{O}}_P \simeq \frac{\mathbb{C}[[u_1, \dots, u_7]]}{I_G}.$$

Let now calculate the tangent cone in  $Q_0$  in order to understand which kind of singularity occurs in  $Q_0$ . With [4] aid we find that this local cone is:

$$\text{Spec}\left(\frac{\mathbb{C}[[u_1, \dots, u_7]]}{I}\right)$$

where  $I$  is the ideal generated by the following polynomials:

$$u_5^2 - u_4u_6$$

$$u_4u_7 - u_5u_6$$

$$u_6^2 - u_5u_7$$

$$u_3u_4 + u_2u_5 + u_1u_6$$

$$u_3u_5 + u_2u_6 + u_1u_7.$$

The degree of the variety  $V(I) \subset \mathbb{P}^6$  is 5, this means that  $Q_0$  is a singular point of multiplicity 5.

What we want to do now is to describe the singular locus of the local description. Let us start to calculate the Jacobian of  $V(I_G)$ , what we find is the following  $5 \times 7$  matrix:

$$J_G := \begin{pmatrix} u_2^2 - 8u_1u_3 & 2u_1u_2 & -4u_1^2 & -3u_6 & 6u_5 & -3u_4 & 0 \\ -4u_2u_3 & 3u_2^2 - 4u_1u_3 & -4u_1u_2 & 3u_7 & -3u_6 & -3u_5 & 3u_4 \\ -4u_3^2 & 2u_2u_3 & u_2^2 - 8u_1u_3 & 0 & -3u_7 & 6u_6 & -3u_5 \\ u_6 & u_5 & u_4 & u_3 & u_2 & u_1 & 0 \\ u_7 & u_6 & u_5 & 0 & u_3 & u_2 & u_1 \end{pmatrix}$$

Local equations define a fourfold, so we have to find the locus where the dimension of  $\text{Ker}(J_G)$  is at least 5. In order to do it we calculate the minimal system of generators of all  $3 \times 3$  minors of  $J_G$ , we intersect the corresponding variety with  $V(I_G)$ , we find a minimal base of generators of the ideal corresponding to this intersection and we compute its radical; the polynomials we find define, after suitable change of coordinates, the (reduced) variety of singular locus

$V(I_S)$ , where  $I_S = (u_6^2 - u_5u_7, u_5u_6 - u_4u_7, u_5^2 - u_4u_6, u_3u_6 - u_2u_7, u_3u_5 - u_1u_7, u_2u_6 - u_1u_7, u_3u_4 - u_1u_6, u_2u_5 - u_1u_6, u_2u_4 - u_1u_5, u_2^2 - u_1u_3, u_3^3 - u_7^2, u_2u_3^2 - u_6u_7, u_1u_3^2 - u_5u_7, u_1u_2u_3 - u_4u_7, u_1^2u_3 - u_4u_6, u_1^2u_2 - u_4u_5, u_1^3 - u_4^2)$ . We verified that the only one singular point of  $V(I_S)$  is the origin. Now, let us consider the map from  $\mathbb{C}^2$  to  $\mathbb{C}^7$  such that:

$$(4) \quad (t, s) \mapsto (t^2, ts, s^2, t^3, t^2s, ts^2, s^3).$$

This is the parametrization of  $V(I_S)$ ; as we have already done we can find relations between these polynomials and verify that the ideal we get is equal to  $I_S$ . Now we can consider the following smooth parametrization from  $\mathbb{C}^2$  to  $\mathbb{C}^9$ :

$$(t, s) \mapsto (t, s, t^2, ts, s^2, t^3, t^2s, ts^2, s^3)$$

(which is nothing but the graph of (4)) whose projective closure is the Veronese surface  $v_3(\mathbb{P}^2) = V_{2,3}$  where  $v_3 : (\mathbb{P}^2)^* \rightarrow (\mathbb{P}^9)^*$ ,  $v_3(L) = L^3$ .

What we want to find now is the tangent cone in  $Q_0$  seen inside the singular locus. Using [4] we find that its corresponding ideal  $\tilde{I}_C$  is generated by following polynomials:

$$\begin{matrix} u_7^2 & u_6u_7 & u_5u_7 & u_4u_7 & u_4u_6 \\ u_4u_5 & u_4^2 & u_6^2 & u_5u_6 & u_5^2 \\ u_3u_6 - u_2u_7 & u_3u_5 - u_1u_7 & u_2u_6 - u_1u_7 & u_3u_4 - u_1u_6 & u_2u_5 - u_1u_6 \\ u_2u_4 - u_1u_5 & u_2^2 - u_1u_3 & & & \end{matrix}$$

The ideal  $\tilde{I}_C$  has multiplicity 4 (the corresponding variety has degree four) and its radical is the following ideal:

$$I_C = (u_2^2 - u_3u_1, u_4, u_5, u_6, u_7).$$

Then  $V(I_C)$  is a cone and  $V(\tilde{I}_C)$  is a double cone.

This gives the description of the singularity at one of the 81 3-torsion points.  $\square$

#### 4. Degree of $\text{Kum}_3(A)$ .

To find the degree of  $\text{Kum}_3(A)$ , we have to recall some general facts about theta divisors.

##### 4.1 The Riemann theta divisor.

Let  $X$  be a curve of genus  $g$  and  $\Theta_{\text{Jac}(X)}$  is the *Riemann theta divisor*. It is known that it is an ample divisor and

$$\dim |r\Theta_{\text{Jac}(X)}| = r^g - 1$$

(see [6] Theorem p. 317). Recall that for any fixed point  $q_0 \in X$  there exists an isomorphism:

$$\psi_{g-1,0} : \text{Pic}^{g-1}(X) \rightarrow \text{Jac}(X) = \text{Pic}^0(X).$$

The set  $W_{g-1}$  of effective line bundles of degree  $g-1$  is a divisor in  $\text{Pic}^{g-1}(X)$  denoted by  $\Theta_{\text{Pic}^{g-1}(X)}$ . By Riemann's Theorem there exists a divisor  $k$  of degree 0 such that:

$$\psi_{g-1,0}(\Theta_{\text{Pic}^{g-1}(X)}) = \Theta_{\text{Jac}(X)} - k.$$

In a similar way we can define the *generalized theta divisor* as follows:

$$\Theta_{\text{SU}_X(r,L)}^{\text{gen}} = \{E \in \text{Pic}^{g-1}(X) : h^0(E \otimes L) > 0\}.$$

It is known that

$$\text{Pic}(\text{SU}_X(r, L)) = \mathbb{Z}\Theta_{\text{SU}_X(r,L)}^{\text{gen}},$$

and there exists a canonical isomorphism:

$$|r\Theta_{\text{Pic}^{g-1}(X)}| \simeq |\Theta_{\text{SU}_X(r)}^{\text{gen}}|^*$$

(see [2]).

#### 4.2 Degree of $\text{Kum}_3(A)$

Let us consider the  $(2 : 1)$ -map

$$\phi_3 : \text{SU}_3(X) \longrightarrow |3\Theta_{\text{Pic}^1(X)}| \simeq |\Theta_{\text{SU}_X(3)}^{\text{gen}}|^*$$

$$E \longmapsto D_E = \{L \in \text{Pic}^1(X) : h^0(E \otimes L) > 0\}.$$

**Definition 4.1.**  $\Theta_\eta := \{E \in \text{SU}_X(3) : h^0(E \otimes \eta) > 0\} \subset \text{SU}_X(3)$  where  $\eta$  is a fixed divisor in  $\text{Pic}^1(X)$ .

**Observation:**  $\phi_3(\Theta_\eta) = H_\eta \subset |3\Theta_{\text{Pic}^1(X)}|$  and  $H_\eta$  is a hyperplane. Since  $\phi_3|_{\text{Kum}_3(A)} : \text{Kum}_3(A) \rightarrow \phi_3(\text{Kum}_3(A))$  is a  $(1 : 1)$ -map (it is a well known fact but we will see it in the next section), we have that  $\Theta_\eta \cap \text{Kum}_3(X) \simeq H_\eta \cap \phi_3(\text{Kum}_3(X))$ . In order to study the degree of  $\text{Kum}_3(A)$  we have to take four generic divisors  $\eta_1, \dots, \eta_4 \in \text{Pic}^1(X)$  and consider the respective  $\Theta_{\eta_1}, \dots, \Theta_{\eta_4} \subset \text{SU}_X(3)$ . The intersection  $\Theta_{\eta_i} \cap \text{Kum}_3(A)$  is equal to  $\{L_a \oplus L_b \oplus L_{-a-b} \in \text{Kum}_3(X) : h^0(L_a \oplus L_b \oplus L_{-a-b} \otimes \eta_i) > 0\} = \{L_a \oplus L_b \oplus L_{-a-b} \in \text{Kum}_3(A) : h^0(L_a \otimes \eta_i) > 0\} \cup \{L_a \oplus L_b \oplus L_{-a-b} \in \text{Kum}_3(A) : h^0(L_b \otimes \eta_i) > 0\} \cup \{L_a \oplus L_b \oplus L_{-a-b} \in \text{Kum}_3(A) : h^0(L_{-a-b} \otimes \eta_i) > 0\}$  for all  $i = 1, \dots, 4$ . If  $L_a \oplus L_b \oplus L_{-a-b}$  is a generic element of  $\text{Kum}_3(A)$  and  $p$  is the  $(6 : 1)$ -covering of  $\text{Kum}_3(A)$  defined as in (1), then  $p^{-1}(L_a \oplus L_b \oplus L_{-a-b}) \subset A^2$  is a set of 6 points. It's easy to see that  $p((a, b)) \in \Theta_{\eta_i} \cap \text{Kum}_3(X)$  if and only if or  $h^0(L_a \otimes \eta_i) > 0$  or  $h^0(L_b \otimes \eta_i) > 0$  or  $h^0(L_{-a-b} \otimes \eta_i) > 0$  where  $(a, b) \in A^2$  and  $L_a, L_b, L_{-a-b} \in \text{Pic}^0(X)$  are three line bundles respectively associated to  $a, b, -a - b \in A$ .

Let us recall Jacobi's Theorem ([6] page: 235):

**Jacobi's Theorem:** *Let  $X$  be a curve of genus  $g$ ,  $q_0 \in X$  and  $\omega_1, \dots, \omega_g$  a basis for  $H^0(X, \Omega^1)$ . For any  $\lambda \in \text{Jac}(X)$  there exist  $g$  points  $p_1, \dots, p_g \in X$  such that*

$$\mu\left(\sum_{i=1}^g (p_i - q_0)\right) = \lambda,$$

where

$$\mu : \text{Div}^0(X) \rightarrow \text{Jac}(X)$$

$$\sum_i (p_i - q_i) \mapsto \left( \sum_i \int_{q_i}^{p_i} \omega_1, \dots, \sum_i \int_{q_i}^{p_i} \omega_g \right).$$

Since  $\text{Jac}(X)$  is isomorphic to  $\text{Pic}^0(X)$ , this theorem has the following two corollaries:

1. if  $q_0$  is a fixed point of  $C$ , then for all  $L_a \in \text{Pic}^0(X)$ , there are two points  $P_1, P_2$  in  $X$  such that  $L_a \simeq \mathcal{O}_X(P_1 + P_2 - 2q_0)$ ;
2. Consider the isomorphism

$$\psi_{1,0} : \text{Pic}^1(X) \xrightarrow{\sim} \text{Pic}^0(X)$$

$$\eta \mapsto \eta \otimes \mathcal{O}_X(-q_0).$$

For every  $i = 1, \dots, 4$  there are  $q_{i_1}, q_{i_2} \in C$  such that  $\eta_i \simeq \mathcal{O}_X(q_{i_1} + q_{i_2} - q_0)$ .

Now these two facts imply that  $h^0(L_a \otimes \eta_i) > 0$  if and only if  $h^0(\mathcal{O}_X(P_1 + P_2 - 2q_0) \otimes \mathcal{O}_X(q_{i,1} + q_{i,2} - q_0)) > 0$ , and this happens if and only if  $h^0(\mathcal{O}_X(P_1 + P_2 + q_{i,1} + q_{i,2} - 3q_0)) > 0$ .

**Notations:**  $\Theta_{-k}$  is a translate of theta divisor by  $k \in \text{Pic}^0(X)$ .

By Riemann's Singularity Theorem (see [6], p. 348) the dimension  $h^0(\mathcal{O}_X(P_1 + P_2 + q_{i,1} + q_{i,2} - 3q_0))$  is equal to the multiplicity of  $\psi_{1,0}(P_1 + P_2 + q_{i,1} + q_{i,2} - 3q_0)$  in  $\Theta_{-k}$  (by a suitable  $k \in \text{Pic}^0(X)$ ), i.e. it is equal to the multiplicity of  $(P_1 + P_2 + q_{i,1} + q_{i,2} - 4q_0)$  in  $\Theta_{-k}$ . It follows from this fact that  $h^0(\mathcal{O}_X(P_1 + P_2 + q_{i,1} + q_{i,2} - 3q_0))$  is greater than zero if and only if  $(P_1 + P_2 + q_{i,1} + q_{i,2} - 4q_0) \in \Theta_{-k}$ .

**Notations:**

$$\begin{aligned} \Theta_i &:= \Theta_{-k-\eta_i+q_0}; \\ R_i &:= \{(a, b) \in A^2 : (a + b) \in \{-\Theta_i\}\}; \\ \Xi_i &:= (\Theta_i \times A) \cup (A \times \Theta_i) \cup R_i. \end{aligned}$$

Now  $(P_1 + P_2 + q_{i,1} + q_{i,2} - 4q_0) \in \Theta_{-k}$  iff  $P_1 + P_2 - 2q_0 \in \Theta_i$  which is equivalent to say that  $L_a$  belongs to  $\Theta_i$ , but this implies that  $p((a, b)) \in \Theta_{\eta_i} \cap \text{Kum}_3(A)$  if and only if  $L_a \in \Theta_i$  or  $L_b \in \Theta_i$  or  $L_{-a-b} \in \Theta_i$  (or equivalently  $L_{a+b}$  belongs to  $\{-\Theta_i\}$ ), i.e.  $(a, b) \in \Xi_i$ .

Therefore we can conclude:

$(a, b) \in A^2$  is such that  $p((a, b)) \in \text{Kum}_3(A) \cap \Theta_{\eta_i}$ ,  $i = 1, \dots, 4$  if and only if  $(a, b) \in \Xi_i$ .

The last conclusion together with the observation that  $\sharp(pr^{-1}(L_a \oplus L_b \oplus L_{-a-b})) = 6$  gives the following proposition:

**Proposition 4.2.**  $\text{deg}(\text{Kum}_3(A)) = \frac{1}{6}(\sharp(\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4))$ .

*Proof.*  $\sharp(\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4) = 6 \cdot \sharp(\text{Kum}_3(A) \cap \Theta_{\eta_1} \cap \Theta_{\eta_2} \cap \Theta_{\eta_3} \cap \Theta_{\eta_4}) = 6 \cdot \text{deg}(\text{Kum}_3(A))$ .  $\square$

**Notations:**

$$\begin{aligned} R_j^{a,i} &= \{(a, b) \in A^2 : a \in \Theta_i \text{ and } (a + b) \in \{-\Theta_j\}\}, \\ R_j^{b,i} &= \{(a, b) \in A^2 : b \in \Theta_i \text{ and } (a + b) \in \{-\Theta_j\}\} \text{ and} \\ R_{1,2} &= \{(a, b) \in A^2 : (a + b) \in \{-\Theta_1\} \cap \{-\Theta_2\}\}. \end{aligned}$$

Instead of computing directly  $\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4$ , we will compute  $(\Xi_1 \cap \Xi_2) \cap$

$(\Xi_3 \cap \Xi_4)$ :

$$\begin{aligned} \Xi_1 \cap \Xi_2 &= ((\Theta_1 \cap \Theta_2) \times A) \cup (A \times (\Theta_1 \cap \Theta_2)) \cup (\Theta_1 \times \Theta_2) \cup \\ &\quad (\Theta_2 \times \Theta_1) \cup (R_b^{a,1}) \cup (R_2^{b,1}) \cup (R_1^{a,2}) \cup (R_1^{b,2}) \cup (R_{1,2}). \\ \Xi_3 \cap \Xi_4 &= ((\Theta_3 \cap \Theta_4) \times A) \cup (A \times (\Theta_3 \cap \Theta_4)) \cup (\Theta_3 \times \Theta_4) \cup \\ &\quad (\Theta_4 \times \Theta_3) \cup (R_b^{a,3}) \cup (R_4^{b,3}) \cup (R_3^{a,4}) \cup (R_3^{b,4}) \cup (R_{3,4}). \end{aligned}$$

At the end we will obtain that  $\sharp(\Xi_1 \cap \Xi_2 \cap \Xi_3 \cap \Xi_4) = 216$  (see also tables 1. and 2.) and so:

**Proposition 4.3.**  $\text{deg}(\text{Kum}_3(A)) = 36$ .

*Proof.* In the following two tables we write at place  $(i, j)$  the cardinality of intersection of the subset of  $\Xi_1 \cap \Xi_2$  which we write at the place  $(0, j)$ , with the subset of  $\Xi_3 \cap \Xi_4$  which we write at the place  $(i, 0)$ .

$\cap$	$(\Theta_1 \cap \Theta_2) \times A$	$A \times (\Theta_1 \cap \Theta_2)$	$\Theta_1 \times \Theta_2$	$\Theta_2 \times \Theta_1$
$(\Theta_3 \cap \Theta_4) \times A$	0	4	0	0
$A \times (\Theta_3 \cap \Theta_4)$	4	0	0	0
$\Theta_3 \times \Theta_4$	0	0	4	4
$\Theta_4 \times \Theta_3$	0	0	4	4
$R_4^{a,3}$	0	4	4	4
$R_3^{a,4}$	0	4	4	4
$R_4^{b,3}$	4	0	4	4
$R_3^{b,4}$	4	0	4	4
$R_{3,4}$	4	4	4	4

Table 1.

In order to be more clear we show some cases:

$\mathbf{R}_2^{a,1} \cap \mathbf{R}_4^{b,3}$ :  $R_2^{a,1} \cap R_4^{b,3} = \{(a, b) \in A^2 : a \in \Theta_1 \text{ and } b \in \Theta_3 \text{ and } (a + b) \in \{-\Theta_2\} \cap \{-\Theta_4\}\}$ . Recall that  $\Theta_i \cdot \Theta_j = 2$ . So  $(a + b) \in \{k_1, k_2\}$  where  $\{k_1, k_2\} = \{-\Theta_2\} \cap \{-\Theta_4\}$ . Fix for a moment  $(a+b) = k_1$ . If we translate  $\Theta_1$  and  $\Theta_3$  by  $-k_1$  we get that  $a \in (\Theta_1)_{-k_1}$ ,  $b \in (\Theta_3)_{-k_1}$  and  $a + b = 0$ , then  $b$  must be equal to  $-a$  and  $a \in ((\Theta_1)_{-k_1}) \cap ((-\Theta_3)_{+k_1})$ . Then for

$\cap$	$R_2^{a,1}$	$R_1^{a,2}$	$R_2^{b,1}$	$R_1^{b,2}$	$R_{1,2}$
$(\Theta_3 \cap \Theta_4) \times A$	0	0	4	4	4
$A \times (\Theta_3 \cap \Theta_4)$	4	4	0	0	4
$\Theta_3 \times \Theta_4$	4	4	4	4	4
$\Theta_4 \times \Theta_3$	4	4	4	4	4
$R_4^{a,3}$	4	4	4	4	0
$R_3^{a,4}$	4	4	4	4	0
$R_4^{b,3}$	4	4	4	4	0
$R_3^{b,4}$	4	4	4	4	0
$R_3^{b,4}$	4	4	4	4	0
$R_{3,4}$	0	0	0	0	0

Table 2.

fixed  $a+b$  the couple  $(a, b)$  has to belong to  $\{(h_1, -h_1), (h_2, -h_2)\}$  where  $((\Theta_1)_{+k_1}) \cap ((-\Theta_3)_{-k_1}) = \{h_1, h_2\}$ . Therefore  $\sharp(R_2^{a,1} \cap R_4^{b,3}) = 2 \cdot 2 = 4$ .

$(\Theta_1 \times \Theta_2) \cap R_{3,4}$ :  $(\Theta_1 \times \Theta_2) \cap R_{3,4} = \{(a, b) \in A^2 : a \in \Theta_1, b \in \Theta_2 \text{ and } (a + b) \in \{-\Theta_3\} \cap \{-\Theta_4\}\}$ . Then, as in the previous case, we have  $\sharp((\Theta_1 \times \Theta_2) \cap R_{3,4}) = 4$ .

$R_2^{a,1} \cap ((\Theta_3 \cap \Theta_4) \times A)$ :  $R_2^{a,1} \cap ((\Theta_3 \cap \Theta_4) \times A) = \{(a, b) \in A^2 : a \in \Theta_1 \cap \Theta_3 \cap \Theta_4, (a + b) \in \{-\Theta_2\}\}$ , but since  $\Theta_i$  are generic curves on a surface, their intersection two by two is the empty set, then  $\sharp(R_2^{a,1}) \cap ((\Theta_3 \cap \Theta_4) \times A) = 0$ .  $\square$

### 4.3 The degree of $\text{Sing}(\text{Kum}_3(A))$

As we have already seen, the singular locus of  $\text{Kum}_3(A)$  is a surface. What we want to do now is to compute its degree. We use the notation from the previous section.

Let us fix two divisors  $\Xi_1$  and  $\Xi_2$  in  $A^2$ . We denote by  $\Delta$  the diagonal of  $A \times A$ .

**Proposition 4.4.**  $\text{deg}(\text{Sing}(\text{Kum}_3(A))) = \sharp(\Xi_1 \cap \Xi_2 \cap \Delta)$ .

*Proof.* It is sufficient to consider the restriction to  $\Delta$  of the map  $p$  defined as in (1) and get out the  $(1 : 1)$ -map  $p|_\Delta : \Delta \rightarrow \text{Sing}(\text{Kum}_3(A))$ .  $\square$

**Proposition 4.5.**  $\deg(\text{Sing}(\text{Kum}_3(A))) = 42$ .

*Proof.* The following table is used in the same way as we used Table 1 and Table 2 in the previous section:

$\cap$	$\Delta$
$(\Theta_1 \cap \Theta_2) \times A$	2
$A \times (\Theta_1 \cap \Theta_2)$	/
$\Theta_1 \times \Theta_2$	/
$\Theta_2 \times \Theta_1$	/
$R_2^{a,1}$	4
$R_1^{a,2}$	4
$R_2^{b,1}$	/
$R_1^{b,2}$	/
$R_{1,2}$	32

Table 3.

The following list describes Table 3:

- $\Delta \cap A \times (\Theta_1 \cap \Theta_2)$ : we have not considered the intersection points between  $\Delta$  and  $A \times (\Theta_1 \cap \Theta_2)$ ,  $\Theta_1 \times \Theta_2$ ,  $\Theta_2 \times \Theta_1$  because we have already counted them in  $((\Theta_1 \cap \Theta_2) \times A) \cap \Delta$ .
- $\Delta \cap R_2^{b,1}$ : the previous argument can be used for  $\Delta \cap R_2^{b,1}$  and  $\Delta \cap R_1^{b,2}$ : we have already counted these intersection points respectively in  $R_2^{a,1}$  and in  $R_1^{a,2}$ .
- $R_2^{a,1} \cap \Delta$ : we have now to show that  $\sharp(R_2^{a,1} \cap \Delta) = 4$ . The set  $R_2^{a,1} \cap \Delta$  is  $\{(a, a) \in A \times A \mid a \in \Theta_1, 2a \in (-\Theta_2)\}$  which is equal to  $\{(a, a) \in A \times A : 2a \in ((-\Theta_2) \cap (2 \cdot \Theta_1)) \text{ and } a \in \Theta_1\}$ . Let now  $L_1$  be the line bundle on  $A$  associated to  $\Theta_1$ . The line bundle  $L_1^2$  is associated to  $(2 \cdot \Theta_1)$  and its divisor is linearly equivalent to  $2\Theta_1$ . As a consequence of this fact we have that  $2a \in (2\Theta_1 \cap (-\Theta_2))$  then  $\sharp\{2\Theta_1 \cap (-\Theta_2)\} = 4$ . Now, since the map from  $\Theta_1$  to  $(2 \cdot \Theta_1)$  is  $1 : 1$  we get the conclusion.
- $R_{1,2} \cap \Delta$ : finally we have that  $(R_{1,2} \cap \Delta)$  is equivalent to the set  $\{a \in A \mid 2a \in ((-\Theta_1) \cap (-\Theta_2))\}$  whose cardinality is 32.  $\square$

### 5. On action of the hyperelliptic involution and $\text{Kum}_3(A)$ .

Let  $X$  be a curve of genus 2. Consider the degree 2 map:

$$\phi_3 : \text{SU}_X(3) \xrightarrow{2:1} \mathbb{P}^8 = |3\Theta_{\text{Pic}^1(X)}|$$

$$E \longmapsto D_E = \{L \in \text{Pic}^1(X) / h^0(E \otimes L) > 0\}$$

(see [7]). Let  $\tau'$  be the involution on  $\text{SU}_X(3)$  acting by the duality:

$$\tau'(E) = E^*$$

and  $\tau$  the hyperelliptic involution on  $\text{Pic}^1(X)$ :

$$\tau(L) = \omega_X \otimes L^{-1}.$$

We will use the following well known relation:

$$\tau \circ \phi_3(E) = \phi_3 \circ \tau'(E).$$

On  $\text{SU}_X(3)$  there is also the hyperelliptic involution  $h^*$ :

$$E \mapsto h^*(E)$$

induced by the hyperelliptic involution  $h$  of the curve  $X$ . We define  $\sigma := \tau' \circ h^*$ . It is the involution which gives the double covering of  $\text{SU}_X(3)$  on  $\mathbb{P}^8$ . The fixed locus of  $\sigma$  is obviously contained in  $\text{SU}_X(3)$  and we recall:

$$(5) \quad \phi_3(\text{Fix}(\sigma)) = \text{Coble sextic hypersurface}$$

(see [7]). By definition, the strictly semi-stable locus  $\text{SU}_X(3)^{ss}$  of  $\text{SU}_X(3)$  consists of isomorphism classes of split rank 3 semi-stable vector bundles of determinant  $\mathcal{O}_X$ . Its points can be represented by the vector bundles of the form  $F \oplus L$  or  $L_a \oplus L_b \oplus L_c$  with trivial determinant where  $L, L_a, L_b, L_c$  are line bundles and  $F$  is a rank 2 vector bundle. We want to consider the elements of the form  $L_a \oplus L_b \oplus L_c$  (those belonging to  $\text{Kum}_3(A)$ ) and actions of previous involutions on them:

- $\tau'(L_a \oplus L_b \oplus L_c) = (L_a \oplus L_b \oplus L_c)^* = L_{-a} \oplus L_{-b} \oplus L_{-c}$ ;
- $\tau'(h^*(L_a \oplus L_b \oplus L_c)) = L_a \oplus L_b \oplus L_c$ .

This implies that  $\sigma(\text{Kum}_3(A)) = \text{Kum}_3(A) \subset \text{SU}_X(3)$  which means that  $\text{Kum}_3(A) \subset \text{Fix}(\sigma)$  and then  $\phi_3(\text{Kum}_3(X)) \subset \text{Coble sextic}$  (see 5).

Let us now consider rank 2 semistable vector bundles of trivial determinant:  $\text{SU}_X(2)$ . If we take its symmetric square, we obtain a semisable rank three vector bundle with trivial determinant:

$$\text{SU}_X(2) \rightarrow \text{SU}_X(3); \quad E \mapsto \text{Sym}^2(E).$$

We want to study the action of involutions defined on the beginning of this paragraph on  $\text{Sym}^2(E)$  with  $E \in \text{SU}_X(2)$ . Since  $\text{Sym}^2(E)^* = \text{Sym}^2(E) = h^*(\text{Sym}^2(E))$ , then  $\sigma(\text{Sym}^2(E)) = \text{Sym}^2(E) \subset \text{SU}_X(3)$ , so  $\text{Sym}^2(\text{SU}_X(2)) \subset \text{Fix}(\sigma)$ , and, again by (5),  $\phi_3(\text{Sym}^2(\text{SU}_X(2))) \subset \text{Coble sextic}$ .

Now we want to see the action of  $\tau$  on  $|3\Theta_{\text{Pic}^1(X)}|$ . It is known that  $\text{Fix}(\tau) = \mathbb{P}^4 \sqcup \mathbb{P}^3$ .

**Notations:** We denote by  $\mathbb{P}_\tau^3$  and  $\mathbb{P}_\tau^4$ , respectively, the  $\mathbb{P}^3$  and the  $\mathbb{P}^4$  which are fixed by action of  $\tau$ .

Since the image of  $\text{Sym}^2(\text{SU}_X(2))$  by  $\phi_3$  in  $\mathbb{P}^8$  has dimension 3 and also  $\phi_3(\text{Sym}^2(\text{SU}_X(2))) \subset \text{Fix}(\tau)$ , we obtain

$$\phi_3(\text{Sym}^2(\text{SU}_X(2))) \subset \mathbb{P}_\tau^4.$$

Let  $L_a \oplus L_{-a}$  be an element of  $\text{Kum}_2(X) \subset \text{SU}_X(2)$ , then  $\text{Sym}^2(L_a \oplus L_{-a}) = L_{2a} \oplus L_{-2a} \oplus \mathcal{O} \in \text{Kum}_3(A) \subset \text{SU}_X(3)$ . It means that  $\text{Sym}^2(\text{Kum}_2(A)) \subset \text{Kum}_3(A)$ .

**Observation:** Since  $\{L_{2a} \oplus L_{-2a} \oplus \mathcal{O} \in \text{SU}_X(3)\}$  is isomorphic to  $S^2(\{L_a \oplus L_{-a}\})$ , we can view  $\{L_{2a} \oplus L_{-2a} \oplus \mathcal{O} \in \text{SU}_X(3)\}$  as the image of  $\text{Kum}_2(A)$  inside  $\text{SU}_X(3)$  under the symmetric square map. Moreover it follows from the surjectivity of the multiplication by 2 map  $[2] : A \rightarrow A$  that the image of  $\text{Kum}_2(A)$  in  $\text{SU}_X(3)$  is isomorphic to  $\text{Kum}_2(A)$ .

We have already observed that  $\phi_3|_{\text{Kum}_3(A)}$  is a  $(1 : 1)$ -map on the image; this fact allows us to view  $\phi_3(\text{Kum}_3(A))$  as the  $\text{Kum}_3(A)$  in  $|3\Theta_{\text{Pic}^1(X)}|$ . For the same reason we can view  $\phi_3(\text{Sym}^2(\text{SU}_X(2)))$  as  $\text{Kum}_2(A) \subset |3\Theta_{\text{Pic}^1(X)}|$ . Using this language we can say that  $\text{Kum}_2(A)$  is left fixed by the action of  $\tau$  in  $\text{Kum}_3(A) \subset |3\Theta_{\text{Pic}^1(X)}|$  because  $|3\Theta_{\text{Pic}^1(X)}| \supset \phi_3(\text{Kum}_3(A)) \supset \phi_3(\text{Sym}^2(\text{SU}_X(2))) = \text{Kum}_2(A) \subset \mathbb{P}^4 \subset \text{Fix}(\tau) \subset |3\Theta_{\text{Pic}^1(X)}^1|$ .

**Proposition 5.1.**  $\text{Fix}(\tau) \cap \phi_3(\text{Kum}_3(A)) = \phi_3(\text{Sym}^2(\text{Kum}_2(A)))$ .

*Proof.* By definition  $\tau(L_a \oplus L_b \oplus L_c) = L_{-a} \oplus L_{-b} \oplus L_{-c}$  then  $L_a \oplus L_b \oplus L_c$  belongs to  $\text{Fix}(\tau)$  if and only if  $\{a, b, c\} = \{-a, -b, -c\}$ . Let  $P$  belong to  $\{-a, -b, -c\}$  and  $a = P$ .

- If  $P$  is different from  $-a$ , suppose that  $P = -c$ , then  $\{-a, -b, -c\} = \{-a, -b, a\}$ ; moreover  $a + b + c = 0$  because  $L_a \oplus L_b \oplus L_c \in \text{Kum}_3(A)$ , then  $b = 0$ .
- Now, if  $P = -a$  or, equivalently  $a = -a$ , then  $a = 0$  and  $b = -c$ .

In both cases  $L_a \oplus L_b \oplus L_c \in \text{Kum}_3(A)$  such that  $\tau(L_a \oplus L_b \oplus L_c) = L_a \oplus L_b \oplus L_c$  are of the form  $L_a \oplus L_{-a} \oplus L_0$ . This means that they belong to  $\text{Kum}_2(A) \subset |3\Theta_{\text{Pic}^1(X)}|$ .  $\square$

The previous proposition tells us also that  $\mathbb{P}_\tau^3 \cap \text{Kum}_3 A = \emptyset$ . So the projection of  $\text{Kum}_3(A) \subset |3\Theta_{\text{Pic}^1(X)}|$  from  $\mathbb{P}_\tau^3$  to  $\mathbb{P}_\tau^4$  is a morphism. It would be interesting to find its degree.

Our final observation is the following.

**Proposition 5.2.**  $\text{Sing}(\text{Kum}_3(A)) \cap \text{Kum}_2(A) = \text{Sing}(\text{Kum}_2(A))$

*Proof.* Points of  $\text{Kum}_2(A) \subset \text{Kum}_3(A)$  are of the form  $(P, -P, 0)$ . Singular points of  $\text{Kum}_3(A)$  are those which have at least two equal components, then  $\text{Sing}(\text{Kum}_3(A)) \cap \text{Kum}_2(A) = \{(P, -P, 0)\}$  where  $2P = 0$  that are exactly the 15 points of 2-torsion and one more point  $(\mathcal{O}_X, \mathcal{O}_X, \mathcal{O}_X)$  which are singularities of the usual  $\text{Kum}_2(A)$ . This implies that  $\sharp(\text{Sing}(\text{Kum}_3(A)) \cap \text{Kum}_2(A)) = 16$  and  $\text{Sing}(\text{Kum}_3(A)) \cap \text{Kum}_2(A) = \text{Sing}(\text{Kum}_2(A))$ .  $\square$

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