Computing symmetric rank for symmetric tensors

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Abstract

We consider the problem of determining the symmetric tensor rank for symmetric tensors with an algebraic geometry approach. We give algorithms for computing the symmetric rank for $2 \times \cdots \times 2$ tensors and for tensors of small border rank. From a geometric point of view, we describe the symmetric rank strata for some secant varieties of Veronese varieties.

Key words: Symmetric tensor, tensor rank, secant variety.

1. Introduction

In this paper we study problems related to how to represent symmetric tensors, this is a kind of question which is relevant in many applications as in Electrical Engineering (Antenna Array Processing (Albera et al., 2005), (Dogan, Mendel, 1995) and Telecommunications (Chevalier, 1999), (De Lathauwer, Castaing, 2007)); in Statistics (cumulant

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tensors, see (McCullagh, 1987)), or in Data Analysis (Independent Component Analysis (Comon, 1992), (Jiang, Sidiropoulos, 2004)). For other applications see also (Comon, 2002), (Comon, Rajih, 2006), (De Lathauwer et al., 2000), (Sidiropoulos et al., 2000).

Let t be a symmetric tensor $t \in S^dV$, where V is an $(n+1)$-dimensional vector space; the minimum integer r such that t can be written as the sum of r elements of the type $v^{\otimes d} \in S^dV$ is called the *symmetric rank* of t (Definition 1).

In most applications it turns out that the knowledge of the symmetric rank is quite useful, e.g. the symmetric rank of a symmetric tensor extends the Singular Value Decomposition (SVD) problem for symmetric matrices (see (Golub et al., 1983)).

It is quite immediate to see that we can associate a homogeneous polynomial in $K[x_0, ..., x_n]_d$ to any symmetric tensor $t \in S^dV$ (see 3.1). It is a very classical algebraic problem (inspired by a number theory problem posed by Waring in 1770, see (Waring, 1991)), to determine which is the minimum integer r such that a generic form of degree d in $n+1$ variables can be written as a sum of r d-th powers of linear forms. This problem, known as the Big Waring Problem, is equivalent to determining the symmetric rank of t.

If we regard $\mathbb{P}^{\binom{n+d}{d}-1}$ as $\mathbb{P}(K[x_0, \ldots, x_n]_d)$, then the Veronese variety $X_{n,d} \subset \mathbb{P}^{\binom{n+d}{d}-1}$ is the variety that parameterizes those polynomials that can be written as d-th powers of a linear form (see Remark 4). When we view $\mathbb{P}^{\binom{n+d}{d}-1}$ as $\mathbb{P}(S^dV)$, where V is an $(n+1)$-dimensional vector space, the Veronese variety parameterizes projective classes of symmetric tensors of the type $v^{\otimes d} \in S^dV$ (see Definition 3).

The set that parameterizes tensors in $\mathbb{P}(S^dV)$ of a given symmetric rank is not a closed variety. If we consider $\sigma_r(X_{n,d})$, the r-th secant variety of $X_{n,d}$ (see Definition 7), this is the smallest variety containing all tensors of symmetric rank r, and this for all r up to the "typical rank", i.e. the first r for which $\sigma_r(X_{n,d}) = \mathbb{P}(S^dV)$. The smallest r such that $T \in \sigma_r(X_{n,d})$ is called the *symmetric border rank* of T (Definition 14). This shows that, from a geometric point of view, it seems more natural to study the symmetric border rank of tensors rather than the symmetric rank.

A geometric formulation of the Waring problem for forms asks which is the symmetric border rank of a generic symmetric tensor of S^dV. This problem was completely solved by J. Alexander and A. Hirschowitz who computed the dimensions of $\sigma_r(X_{n,d})$ for any r, n, d (see (Alexander, Hirschowitz, 1995) for the original proof and (Brambilla, Ottaviani, 2008) for a recent proof).

Although the dimensions of the $\sigma_r(X_{n,d})$'s are now all known, the same is not true for their defining equations: in general for all $\sigma_r(X_{n,d})$'s the equations coming from catalecticant matrices (Definition 17) are known, but in many cases they are not enough to describe their ideal; only in a few cases our knowledge is complete (see for example (Kanev, 1999), (Iarrobino, Kanev, 1999), (Catalisano et al., 2008), (Ottaviani, 2009) and (Landsberg, Ottaviani, -)). The knowledge of equations which define $\sigma_r(X_{n,d})$, at least set-theoretically, would give the possibility to compute the symmetric border rank for any tensor in S^dV.

A first efficient method to compute the symmetric rank of a symmetric tensor in $\mathbb{P}(S^dV)$ when $\dim(V) = 2$ is due to Sylvester (Sylvester, 1886). More than one version of that algorithm is known (see (Sylvester, 1886), (Brachat et al., 2009), (Comas, Seiguer, 2001)). In Section 3 we present a new version of that algorithm, which gives the symmetric rank of a tensor without passing through an explicit decomposition of it. The advantage of not giving an explicit decomposition is that this allows to much improve the speed of the algorithm. Finding explicit decompositions is a very interesting open problem (see also (Brachat et al., 2009) and (Landsberg, Teitler, 2009) for a study of the case $\dim(V) \geq 2$).

The aim of this paper is to explore a "projective geometry view" of the problem of finding what are the possible symmetric ranks of a tensor once its symmetric border rank is given. This amounts to determining the symmetric rank strata of the varieties $\sigma_r(X_{n,d})$. We do that in the following four cases: $\sigma_r(X_{1,d})$ (for any r and d, see also (Brachat et al., 2009), (Comas, Seiguer, 2001), (Landsberg, Teitler, 2009) and (Sylvester, 1886)); $\sigma_2(X_{n,d})$, $\sigma_3(X_{n,d})$ (any n,d, see Section 4); $\sigma_r(X_{2,4})$, for $r \leq 5$. In the first three cases we also give an algorithm to compute the symmetric rank. Some of these results were known or partially known, with different approaches and different algorithms, e.g in (Landsberg, Teitler, 2009) bounds on the symmetric rank are given for tensors in $\sigma_3(X_{n,d})$, while the possible values of the symmetric rank on $\sigma_3(X_{2,3})$ can be found in (Brachat et al., 2009), where an algorithm to find the decomposition is given. In Section 3 we also study the rank of points on $\sigma_2(\Gamma_{d+1}) \subset \mathbb{P}^d$, with respect to an elliptic normal curve Γ_{d+1}; for $d = 3$, Γ_4 gives another example (besides rational normal curves) of a curve $C \subset \mathbb{P}^n$ for which there are points of C-rank equal to n.

2. Preliminaries

We will always work with finite dimensional vector spaces defined over an algebraically closed field K of characteristic 0.

Definition 1. Let V be a finite dimensional vector space. The symmetric rank $s\mathrm{rk}(t)$ of a symmetric tensor $t \in S^dV$ is the minimum integer r such that there exist $v_1, \ldots, v_r \in V$ such that $t = \sum_{j=1}^{r} v_j^{\otimes d}$.

Notation 2. From now on we will indicate with T the projective class of a symmetric tensor $t \in S^dV$, i.e. if $t \in S^dV$ then $T = [t] \in \mathbb{P}(S^dV)$. We will write that an element $T \in \mathbb{P}(S^dV)$ has symmetric rank equal to r meaning that there exists a tensor $t \in S^dV$ such that $T = [t]$ and $s\mathrm{rk}(t) = r$.

Definition 3. Let V be a vector space of dimension $n+1$. The Veronese variety $X_{n,d} = \nu_d(\mathbb{P}(V)) \subset \mathbb{P}(S^dV) = \mathbb{P}^{\binom{n+d}{d}-1}$ is the variety given by the embedding ν_d defined by the complete linear system of hypersurfaces of degree d in $\mathbb{P}^n$.

Veronese varieties parameterize projective classes of symmetric tensors in S^dV of symmetric rank 1. Actually $T \in X_{n,d}$ if and only if there exists $v \in V$ such that $t = v^{\otimes d}$.

Remark 4. Let V be a vector space of dimension n and let $l \in V^*$ be a linear form. Now define $\nu_d : \mathbb{P}(V^*) \to \mathbb{P}(S^dV^*)$ as $\nu_d([l]) = [l^d] \in \mathbb{P}(S^dV^*)$. The image of this map is indeed the d-uple Veronese embedding of $\mathbb{P}(V^*)$.

Remark 5. Remark 4 shows that, if V is an n-dimensional vector space, then, given a basis for V, we can associate to any symmetric tensor $t \in S^dV$ of symmetric rank r a homogeneous polynomial of degree d in $n+1$ variables that can be written as a sum of r d-th power of linear forms (see 3.1).

Notation 6. If $v_1, \dots, v_s$ belong to a vector space V, we will denote with $< v_1, \dots, v_s >$ the subspace spanned by them. If $P_1, \dots, P_s$ belong to a projective space $\mathbb{P}^n$ we will use the same notation $< P_1, \dots, P_s >$ to denote the projective subspace generated by them.

Definition 7. Let $X \subset \mathbb{P}^N$ be a projective variety of dimension n. We define the s-th secant variety of X as follows:

$$\sigma_s(X) := \overline{\bigcup_{P_1,\dots,P_s \in X} < P_1, \dots, P_s >}.$$

Notation 8. We will indicate with $\sigma_s^0(X)$ the set $\bigcup_{P_1,\dots,P_s \in X} < P_1, \dots, P_s >$.

Notation 9. With $G(k, V)$ we denote the Grassmannian of k-dimensional subspaces of a vector space V, and with $\mathbb{G}(k-1, \mathbb{P}(V))$ we denote the $(k-1)$-dimensional projective subspaces of the projective space $\mathbb{P}(V)$.

Remark 10. Let $X \subset \mathbb{P}^N$ be a non degenerate smooth variety. If $P \in \sigma_r^0(X) \setminus \sigma_{r-1}^0(X)$ then the minimum number of distinct points $P_1, ..., P_s \in X$ such that $P \in < P_1, ..., P_s >$ is obviously r, which is achieved on $\sigma_r^0(X)$. We want to study what is that minimum number in $\sigma_r(X) \setminus (\sigma_r^0(X) \cup \sigma_{r-1}(X))$.

Proposition 11. Let $X \subset \mathbb{P}^N$ be a non degenerate smooth variety. Let H_r be the irreducible component of the Hilbert scheme of 0-dimensional schemes of degree r of X containing r distinct points, and assume that for each $y \in H_r$, the corresponding subscheme Y of X imposes independent conditions to linear forms. Then for each $P \in \sigma_r(X) \setminus \sigma_r^0(X)$ there exists a 0-dimensional scheme $Z \subset X$ of degree r such that $P \in < Z > \cong \mathbb{P}^{r-1}$.

Conversely if there exists $Z \in H_r$ such that $P \in < Z >$, then $P \in \sigma_r(X)$.

Proof. Let us consider the map $\phi : H_r \to \mathbb{G}(r-1, \mathbb{P}^N)$, $\phi(y) =< Y >$. The map ϕ is well defined since $\dim < Y >= r-1$ for all $y \in H_r$ by assumption. Hence $\phi(H_r)$ is closed in $\mathbb{G}(r-1, \mathbb{P}^N)$.

Now let $\mathcal{I} \subset \mathbb{P}^N \times \mathbb{G}(r-1, \mathbb{P}^N)$ be the incidence variety, and p, q its projections on $\mathbb{P}^N$ and on $\mathbb{G}(r-1, \mathbb{P}^N)$ respectively; then, $A := pq^{-1}(\phi(H_r))$ is closed in $\mathbb{P}^N$. Moreover, A is irreducible since H_r is irreducible, so $\sigma_r^0(X)$ is dense in A. Hence $\sigma_r(X) = \overline{\sigma_r^0(X)} = A$. $\square$

In the following we will use Proposition 11 when $X = X_{n,d}$, a Veronese variety, in many cases.

Remark 12. Let $n = 1$; in this case the Hilbert scheme of 0-dimensional schemes of degree r of $X = X_{1,d}$ is irreducible; moreover, for all y in the Hilbert scheme, Y imposes independent conditions to forms of any degree.

Also for $n = 2$ the Hilbert scheme of 0-dimensional schemes of degree r of $X = X_{2,d}$ is irreducible. Moreover, in the cases that we will study here, r is always small enough with respect to d, so to imply that all the elements in the Hilbert scheme impose independent conditions to forms of degree d.

Hence in the two cases above $P \in \sigma_r(X)$ if and only if there exists a scheme $Z \subset X$ of degree r such that $P \in < Z > \simeq \mathbb{P}^{r-1}$.

Now we give an example which shows that an $(r-1)$-dimensional linear space contained in $\sigma_r(X)$ is not always spanned by a 0-dimensional scheme of X of degree r. Let $n = 2$, $d = 6$, and consider $X = X_{2,6} = \nu_6(\mathbb{P}^2) \subset \mathbb{P}^{27}$ The first r for which $\sigma_r(X)$ is the whole of $\mathbb{P}^{27}$ is 10; we will consider $\sigma_8(X) \subset \mathbb{P}^{27}$. Let $Z \in \mathbb{P}^2$ be a scheme which is the union of 8 distinct points on a line $L \subset \mathbb{P}^2$. The curve $\nu_6(L)$ is a rational normal curve C_6 in its span $< C_6 >\cong \mathbb{P}^6$, so $\dim < \nu_6(Z) >= 6$. Moreover, since Z imposes only 6 conditions to curves of degree six in $\mathbb{P}^2$, then $\nu(Z)$ does not impose independent conditions to linear forms in $\mathbb{P}^{27}$. Now every linear 7-dimensional space $\Pi \subset \mathbb{P}^{27}$ containing C_6, meets X along C_6 and no other point; hence there does not exist a 0-dimensional scheme B of degree 8 on X such that $< B >\supset < \nu_6(Z) >$ and $< B >= \Pi$. On the other hand, consider a 1-dimensional flat family whose generic fiber Y is the union of 8 distinct points on X (hence $\dim < Y >= 7$) and such that $\nu(Z)$ is a special fiber of the family. If we consider the closure of the corresponding family of linear spaces with generic fiber $< Y >$, this is still is a 1-dimensional flat family, so it has to have a linear space $\Pi_0 \cong \mathbb{P}^7$ as special fiber. Hence $\sigma_8(X)$ contains linear spaces of dimension 7 as Π_0, such that $< \nu_6(Z) >\subset \Pi_0$, but for no subscheme Y' of degree 8 on X we have $\Pi_0 =< Y' >$.

Remark 13. A tensor $t \in S^dV$ with $\dim(V) = n + 1$ has symmetric rank r if and only if $T \in \sigma_r^0(X_{n,d})$ and, for any $s < r$, we have that $T \notin \sigma_s^0(X_{n,d})$. In fact by definition of symmetric rank of an element $T \in S^dV$, there should exist r elements (and no less) $T_1, \ldots, T_r \in X_{n,d}$ corresponding to tensors $t_1, \ldots, t_r$ of symmetric rank one such that $t = \sum_{i=1}^r t_i$. Hence $T \in \sigma_r^0(X_{n,d}) \setminus \sigma_{r-1}^0(X_{n,d})$.

Definition 14. If $T \in \sigma_s(X_{n,d}) \setminus \sigma_{s-1}(X_{n,d})$, we say that t has symmetric border rank s, and we write $\underline{\mathrm{srk}}(t) = s$.

Remark 15. The symmetric border rank of $t \in S^dV$, with $\dim(V) = n+1$, is the smallest s such that $T \in \sigma_s(X_{n,d})$. Therefore $s\mathrm{rk}(t) \geq \underline{\mathrm{srk}}(t)$. Moreover if $T \in \sigma_s(X_{n,d}) \setminus \sigma_s^0(X_{n,d})$ then $s\mathrm{rk}(t) > s$.

The following notation will turn out to be useful in the sequel.

Notation 16. We will indicate with $\sigma_{b,r}(X_{n,d}) \subset \mathbb{P}(S^dV)$ the set:

$$\sigma_{b,r}(X_{n,d}) := \{T \in \sigma_b(X_{n,d}) \setminus \sigma_{b-1}(X_{n,d}) | \mathrm{srk}(T) = r\},$$

i.e. the set of the points in $\mathbb{P}(S^dV)$ corresponding to symmetric tensor whose symmetric border rank is b and whose symmetric rank is r.

It is not easy to get a geometric description of the loci $\sigma_{b,r}(X_{n,d})$'s; we think that (when the base field is algebrically closed) they should be locally closed (when $n = 1$, i.e. for rational normal curves, this follows from Corollary 26), but we have no general reference for that.

3. Two dimensional case

In this section we will restrict to the case of a 2-dimensional vector space V. We first describe the Sylvester algorithm which gives the symmetric rank of a symmetric tensor $t \in S^d V$ and a decomposition of t as a sum of $r = s\mathrm{rk}(t)$ symmetric tensors of symmetric rank one (see (Sylvester, 1886) (Comas, Seiguer, 2001), (Brachat et al., 2009)). Then we give a geometric description of the situation and a slightly different algorithm which produces the symmetric rank of a symmetric tensor in $S^d V$ without giving explicitly its decomposition. This algorithm makes use of a result (see Theorem 23) which describes the rank of tensors on the secant varieties of the rational normal curve $C_d = X_{1,d}$; this Theorem has been proved in the unpublished paper (Comas, Seiguer, 2001) (see also (Landsberg, Teitler, 2009)); here we give a proof which uses only classical projective geometry.

Moreover we extend part of that result to elliptic normal curves, see Theorem 28.

3.1. The Sylvester algorithm

Let $p \in K[x_0, x_1]_d$ be a homogeneous polynomial of degree d in two variables: $p(x_0, x_1) = \sum_{k=0}^d a_k x_0^k x_1^{d-k}$; then we can associate to the form p a symmetric tensor $t \in S^d V \simeq K[x_0, x_1]_d$ where $t = (b_{i_1,\dots,i_d})_{i_j \in \{0,1\}; j=1,\dots,d}$, and $b_{i_1,\dots,i_d} = \binom{d}{k}^{-1} \cdot a_k$ for any d-uple $(i_1, \dots, i_d)$ containing exactly k zeros. This correspondence is clearly one to one:

$$\begin{array}{rcl} K[x_0,x_1]_d & \leftrightarrow & S^d V \\ \sum_{k=0}^d a_k x_0^k x_1^{d-k} & \leftrightarrow & (b_{i_1,\dots,i_d})_{i_j=0,1;\ j=1,\dots,d} \end{array} \tag{1}$$

with $(b_{i_1,\dots,i_d})$ as above.

The algorithm uses Catalecticant matrices, which are matrices that we can associate to a polynomial $p(x_0, x_1) = \sum_{k=0}^d a_k x_0^k x_1^{d-k}$, or to the symmetric tensor t associated to it. We give below the definition of Catalecticant matrices $M^n_{d-r,r}$ and $M_{d-r,r}(t)$ in the general case, see (Geramita , 1999) or (Kanev, 1999); $M_{d-r,r}(t)$ is also called Hankel matrix in (Brachat et al., 2009)).

Definition 17. Let $R = k[x_0, ..., x_n]$ and $i, j, d \in \mathbb{N}$ with $i + j = d$. Consider the bilinear map given by multiplication:

$$R_i \times R_j \to R_d.$$

If we fix in R_i,R_j the natural bases given by monomials (say in lex order), the map above can be represented by a $(\binom{n+i}{n}) \times (\binom{n+j}{n})$ matrix A. The (i, j)-catalecticant Matrix of R: $M^n_{i,j}$ is the $(\binom{n+i}{n}) \times (\binom{n+j}{n})$ matrix whose entries are the indeterminates $z_{\underline{\alpha}}$, $\underline{\alpha} = (\alpha_0, ..., \alpha_n) \in \mathbb{N}^n$, with $|\underline{\alpha}| = d$. For each entry $m_{u,v}$ of $M^n_{i,j}$, we have $m_{u,v} = z_{\underline{\alpha}}$ if the entry $a_{u,v}$ in A is associated to multiplication of two monomials which yields $\underline{x}^{\underline{\alpha}} = x_0^{\alpha_0} ... x_n^{\alpha_n}$.

Example. Let $n = d = 2$, $i = j = 1$; we get:

$$M^2_{1,1} = \begin{pmatrix} z_{2,0,0} & z_{1,1,0} & z_{1,0,1} \\ z_{1,1,0} & z_{0,2,0} & z_{0,1,1} \\ z_{1,0,1} & z_{0,1,1} & z_{0,0,2} \end{pmatrix}.$$

If we consider the new variables as coordinates in $\mathbb{P}^N$, $N = \binom{n+d}{n} - 1$, it is well known that the ideal of the 2×2 minors of $M^n_{i,j}$ is the defining ideal of the Veronese variety $X_{n,d} = \nu_d(\mathbb{P}^n)$.

Now consider a form $p \in R_d$. The (i,j)-catalecticant Matrix of p, $M_{i,j}(p)$, is the numerical matrix which yields:

$$(x_0^{\beta_0}, ..., x_n^{\beta_n}) \cdot M_{i,j}(p) \cdot {}^t(x_0^{\gamma_0}, ..., x_n^{\gamma_n}) = f(x_0, ..., x_n),$$

where $\{x_0^{\beta_0}, ..., x_n^{\beta_n}\}$, $\{x_0^{\gamma_0}, ..., x_n^{\gamma_n}\}$ are the bases for R_i,R_j, respectively.

Since we are more interested in tensors, we will always write $M_{i,j}(t)$ or $M_{i,j}(T)$, instead of $M_{i,j}(p)$, where t is the symmetric tensor associated to p (as we did at the beginning of the section in the 2-dimensional case), and T is its projective class in $\mathbb{P}(S^d(V))$.

Remark 18. When considering the two dimensional case, it is easier to describe $M_{i,j}(p)$ more explicitely. Let $p(x_0, x_1) = \sum_{k=0}^d a_k x_0^k x_1^{d-k}$, and $t = (b_{i_1}, ..., b_{i_d})_{i_j=0,1;\ j=1,...,d} \in S^d V$ be the symmetric tensor associated to p, as we did at the beginning of section. Then the Catalecticant matrix $M_{d-r,r}(t)$ associated to t (or to p) is the $(d-r+1) \times (r+1)$ matrix with entries: $c_{i,j} = \binom{d}{i}^{-1} a_{i+j-2}$ with $i = 1, \ldots, d-r$ and $j = 1, \ldots, r$.

We describe here the version of the Sylvester algorithm that can be found in (Sylvester, 1886), (Comas, Seiguer, 2001), or (Brachat et al., 2009):

Algorithm 1. Input: A binary form $p(x_0, x_1)$ of degree d or, equivalently, its associated symmetric tensor t.

Output: A decomposition of p as $p(x_0, x_1) = \sum_{j=1}^r \lambda_j l_j(x_0, x_1)^d$ with $\lambda_j \in K$ and $l_j \in K[x_0, x_1]_1$ for $j = 1, \ldots, r$ with r minimal.

(1) Initialize $r = 0$;
(2) Increment $r \leftarrow r + 1$;
(3) If the rank of the matrix $M_{d-r,r}(t)$ is maximum, then go to step 2;
(4) Else compute a basis $\{l_1, \ldots, l_h\}$ of the right kernel of $M_{d-r,r}$;
(5) Specialization:
  - Take a vector q in the right kernel of $M_{d-r,r}(t)$, e.g. $q = \sum_i \mu_i l_i$;
  - Compute the roots of the associated polynomial $q(x_0, x_1) = \sum_{h=0}^r q_h x_0^h x_1^{d-h}$ and denote them by (α_j, β_j), where $|\alpha_j|^2 + |\beta_j|^2 = 1$;
  - If the roots are not distinct in $\mathbb{P}^1$, go to step 2;
  - Else if $q(x_0, x_1)$ admits r distinct roots then compute coefficients λ_j, $1 \le j \le r$, by solving the linear system below:

$$\begin{pmatrix} \alpha_1^d & \cdots & \alpha_r^d \\ \alpha_1^{d-1}\beta_1 & \cdots & \alpha_r^{d-1}\beta_r \\ \alpha_1^{d-2}\beta_1^2 & \cdots & \alpha_r^{d-2}\beta_r^2 \\ \vdots & \vdots & \vdots \\ \beta_1^d & \cdots & \beta_r^d \end{pmatrix} \lambda = \begin{pmatrix} a_0 \\ 1/d a_1 \\ \binom{d}{2}^{-1} a_2 \\ \vdots \\ a_d \end{pmatrix};$$

(6) The required decomposition is $p(x_0, x_1) = \sum_{j=1}^r \lambda_j l_j(x_0, x_1)^d$, where $l_j(x_0, x_1) = (\alpha_j x_1 + \beta_j x_2)$.

3.2. Geometric description

If V is a two dimensional vector space, there is a well known isomorphism between $\bigwedge^{d-r+1}(S^dV)$ and $S^{d-r+1}(S^rV)$,(see (Murnaghan, 1938)). When $d \geq r$ such isomorphism can be interpreted in terms of projective algebraic varieties; it allows to view the $(d-r+1)$-uple Veronese embedding of $\mathbb{P}^r$, as the set of $(r-1)$-dimensional projective subspaces of $\mathbb{P}^d$ that are r-secant to the rational normal curve. The description of this result, via coordinates, was originally given by A. Iarrobino, V. Kanev (see (Iarrobino, Kanev, 1999)). We give here the description appeared in (Arrondo, Bernardi, 2009) (Lemma 2.1) (Notations as in 9).

Lemma 19. Consider the map $\phi_{r,d-r+1} : \mathbb{P}(K[t_0,t_1]_r) \to G(d-r+1, K[t_0,t_1]_d)$ that sends the class of $p_0 \in K[t_0,t_1]_r$ to the $(d-r+1)$-dimensional subspace of $K[t_0,t_1]_d$ of forms of the type p_0q, with $q \in K[t_0,t_1]_{d-r}$. Then the following hold:
(i) The image of $\phi_{r,d-r+1}$, after the Plücker embedding of $G(d-r+1, K[t_0,t_1]_d)$, is the Veronese variety $X_{r,d-r+1}$.
(ii) Identifying, by duality, $G(d-r+1, K[t_0,t_1]_d)$ with the Grassmann variety of subspaces of dimension $r-1$ in $\mathbb{P}(K[t_0,t_1]_d^*)$, the above Veronese variety is the set of r-secant spaces to a rational normal curve $C_d \subset \mathbb{P}(K[t_0,t_1]_d^*)$.

Proof. Write $p_0 = u_0t_0^r + u_1t_0^{r-1}t_1 + \cdots + u_rt_1^r$. Then a basis of the subspace of $K[t_0,t_1]_d$ of forms of the type p_0q is given by:

$$\begin{array}{l} u_0t_0^d + \cdots + u_rt_0^{d-r}t_1^r \\ \quad u_0t_0^{d-1}t_1 + \cdots + u_rt_0^{d-r-1}t_1^{r+1} \\ \qquad \ddots \\ \qquad\quad u_0t_0^rt_1^{d-r} + \cdots + u_rt_1^d. \end{array} \tag{2}$$

The coordinates of these elements with respect to the basis $\{t_0^d, t_0^{d-1}t_1, \ldots, t_1^d\}$ of $K[t_0,t_1]_d$ are thus given by the rows of the matrix

$$\begin{pmatrix} u_0 & u_1 & \ldots & u_r & 0 & \ldots & 0 & 0 \\ 0 & u_0 & u_1 & \ldots & u_r & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & \vdots \\ 0 & \ldots & 0 & u_0 & u_1 & \ldots & u_r & 0 \\ 0 & \ldots & 0 & 0 & u_0 & \ldots & u_{r-1} & u_r \end{pmatrix}.$$

The standard Plücker coordinates of the subspace $\phi_{r,d-r+1}([p_0])$ are the maximal minors of this matrix. It is known (see for example (Arrondo, Paoletti , 2005)), that these minors form a basis of $K[u_0,\ldots,u_r]_{d-r+1}$, so that the image of ϕ is indeed a Veronese variety, which proves (i).

To prove (ii), we recall some standard facts from (Arrondo, Paoletti , 2005). Take homogeneous coordinates $z_0,\ldots,z_d$ in $\mathbb{P}(K[t_0,t_1]_d^*)$ corresponding to the dual basis of $\{t_0^d, t_0^{d-1}t_1, \ldots, t_1^d\}$. Consider $C_d \subset \mathbb{P}(K[t_0,t_1]_d^*)$ the standard rational normal curve with respect to these coordinates. Then, the image of $[p_0]$ by $\phi_{r,d-r+1}$ is precisely the r-secant space to C_d spanned by the divisor on C_d induced by the zeros of p_0. This completes the proof of (ii). $\square$

Since $\dim(V)=2$, the Veronese variety of $\mathbb{P}(S^dV)$ is the rational normal curve $C_d \subset \mathbb{P}^d$. Hence, a symmetric tensor $t \in S^dV$ has symmetric rank r if and only if r is the minimum integer for which there exists a $\mathbb{P}^{r-1}=\mathbb{P}(W)\subset \mathbb{P}(S^dV)$ such that $T\in \mathbb{P}(W)$ and $\mathbb{P}(W)$ is r-secant to the rational normal curve $C_d \subset \mathbb{P}(S^dV)$ in r distinct points.
Consider the maps:

$$\mathbb{P}(K[t_0,t_1]_r) \stackrel{\phi_{r,d-r+1}}{\to} \mathbb{G}(d-r,\mathbb{P}(K[t_0,t_1]_d)) \stackrel{\alpha_{r,d-r+1}}{\simeq} \mathbb{G}(r-1,\mathbb{P}(K[t_0,t_1]_d)^*). \qquad (3)$$

Clearly, since $\dim(V)=2$, we can identify $\mathbb{P}(K[t_0,t_1]_d)^*$ with $\mathbb{P}(S^dV)$, hence the Grassmannian $\mathbb{G}(r-1,\mathbb{P}(K[t_0,t_1]_d)^*)$ can be identified with $\mathbb{G}(r-1,\mathbb{P}(S^dV))$.
Now, by Lemma 19, a projective subspace $\mathbb{P}(W)$ of $\mathbb{P}(K[t_0,t_1]_d)^* \simeq \mathbb{P}(S^dV)\simeq \mathbb{P}^d$ is r-secant to $C_d\subset \mathbb{P}(S^dV)$ in r distinct points if and only if it belongs to $\mathrm{Im}(\alpha_{r,d-r+1}\circ \phi_{r,d-r+1})$ and the preimage of $\mathbb{P}(W)$ via $\alpha_{r,d-r+1}\circ\phi_{r,d-r+1}$ is a polynomial with r distinct roots.
Therefore, a symmetric tensor $t\in S^dV$ has symmetric rank r if and only if r is the minimum integer for which:

(1) T belongs to an element $\mathbb{P}(W)\in \mathrm{Im}(\alpha_{r,d-r+1}\circ\phi_{r,d-r+1})\subset \mathbb{G}(r-1,\mathbb{P}(S^dV))$,
(2) there exists a polynomial $p_0\in K[t_0,t_1]_r$ such that $\alpha_{r,d-r+1}(\phi_{r,d-r+1}([p_0]))=\mathbb{P}(W)$ and p_0 has r distinct roots.

Fix the natural basis $\Sigma=\{t_0^d, t_0^{d-1}t_1,\ldots,t_1^d\}$ in $K[t_0,t_1]_d$. Let $\mathbb{P}(U)$ be a $(d-r)$-dimensional projective subspace of $\mathbb{P}(K[t_0,t_1]_d)$. The proof of Lemma 19 shows that $\mathbb{P}(U)$ belongs to the image of $\phi_{r,d-r+1}$ if and only if there exist $u_0,\ldots,u_r\in K$ such that $U=< p_1,\ldots,p_{d-r+1}>$ with $p_1=(u_0,u_1,\ldots,u_r,0,\ldots,0)_\Sigma$, $p_2=(0,u_0,u_1,\ldots,u_r,0,\ldots,0)_\Sigma$, . . . , $p_{d-r+1}=(0,\ldots,0,u_0,u_1,\ldots,u_r)_\Sigma$.
Now let $\Sigma^*=\{z_0,\ldots,z_d\}$ be the dual basis of Σ. Therefore there exists a $W\subset S^dV$ such that $\mathbb{P}(W)=\alpha_{r,d-r+1}(\mathbb{P}(U))$ if and only if $W=H_1\cap\cdots\cap H_{d-r+1}$ and the H_i's are as follows:

$$\begin{aligned} H_1 &: u_0z_0+\cdots+u_rz_r=0\\ H_2 &: \quad u_0z_1+\cdots+u_rz_{r+1}=0\\ &\qquad\qquad \ddots\\ H_{d-r+1} &: \qquad u_0z_{d-r}+\cdots+u_rz_d=0. \end{aligned}$$

This is sufficient to conclude that $T\in\mathbb{P}(S^dV)$ belongs to an $(r-1)$-dimensional projective subspace of $\mathbb{P}(S^dV)$ that is in the image of $\alpha_{r,d-r+1}\circ\phi_{r,d-r+1}$ defined in (3) if and only if there exist $H_1,\ldots,H_{d-r+1}$ hyperplanes in S^dV as above such that $T\in H_1\cap\ldots\cap H_{d-r+1}$.
Given $t=(a_0,\ldots,a_d)_{\Sigma^*}\in S^dV$, $T\in H_1\cap\ldots\cap H_{d-r+1}$ if and only if the following linear system admits a non trivial solution:

$$\begin{cases} u_0a_0+\cdots+u_ra_r=0\\ u_0a_1+\cdots+u_ra_{r+1}=0\\ \vdots\\ u_0a_{d-r}+\cdots+u_ra_d=0. \end{cases}$$

If $d-r+1 < r+1$ this system admits an infinite number of solutions.
If $r \leq d/2$, it admits a non trivial solution if and only if all the maximal $(r+1)$-minors of the following $(d-r+1)\times(r+1)$ catalecticant matrix, defined in Definition 17, vanish :

$$M_{d-r,r} = \begin{pmatrix} a_0 & \cdots & a_r \\ a_1 & \cdots & a_{r+1} \\ \vdots & & \vdots \\ a_{d-r} & \cdots & a_d \end{pmatrix}.$$

The following three remarks contain results on rational normal curves and their secant varieties that are classically known and that we will need in our description.

Remark 20. The dimension of $\sigma_r(C_d)$ is the minimum between $2r-1$ and d. Actually $\sigma_r(C_d) \subsetneq \mathbb{P}^d$ if and only if $1 \leq r < \lceil \frac{d+1}{2} \rceil$.

Remark 21. An element $T \in \mathbb{P}^d$ belongs to $\sigma_r(C_d)$ for $1 \leq r < \lceil \frac{d+1}{2} \rceil$ if and only if the catalecticant matrix $M_{r,d-r}$ defined in Definition 17 does not have maximal rank.

Remark 22. Any divisor $D \subset C_d$, with $\deg D \leq d+1$, is such that $\dim < D >= \deg D - 1$.

The following result has been proved by G. Comas and M. Seiguer in the unpublished paper (Comas, Seiguer, 2001) (see also (Landsberg, Teitler, 2009)), and it describes the structure of the stratification by symmetric rank of symmetric tensors in S^dV with $\dim(V) = 2$. The proof we give here is a strictly "projective geometry" one.

Theorem 23. *Let V be a 2-dimensional vector space and $X_{1,d} = C_d \subset \mathbb{P}(S^dV)$, be the rational normal curve, parameterizing decomposable symmetric tensors ($C_d = \{T \in \mathbb{P}(S^dV) \mid \mathrm{srk}(T) = 1\}$), i.e. homogeneous polynomials in $K[t_0,t_1]_d$ which are d-th powers of linear forms. Then:*

$$\forall\, r,\ 2 \leq r \leq \left\lceil \frac{d+1}{2} \right\rceil : \qquad \sigma_r(C_d) \setminus \sigma_{r-1}(C_d) = \sigma_{r,r}(C_d) \cup \sigma_{r,d-r+2}(C_d)$$

where $\sigma_{r,r}(C_d)$ and $\sigma_{r,d-r+2}(C_d)$ are defined in Notation 16.

Proof. Of course, for all $t \in S^dV$, if $\mathrm{srk}(t) = r$, with $r \leq \lceil \frac{d+1}{2} \rceil$, we have $T \in \sigma_r(C_d) \setminus \sigma_{r-1}(C_d)$. Thus we have to consider the case $\mathrm{srk}(t) > \lceil \frac{d+1}{2} \rceil$, which can happen only if $T \in \sigma_r(C_d) \setminus \sigma_{r-1}(C_d)$ and $\mathrm{srk}(t) > r$, i.e. $T \notin \sigma_r^0(C_d)$.

If a point in $K[t_0,t_1]_d^*$ represents a tensor t with $\mathrm{srk}(t) > \lceil \frac{d+1}{2} \rceil$, then we want to show that $\mathrm{srk}(t) = d-r+2$, where r is the minimum integer such that $T \in \sigma_r(C_d)$, $r \leq \lceil \frac{d+1}{2} \rceil$.

First let us consider the case $r = 2$. Let $T \in \sigma_2(C_d) \setminus C_d$. If $\mathrm{srk}(t) > 2$, then T lies on a line t_P, tangent to C_d at a point P (this is because T has to lie on a $\mathbb{P}^1$ which is the image of a non-reduced form of degree 2: $p_0 = l^2$ with $l \in K[x_0,x_1]_1$, otherwise $\mathrm{srk}(t) = 2$). We want to show that $\mathrm{srk}(t) = d$. If $\mathrm{srk}(t) = r < d$, there would exist distinct points $P_1, \ldots, P_{d-1} \in C_d$, such that $T \in < P_1, \ldots, P_{d-1} >$; in this case the hyperplane

$H =< P_1, \dots, P_{d-1}, P >$ would be such that $t_P \subset H$, but this is a contradiction, since $H \cap C_d = 2P + P_1 + \cdots + P_{d-1}$ has degree $d+1$.

Notice that $s\mathrm{rk}(t) = d$ is possible, since obviously there is a $(d-1)$-space (i.e. a hyperplane) through T cutting d distinct points on C_d (any generic hyperplane through T will do). This also shows that d is the maximum possible rank.

Now let us generalize the procedure above; let $T \in \sigma_r(C_d) \setminus \sigma_{r-1}(C_d)$, $r \leq \lceil \frac{d+1}{2} \rceil$; we want to prove that if $s\mathrm{rk}(t) \neq r$, then $s\mathrm{rk}(t) = d-r+2$. Since $s\mathrm{rk}(t) > r$, we know that T must lie on a $\mathbb{P}^{r-1}$ which cuts a non-reduced divisor $Z \in C_d$ with $\deg(Z) = r$; therefore there is a point $P \in C_d$ such that $2P \in Z$. If we had $s\mathrm{rk}(t) \leq d-r+1$, then T would be on a $\mathbb{P}^{d-r}$ which cuts C_d in distinct points $P_1, \dots, P_{d-r+1}$; if that were true the space $< P_1, \dots, P_{d-r+1}, Z-P >$ would be $(d-1-\deg(Z-2P) \cap \{P_1, \dots, P_{d-r+1}\})$-dimensional and cut $P_1 + \cdots + P_{d-r+1} + Z - (Z-2P) \cap \{P_1, \dots, P_{d-r+1}\}$ on C_d, which is impossible.

So we got $s\mathrm{rk}(t) \geq d-r+2$; now we have to show that the rank is actually $d-r+2$. Let's consider the divisor $Z - 2P$ on C_d; we have $\deg(Z-2P) = r-2$, and the space $\Gamma =< Z-2P, T >$ which is $(r-2)$-dimensional since $< Z-2P >$ does not contain T (otherwise $T \in \sigma_{r-2}(C_d)$). We will be finished if we show that the generic divisor of the linear series cut on C_d by the hyperplanes containing Γ is reduced.

If it is not, there should be a fixed non-reduced part of the series, i.e. there should exist at least a fixed divisor of type $2Q$. If this is the case, each hyperplane through Γ would contain $2Q$, hence $2Q \subset \Gamma$, which is impossible, since we would have $\deg(\Gamma \cap C_d) = r$, while $\dim \Gamma = r-2$.

Thus $s\mathrm{rk}(t) = d-r+2$, as required. □

Remark 24. (Rank for monomials) In the proof above we have used the fact that (see Proposition 11) if t is a symmetric tensor such that $T \in \sigma_r(C_d) \setminus \sigma_{r-1}(C_d)$, and $T \notin \sigma_r^0(C_d)$, then there exists a non reduced 0-dimensional scheme $Z \subset \mathbb{P}^d$, which is a divisor of degree r on C_d, such that $T \in < Z >$. Let $Z = m_1 P_1 + \cdots + m_s P_s$, with $P_1, \dots, P_s$ distinct points on the curve, $m_1 + \cdots + m_s = r$ and $m_i \geq 2$ for at least one value of i. Then t^* can be written as

$$t^* = l_1^{d-m_1+1} f_1 + \cdots + l_s^{d-m_s+1} f_s$$

where $l_1, \dots, l_s$ are homogeneous linear forms in two variables and each f_i is a homogeneous form of degree $m_i - 1$ for $i = 1, \dots, s$.

In the theorem above it is implicitly proved that each form of this type has symmetric rank $d-r+2$. In particular, every monomial of type $x^{d-s}y^s$ is such that

$$s\mathrm{rk}(x^{d-s}y^s) = \max\{d-s+1, s+1\}.$$

Notation 25. For all smooth projective varieties $X, Y \subset \mathbb{P}^d$, we denote with $\tau(X)$ the *tangential variety* to X, i.e. the closure of the union of all its projective embedded tangent spaces at its points, and with $J(X,Y)$, the *join* of X and Y, i.e. the closure of the union of all the lines $< x, y >$, for $x \in X$ and $y \in Y$.

From the proof of Theorem 23, we can also deduce the following result which describes the strata of high rank on each $\sigma_r(C_d)$:

Corollary 26. Let $C_d \subset \mathbb{P}^d$, $d > 2$; then we have:
- $\sigma_{2,d}(C_d) = \tau(C_d) \setminus C_d$;
- For all r, with $3 \leq r < \frac{d+2}{2}$: $\sigma_{r,d-r+2}(C_d) = J(\tau(C_d), \sigma_{r-2}(C_d)) \setminus \sigma_{r-1}(C_d)$.

3.3. A result on elliptic normal curves.

We can use the same kind of construction we used for rational normal curves to prove the following result on elliptic normal curves.

Notation 27. If $\Gamma_{d+1} \subset \mathbb{P}^d$, with $d \geq 3$, is an elliptic normal curve, and $T \in \mathbb{P}^d$, we say that T has rank r with respect to Γ_{d+1} and we write $r = \mathrm{rk}_{\Gamma_{d+1}}(T)$, if r is the minimum number of points of Γ_{d+1} such that T depends linearly on them.

In the following the $\sigma_{i,j}(\Gamma_{d+1})$'s are defined as in Notation 16, but with respect to Γ_{d+1}, i.e. $\sigma_{i,j}(\Gamma_{d+1}) = \{T \in \mathbb{P}^d | \mathrm{rk}_{\Gamma_{d+1}}(t) = j, T \in \sigma_i(\Gamma_{d+1})\}$.

Theorem 28. *Let $\Gamma_{d+1} \subset \mathbb{P}^d$, $d \geq 3$, be an elliptic normal curve, then:*
- *When $d = 3$, we have :* $\quad \sigma_2(\Gamma_4) \setminus \Gamma_4 = \sigma_{2,2}(\Gamma_4) \cup \sigma_{2,3}(\Gamma_4)$; *(here $\sigma_2(\Gamma_4) = \mathbb{P}^3$).*
- *For $d \geq 4$:* $\quad \sigma_2(\Gamma_{d+1}) \setminus \Gamma_{d+1} = \sigma_{2,2}(\Gamma_{d+1}) \cup \sigma_{2,d-1}(\Gamma_{d+1})$.

Moreover $\sigma_{2,3}(\Gamma_4) = \{T \in \tau(\Gamma_4) \mid$ two tangent lines to Γ_4 meet in $T\}$.

Proof. First let $d \geq 4$; let $T \in \sigma_2(\Gamma_{d+1}) \setminus \Gamma_{d+1}$. If $\mathrm{rk}_{\Gamma_{d+1}}(T) > 2$, it means that T lies on a line t_P, tangent to Γ_{d+1} at a point P. We want to show that $\mathrm{rk}_{\Gamma_{d+1}}(T) = d - 1$. First let us check that we cannot have $\mathrm{rk}_{\Gamma_{d+1}}(T) = r < d - 1$. In fact, if that were the case, there would exist points $P_1, \ldots, P_{d-2} \in \Gamma_{d+1}$, such that $T \in < P_1, \ldots, P_{d-2} >$; in this case the space $< P_1, \ldots, P_{d-2}, P >$ would be $(d-2)$-dimensional, and such that $< P_1, \ldots, P_{d-2}, 2P > = < P_1, \ldots, P_{d-2}, P >$, since T is on $< P_1, \ldots, P_{d-2} >$, so the line $< 2P > = t_P$ is in $< P_1, \ldots, P_{d-2}, P >$ already. But this is a contradiction, since $< P_1, \ldots, P_{d-2}, 2P >$ has to be $(d-1)$-dimensional (on Γ_{d+1} every divisor of degree at most d imposes independent conditions to hyperplanes).

Now we want to check that $\mathrm{rk}_{\Gamma_{d+1}}(T) \leq d - 1$. We have to show that there exist $d - 1$ distinct points $P_1, \ldots, P_{d-1}$ on Γ_{d+1}, such that $T \in < P_1, \ldots, P_{d-1} >$. Consider the hyperplanes in $\mathbb{P}^d$ containing the line t_P; they cut a g^{d-2}_{d+1} on Γ_{d+1}, which is made of the fixed divisor $2P$, plus a complete linear series g^{d-2}_{d-1}, which is of course very ample; among the divisors of this linear series, the ones which span a $\mathbb{P}^{d-2}$ containing T form a sub-series g^{d-3}_{d-1}, whose generic element is smooth (this is always true for a subseries of codimension one of a very ample linear series), hence it is made of $d - 1$ distinct points whose span contains T, as required.

Now let $d = 3$; obviously $\sigma_2(\Gamma_4) = \mathbb{P}^3$; if we have a point $T \in (\sigma_2(\Gamma_4) \setminus \Gamma_4)$, then T is on a tangent line t_P of the curve. Consider the planes through t_P; they cut a g^1_2 on Γ_4 outside $2P$; each divisor D of such g^1_2 spans a line which meets t_P in a point ($< D > + < 2P >$ is a plane in $\mathbb{P}^3$), so the g^1_2 defines a $2:1$ map $\Gamma_4 \to t_P$ which, by Hurwitz theorem, has four ramification points. Hence for a generic point of t_P there is a secant line through it (i.e. it lies on $\sigma_{2,2}(\Gamma_4)$), but for those special points no such line exists (namely, for the points in which two tangent lines at Γ_4 meet), hence those points have $\mathrm{rk}_{\Gamma_4} = 3$ (a generic hyperplane through one point cuts 4 distinct points on Γ_4, and three of them span it). $\square$

Remark 29. Let $T \in \mathbb{P}^d$ and $C \subset \mathbb{P}^d$ be a smooth curve not contained in a hyperplane. It is always true that $\mathrm{rk}_C(T) \leq d$. E.g. if C is the rational normal curve $C = C_d \subset \mathbb{P}^d$, this maximum value of the rank can be attained by a tensor T, and this is precisely the case when T belongs to $\tau(C_d) \setminus C_d$, see Theorem 23). Actually Theorem 28 shows that, if $d = 3$, then there are tensors of $\mathbb{P}^3$ whose rank with respect to an elliptic normal curve

$\Gamma_4 \subset \mathbb{P}^3$ is precisely 3. In the very same way, one can check that the same is true for a rational (non-normal) quartic curve $C_4 \subset \mathbb{P}^3$. For the case of space curves, several other examples can be found in (Piene, 1981).

3.4. Simplified version of The Sylvester Algorithm

Theorem 23 allows to get a simplified version of the Sylvester algorithm (see also (Comas, Seiguer, 2001)), which computes only the symmetric rank of a symmetric tensor, without computing the actual decomposition.

Algorithm 2. The (Sylvester) Symmetric Rank Algorithm:

Input: The projective class T of a symmetric tensor $t \in S^dV$ with $\dim(V) = 2$
Output: $s\mathrm{rk}(t)$.

(1) Initialize $r = 0$;
(2) Increment $r \leftarrow r + 1$;
(3) Compute $M_{d-r,r}(t)$'s $(r+1) \times (r+1)$-minors; if they are not all equal to zero then go to step 2; else, $T \in \sigma_r(C_d)$ (notice that this happens for $r \leq \lceil \frac{d+1}{2} \rceil$); go to step 4.
(4) Choose a solution $(\overline{u}_0, \ldots, \overline{u}_d)$ of the system $M_{d-r,r}(t) \cdot (u_0, \ldots, u_r)^t = 0$. If the polynomial $\overline{u}_0 t_0^d + \overline{u}_1 t_0^{d-1} t_1 + \cdots + \overline{u}_r t_1^r$ has distinct roots, then $s\mathrm{rk}(t) = r$, i.e. $T \in \sigma_{r,r}(C_d)$, otherwise $s\mathrm{rk}(t) = d - r + 2$, i.e. $T \in \sigma_{r,d-r+2}(C_d)$.

4. Beyond dimension two

The maps in (3) have to be reconsidered when working on $\mathbb{P}^n$, $n \geq 2$, and with secant varieties to the Veronese variety $X_{n,d} \subset \mathbb{P}^N$, $N = \binom{d+n}{n} - 1$. Now a polynomial in $K[x_0, \ldots, x_n]_r$ gives a divisor, which is not a 0-dimensional scheme, so the previous construction would not give $(r-1)$-spaces which are r-secant to the Veronese variety.

Actually in this case, when following the construction in (3), we associate to a polynomial $f \in K[x_0, \ldots, x_n]_r$, the degree d part of the principal ideal (f), i.e. the vector space $(f)_d \subset K[x_0, \ldots, x_n]_d$, which is $\binom{d-r+n}{n}$-dimensional. Then, working by duality as before, we get a linear space in $\mathbb{P}^N$ which has dimension $\binom{d+n}{n} - \binom{d-r+n}{n} - 1$ and it is the intersection of the hyperplanes containing the image $\nu_d(F) \subset \nu_d(\mathbb{P}^n)$ of the divisor $F = \{f = 0\}$ where ν_d is the Veronese map defined in Notation 4.

Since the condition for a point in $\mathbb{P}^N$ to belong to such a space is given by the annihilation of the maximal minors of the catalecticant matrix $M^{(n)}_{d-r,r}$, this shows that such minors define in $\mathbb{P}^N$ a variety which is the union of the linear spaces spanned by the images of the divisors (hypersurfaces in $\mathbb{P}^n$) of degree r on the Veronese $X_{n,d}$ (see (Gherardelli, 1996)).

In order to consider linear spaces which are r-secant to $X_{n,d}$, we will change our approach by considering the Hilbert scheme of 0-dimensional subschemes of degree r in $\mathbb{P}^n$, $Hilb_r(\mathbb{P}^n)$, instead of $K[x_0, \ldots, x_n]_r$:

$$Hilb_r(\mathbb{P}^n) \overset{\phi}{\dashrightarrow} \vec{G}\left(\binom{d+n}{n} - r, K[x_0, \dots, x_n]_d\right) \overset{\beta}{\to}$$

$$\overset{\beta}{\to} \mathbb{G}\left(\binom{d+n}{n} - r - 1, \mathbb{P}(K[x_0, \dots, x_n]_d)\right) \overset{\alpha}{\to} \mathbb{G}(r-1, \mathbb{P}(K[x_0, \dots, x_n]_d)^*). \tag{4}$$

The map ϕ in (4) sends a scheme Z, with $\deg(Z) = r$, to the vector space $(I_Z)_d$; it is defined in the open set which parameterizes the schemes Z which impose independent conditions to forms of degree d. The isomorphism β is the identification between the vectorial and projective Grassmannians, while α is given by duality.

As in the case $n = 1$, the final image in the above sequence of maps gives the $(r-1)$-spaces which are r-secant to the Veronese variety in $\mathbb{P}^N \cong \mathbb{P}(K[x_0, \dots, x_n]_d)^*$; moreover such a space cuts the image of Z on the Veronese.

Notation 30. From now on we will always use the notation Π_Z to indicate the projective linear subspace of dimension $r-1$ in $\mathbb{P}(S^dV)$, with $\dim(V) = n+1$, generated by the image of a 0-dimensional scheme $Z \subset \mathbb{P}^n$ of degree r via Veronese embedding.

4.1. The chordal varieties to Veronese varieties

Here we describe $\sigma_r(X_{n,d})$ for $r = 2$ and $n, d \geq 1$. More precisely we give a stratification of $\sigma_r(X_{n,d})$ in terms of the symmetric rank of its elements. We will end with an algorithm that allows to determine if an element belongs to $\sigma_2(X_{n,d})$ and, if this is the case, to compute $s\mathrm{rk}(t)$.

We premit a remark that will be useful in the sequel.

Remark 31. When a form $f \in K[x_0, \dots, x_n]$ can be written using less variables (i.e. $f \in K[l_0, \dots, l_m]$, for $l_j \in K[x_0, \dots, x_n]_1$, $m < n$) then the symmetric rank of the symmetric tensor associated to f (with respect to $X_{n,d}$) is the same one as the one with respect to $X_{m,d}$, (e.g. see (Lim, De Silva, 2008), (Landsberg, Teitler, 2009)). In particular, when a tensor is such that $T \in \sigma_r(X_{n,d}) \subset \mathbb{P}(S^dV)$, $\dim(V) = n+1$, then, if $r < n+1$, there is a subspace $W \subset V$ with $\dim(W) = r$ such that $T \in \mathbb{P}(S^dW)$; i.e. the form corresponding to T can be written with respect to r variables.

Theorem 32. *Any $T \in \sigma_2(X_{n,d}) \subset \mathbb{P}(V)$, with $\dim(V) = n+1$, can only have symmetric rank equal to 1, 2 or d. More precisely:*

$$\sigma_2(X_{n,d}) \setminus X_{n,d} = \sigma_{2,2}(X_{n,d}) \cup \sigma_{2,d}(X_{n,d}),$$

moreover $\sigma_{2,d}(X_{n,d}) = \tau(X_{n,d}) \setminus X_{n,d}$.

Here $\sigma_{2,2}(X_{n,d})$ and $\sigma_{2,d}(X_{n,d})$ are defined in Notation 16 and $\tau(X_{n,d})$ is defined in Notation 25.

Proof. The theorem is actually a quite direct consequence of Remark 31 and of Theorem 23, but let us describe the geometry in some detail. Since $r = 2$, every $Z \in Hilb_2(\mathbb{P}^n)$ is the complete intersection of a line and a quadric, so the structure of I_Z is well known: $I_Z = (l_1, \dots, l_{n-1}, q)$, where $l_i \in R_1$, linearly independent, and $q \in R_2 - (l_1, \dots, l_{n-1})_2$.

If $T \in \sigma_2(X_{n,d})$ we have two possibilities; either $s\mathrm{rk}(T) = 2$ (i.e. $T \in \sigma_2^0(X_{n,2})$), or $s\mathrm{rk}(T) > 2$ i.e. T lies on a tangent line Π_Z to the Veronese, which is given by the image of a scheme Z of degree 2, via the maps (4). We can view T in the projective linear space $H \cong \mathbb{P}^d$ in $\mathbb{P}(S^dV)$ generated by the rational normal curve $C_d \subset X_{n,d}$, which is the image of the line L defined by the ideal $(l_1, \ldots, l_{n-1})$ in $\mathbb{P}^n$ with $l_1, \ldots, l_{n-1} \in V^*$ (i.e. $L \subset \mathbb{P}^n$ is the unique line containing Z); hence we can apply Theorem 23 in order to get that $\mathrm{rk}_{C_d}(T) = d$.

Hence, by Remark 31, we have $s\mathrm{rk}(T) = d$. $\square$

Remark 33. Let us check that the annihilation of the (3×3)-minors of the first two catalecticant matrices, $M_{d-1,1}$ and $M_{d-2,2}$ determines $\sigma_2(X_{n,d})$ (actually such minors are the generators of $I_{\sigma_2(X_{n,d})}$, see (Kanev, 1999)).

Following the construction before Theorem 3.3, we can notice that the linear spaces defined by the forms $l_i \in V^*$ in the ideal I_Z, are such that their coefficients are the solutions of a linear system whose matrix is given by the catalecticant matrix $M_{d-1,1}$ defined in Definition 17 (where the a_i's are the coefficients of the polynomial defined by t); since the space of solutions has dimension $n-1$, we get $\mathrm{rk}(M_{d-1,1}) = 2$. When we consider the quadric q in I_Z, instead, the analogous construction gives that its coefficients are the solutions of a linear systems defined by the catalecticant matrix $M_{d-2,2}$, and the space of solutions has to give q and all the quadrics in $(l_1, \ldots, l_{n-1})_2$, which are $\binom{n}{2} + 2n - 1$, hence $\mathrm{rk}(M_{d-2,2}) = \binom{n+2}{2} - (\binom{n}{2} + 2n) = 2$.

Therefore we can write down an algorithm to test if an element $T \in \sigma_2(X_{n,d})$ has symmetric rank 2 or d.

Algorithm 3. Algorithm for the symmetric rank of an element of $\boldsymbol{\sigma_2(X_{n,d})}$

Input: The projective class T of a symmetric tensor $t \in S^dV$, with $\dim(V) = n + 1$;
Output: $T \notin \sigma_2(X_{n,d})$, or $T \in \sigma_{2,2}(X_{n,d})$, or $T \in \sigma_{2,d}(X_{n,d})$, or $T \in X_{n,d}$.

(1) Consider the homogeneous polynomial associated to t as in (3.1) and rewrite it with the minimum possible number of variables (methods are described in (Carlini, 2005) or (Oldenburger, 1934)), if this number is 1 then $T \in X_{n,d}$; if it is > 2 then $T \notin \sigma_2(X_{n,d})$, otherwise T can be viewed as a point in $\mathbb{P}(S^dW) \cong \mathbb{P}^d \subset \mathbb{P}(S^dV)$, and $\dim(W) = 2$, so go to step 2.
(2) Apply the Algorithm 2 to conclude.

4.2. Varieties of secant planes to Veronese varieties

In this section we give a stratification of $\sigma_3(X_{n,d}) \subset \mathbb{P}(S^dV)$ with $\dim(V) = n+1$ via the symmetric rank of its elements.

Lemma 34. Let $Z \subset \mathbb{P}^n$, $n \geq 2$, be a 0-dimensional scheme, with $\deg(Z) \leq 2d+1$. A necessary and sufficient condition for Z to impose independent conditions to hypersurfaces of degree d is that no line $L \subset \mathbb{P}^n$ is such that $\deg(Z \cap L) \geq d + 2$.

Proof. The statement was probably classically known, we prove it here for lack of a precise reference. Notice that $h^0(\mathcal{O}_{\mathbb{P}^n}(d)) = \binom{d+n}{d} \geq 2d + 1$, so what we have to prove is that, for Z as in the statement, if there exists no line L such that $\deg L \cap Z \geq d+2$, then $h^1(\mathcal{I}_Z(d)) = 0$. Let us work by induction on n and d; if $d = 1$ the statement is trivial; so let us suppose that $d \geq 2$. Let us consider the case $n = 2$ first. If there is a line L

which intersects Z with multiplicity $\geq d+2$, then trivially Z cannot impose independent condition to curves of degree d, since the fixed line imposes $d+1$ conditions, hence we have already missed one. So, suppose that there exists no such line, and let L be a line such that $Z \cap L$ is as big as possible (hence $2 \leq \deg(Z \cap L) \leq d+1$). Let the *Trace* of Z on L, $Tr_L Z$, be the schematic intersection $Z \cap L$ and the *Residue* of Z with respect to L, $Res_L Z$, be the scheme defined by $(I_Z : I_L)$. Notice that $\deg(Tr_L Z) + \deg(Res_L Z) = \deg Z$. We have the following exact sequence of ideal sheaves:

$$0 \to \mathcal{I}_{Res_L Z}(d-1) \to \mathcal{I}_Z(d) \to \mathcal{I}_{Tr_L Z}(d) \to 0.$$

Then no line can intersect $Res_L Z$ with multiplicity $\geq d+1$, because $\deg(Z) \leq 2d+1$ and L is a line with maximal intersection with Z; so if $\deg(L' \cap res_L Z) = d+1$, we'd have that also $\deg(L \cap Z) = d+1$, which is impossible because it would give $2d+1 \geq \deg Z \geq \deg(Tr_L Z) + \deg(L' \cap Res_L Z) = 2d+2$. Since $\deg(Res_L Z) \leq 2d+1$, we have $h^1(\mathcal{I}_{Res_L Z}(d-1)) = 0$, by induction on d. On the other hand, we have $h^1(\mathcal{I}_{Tr_L Z}(d)) = h^1(\mathcal{O}_{\mathbb{P}^1}(d - \deg(Tr_L Z))) = 0$, hence also $h^1(\mathcal{I}_Z(d)) = 0$, i.e. Z imposes independent conditions to curves of degree d.

With the case $n = 2$ done, let us finish by induction on n. Consider $n \geq 3$, if there is a line L which intersects Z with multiplicity $\geq d+2$, we can conclude again that Z does not impose independent conditions to forms of degree d, as in the case $n = 2$. Otherwise, consider a hyperplane H, with maximum multiplicity of intersection with Z, and consider the exact sequence:

$$0 \to \mathcal{I}_{Res_H Z}(d-1) \to \mathcal{I}_Z(d) \to \mathcal{I}_{Tr_H Z}(d) \to 0.$$

We have $h^1(\mathcal{I}_{Res_H Z}(d-1)) = 0$, by induction on d, and $h^1(\mathcal{I}_{Tr_H Z}(d)) = 0$, by induction on n, so we get that $h^1(\mathcal{I}_Z(d)) = 0$ again, and we are done. $\square$

Remark 35. Notice that if $\deg L \cap Z$ is exactly $d+1+k$, then the dimension of the space of curves of degree d through Z increases exactly by k with respect to the generic case.

In the sequel we will need the following definition.

Definition 36. A *t-jet* is a 0-dimensional scheme $J \subset \mathbb{P}^n$ of degree t with support at a point $P \in \mathbb{P}^n$ and contained in a line L; namely the ideal of J is of type: $I_P^t + I_L$, where $L \subset \mathbb{P}^n$ is a line containing P. We will say that $J_1, \ldots, J_s$ are generic t-jets in $\mathbb{P}^n$ if for each $i = 1, ..., s$, we have $I_{J_i} = I_{P_i}^t + I_{L_i}$, the points $P_1, \ldots, P_s$ are generic in $\mathbb{P}^n$ and $\{L_1, \ldots, L_s\}$ is generic among all the sets of s lines with $P_i \in L_i$.

Theorem 37. *Let $d \geq 3$, $X_{n,d} \subset \mathbb{P}(S^d V)$. Then:*
$\sigma_3(X_{n,3}) \setminus \sigma_2(X_{n,3}) = \sigma_{3,3}(X_{n,3}) \cup \sigma_{3,4}(X_{n,3}) \cup \sigma_{3,5}(X_{n,3})$, while, for $d \geq 4$:
$\sigma_3(X_{n,d}) \setminus \sigma_2(X_{n,d}) = \sigma_{3,3}(X_{n,d}) \cup \sigma_{3,d-1}(X_{n,d}) \cup \sigma_{3,d+1}(X_{n,d}) \cup \sigma_{3,2d-1}(X_{n,d})$.
The $\sigma_{b,r}(X_{n,d})$'s are defined in Notation 16.

Proof. For any scheme $Z \in Hilb_3(\mathbb{P}(V))$ there exists a subspace $U \subset V$ of dimension 3 such that $Z \subset \mathbb{P}(U)$. Hence, when we make the construction in (4) we get that Π_Z is always a $\mathbb{P}^2$ contained in $\mathbb{P}(S^d U)$ and $\nu_d(\mathbb{P}(U))$ is a Veronese surface $X_{2,d} \subset \mathbb{P}(S^d U) \subset \mathbb{P}(S^d V)$. Therefore, by Remark 31, it is sufficient to prove the statement for $X_{2,d} \subset \mathbb{P}(S^d U)$.

First we will consider the case when there is a line L such that $Z \subset L$. In this case, let $C_d = \nu_d(L)$; we get that $T \in \sigma_3(C_d)$, hence either $T \in \sigma_{3,3}(C_d)$ (so $T \in \sigma_{3,3}(X_{2,d})$), or (only when $d \geq 4$) $T \in \sigma_{3,d-1}(C_d)$, hence $s\mathrm{rk}(T) \leq d-1$. The symmetric rank of T is actually $d-1$ by Remark 31.

Now we let Z not to be on a line; the scheme $Z \in Hilb_3(\mathbb{P}^n)$ can have support on 3 , 2 distinct points or on one point.

If $Supp(Z)$ is the union of 3 distinct points then clearly Π_Z, that is the image of Z via (4), intersects $X_{2,d}$ in 3 different points and hence any $T \in \Pi_Z$ has symmetric rank precisely 3, so $T \in \sigma_{3,3}(X_{2,d})$.

If $Supp(Z) = \{P, Q\}$ with $P \neq Q$, then the scheme Z is the union of a simple point, Q, and of a 2-jet J (see Definition 36) at P. The structure of 2-jet on P implies that there exists a line $L \subset \mathbb{P}^n$ whose intersection with Z is a 0-dimensional scheme of degree 2. Hence $\Pi_Z =< T_{\nu_d(P)}(C_d), \nu_d(Q) >$ where $T_{\nu_d(P)}(C_d)$ is the projective tangent line at $\nu_d(P)$ to $C_d = \nu_d(L)$. Since $T \in \Pi_Z$, the line $< T, \nu_d(Q) >$ intersects $T_{\nu_d(P)}(C_d)$ in a point $Q' \in \sigma_2(C_d)$. From Theorem 23 we know that $s\mathrm{rk}(Q') = d$. We may assume that $T \neq Q'$ because otherwise T should belong to $\sigma_2(X_{2,d})$.

We have $Q \notin L$ because Z is not in a line, so T can be written as a combination of a tensor of symmetric rank d and a tensor of symmetric rank 1, hence $s\mathrm{rk}(t) \leq d+1$. If $s\mathrm{rk}(t) = d$, then there should exist $Q_1, \ldots, Q_d \in X_{2,d}$ such that $T \in < Q_1, \ldots, Q_d >$; notice that $Q_1, \ldots, Q_d$ are not all on C_d, otherwise $T \in \sigma_2(X_{2,d})$. Let $P_1, \ldots, P_d$ be the pre-image via ν_d of $Q_1, \ldots, Q_d$; then $P_1, \ldots, P_d$ together with J and Q should not impose independent conditions to curves of degree d, so, by Lemma 34, either $P_1, \ldots, P_d, J$ are on L, or $P_1, \ldots, P_d, P, Q$ are on a line L'. The first case is not possible, since $Q_1, \ldots, Q_d$ are not all on C_d. In the other case notice that, by Lemma 34 and the Remark 35, we should have that $< Q_1, \ldots, Q_d, T_{\nu_d(P)}(C_d), \nu_d(Q) > \cong \mathbb{P}^d$, but since $< Q_1, \ldots, Q_d >$ and $< T_{\nu_d(P)}(C_d), \nu_d(Q) >$ have $T, \nu_d(P)$ and $\nu_d(Q)$ in common, they generate a $(d-1)$-dimensional space, but this is a contradiction. Hence $s\mathrm{rk}(t) = d+1$.

This construction also shows that $T \in \sigma_{3,d+1}(X_{2,d})$, and that there exists $W \subset V$ with $\dim(W) = 2$ and $l_1, \ldots, l_d \in W^*$ and $l_{d+1} \in V^*$ such that $t = l_1^d + \cdots + l_d^d + l_{d+1}^d$ and $t = [T]$.

If $Supp(Z)$ is only one point $P \in \mathbb{P}^2$, then Z can only be one of the following: either Z is 2-fat point (i.e. I_Z is I_P^2), or there exists a smooth conic containing Z.
If Z is a 2-fat point then Π_Z is the tangent space to $X_{2,d}$ at $\nu_d(P)$, hence if $T \in \Pi_Z$, then the line $< \nu_d(P), T >$ turns out to be a tangent line to some rational normal curve of degree d contained in $X_{2,d}$, hence in this case $T \in \sigma_2(X_{2,d})$.
If there exists a smooth conic $C \subset \mathbb{P}^2$ containing Z, write $Z = 3P$ and consider $C_{2d} = \nu_d(C)$, hence $T \in \sigma_3(C_{2d})$, therefore by Theorem 23 $\mathrm{rk}_{C_{2d}}(T)$, hence $s\mathrm{rk}(t) \leq 2d-1$. Suppose that $s\mathrm{rk}(t) \leq 2d-2$, hence there exist $P_1, \ldots, P_{2d-2} \in \mathbb{P}^2$ distinct points that are neither on a line nor on a conic containing $3P$, such that $T \in \Pi_{Z'}$ with $Z' = P_1 + \cdots + P_{2d-2}$ and $Z + Z' = 3P + P_1 + \cdots + P_{2d-2}$ doesn't impose independent conditions to the planes curves of degree d. Now, by Lemma 34 we get that $3P + P_1 + \cdots + P_{2d-2}$ doesn't impose independent conditions to the plane curves of degree d if and only if there exists a line $L \subset \mathbb{P}^2$ such that $\deg((Z + Z') \cap L) \geq d+2$. Observe that Z' cannot have support contained in a line because otherwise $T \in \sigma_2(X_{2,d})$. Moreover $Z + Z'$ cannot have support on a conic $C \subset \mathbb{P}^2$ because in that case T would have rank $2d-1$ with respect to $\nu_d(C) = C_{2d}$, while $\mathrm{rk}_{C_{2d}}(T)$.
Assume that $\deg((Z + Z') \cap L) = d+2$ first; we have to check the following cases:

(1) There exist $P_1, \dots, P_{d+2} \in Z'$ on a line $L \subset \mathbb{P}^2$;
(2) There exist $P_1, \dots, P_{d+1} \in Z'$ such that together with $P = Supp(Z)$ they are on the same line $L \subset \mathbb{P}^2$;
(3) There exist $P_1, \dots, P_d \in Z'$ such that together with the 2-jet $2P$ they are on the same line $L \subset \mathbb{P}^2$.

Case 1. Let $P_1, \dots, P_{d+2} \in L \subset \mathbb{P}^2$, then $\nu_d(L) = C_d \subset \mathbb{P}^d \subset \mathbb{P}^N$ with $N = \binom{d+2}{2} - 1$. Clearly $T \in \Pi_Z \cap \Pi_{Z'}$, then $\dim(\Pi_Z + \Pi_{Z'}) \leq \dim(\Pi_Z) + \dim(\Pi_{Z'})$, moreover $\Pi_{Z'}$ doesn't have dimension $2d-3$ as expected because $\nu_d(P_1), \dots, \nu_d(P_{d+2}) \in C_d \subset \mathbb{P}^d$, hence $\dim(\Pi_{Z'}) \leq 2d-4$ and $\dim(\Pi_Z + \Pi_{Z'}) \leq 2d-2$. But this is not possible because $Z + Z'$ imposes to the plane curves of degree d only one condition less then the expected, hence $\dim(I_{Z+Z'}(d)) = \binom{d+1}{2} - d + 1$ and then $\dim(\Pi_Z + \Pi_{Z'}) = 2d-1$, that is a contradiction.

Case 2. Let $P_1, \dots, P_{d+1}, P \in L \subset \mathbb{P}^2$, then $\nu_d(P_1), \dots, \nu_d(P_{d+1}), \nu_d(P) \in \nu_d(L) = C_d$. Now $\Pi_Z \cap \Pi_{Z'} \supset \{\nu_d(P), T\}$, then again $\dim(\Pi_Z + \Pi_{Z'}) \leq 2d-2$.

Case 3. Let $P_1, \dots, P_d, 2P \in L \subset \mathbb{P}^2$, as previously $\nu_d(P_1), \dots, \nu_d(P_{d+1}), \nu_d(2P) \in \nu_d(L) = C_d$, then now $T_{\nu_d(P)}(C_d)$ is contained in $< C_d > \cap \Pi_Z$. Since $< \nu_d(P_1), \dots, \nu_d(P_d) >$ is a hyperplane in $< C_d >= \mathbb{P}^d$, it will intersect $T_{\nu_d(P)}(C_d)$ in a point Q different form $\nu_d(P)$. Again $\dim(\Pi_Z \cap \Pi_{Z'}) \geq 1$ and then $\dim(\Pi_Z + \Pi_{Z'}) \leq 2d-2$.

When $\deg((Z+Z') \cap L) = d+k+1$, $k > 1$, we can conclude in the same way, by using Remark 35. □

Now we are almost ready to present an algorithm which allows to indicate if a projective class of a symmetric tensor in $\mathbb{P}^{\binom{n+d}{d}-1}$ belongs to $\sigma_3(X_{n,d})$, and in this case to determine its rank. Before giving the algorithm we need to recall a result about $\sigma_3(X_{2,3})$:

Remark 38. The secant variety $\sigma_3(X_{2,3}) \subset \mathbb{P}^9$ is a hypersurface and its defining equation it is the "Aronhold (or Clebsch) invariant" (for an explicit expression see e.g. (Ottaviani, 2009)). When $d \geq 4$, instead, $\sigma_3(X_{2,3})$ is defined (at least scheme theoretically) by the (4×4)-minors of $M_{d-2,2}$, see (Landsberg, Ottaviani, 2009).

Notice also that there is a very direct and well known way of getting the equations for the secant variety $\sigma_s(X_{n,d})$, which we describe in the next remark. The problem with this method is that it is computationally very inefficient, and it can be worked out only in very simple cases.

Remark 39. Let $T = [t] = \left[z_0, \dots, z_{\binom{n+d}{d}}\right] \in \mathbb{P}(S^d(V))$, where V is an $(n+1)$-dimensional vector space. T is an element of $\sigma_s(X_{n,d})$ if there exist $P_i = [x_{0,i}, \dots, x_{n,i}] \in \mathbb{P}^n = \mathbb{P}(V)$, $i = 1, \dots, s$, and $\lambda_1, \dots, \lambda_s \in K$, such that $t = \lambda_1 w_1 + \cdots + \lambda_s w_s$, where $[w_i] = \nu_d(P_i) \in \mathbb{P}^{\binom{n+d}{d}-1} = \mathbb{P}(S^d V)$, $i = 1, \dots, s$ (i.e. $[w_i] = [x_{0,i}^d, x_{0,i}^{d-1} x_1, \dots, x_{n,i}^d]$).

This can be expressed via the following system of equations:

$$\begin{cases} z_0 = \lambda_1 x_{0,1}^d + \cdots + \lambda_s x_{0,s}^d \\ z_1 = \lambda_1 x_{0,1}^{d-1} x_{1,1} + \cdots + \lambda_s x_{0,s}^{d-1} x_{1,s} \\ \vdots \\ z_{\binom{n+d}{d}-1} = \lambda_1 x_{n,1}^d + \cdots + \lambda_s x_{s,s}^d \end{cases}.$$

Now consider the ideal $I_{s,n,d}$ defined by the above polynomials in the weighted coordinate ring

$$R = K\left[x_{0,1},\dots,x_{n,1};\dots;x_{0,s},\dots,x_{n,s};\lambda_1,\dots,\lambda_s;z_0,\dots,z_{\binom{n+d}{d}-1}\right]$$

where the z_i's have degree $d+1$:

$$I_{s,n,d} = (z_0-\lambda_1 x_{0,1}^d+\dots+\lambda_s x_{0,s}^d, z_1-\lambda_1 x_{0,1}^{d-1}x_{1,1}+\dots+\lambda_s x_{0,s}^{d-1}x_{1,s},\dots,z_{\binom{n+d}{d}-1}-\lambda_1 x_{n,1}^d+\dots+\lambda_s x_{s,s}^d).$$

Now eliminate from $I_{s,n,d}$ the variables λ_i's and $x_{j,i}$'s, $i=1,\dots,s$ and $j=0,\dots,n$. The elimination ideal $J_{s,n,d}\subset K\left[z_0,\dots,z_{\binom{n+d}{d}-1}\right]$ that we get from this process is an ideal of $\sigma_s(X_{n,d})$.

Obviously $J_{s,n,d}$ contains all the $(s+1)\times(s+1)$ minors of the catalecticant matrix of order $r\times(d-r)$ (if they exist).

Algorithm 4. Algorithm for the symmetric rank of an element of $\sigma_3(\mathbf{X_{n,d}})$

Input: The projective class T of a symmetric tensor $t\in S^dV$, with $\dim(V)=n+1$;
Output: $T\notin\sigma_3(X_{n,d})$ or $T\in\sigma_2(X_{n,d})$ or $T\in\sigma_{3,3}(X_{n,d})$ or $T\in\sigma_{3,d-1}(X_{n,d})$ or $T\in\sigma_{3,d+1}(X_{n,d})$ or $T\in\sigma_{3,2d-1}$.

(1) Run the first step of Algorithm 3. If only one variable is needed, then $T\in X_{n,d}$; if two variables are needed, then $T\in\sigma_3(X_{n,d})$ and use Algorithm 3 to determine $s\mathrm{rk}(T)$. If the number of variables is greater than 3, then $T\notin\sigma_3(X_{n,d})$. Otherwise (three variables) consider $t\in S^d(W)$, with $\dim(W)=3$ and go to next step;
(2) If $d=3$, evaluate the Aronhold invariant (see 38) on T, if it is zero on T then $T\in\sigma_3(X_{2,3})$ and go to step 3; otherwise $T\notin\sigma_3(X_{2,3})$. If $d\geq 4$, evaluate $\mathrm{rk}M_{2,d-2}(T)$; if $\mathrm{rk}M_{2,d-2}(T)\geq 4$, then $T\notin\sigma_3(X_{2,d})$; otherwise $T\in\sigma_3(X_{2,d})$ and go to step 3.
(3) Consider the space $S\subset K[x_0,x_1,x_2]_2$ of the solutions of the system $M_{2,d-2}(T)\cdot(b_{0,0},\dots,b_{2,2})^t=0$. Choose three generators F_1,F_2,F_3 of S.
(4) Compute the radical ideal I of the ideal (F_1,F_2,F_3) (this can be done e.g. with CoCoA). Since $\dim(W)=3$, i.e. 3 variables were needed, F_1,F_2,F_3 do not have a common linear factor.
(5) Consider the generators of I. If there are two linear forms among them, then $T\in\sigma_{3,2d-1}(X_{n,d})$, if there is only one linear form then $T\in\sigma_{3,d+1}(X_{n,d})$, if there are no linear forms then $T\in\sigma_{3,3}(X_{n,d})$.

4.3. Secant varieties of $X_{2,3}$

In this section we describe all possible symmetric ranks that can occur in $\sigma_s(X_{2,3})$ for any $s\geq 1$.

Theorem 40. *Let U be a 3-dimensional vector space. The stratification of the cubic forms of $\mathbb{P}(S^3U^*)$ with respect to symmetric rank is the following:*
- $X_{2,3}=\{T\in\mathbb{P}(S^3U)\mid s\mathrm{rk}(T)=1\}$;
- $\sigma_2(X_{2,3})\setminus X_{2,3}=\sigma_{2,2}(X_{2,3})\cup\sigma_{2,3}(X_{2,3})$;
- $\sigma_3(X_{2,3})\setminus\sigma_2(X_{2,3})=\sigma_{3,3}(X_{2,3})\cup\sigma_{3,4}(X_{2,3})\cup\sigma_{3,5}(X_{2,3})$;
- $\mathbb{P}^9\setminus\sigma_3(X_{2,3})=\sigma_{4,4}(X_{2,3})$;

where $\sigma_{s,m}(X_{2,3})$ is defined as in Notation 16.

Proof. We only need to prove that $\mathbb{P}^9 \setminus \sigma_3(X_{2,3}) = \sigma_{4,4}(X_{2,3})$ because $X_{2,3}$ is by definition the set of symmetric tensors of symmetric rank 1 and the cases of $\sigma_2(X_{2,3})$ and $\sigma_3(X_{2,3})$ are consequences of Theorem 32 and Theorem 37 respectively.

So now we show that all symmetric tensors in $\mathbb{P}^9 \setminus \sigma_3(X_{2,3})$ are of symmetric rank 4. Clearly, since they do not belong to $\sigma_3(X_{2,3})$, they have symmetric rank ≥ 4; hence we need to show that their symmetric rank is actually less or equal than 4. Let $T \in \mathbb{P}^9 \setminus \sigma_3(X_{2,3})$ and consider the system $M_{2,1} \cdot (b_{0,0}, \ldots, b_{2,2})^T = 0$. The space of solutions of this system gives a vector space of conics which has dimension 3; moreover it is not the degree 2 part of any ideal representing a 0-dimensional scheme of degree 3 (otherwise we'd have $T \in \sigma_3(X_{2,3})$), hence the generic solution of that system is a smooth conic. Therefore in the space of the cubics through T, there is a subspace given by $< C \cdot x_0, C \cdot x_1, C \cdot x_2 >$ where C is indeed a smooth conic given by the previous system. Hence, if C_6 is the image of C via the Veronese embedding ν_3, we have that $T \in < C_6 >$, in particular $T \in \sigma_4(C_6) \setminus \sigma_3(C_6)$, therefore $s\mathrm{rk}(t) \leq 6 - 4 + 2 = 4$. $\square$

4.4. Secant varieties of $X_{2,4}$

We recall that the k-th osculating variety to $X_{n,d}$, denoted by $\mathcal{O}_{k,n,d}$, is the closure of the union of the k-osculating planes to the Veronese variety $X_{n,d}$, where the k-osculating plane $\mathcal{O}_{k,n,d,P}$ at the point $P \in X_{n,d}$ is the linear space generated by the k-th infinitesimal neighborhood $(k+1)P$ of P on $X_{n,d}$ (see for example (Bernardi et al., 2007) 2.1, 2.2). Hence for example the first osculating variety is the tangential variety.

Lemma 41. The second osculating variety $\mathcal{O}_{2,2,4}$ of $X_{2,4}$ is contained in $\sigma_4(X_{2,4})$.

Proof. Let T be a generic element of $\mathcal{O}_{2,2,4} \subset \mathbb{P}(S^4V)$ with $\dim(V) = 3$. Hence $T = l^2\mathcal{C}$ where l and $\mathcal{C}$ are a linear and a quadratic generic forms respectively of $\mathbb{P}(S^4V)$ regarded as a projectivization of the homogeneous polynomials of degree 4 in 3 variables, i.e. $K[x,y,z]_4$ (see (Bernardi et al., 2007)). We can always assume that $l = x$ and $\mathcal{C} = a_{0,0}x^2 + a_{0,1}xy + a_{0,2}xz + a_{1,1}y^2 + a_{1,2}yz + a_{2,2}z^2$. The catalecticant matrix $M_{2,2}$ (defined in general in Definition 17) for a plane quartic $a_{0000}x^4 + a_{0001}x^3y + \cdots + a_{2222}z^4$ is the following:

$$M_{2,2} = \begin{pmatrix} a_{0000} & a_{0001} & a_{0002} & a_{0011} & a_{0012} & a_{0022} \\ a_{0001} & a_{0011} & a_{0012} & a_{0111} & a_{0112} & a_{0122} \\ a_{0002} & a_{0012} & a_{0022} & a_{0112} & a_{0122} & a_{0222} \\ a_{0011} & a_{0111} & a_{0112} & a_{1111} & a_{1112} & a_{1122} \\ a_{0012} & a_{0112} & a_{0122} & a_{1112} & a_{1122} & a_{1222} \\ a_{0022} & a_{0122} & a_{0222} & a_{1122} & a_{1222} & a_{2222} \end{pmatrix}$$

hence in the specific case of the quartic above $l^2\mathcal{C} = x^2(a_{0,0}x^2 + a_{0,1}xy + a_{0,2}xz + a_{1,1}y^2 +$

$a_{1,2}yz + a_{2,2}z^2$) it becomes:

$$M_{2,2}(T) = \begin{pmatrix} a_{0000} & a_{0001} & a_{0002} & a_{0011} & a_{0012} & a_{0022} \\ a_{0001} & a_{0011} & a_{0012} & 0 & 0 & 0 \\ a_{0002} & a_{0012} & a_{0022} & 0 & 0 & 0 \\ a_{0011} & 0 & 0 & 0 & 0 & 0 \\ a_{0012} & 0 & 0 & 0 & 0 & 0 \\ a_{0022} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

that clearly has rank less or equal than 4. Since the ideal of $\sigma_4(X_{2,4})$ is generated by the (5×5)-minors of $M_{2,2}$, e.g. see (Landsberg, Ottaviani, 2010), we have that $\mathcal{O}_{2,2,4} \subset \sigma_4(X_{2,4})$. $\square$

Lemma 42. If $Z \in Hilb_4(\mathbb{P}^2)$ and Z is contained in a line, then $r = s\mathrm{rk}(T) \leq 4$ for any $T \in \Pi_Z$, where Π_Z is defined in Notation 30, and T belongs either to $\sigma_2(X_{2,4})$ or to $\sigma_3(X_{2,4})$. Moreover there exists W of dimension 2 and $l_1, \ldots, l_r \in S^1W^*$ such that $t = l_1^4 + \cdots + l_r^4$ with $r \leq 4$.

Proof. If there exists a 2-dimensional subspace $W \subset V$ with $\dim(V) = 3$ such that $Supp(Z) \subset \mathbb{P}(W)$ then any $T \in \Pi_Z \subset \mathbb{P}(S^4V)$ belongs to $\sigma_4(\nu_4(\mathbb{P}(W))) \simeq \mathbb{P}^4$, therefore $s\mathrm{rk}(T) \leq 4$. If $s\mathrm{rk}(T) = 2, 4$ then $T \in \sigma_2(X_{2,4})$, otherwise $T \in \sigma_3(X_{2,4})$. $\square$

Lemma 43. If $Z \subset Hilb_4(\mathbb{P}^2)$ and there exists a smooth conic $C \subset \mathbb{P}^2$ such that $Z \subset C$, then any $T \in \Pi_Z$, with $T \notin \sigma_3(X_{2,4})$, is of symmetric rank 4 or 6.

Proof. Clearly $T \in \sigma_4(\nu_4(C))$ and $\nu_4(C)$ is a rational normal curve of degree 8, then $s\mathrm{rk}(T) \leq 6$. If $\sharp\{Supp(Z)\} = 4$ then $s\mathrm{rk}(T) = 4$. Otherwise $s\mathrm{rk}(T)$ cannot be less or equal than 5 because there would exists a 0-dimensional scheme $Z' \subset \mathbb{P}^2$ made of 5 distinct points such that $T \in \Pi_{Z'}$, then $Z + Z'$ should not impose independent conditions to plane curves of degree 4. In fact by Lemma 34 the scheme $Z + Z'$ doesn't impose independent conditions to the plane quartic if and only if there exists a line $M \subset \mathbb{P}^2$ such that $\deg((Z + Z') \cap M) \geq 6$. If $\deg((Z') \cap M) \geq 5$ then $T \in \sigma_2(X_{2,4})$ or $T \in \sigma_3(X_{2,4})$. Hence assume that $\deg((Z + Z') \cap M) \geq 6$ and $\deg((Z') \cap M) < 5$. Consider first the case $\deg((Z + Z') \cap M) = 6$. Then $\deg((Z') \cap M) = 4$ and $\deg((Z) \cap M) = 2$. We have that $\Pi_{Z+Z'}$ should be a $\mathbb{P}^7$ but actually it is at most a $\mathbb{P}^6$ in fact $\Pi_{(Z+Z')\cap M} = \mathbb{P}^4$ because $< \nu_4(M) >= \mathbb{P}^4$, moreover $T \in \Pi_Z \cap \Pi_{Z'}$ hence $\Pi_{Z+Z'}$ is at most a $\mathbb{P}^6$. Analogously if $\deg((Z+Z') \cap M) = 7$ (it cannot be more) one can see that $\Pi_{Z+Z'}$ should have dimension 6 but it must have dimension strictly less than 6. $\square$

Theorem 44. *The s-th secant varieties to $X_{2,4}$, up to $s = 5$, are described in terms of symmetric ranks as follows:*
- $X_{2,4} = \{T \in S^4V \mid s\mathrm{rk}(T) = 1\}$;
- $\sigma_2(X_{2,4}) \setminus X_{2,4} = \sigma_{2,2}(X_{2,4}) \cup \sigma_{2,4}(X_{2,4})$;
- $\sigma_3(X_{2,4}) \setminus \sigma_2(X_{2,4}) = \sigma_{3,3}(X_{2,4}) \cup \sigma_{3,5}(X_{2,4}) \cup \sigma_{3,7}(X_{2,4})$;

- $\sigma_4(X_{2,4}) \setminus \sigma_3(X_{2,4}) = \sigma_{4,4}(X_{2,4}) \cup \sigma_{4,6}(X_{2,4}) \cup \sigma_{4,7}(X_{2,4})$;
- $\sigma_5(X_{2,4}) \setminus \sigma_4(X_{2,4}) = \sigma_{5,5}(X_{2,4}) \cup \sigma_{5,6}(X_{2,4}) \cup \sigma_{5,7}(X_{2,4})$.

Proof. By definition of $X_{n,d}$ we have that $X_{2,4}$ is the variety parameterizing symmetric tensors of S^4V having symmetric rank 1 and the cases of $\sigma_2(X_{2,4})$ and $\sigma_3(X_{2,4})$ are consequences of Theorem 32 and Theorem 37 respectively.

Now we study $\sigma_4(X_{2,4}) \setminus \sigma_3(X_{2,4})$. Let $Z \in Hilb_4(\mathbb{P}^2)$ and $T \in \Pi_Z$ be defined as in Notation 30.

- Let Z be contained in a line L; then by Lemma 42 we have that T belongs either to $\sigma_2(X_{2,4})$ or to $\sigma_3(X_{2,4})$.
- Let $Z \subset C$, with C a smooth conic. Then by Lemma 43, $T \in \sigma_{4,4}(X_{2,4})$ or $T \in \sigma_{4,6}(X_{2,4})$.
- If there are no smooth conics containing Z then either there is a line L such that $\deg(Z \cap L) = 3$, or I_Z can be written as (x^2, y^2). We study separately those two cases.

(1) In the first case the ideal of Z in degree 2 can be written either as $< x^2, xy >$ or $< xy, xz >$.

If $(I_Z)_2 =< x^2, xy >$ then it can be seen that the catalecticant matrix of T is

$$M_{2,2}(T) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{0222} \\ 0 & 0 & 0 & a_{1111} & a_{1112} & a_{1122} \\ 0 & 0 & 0 & a_{1112} & a_{1122} & a_{1222} \\ 0 & 0 & a_{0222} & a_{1122} & a_{1222} & a_{2222} \end{pmatrix}.$$

Hence, for a generic such T, we have that $T \notin \sigma_3(X_{2,4})$ since the rank of $M_{2,2}(T)$ is 4, while it has to be 3 for points in $\sigma_3(X_{2,4})$. In this case if Z has support in a point then I_Z can be written as (x^2, xy, y^3) and the catalecticant matrix defined in Definition 17 evaluated in T turns out to be:

$$M_{2,2}(T) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{0222} \\ 0 & 0 & 0 & 0 & 0 & a_{1122} \\ 0 & 0 & 0 & 0 & a_{1122} & a_{1222} \\ 0 & 0 & a_{0222} & a_{1122} & a_{1222} & a_{2222} \end{pmatrix}$$

that clearly has rank less or equal than 3. Hence $T \in \sigma_3(X_{2,4})$.

Otherwise Z is either made of two 2-jets or one 2-jet and two simple points. In both cases denote by R the line $y = 0$. We have $\deg(Z \cap R) = 2$. Thus Π_Z is the sum of the linear space $\Pi_{Z\cap L} \simeq \mathbb{P}^2$ and $\Pi_{Z\cap R} \simeq \mathbb{P}^1$. Hence $T = Q + Q'$ for suitable $Q \in \Pi_{Z\cap L}$ and $Q' \in \Pi_{Z\cap R}$. Since $Q \in \sigma_3(\nu_4(L))$ and Q' is in a tangent line to $\nu_4(R)$ we have that $s\mathrm{rk}(T) \leq 7$. Working as in Lemma 43 we can prove that $s\mathrm{rk}(T) = 7$.

Eventually if $(I_Z)_2$ can be written as (xy, xz) then Z is made of a subscheme Z' of degree 3 on the line L and a simple point $P \notin L$. In this case $s\mathrm{rk}(T) = 4$ since $\Pi_Z =< \Pi_{Z'}, \nu_4(P) >$ and any element in $\Pi_{Z'}$ has symmetric rank ≤ 3 (since it is on $\sigma_3(\nu_4(L))$).

(2) In the last case we have that I_Z can be written as (x^2, y^2). If we write the catalecticant matrix defined in Definition 17 evaluated in T we get the following matrix:

$$M_{2,2}(T) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{0122} \\ 0 & 0 & 0 & 0 & a_{0122} & a_{0222} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{0122} & 0 & 0 & a_{1222} \\ 0 & a_{0122} & a_{0222} & 0 & a_{1222} & a_{2222} \end{pmatrix}.$$

Clearly if $a_{0122} = 0$ the rank of $M_{2,2}(T)$ is three, hence such a T belongs to $\sigma_3(X_{2,4})$, otherwise we can make a change of coordinates (that corresponds to do a Gauss elimination on $M_{2,2}(T)$) that allows to write the above matrix as follows:

$$M_{2,2}(T) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{0122} \\ 0 & 0 & 0 & 0 & a_{0122} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{0122} & 0 & 0 & 0 \\ 0 & a_{0122} & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This matrix is associated to a tensor $t \in S^4V$, with $\dim(V) = 3$, that can be written as the polynomial $t(x_0, x_1, x_2) = x_0x_1x_2^2$. Now $s\mathrm{rk}(t) = 6$ (see (Landsberg, Teitler, 2009), Proposition 11.9).

We now study $\sigma_5(X_{2,4}) \setminus \sigma_4(X_{2,4})$, so in the following we assume $T \notin \sigma_4(X_{2,4})$, which implies $s\mathrm{rk}(T) \geq 5$. We have to study the cases with $\deg(Z) = 5$, i.e., $Z \in Hilb_5(\mathbb{P}^2)$. The scheme Z is hence always contained in a conic, which can be a smooth conic, the union of 2 lines or a double line. In the last two cases, Z might be contained in a line; we now distinguish the various cases according to these possibilities.

- Z is contained in a line L: $\Pi_Z \cong \mathbb{P}^4$ is spanned by the rational curve $\nu(L) = C_4$, hence $s\mathrm{rk}(T) \leq 4$, against assumptions.
- Z is contained in a smooth conic C. Hence Π_Z is spanned by the subscheme $\nu(Z)$ of the rational curve $\nu(C) = C_8$, so that $T \in \sigma_5(C_8)$ and by Theorem 23 $s\mathrm{rk}(T) = 5$.
- Z is contained in the union of two lines L and R. We say that Z is of type (i, j) if $\deg(Z \cap L) = i$ and $\deg(Z \cap R) = j$ and for any other couple of lines in the ideal of Z the degree of the intersections is not smaller. Four different cases can occur: Z is of

type $(3,2)$, in which case $Z \cap L \cap R = \emptyset$, Z is of type $(3,3)$ or $(4,2)$, and in these two cases Z, L and R meet in a point P, Z is of type $(4,1)$, in which case R is not unique. We set $C_4 = \nu(L)$, $C'_4 = \nu(R)$, $O = \nu(P)$, $\Pi_L =< \nu(Z \cap L) >$ and $\Pi_R =< \nu(Z \cap R) >$.

- Z is of type $(4,1)$. Hence Π_Z is sum of the linear space $\Pi_L \subseteq \sigma_4(C_4)$ and the point $Q = \Pi_R \in X_{2,4}$, so that $T = Q' + Q$ for a suitable $Q' \in \sigma_4(C_4)$, and since $s\mathrm{rk}(Q') \leq 4$ by Theorem 23, we get $s\mathrm{rk}(Q') \leq 5$.
- Z is of type $(3,2)$. Hence Π_Z is sum of the linear spaces $\Pi_L \cong \mathbb{P}^2$ and the line Π_R, so that $T = Q' + Q$ for suitable $Q \in \Pi_L \subseteq \sigma_3(C_4)$ and $Q' \in \Pi_R \subseteq \sigma_2(C'_4)$. Since $s\mathrm{rk}(Q) \leq 3$ and $s\mathrm{rk}(Q') \leq 4$, we get $s\mathrm{rk}(Q) \leq 7$.
- Z is of type $(3,3)$. Hence Π_Z is sum of the linear spaces $\Pi_L \cong \mathbb{P}^2$ and $\Pi_R \cong \mathbb{P}^2$ meeting at one point, so that $T = Q' + Q$ for suitable $Q \in \Pi_L \subseteq \sigma_3(C_4)$ and $Q' \in \Pi_R \subseteq \sigma_3(C'_4)$. Since $s\mathrm{rk}(Q) \leq 3$ and $s\mathrm{rk}(Q') \leq 3$, we get $s\mathrm{rk}(T) \leq 6$. Moreover if Z has support on 4 points, we see that $s\mathrm{rk}(T) = 6$, using the same kind of argument as in Lemma 43.
- Z is of type $(4,2)$. In this case $(I_Z)_2$ can be written as $< xy, x^2 >$, then working as above we can see that the catalecticant matrix $M_{2,2}(T)$ has rank 4. Since at least set theoretically $I(\sigma_4(X_{2,4}))$ is generated by the 5×5 minors of $M_{2,2}$, we conclude that such T belongs to $\sigma_4(X_{2,4})$.

- Z is contained in a double line. We distinguish the following cases:
  - The support of Z is a point P, i.e. the ideal of Z is either of type (x^3, x^2y, y^2) or, in affine coordinates, $(x - y^2, y^4) \cap (x^2, y)$. In the first case Z is contained in the 3-fat point supported on P, so that Π_Z is contained in in the second osculating variety and by Lemma 41 $T \in \sigma_4(X_{2,4})$.
    In the second case it easy to see that the homogeneous ideal contains x^2, xy^2 and y^4 and this fact forces the catalecticant matrix $M_{2,2}(T)$ to have rank smaller or equal to 4. Hence $T \in \sigma_4(X_{2,4})$.
  - The support of Z consists of two points, i.e. the ideal of Z is of type $(x^2, y^2) \cap (x-1, y)$ or $(x^2, xy, y^2) \cap (x - 1, y^2)$.
    In the first case Z is union of a scheme Y of degree 4 and of a point P, hence Π_Z is sum of the linear spaces Π_Y and Π_P, so that $T = Q + \nu(P)$ for suitable $Q \in \Pi_Y$. The above description of the case corresponding to I_Z of the type (x^2, y^2) shows that either $Q \in \sigma_3(X_{2,4})$ or $s\mathrm{rk}(Q) = 6$. Now if $Q \in \sigma_3(X_{2,4})$ then clearly $T \in \sigma_4(X_{2,4})$, if $s\mathrm{rk}(Q) = 6$ then $s\mathrm{rk}(T) = 7$.
    In the second case Z is union of a jet and of a 2-fat point, hence Π_Z is sum of two linear spaces, each of them is contained in a tangent space of $X_{2,4}$ at a different point, so that $T = Q + Q'$ with Q, Q' contained in the tangential variety; then both Q and Q' belongs to $\sigma_2(X_{2,4})$ hence $T \in \sigma_4(X_{2,4})$.
  - The support of Z consists of three points, i.e. the ideal of Z is of type $(x, y) \cap ((x^2 - 1), y^2)$. Let P_1, P_2, P_3 be the points supporting Z, with η_1, η_2 jets such that $Z = \eta_1 \cup \eta_2 \cup P_3$. There exists a smooth conic C containing $\eta_1 \cup \eta_2$, and $\nu(C)$ is a C_8. Then Π_Z is the sum of $\nu(P_3)$ and of the linear space $< \nu(\eta_1), \nu(\eta_2) >$, so that $T = Q + \nu(P_3)$ for a suitable $Q \in \sigma_4(C_8)$, with $s\mathrm{rk}(Q) \leq 6$, so we get $s\mathrm{rk}(T) \leq 7$.

□

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