Numerical approximation of parabolic problems by means of residual distribution schemes

Remi Abgrall, Guillaume Baurin, Arnaud Krust, Dante De Santis, Mario Ricchiuto

INRIA

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Numerical approximation of parabolic problems by means of residual distribution schemes

Rémi Abgrall — Guillaume Baurin — Arnaud Krust — Dante de Santis — Mario Ricchiuto

N° 7824

Décembre 2011

Thème NUM

ISSN 0249-6399 ISRN INRIA/RR--7824--FR+ENG

INSTITUT NATIONAL
DE RECHERCHE
EN INFORMATIQUE
ET EN AUTOMATIQUE

INRIA

centre de recherche
BORDEAUX - SUD OUEST

Numerical approximation of parabolic problems by means of residual distribution schemes

Rémi Abgrall* , Guillaume Baurin† , Arnaud Krust‡ , Dante de Santis§ , Mario Ricchiuto¶

Thème NUM — Systèmes numériques
Équipes-Projets Bacchus

Rapport de recherche n° 7824 — Décembre 2011 — 22 pages

Abstract: We are interested in the numerical approximation of steady scalar convection diffusion problems by mean of high order schemes called Residual Distribution (RD). In the inviscid case, one can develop non linear RD that are non oscillatory, even in the case of very strong shocks, while having the most possible compact stencil, on hybrid unstructured meshes. This paper proposes and compare several extension of these schemes for the convection diffusion problem. This methodology, in particular in term of accuracy, is evaluated on several problems, some of which having exact solutions.

Key-words: Problèmes de convection diffusion, schémas distribuant le résidu, méthods d'éléments finis, maillages non struturés

* Equipe-projet Bacchus, Institut de Mathématiques, INRIA et Université de Bordeaux, 33 405 Talence cedex
† SNECMA et Equipe-projet Bacchus, Institut de Mathématiques, INRIA et Université de Bordeaux, 33 405 Talence cedex
‡ Equipe-projet Bacchus, Institut de Mathématiques, INRIA et Université de Bordeaux, 33 405 Talence cedex
§ Equipe-projet Bacchus, Institut de Mathématiques, INRIA et Université de Bordeaux, 33 405 Talence cedex
¶ Equipe-projet Bacchus, Institut de Mathématiques, INRIA et Université de Bordeaux, 33 405 Talence cedex

Centre de recherche INRIA Bordeaux – Sud Ouest
Domaine Universitaire - 351, cours de la Libération 33405 Talence Cedex
Téléphone : +33 5 40 00 69 00

Approximation numérique de problèmes parabolique au moyen de schémas distribuant le résidu.

Résumé : We are interested in the numerical approximation of steady scalar convection diffusion problems by mean of high order schemes called Residual Distribution (RD). In the inviscid case, one can develop non linear RD that are non oscillatory, even in the case of very strong shocks, while having the most possible compact stencil, on hybrid unstructured meshes. This paper proposes and compare several extension of these schemes for the convection diffusion problem. This methodology, in particular in term of accuracy, is evaluated on several problems, some of which having exact solutions.

Mots-clés : Convection diffusion, residual distribution schemes, finite element methods, non structured meshes

We are interested in the approximation of convection diffusion problems such as

$$\begin{array}{ll} \text{div } f(u) = \text{div } (\mathbf{D}\nabla u), & x \in \Omega \\ u = g_S & \text{on } \Gamma_S \subset \partial\Omega \\ u = g_w & \text{on } \Gamma_W \subset \partial\Omega \end{array} \tag{1}$$

where $f(u)$ is a C^1 function (the flux) and $\mathbf{D}$ is a $d \times d$ matrix which symmetric part $(\mathbf{D} + \mathbf{D}^T)/2$ is positive definite. In (1), see figure 1, $\partial\Omega$ is the boundary of $\Omega \subset \mathbb{R}^d$, $\Gamma_s \cup \Gamma_w = \Omega$. On Γ_S, we apply Dirichlet (strong) boundary conditions, the subset Γ_W is included in the outflow boundary

$$\Gamma^- = \{M \in \Gamma, \nabla_u f(u). \cdot \vec{n} > 0, \quad \vec{n} \text{ outward unit normal}\}.$$

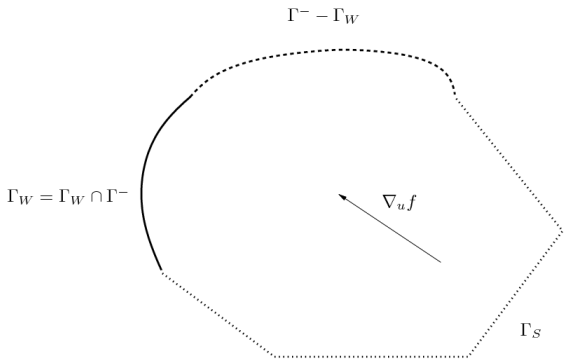


Figure 1: Weak and strong Bcs locations. The inflow boundary is the union of Γ_W and $\Gamma^- - \Gamma_W$.

The ultimate goal is the approximation of the Navier Stokes equations. The numerical setting is the following. The domain Ω is discretized by means of an unstructured grid $\mathcal{T}_h$, and to fix ideas, we assume that $\Omega \subset \mathbb{R}^2$ and that the elements of the mesh $\mathcal{T}_h$ are triangles. None of these two assumptions is essential by any mean. In the setting of this paper, we seek an approximation to the values of the solution of (1) at the vertices of the mesh and at additional points that correspond to the Lagrange points associated to the standard equispaced Lagrange interpolation. For example, in the third order case, in addition to the vertices, we also consider the mid points of the edges, in the fourth order case we consider vertices, plus (in 2d) cell centers and two additional (equispaced) edge points, etc. etc. Note that other type of degrees of freedom can be considered (cf. for example [7]). The key point here is that one is able to reconstruct a continuous interpolant u^h from the degrees of freedom. Here, $u^h \in V^h$, where

$$V^h = \{u \in H^1(\Omega), u_K \in \mathbb{P}^k(K) \text{ for any element } K\}.$$

We focus on C^0 continuous approximations and on the steady problem (1). However our discussion can be generalized to discontinuous approximations (cf. [6, 2]), and to unsteady problems (cf. [10, 20, 18]).

We are interested in the approximation of (1) using Residual Distribution (RD) schemes. If one denotes by $\{\sigma_\ell\}_{\ell=1,\ldots,n_{\text{dof}}}$ the set of degrees of freedom, a

RD scheme writes as: for any σ_ℓ, find $u^h \in V^h$ such that

$$\sum_{K,\sigma_\ell \in K} \phi^K_{\sigma_\ell}(u^h) = 0 \tag{2a}$$

where the residuals $\phi^K_{\sigma_\ell}(u^h)$ in the case $\mathbf{D} \equiv 0$ satisfy the following conservation relation:

$$\sum_{\sigma \in K} \phi^K_\sigma = \int_{\partial K} f^h(u^h) \cdot \vec{n} dl \tag{2b}$$

where $f^h(u^h)$ is a convergent interpolant of $f(u)$ (for example $f(u^h)$ or the Lagrange interpolant of $f(u)$). The approximation of the non diffusive problem $\mathbf{D} \equiv 0$ (with slightly modified boundary conditions) is now a standard matter, even for high order of accuracy. The deal is to extend theses approximations to the viscous problem (1). In classical RD schemes, we approximate (1) where $\mathbf{D} \equiv 0$ with suitable boundary conditions. The schemes are formally high order accurate and the focus is on the L^∞ stability of the scheme. Here, we wish to keep the accuracy property without sacrificing the L^∞ stability, i.e. the shock capturing capability of the numerical scheme.

The numerical approximation of (1) has already been considered by several authors. It is not, however, a completely trivial matter. Indeed, this class of schemes has originally been devised for (steady) transport problems, based on a genuinely multidimensional upwind approach. Among the first contributions, one may quote the work of P.L. Roe [21]. Some connections with more classical schemes, as the streamline diffusion method by Hughes, Johnson and co-authors, have soon been made [12]. However, the main problem is that, even though some residual distribution schemes can be recast as a particular class stabilized finite elements with emphasis on L^∞ stability, there is no clear general framework allowing to choose the test functions in order to recover a traditional variational statement. The main reason of this problem is related to the underlying formulation: everything is seen from a discrete point of view, and emphasis is put on the point-wise behavior of the residual (which is natural given the focus on L^∞ stability). Indeed the same remark applies to variants of the method not aiming at approximating point values of the solution, as in [7]. In this case, the local discrete (point-wise) residuals are replaced by residuals for polynomial coefficient sets for which once again a maximum principle is sought for.

The situation is the following: the PDE (1) (for $\mathbf{D} \equiv 0$ and the suitable boundary conditions) is approximated on the degrees of freedoms, and the approximation writes as a large system of possibly non linear algebraic equations. The question is how to interpret this system into a variational formulation with properly choses functional spaces. Unfortunately, there is not a unique way to obtain such a formulation, as pointed out in [8]. The next step should be to use a variational formulation to approximate (1). The task is not a simple one, being quite hard to find out what the "good" variational interpretation is, as we will show shortly.

There has been already quite a lot of work on the approximation of (1) by means of residual distribution. In addition to early works where the RD scheme for the non viscous (1) was coupled, for $\mathbb{P}^1$ elements, to the Galerkin approximation of the viscous terms, one may quote the work of Caraeni [11], and more recently that of Villedieu et al. [19, 22] where this path has been explored

further for scalar problems and for the Navier Stokes equations. A different approach is being pursued by Nishikawa [17], for second order of accuracy, based on a re-interpretation of the solution of (1) as the steady solution of a relaxation (hyperbolic) system. The difference between our approach and [11] is that our technique is probably simpler, and the memory footprint is probably less important. Careni's method relies on the use of a gradient reconstruction that has the flavor of of what is done in high order finite volume methods, and thus has a much wider stencil, especially when higher accuracy is sought for. We believe our method to be more systematic. The difference between [19, 22] is that our method is able to handle very large gradients. In [17], only second order accuracy is sought for.

In this paper, we are interested in gaining a better understanding of the ideas discussed in the above references, in order to be able to handle (1) in a more general setting. In particular, we are interested in a formal accuracy higher than second, and to discretizations that are L^∞ stable, at least in the limit $\mathbf{D} \to 0$.

The presentation is organized as follows. We first recall a remark of [8] in the $\mathbb{P}^1$ case (second order of accuracy). This remark allows to explain why a simple coupling between the L^∞ stable RD scheme and a simple Galerkin approximation of the viscous term is still a residual method. Then we show, by a counter example, that this remark cannot be generalized (it only works on $\mathbb{P}^1$ elements), so that something else has to be done. To achieve that, we first revisit the formulation of the non viscous scheme, by generalizing relations (4), in particular the conservation relation (4a). We also consider the approach of Nishikawa to which we apply the technique developed in [4] to upgrade the formal accuracy. Numerical tests are performed to check the numerical accuracy, in particular with respect to variations of the values of the viscosity, for both methods. We also investigate the non oscillatory properties of the two methods.

1 Some basic information about RD schemes in the case $\mathbf{D} \equiv 0$.

The problem is to find an approximate solution of (1) where the boundary conditions are

$$u = g \text{ on } \Gamma_- = \{M \in \partial\Omega, \nabla_u f(u) \cdot \vec{n} < 0\} \tag{3}$$

and $\vec{n}$ is the inward normal to $\partial\Omega$ at $M \in \Omega$.

Let us denote by $\{\sigma\}$ the set of degrees of freedom that are necessary to represent

$$V_h = \{v \in L^2(\Omega), \text{ for any } K \in \mathcal{T}_h, v_{|K} \in \mathbb{P}^r(K)\} \cap \{v \in L^2, v_{|\Gamma_-} = g\}$$

the set of functions where we are looking for a solution. There is some abuse of notations in this "definition". Since the elements of $v \in V_h$ are polynomials of degree r in any triangle of $\mathcal{T}_h$, the degrees of freedom are, in this paper, the solution values at Lagrange points : the vertices for $\mathbb{P}^1$, the vertices and the edge mid-points for $\mathbb{P}^2$, etc.

We define two types of residuals: given $u^h \in V_h$,

1. For any $K \in \mathcal{T}_h$, the element residuals ϕ_σ^K for any $\sigma \in K$. They must sum up to the total residual ϕ^K,

$$\sum_{\sigma \in K} \phi_\sigma^K = \phi^K := \int_{\partial K} f(u^h) \cdot \vec{n} dl. \tag{4a}$$

2. For any boundary edge $\Gamma \subset \Gamma_-$, the edge residuals ϕ_σ^Γ for any $\sigma \in \Gamma$. They must sum up to the total edge residual ϕ^Γ,

$$\sum_{\sigma \in \Gamma} \phi_\sigma^\Gamma = \phi^\Gamma := \int_\Gamma \left[f(u^h) \cdot \vec{n} - \hat{f}(u^h, g, \vec{n}) \right] dl. \tag{4b}$$

   In (4b), $\hat{f}$ is a consistent numerical flux.

The conditions (4) enable to prove a Lax Wendroff theorem, see [5]. The total residuals and total edge residuals are evaluated by numerical quadrature and, if one looks at the proofs in [5], we see that, if achieved, convergence to a weak solution ensured as soon as the result of the edge numerical quadrature used to evaluate (4) depends on the edges only, and not on the element the edge belongs to: we only need the normal flux continuity.

Equation (1) with (3) is approximated by: find $u^h \in V^h$ such that for any $\sigma \in \mathcal{T}^h$,

$$\sum_{K, \sigma \in K} \phi_\sigma^K + \sum_{\Gamma, \sigma \in \Gamma} \phi_\sigma^\Gamma = 0. \tag{5}$$

Following our conventions, only the first sum exists for internal degrees of freedom while the second term appears only if σ belongs to the boundary.

The next question is how to define in practice the sub-residuals. The first requirement is about accuracy. In [5], it is shown that if

$$\phi_\sigma^\Sigma = O(h^{d_\Sigma + r}),$$

for any mesh entity Σ (element or edge), d_Σ denoting its dimension[1], then the scheme is $r+1$ accurate, provided of course as usual that the mesh is regular in the finite element sense. We recall later in the text how this can be achieved in practice.

The second question is about stability. To do that, the standard technique is to compare the sign of the residuals with that of a monotone scheme. In [5], following a path initiated by Roe and others, we show how it is possible to get *simultaneously* monotonicity preservation and accuracy. Unfortunately, in general the technique proposed in [5] does not converge to a steady state. The main reason is that the nonlinear mechanism at the basis of the construction only involves the preservation of the signs of the discretization coefficients in order to guarantee a discrete maximum principle. A priori, no physical principle is involved in the process. This might even lead locally to a *downind* discretization, prone to instabilities, or anyways to the appearance of mild spurious modes not correctly transported by the numerical scheme. A typical footprint of this behavior is a staircase appearance of the numerical solution. These difficulties are analyzed and solved in [1] for $\mathbb{P}^1$ elements and in [3, 4] for higher degree

[1] In 2d $d_\Sigma = 1$ for an edge Γ and $d_\Sigma = 2$ for an element K.

polynomials. The interested reader can consult these references for details. The key element is to modify the residuals $\phi_i^K(u^h)$ obtained following the approach of [5] as follows

$$\phi_i^{K\star}(u^h) = \phi_i^K(u^h) + \overbrace{h_K \sum_{x_q \in K} \Big(\nabla_u f(u^h)(x_q)\cdot\nabla\varphi_i\Big)\tau\Big(\nabla_u f(u^h)(x_q)\cdot\nabla u^h(x_q)\Big)}^{\mathcal{F}}, \tag{6}$$

The *filter* $\mathcal{F}$ in (6) has the flavor of

$$h_K \int_K \big(\nabla_u f(u^h)\nabla\varphi_i\big)\tau\big(\nabla_u f(u^h)\cdot\nabla u^h\big)dx$$

evaluated via numerical quadrature. In [3], an analysis is conducted to understand what are the relations between the last integral and the filter $\mathcal{F}$. It is shown that the quadrature does not need to be consistent, but, since the role of this term is to act as a dissipation that vanishes on exact solutions, the paper gives a criterion to choose the points x_q for triangular/tet and quad/hex elements. In the linear case, the centroid of the cell K is fine for triangular elements, while in the quadratic (triangle) case, the triangle vertices are fine. Other kind of elements are analyzed in that reference. The parameter τ (a matrix in the system case) is also discussed, but the quality of solution has no real dependency on τ in practice: its role is mainly to satisfy some dimensional consistency principle.

The modification in (6) looks very much with the stabilization term in the SUPG scheme, which itself has a lot to do with the artificial dissipation of the Lax Wendroff scheme. However, its role is *very* different: without this term, the scheme is perfectly stable in the L^∞ norm, but if one implements an iterative method to solve the system (5), the method will not converge in general. The role of $\mathcal{F}$ is to enable the iterative convergence, it is an experimental fact that the L^∞ is not degraded in practice.

2 Approximation of (1) in the $\mathbb{P}^1$ with triangle elements case.

In the $\mathbb{P}^1$ case, the degrees of freedom are simply the vertices of the mesh. To make things simpler, we assume $\mathbf{D} = \varepsilon\,\mathrm{Id}$. In the case, $\varepsilon = 0$, the RD scheme (2a) for (1) would write: for any mesh point i,

$$\sum_{K, i\in K} \phi_i^K(u^h) = 0 \tag{7}$$

where the residuals are subjected to the conservation condition (2b)

$$\sum_{i\in K} \phi_i^K(u^h) = \phi^K := \int_{\partial K} f^h(u^h)\cdot\vec{n}dl \tag{8}$$

In the following we shall assume that the flux is linear, that is $f(u) = \vec{\lambda}u$. The analysis can be generalized to non linear flux provided that $f^h(u^h)$ is the

Lagrange interpolant of $f(u)$; this is what we do in practice. In the second order case, the residual have the form

$$\phi_i^K = \beta_i^K \phi^K \tag{9}$$

where $\{\beta_i^K\}$ is uniformly bounded and constructed by various means.

Using the standard $\mathbb{P}^1$ shape function φ_i, we can rewrite ϕ^K in a Petrov Galerkin manner,

$$\phi_i^K = \int_K \varphi_i \nabla \cdot f^h(u^h) dx + \int_K (\beta_i^K - \frac{1}{3}) \nabla \cdot f^h(u^h) := \int_K \omega_i^K \nabla \cdot f^h(u^h)$$

because $f^h(u^h)$ is a linear polynomial and $\int_K \varphi_i dx = |K|/3$ in the case of a triangle. The problem of this formulation is that ω_i is not continuous across edges, and then cannot be used to approximate (1).

In [8], it was noticed that the *same* scheme could be written differently. Denote b_K the hat function that is 0 on ∂K and 1 at the gravity center of K. It is a piecewise linear function that enjoys

$$\int_{\partial K} \nabla b^K \cdot \vec{n} dl = 0 \text{ and } \int_K b^K dx > 0.$$

We can write

$$\phi_i^K = \beta_i^K \phi^K = \int_K \varphi_i \nabla \cdot f^h(u^h) dx + \gamma_i^K \int_K b^K \nabla \cdot f^h(u^h)$$

with

$$\gamma_i^K \int_K b^K dx = (\beta_i^K - \frac{1}{3})|K|,$$

again because u^h and the flux is linear in K.

Now,

$$\omega_i = \varphi_i + \left\{ \begin{array}{ll} \sum_{K, i \in K} \gamma_i^K b^K & \text{if } x \in \text{ support of } \varphi \\ 0 & \text{else.} \end{array} \right. \tag{10}$$

is *a continuous function*, so that it can be used in the variational formulation.

Denoting by $W_h = \text{span}(\omega_i)$ and $V_h = \text{span}(\varphi_i)$, the variational formulation of the problem is (we omit the BCs for short and use some abuses of language): Find $u^h \in V^h$ such that for all $w \in W_h$,

$$-\int_\Omega f^h(u^h) \cdot \nabla w \, dx + \int_\Omega \varepsilon \nabla u^h \cdot \nabla w \, dx = 0$$

Setting $w = \omega_i$ the previous equation, and integrating by parts the first term we get

$$\sum_{K, i \in K} \int_K \left(\omega_i \nabla \cdot f^h(u^h) + \varepsilon \nabla u^h \cdot \nabla \omega_i \right) dx$$

The first term gives back $\beta_i^K \phi^K$. Let us have a look at the second one,

$$\int_K \nabla \omega_i \cdot \nabla u^h \, dx = \int_K \nabla \varphi_i \cdot \nabla u^h \, dx + \gamma_i^K \int_K \nabla b^K \cdot \nabla u^h \, dx.$$

Since ∇u^h is constant, we see that

$$\int_K \nabla b^K \cdot \nabla u^h \, dx = \nabla u^h \cdot \int \nabla b^K \, dx,$$

and by Green formula (which holds due to the continuity of b^K),

$$\int_K \nabla b^K \, dx = \int_{\partial K} b^K \vec{n} \, dx = 0$$

This shows that the variational formulation is : find u^h such that for any i,

$$\sum_{K \ni i} \beta_i^K \phi^K + \varepsilon \int_K \nabla \varphi_i \cdot \nabla u \, dx = 0$$

i.e. the RD scheme on the convection *plus* Galerkin on the diffusion. This is the argument used in [8] to justify the consistency of the above scheme (see however footnote 4 on previous page). The method, however, does not show a uniform accuracy. This is a well known problem of the SUPG scheme which is recovered here with $\beta_i = \frac{1}{3} + k_i \tau$. One has to blend the scheme with a Galerkin approximation, the blending parameter depends on a cell Peclet number, see see [19, 13] for details on the streamline method and the RD schemes.

How can we extend this to higher orders ? The key argument here was that the gradient or the divergence of a linear field is constant, which is only true for linear triangular elements.

3 Extension to higher degrees.

The purpose of this section is to investigate whether or not the technique of the previous section can be extended to higher than second order accurate schemes, in other words, to see wet-her or not the schemes described in [4] can be reinterpreted in a classical variational formulation with continuous test functions. We will see that the answer is no.

We still assume $\mathbf{D} = \varepsilon$ Id. We want to find functions $\gamma_i^K \in H^1(K)$ such that:

1. When we use a $\mathbb{P}^k$ Lagrange interpolant,

$$\int_K (\varphi_i + \gamma_i^K) \nabla \cdot f(u^h) dx = \beta_i^K \int_K \nabla \cdot f(u^h) \, dx, \tag{11a}$$

2. Enable to construct H^1 basis functions:

$$(\gamma_i^K)_{|\partial K} = 0, \tag{11b}$$

3. They have no role on the viscous terms:

$$\int_K \gamma_i^K \Delta u \, dx = \int_K \text{ div } (\gamma_i^K \nabla u) \, dx - \int_K \nabla \gamma_i^K \cdot \nabla u \, dx = 0$$

i.e. because $(\gamma_i^K)_{|\partial K} = 0$,

$$\int_K \nabla \gamma_i^K \cdot \nabla u \, dx = 0 \tag{11c}$$

for any $u \in \mathbb{P}^k(K)$

We can rephrase (11a) as

$$\int_K \gamma_i^K \nabla \cdot f(u^h)\, dx = \beta_i^K \int_K \nabla \cdot f(u^h)\, dx - \int_K \varphi_i \nabla \cdot f(u^h)\, dx. \qquad (12)$$

If the flux f is linear in u^h, the conditions (12)-(11b)-(11c) are affine conditions of the type

$$\ell_p(\gamma_i^K) = a_p,$$

where the linear functional ℓ_p are defined by:

1. From condition (12)

$$\ell_p^1(w) = \int_K w \nabla \cdot f(\varphi_p)\, dx,$$

and

$$a_p = \beta_i^K \int_K \nabla \cdot f(\varphi_p)\, dx - \int_K \varphi_i \nabla \cdot f(\varphi_p)\, dx,$$

2. From condition (11c)

$$\ell_p^2(w) = \int_K \nabla w \cdot \nabla \varphi_p\, dx$$

and $a_p = 0$.

Unfortunately, there is no solution to this problem, in general. Consider the simple 1D case, with quadratic elements. Any element can be mapped onto $[0, 1]$, so we can assume $K = [0, 1]$. In the case of quadratic elements, the Lagrange points are $\xi = 0,\ 1/2,\ 1.$ and thus the Lagrange functions are

$$\varphi_0(x) = (1 - 2x)(1 - x), \quad \varphi_{1/2}(x) = 4x(1 - x), \quad \varphi_1(x) = x(2x - 1)$$

hence

$$\varphi_0'(x) = 4x - 3, \quad \varphi_{1/2}'(x) = 4 - 8x, \quad \varphi_1'(x) = 4x - 1.$$

Since the second derivative of quadratic functions are constant plus the simplification given by (11c) writes $\int_0^1 \gamma_\xi dx = 0$. Using these expressions of φ_0', φ_1' and $\varphi_{1/2}'$ and this relation, (12) becomes:

$$\varphi_0' : \qquad 4\int_0^1 \gamma_\xi x dx = -\beta_\xi - \int_0^1 \varphi_\xi \varphi_0'(x) dx, \qquad (13a)$$

$$\varphi_{1/2}' : \qquad -8\int_0^1 \gamma_\xi x dx = -\int_0^1 \varphi_\xi \varphi_{1/2}'(x) dx, \qquad (13b)$$

$$\varphi_1' : \qquad 4\int_0^1 \gamma_\xi x dx = \beta_\xi - \int_0^1 \varphi_\xi \varphi_1'(x) dx. \qquad (13c)$$

If one takes $\xi = 1/2$, we see (from (13b)) that $\int_0^1 \gamma_\xi x dx = 0$ i.e. $\beta_{1/2}$ is a given *fixed, scheme independent* constant[2]. This already suffices to show that

[2]In fact the unique solution is $\beta_{1/2} = 1/4$

we cannot build γ_i^K for any given scheme.

Let us show that in general, β_ξ can be arbitrary in $[0,1]$. To show that, we consider the limited scheme constructed from the Lax Friedrich's scheme,

$$\phi_\xi = \frac{1}{3}(u_1 - u_0) + \alpha(u_\xi - \bar{u}) \text{ with } \bar{u} = \frac{1}{3}(u_0 + u_{1/2} + u_1)$$

We introduce

$$p = \frac{u_{1/2} - u_0}{u_1 - u_0} \text{ and } q = \frac{u_1 - u_{1/2}}{u_1 - u_0}.$$

We have $p + q = 1$, p can be arbitrary in $\mathbb{R}$. We define the ratios

$$x_0 = \frac{\phi_0}{\phi} = \frac{1}{3} - \alpha\frac{p+1}{3}, \quad x_{1/2} = \frac{\phi_{1/2}}{\phi} = \frac{1}{3} + \alpha\frac{p-q}{3}, \quad x_1 = \frac{\phi_{1/2}}{\phi} = \frac{1}{3} + \alpha\frac{q+1}{3}.$$

If $\alpha = 1$, we have some simplifications

$$x_0 = -\frac{p}{3}, \quad x_{1/2} = \frac{2}{3}p, \quad x_1 = \frac{3-p}{3}.$$

We see that if $p \in [0,3]$, we obtain

$$\beta_0 = 0, \quad \beta_{1/2} = \frac{2p}{3+p} \quad \beta_1 = \frac{3-p}{3+p}.$$

We note that the image of $[0,3]$ by $p \mapsto \dfrac{2p}{3+p}$ is $[0,1]$, i.e. the range of $\beta_{1/2}$ is at least $[0,1]$ which is in contradiction that $\beta_{1/2}$ is the fixed given constant.

This clearly shows that there is no solution to the problem in general, and that something else must be done.

4 Approximation of (1): Variational methods based on gradient reconstruction.

We start again from the formulation (7)-(9). Taking $v^h \in V^h$, we have

$$\sum_i v_i^h \left(\sum_{K \ni i} \phi_i^K + \sum_{\Gamma \ni i} \phi_i^\Gamma \right) = \sum_K \Big(\sum_{j \in K} \beta_j^K v_j^h \Big) \int_{\partial K} f(u^h) \cdot \vec{n} dl + \sum_\Gamma \Big(\sum_{j \in \Gamma} \beta_j^K v_j^h \Big) \int_\Gamma \big(f(u^h) \cdot \vec{n} - \hat{f}(u^h, g, \vec{n}) \big) dl.$$

We introduce W^h the space of the functions that are piecewise constant on the elements K and the mapping

$$\begin{aligned} \pi_\beta^h : & V^h \to W^h \\ & v^h \mapsto \pi_\beta^h(v^h), \text{for all } K, \pi_\beta^h(v^h)_{|K} = \sum_{j \in K} \beta_j^K v_j^h. \end{aligned}$$

In this definition, β stands for the set $\{\beta_j^K\}_{K, j \in K}$.

We can reformulate the RD scheme as finding $u^h \in V^h$ such as for any $v^h \in V^h$,

$$a(u^h, v^h; u^h) = 0$$

with

$$a(u^h, v^h; w^h) = \sum_K \left(\int_{\partial K} \pi^h_{\beta(w^h)}(v^h) f(u^h) \cdot \vec{n} dl - \int_K \nabla \pi^h_{\beta(w^h)}(v^h) \cdot f(u^h) dx \right) \tag{14}$$

where we have indicated the possible dependence of the β_j^Ks in term of w^h by a mechanism that has to be described, we do this later in this section.

We note that the exact solution u, if it is smooth enough, of (1) also satisfies the residual condition

$$a(u, v^h; w^h) = 0$$

for any $v^h, w^h \in V^h$. In fact, u is smooth enough, we can also write

$$\sum_K \int_K \pi^h_{\beta(w^h)}(v^h) \text{div } f(u) dx = 0.$$

To derive the formulation for the viscous problem (1), $\mathbf{D} \not\equiv 0$, we start from this last relation.

Let us turn to the viscous problem (1). If u is the solution of (1), it is known that it is a smooth enough function and we can write for the continuous problem

$$\sum_K \int_K \pi^h_{\beta(w^h)}(v^h) \big[\text{div } \big(f(u) - \mathbf{D}\nabla u \big) \big] dx = 0.$$

By standard calculations, we get

$$\sum_K \int_{\partial K} \pi_h(w^h) f(u) \cdot \vec{n} dl + \int_{\partial K} \pi_h(w^h) \mathbf{D}\nabla u \cdot \vec{n} dx = 0.$$

Since $\nabla u \cdot \vec{n}$ is continuous across any edge of $\mathcal{T}_h$ for a smooth enough solution, and using some average operator[3], we can equivalently rewrite this relation as

$$\sum_K \int_{\partial K} \pi_h(v^h) f(u) \cdot \vec{n} dl + \int_{\partial K} \pi_h(w^h) \{\mathbf{D}\nabla u \cdot \vec{n}\} dx = 0^4 \tag{15}$$

and consider the variational formulation which takes into account the boundary conditions as in (1)

$$\begin{aligned} \sum_K \int_{\partial K} \pi^h_\beta(v^h) f(u) \cdot \vec{n} dl \sum_{\Gamma \subset \Gamma_W} \int_\Gamma \pi^h_\beta(v^h) \big(f(u^h) \cdot \vec{n} - \hat{f}(u^h, g_w, \vec{n}) \big) dl \\ + \int_{\partial K} \pi^h_\beta(v^h) \{\mathbf{D}\nabla u \cdot \vec{n}\} dx \\ + h_K \int_K \Big(\nabla_u f(u^h) \cdot \nabla v^h - \nabla \cdot (\mathbf{D}\nabla v^h) \Big) \Big(\nabla_u f(u^h) \cdot \nabla u^h - \nabla \cdot (\mathbf{D}\nabla u^h) \Big) dx \\ = 0. \end{aligned} \tag{16}$$

In (16) we have made the following assumptions:

[3] In fact we can use any consistent viscous flux $\{D\nabla u \cdot \vec{n}\}(D\nabla u \cdot \vec{n}^-, D\nabla u \cdot \vec{n}^+)$ such that $\{D\nabla u \cdot \vec{n}\}(D\nabla u \cdot \vec{n}, D\nabla u \cdot \vec{n}) = D\nabla u \cdot \vec{n}$

[4] The matrix $\mathbf{D}$ may depend on u, ∇u, etc.

1. The mesh is adapted to the boundary conditions. In particular, the union of the boundary edges on Γ_W is Γ_W exactly. This also means that Ω has a polygonal shape. The more general case can be handle via isoparametric approximation as in [24, 4].

2. u^h and the test functions v^h are sought for in the functional spaces

$$V_h = \{u^h \text{ continuous and defined on } \Omega, u^h_K \in \mathbb{P}^k(K) \text{ for any element K and } u^h_{|\Gamma_S} = g_S\} \quad \text{for } u^h$$
$$V'_h = \{v^h \text{ continuous and defined on } \Omega, v^h_K \in \mathbb{P}^k(K) \text{ for any element K and } u^h_{|\Gamma_S} = 0\} \quad \text{for } v^h$$

The choice of β is free, we choose it so that the convective operator, without the filtering operator, leads to a maximum principle satisfying scheme. To achieve this, in each K, we proceed as follows:

1. We consider the Lax Friedrich residual. For any degree of freedom in K,

$$\phi^{LLF,K}_\sigma = \frac{1}{N_K}\left(\int_{\partial K}\left(f(u^h)\cdot\vec{n} - \{\mathbf{D}\nabla u\cdot\vec{n}\}\right) + \alpha_K(u_\sigma - \overline{u}_K)\right)$$

where α_K is a bound of $\left|\nabla_u f\right|$ on K, N_K the number of degrees of freedom in K and

$$\overline{u}_K = \frac{1}{N_K}\sum_{\sigma\in K} u_\sigma, \tag{17}$$

2. β^K_σ is evaluated from

$$\beta^K_\sigma = \frac{\max(\frac{\phi^{LLF,K}_\sigma}{\phi^K}, 0)}{\sum_{\sigma'\in K}\max(\frac{\phi^{LLF,K}_{\sigma'}}{\phi^K}, 0)}, \tag{18}$$

recall that

$$\phi^K = \int_{\partial K}\left(f(u^h)\cdot\vec{n} - \{\mathbf{D}\nabla u\cdot\vec{n}\}\right).$$

3. The average operator $\{\ .\ \}$ (viscous flux) has to be consistent. The next paragraph is devoted to the description of the reconstructions we have employed in this work.

4. We have added a filtering term similar to (6) but adapted to (1), namely

$$\Phi^{K,\mathrm{F}}_\sigma = \int_K \left(\nabla f_u\cdot\nabla\varphi_\sigma - \nabla\cdot(\mathbf{D}\nabla\varphi_\sigma)\right)\tau\left(\nabla f_u\cdot\nabla u - \nabla\cdot(\mathbf{D}\nabla u)\right) \tag{19}$$

The actual scheme writes

$$\Phi^{LLF-S}_\sigma = \beta^K_\sigma\phi^K + \Phi^{K,\mathrm{F}}_\sigma.$$

Remark 4.1. *We can easily extend this to the case of discontinuous RD scheme, see [6, 2, 14].*

4.0.1 Gradient reconstruction methods.

We have considered a reconstruction techniques based on an L^2 projection, which can be written as follows:

$$\int_\Omega \varphi_\sigma \{\nabla u\} \; d\Omega \; = \sum_{K,\sigma\in K} \; \int_K \varphi_\sigma \; \nabla u|_K \; dx, \quad \forall \sigma \in \mathcal{M}_h \tag{20}$$

where:

$$\{\nabla u\} = \sum_{\sigma\in\mathcal{T}_h} \varphi_\sigma \{\nabla u\}_\sigma \qquad \nabla u|_K = \sum_{\sigma\in K} u_\sigma \nabla\varphi_\sigma \tag{21}$$

For now, the reconstruction space spanned by the $\{\varphi_\sigma\}$ is left unclear, as well as the set of degrees of freedom that are involved here. Since the reconstructed gradient needs to be continuous across edges, the functions φ_σ need to be globally continuous. The first choice is to use the same basis functions as those already in use for the approximation of u on the right hand side of (20). The linear system defined by (20) is not diagonal, so the solution though not difficult to obtain, it is a bit expensive to proceed this way.

A better choice is to use mass lumping. We wish to have an accuracy of order k if k is the polynomial degree. In the $\mathbb{P}^1$, the weights are $1/3$; in the $\mathbb{P}^2$, we use $(\frac{1}{12}, \frac{1}{12}, \frac{1}{12}, \frac{3}{12}, \frac{3}{12}, \frac{3}{12})$ (weights for the 3 vertices and 3 mid-points). In both cases the maximal accuracy on the gradient is met.

It is possible to have exact mass lumping at the price of changing the approximation representation. Following [15, 23], we have tested enriched elements, initially built to allow mass lumping in Galerkin discretizations of time dependent problems. For example, the P^{2+} triangle simply has one additional (w.r.t the standard $\mathbb{P}^2$ triangle) degree of freedom which is located in its centroid (see figure 2). Using the seven points of the enriched element a quadrature formula exact for polynomials of degree 2 can be constructed. We refer to [15, 23] for details.

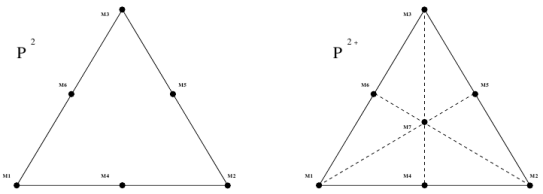


Figure 2: Standard $\mathbb{P}^2$ and enriched $\mathbb{P}^{2+}$ triangles

From the implementation point of view, from (21), we get

$$\{\nabla u\}_\sigma = \frac{\sum_{K,\sigma\in K} |K| \omega_\sigma \nabla u^h(\sigma)}{\sum_{K,\sigma\in K} |K| \omega_\sigma} \tag{22}$$

where the ω_σs are the weight associated to the quadrature formula. We note that, since only a maximal accuracy of order k can be met, we can also apply (22) with $\omega_\sigma = 1$ for any σ. We have also tested this variant.

4.0.2 Dealing with the diffusion dominated case.

The LLFS scheme that used this reconstruction has the expected behavior for small values of the diffusion coefficients. However when the cell Reynolds number $\|\nabla_u f\| h/\nu$ is moderate or small the (grid) convergence rate obtained is considerably less than the $k+1$ value we aim at. When diffusion starts becoming important the convergence curve suffers perturbations and its rate decreases drastically. In a diffusion dominated configuration, the scheme completely fails to converge.

To cure this flaw, we have blended the current limited LLFS scheme with a Lax-Wendroff scheme obtained by rewriting the advection diffusion problem as a system of first order PDEs, as proposed in [17] (cf. section §**??**). In the diffusion dominated case, the LW scheme nodal residuals writes :

$$\Phi_\sigma^{LW} = \frac{1}{N_K}\Phi^K + \mathbf{C}\int_K \nabla\varphi_\sigma \left(\nabla u^K - \{\nabla u\}\right) dK\,, \quad \text{with } \Phi^K = -\int_{\partial K} \{\mathbf{D}\nabla u\cdot\vec{n}\}$$

where we set $\mathbf{C} = 0.5$ when working with triangles. As shown in [17], this term can be related to the Least-Square stabilization associated to the First Order System form of the equation. The Lax-Wendroff scheme reaches respectively second and third order when solving the purely diffusive scalar equation using a $\mathbb{P}^1$ (resp. $\mathbb{P}^2$) formulation.

When solving the advection diffusion equation, we blend the LLFS scheme with the LW scheme. The blending parameter is a functions of the cell Reynolds number $\xi(Re)$. This functions is computed following again [17]. In practice, first we define a modified Lax-Friedrichs scheme, by turning off the LF dissipation in the purely diffusive regime, and apply the limiter as discussed before, so that we get:

$$\Phi_\sigma^{K,LLF-LW} = \frac{1}{N_K}\Phi^K + \xi(Re)\alpha^K\left(u_i - \overline{u}\right)$$

Our choice of the parameter $\xi(Re)$ is

$$\xi^*(Re) = \begin{cases} 1, & \text{if} \quad \xi > 1 - t_r \\ 0, & \text{if} \quad \xi < t_r \\ \xi, & \text{else} \end{cases} \tag{23}$$

with t^r a threshold value to be chosen.

Then we compute the limited residuals $\Phi_\sigma^{K,LLF-LW^*}$ following the procedure (18). We then add the Least-Squares term (19) in advection dominated flows, and the LW Least-Squares term in the diffusion dominated one. The resulting scheme reads :

$$\Phi_\sigma^{K,RG-LLFS-LW^*} = \Phi_\sigma^{K,LLF-LW^*} + \xi(Re)\Phi_\sigma^{K,\mathrm{F}} + (1-\xi(Re))\,\Phi_\sigma^{K,LW}. \tag{24}$$

The parameter ξ that we have used in effective calculations is defined in (23). "RG" stands for reconstructed gradients.

In terms of CPU cost, computing the Lax-Wendroff integral term is not very demanding. The same quadrature formula used for the reconstruction, using only the element's degrees of freedom, has proven to be enough to provide the best convergence rates achievable with this approach. In particular, on $\mathbb{P}^2$ elements, the blended scheme reaches third order at both limits but is still

experiencing a precision loss around $Re \sim 1$ (cf. section §5). Similar conclusions are obtained when using both standard $\mathbb{P}^2$ and enriched $\mathbb{P}^{2+}$ elements.

The choice (23) is such that the diffusive/advection limits of the scheme is reached much before it is the case in the continuous system. This allows to get rid of the computation cost of one of the stabilization terms. In particular, when facing a pure diffusive flow configuration, cutting out the cost associated to the LLFS scheme results in a critical speed-up. In practice, when the Reynolds based function comes close enough from its bounds (0 and 1), we force it to take the limit value. This is obtained by defining the modified blending function :

5 Numerical illustrations.

We present two test cases. In the first one, we reformulate the heat equation as a steady two dimensional problem. Here

$$D = \begin{pmatrix} 0 & 0 \\ 0 & \varepsilon \end{pmatrix}$$

with $\varepsilon > 0$. The matrix D is symmetric but not strictly positive. The second test deals with $D = \varepsilon \mathrm{Id}$.

We then perform a series of test on the standard two-dimensional linear advection diffusion equation to investigate both the accuracy and the non oscillatory properties of the method.

5.1 Accuracy test.

5.1.1 The heat equation revisited.

In order to check the accuracy of the scheme we have considered the following problem :

$$\frac{\partial u}{\partial y} - \varepsilon \frac{\partial^2 u}{\partial x^2} = 0 \tag{25a}$$

on $[0,1]^2$ with the boundary conditions

$$\begin{aligned} u(x,0) &= \sin(\pi x) \text{ on } y = 0 \\ u(x,y) &= \varphi(x,y) \text{ on } x = 0 \text{ and } x = 1 \end{aligned} \tag{25b}$$

where

$$\varphi(x,y) = e^{-\varepsilon y} \sin(\pi x).$$

This is the exact solution of the heat equation (25). We have done the computation of a series of 4 meshes with 10, 20, 40 and 80 edges on each side.

5.1.2 Grid convergence on a truly 2D problem

We are still solving (1) on $\Omega = [0,1]^2$, with

$$f(u) = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} u \text{ and } D = \varepsilon \mathrm{Id},$$

$\log(Re)$	$\log(h)$	L^∞ error		L^2 error	Order
3.47	-0.53	-2.41	-	-2.43	
3.13	-0.84	-3.22	2.57	-3.15	2.30
2.91	-1.09	-4.06	3.42	-3.87	2.94
2.64	-1.36	-4.90	3.06	-4.60	2.65
$\varepsilon = 0.0001$					
$\log(Re)$	$\log(h)$	L^∞ error		L^2 error	Order
2.47	-0.53	-2.42	-	-2.43	-
2.13	-0.84	-3.24	2.62	-3.21	2.47
1.91	-1.09	-4.09	3.48	-3.95	3.06
1.64	-1.36	-4.99	3.24	-4.85	3.25
$\varepsilon = 0.001$					
$\log(Re)$	$\log(h)$	L^∞ error		L^2 error	Order
1.47	-0.53	-2.45	-	-2.52	-
1.13	-0.84	-3.29	2.67	-3.26	2.34
0.91	-1.09	-4.01	2.97	-3.74	1.99
0.64	-1.36	-4.71	2.51	-4.49	2.7
$\varepsilon = 0.01$					
$\log(Re)$	$\log(h)$	L^∞ error		L^2 error	Order
0.47	-0.53	-2.12	-	-2.07	-
0.13	-0.84	-2.56	1.42	-2.55	1.51
-0.09	-1.09	-3.19	2.58	-3.19	2.65
-0.36	-1.36	-4.13	3.40	-3.82	2.27
$\varepsilon = 0.1$					

Table 1: Results for the convergence study of (25) with the model with the RG-LLF scheme.

but the function $g(x, y)$ is now defined by the following exact solution:

$$
\begin{aligned}
g(x,y) &= U_{\text{ex}}(x,y), \qquad \text{for all } x \in \partial\Omega \\
U_{\text{ex}}(x,y) &= -\cos\big(2\pi\eta\big)\exp\left[\frac{1}{2\varepsilon}\xi\big(1-\sqrt{1+(4\pi\varepsilon)^2}\big)\right] \text{ in } [0,1]^2 \\
\xi &= \cos(\frac{\pi}{4})x + \sin(\frac{\pi}{4})y, \\
\nu &= \sin(\frac{\pi}{4})x - \cos(\frac{\pi}{4})y
\end{aligned}
$$

For several values of the parameter ϵ, we conduct a convergence study which uses 8 triangulated meshes, with mesh sizes from $h \approx 1/10$ to $h \approx 1/80$. The study covers a wide range of flow configurations : diffusion dominated ($\log(Re) \ll 0$), advection-diffusion ($\log(Re) \approx 0$) and advection dominated ($\log(Re) \gg 0$). We compare three numerical schemes, all using the reconstructed gradients approach. The first one is the P^2 RG-LLFS scheme, the second one is the P^2 RG-LLFS-LW, and the last one is the P^{2+} RG-LLFS-LW. The results of these mesh convergence studies are presented in Table 2.

We first note that the RG-LLFS scheme experiences convergence problems in the case of the diffusion dominated configuration ($\epsilon = 1$). The tests stop at

the fourth mesh for this scheme. The achieved order of accuracy is close to 3 in the advection dominated configuration and decreases to 2 when $\log(Re) \approx 0$.

The RG-LLFS-LW formulation behaves exactly as the RG-LLFS scheme for both advection-diffusion and advection dominated flows, respectively achieving a precision of 2 and 3. The Lax-Wendroff blending improves the convergence of the scheme in the diffusion dominated case where third order accuracy is now achieved.

The P^{2+} formulation of the RG-LLFS-LW scheme slightly improves the results in the advection-diffusion case, in which we achieve convergence rates larger than 2 (between 2.3 and 3) but still far from optimal and uniform. It is not clear that this minor improvement worth the additional cost of the extra degree of freedom.

5.2 Resolution of steep gradients

We are interested in solving the Smith and Hutton problem

$$\begin{array}{ll} \text{div}\,(\vec{\lambda}u) - \varepsilon\Delta u & x \in \Omega \\ u = g & \text{on } \partial\Omega \end{array}$$

with

$$\vec{\lambda} = (2y(1-x^2), -2x(1-y^2))^T, \qquad \Omega = [-1,1]\times[0,1], \qquad f = 0 \text{ and } g(x,y) = 1+\tanh\left(\delta(1-2\sqrt{\psi})\right),$$

$\psi(x,y) = 1-(1-x^2)(1-y^2)$ and $\delta = 100$. The usual choice for δ is 10, resulting is much less steep gradients. This case enable to check the ability of the scheme to handle very steep solution gradients. As shown in figure 3, no particular problem are encountered. There is no known analytical formula for this problem, hence it is not possible to conduct a convergence study.

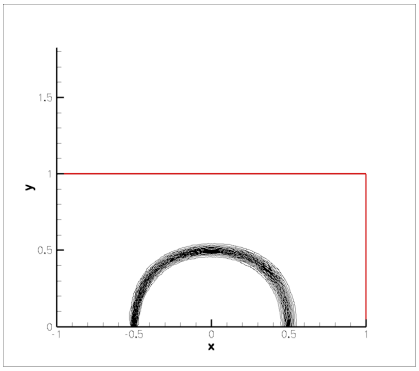
6 Conclusion, future work.

We have explored several discretization of the steady convection diffusion (1) by mean of Residual Distribution schemes. These schemes degenerate to standard Residual Distribution schemes when the diffusion effects vanish and are non oscillatory. We have first shown that the method used in the case of triangular linear element cannot be extended to more general case. It can be shown that the scheme is still consistent (see [16]), but one cannot reach optimal accuracy. Thanks to a reformulation of the inviscid RD scheme as a variational formulation, we can rely on this reformulation to develop a class of schemes adapted to (1). They need a gradient reconstruction, and several options are studied. Several numerical simulations are done and demonstrate the expected accuracy, at least for moderate to high local Reynolds numbers.

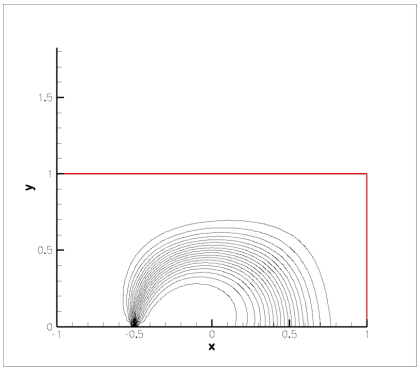
These methods are being extended to the Navier Stokes equations, see [9] for preliminary results.

Acknowledgments

R. Abgrall has been supported in part by the ERC Advanced grant ADDECCO (contract # 226316). A. Krust has been fully supported by the ERC Advanced



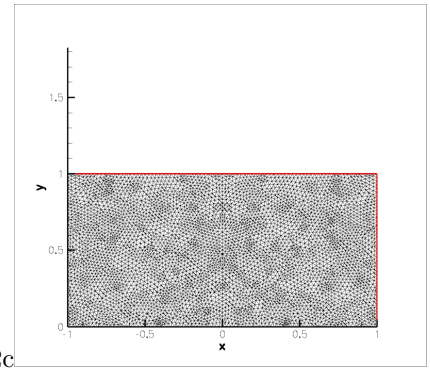
$\nu = 10^{-4}$



$\nu = 10^{-2}$

unknowns : u and reconstructed gradient

multicolumn2c



Mesh (all the degrees of freedom are displayed)

Figure 3: Smith and Hutton case with $\alpha = 100$ for different viscosities.

grant ADDECCO (contract # 226316). Dante de Santis has been supported by the FP7-AAT-2010-RTD-1 STREP IDIHOM. G. Baurin has been supported by SNECMA via a CIFRE grant.

References

[1] R. Abgrall. Essentially non-oscillatory residual distribution schemes for hyperbolic problems. *J. Comput. Phys.*, 214(2):773–808, 2006.

[2] R. Abgrall. A residual distribution method using discontinuous elements for the computation of possibly non smooth flows. *Adv. Appl. Math. Mech.*, 2(1):32–44, 2010.

[3] R Abgrall, A. Larat, M. Ricchiuto, and C. Tavé. A simple construction of very high order non-oscillatory compact schemes on unstructured meshes. *Comput. Fluids*, 38(7):1314–1323, 2009.

		RG-LLFS		RG-LLFS-LW		RG-LLFS-LW (P^{2+})	
$\epsilon = 1$							
$\log(Re)$	$\log(h)$	L^2 error	Order	L^2 error	Order	L^2 error	Order
-1.01	-1.01	-2.58	-	-3.76	-	-3.66	-
-1.31	-1.31	-3.06	1.59	-4.73	3.22	-4.42	2.52
-1.49	-1.49	-3.33	1.58	-4.98	2.56	-4.80	2.38
-1.62	-1.62	-3.39	1.34	-5.37	2.66	-5.28	2.68
-1.71	-1.71	x	x	-5.65	2.70	-5.54	2.69
-1.79	-1.79	x	x	-5.99	2.87	-5.88	2.86
-1.86	-1.86	x	x	-6.23	2.92	-5.98	2.74
-1.92	-1.92	x	x	-6.36	2.88	-6.14	2.75
$\epsilon = 0.01$							
$\log(Re)$	$\log(h)$	L^2 error	Order	L^2 error	Order	L^2 error	Order
0.99	-1.01	-2.36	-	-2.46	-	-2.31	-
0.69	-1.31	-3.06	2.31	-3.13	2.21	-3.26	3.14
0.51	-1.49	-3.47	2.33	-3.49	2.16	-3.59	2.68
0.38	-1.62	-3.73	2.27	-3.73	2.10	-3.81	2.48
0.29	-1.71	-3.91	2.22	-3.93	2.09	-3.98	2.39
0.21	-1.79	-4.02	2.14	-4.07	2.07	-4.11	2.32
0.14	-1.86	-4.10	2.05	-4.20	2.06	-4.25	2.30
0.08	-1.92	-4.18	2.01	-4.32	2.06	-4.37	2.28
$\epsilon = 0.0001$							
$\log(Re)$	$\log(h)$	L^2 error	Order	L^2 error	Order	L^2 error	Order
2.99	-1.01	-2.90	-	-2.94	-	-2.85	-
2.69	-1.31	-3.81	3.00	-3.83	2.95	-3.72	2.89
2.51	-1.49	-4.36	3.05	-4.37	3.01	-4.26	2.96
2.38	-1.62	-4.70	2.98	-4.71	2.94	-4.62	2.93
2.29	-1.71	-4.99	2.99	-5.00	2.95	-4.91	2.94
2.21	-1.79	-5.21	2.97	-5.22	2.93	-5.12	2.92
2.14	-1.86	-5.40	2.96	-5.41	2.93	-5.30	2.90
2.08	-1.92	-5.58	2.96	-5.59	2.93	-5.48	2.91

Table 2: Results obtained with the RG-LLFS P^2 scheme, then with the RG-LLFS-LW P^2 scheme and finally with the P^{2+} enriched element version.

[4] R. Abgrall, M. Ricchiuto, and A. Larat. Construction of very high order residual distribution schemes for steady inviscid flow problems on hybrid unstructured meshes. *J. Comput. Phys.*, 2011.

[5] R. Abgrall and P. L. Roe. High order fluctuation schemes on triangular meshes. *J. Sci. Comput.*, 19(1-3):3–36, 2003. Special issue in honor of the sixtieth birthday of Stanley Osher.

[6] R. Abgrall and C.W. Shu. Development of residual distribution schemes for the discontinuous galerkin method: The scalar case with linear elements. *Communication in Computational Physics*, 5:376–390, 2009.

[7] R. Abgrall and J. Trefilick. An example of high order residual distribution scheme using non-lagrange elements. *J. Sci. Comput.*, 45(1-3):3–25, 2010.

[8] Rémi Abgrall. Residual distribution schemes: current status and future trends. *Comput. Fluids*, 35(7):641–669, 2006.

[9] Remi Abgrall and Dante De Santis. High order residual distribution scheme for Navier-Stokes equations. In AIAA, editor, *20th AIAA Computational Fluid Dynamics Conference*, , Honolulu, Hawaii, États-Unis, June 2011. AIAA, AIAA.

[10] Rémi Abgrall and Mohamed Mezine. Construction of second order accurate monotone and stable residual distribution schemes for unsteady flow problems. *J. Comput. Phys.*, 188(1):16–55, 2003.

[11] D. Caraeni and L. Fuchs. Third-order multidimensional upwind algorithm for flow simulations on unstructured grids. AIAA-2003-3992, 2003.

[12] J.C. Carette, H. Deconinck, H. Paillère, and P.L. Roe. Multidimensional upwinding: Its relation to finite elements. Morgan, K. (ed.) et al., Finite elements in fluids: new trends and applications. Proceedings of the 8th international conference, Barcelona, Spain, September 20-23, 1993. Part I. Barcelona: Centro Internacional de Métodos Numéricos en Ingeniería. 446-456 (1993)., 1993.

[13] Alexandre Ern and Jean-Luc Guermond. *Theory and practice of finite elements.* Applied Mathematical Sciences 159. New York, NY: Springer. xiii , 2004.

[14] Matthew Hubbard. Discontinuous fluctuation distribution. *J. Comput. Phys.*, 227(24):10125–10147, 2008.

[15] S. Jund and S. Salmon. Arbitrary high order finite element schemes and high order mass lumping. *Int. J. Appl. Math. Comput. Sci*, 2007.

[16] Adam Larat. *Conception et Analyse de Schémas Distribuant le Résidu d'Ordre Très Élevé. Application à la Mécanique des Fluides.* These, Université Sciences et Technologies - Bordeaux I, November 2009.

[17] Hiroaki Nishikawa. A first-order system approach for diffusion equation. II: Unification of advection and diffusion. *J. Comput. Phys.*, 229(11):3989–4016, 2010.

[18] M. Ricchiuto and R. Abgrall. Explicit Runge-Kutta residual distribution schemes for time dependent problems: second order case. *J. Comput. Phys.*, 229(16):5653–5691, 2010.

[19] M. Ricchiuto, N. Villedieu, R. Abgrall, and H. Deconinck. On uniformly high-order accurate residual distribution schemes for advection-diffusion. *J. Comput. Appl. Math.*, 215(2):547–556, 2008.

[20] Mario Ricchiuto, Árpád Csík, and Herman Deconinck. Residual distribution for general time-dependent conservation laws. *J. Comput. Phys.*, 209(1):249–289, 2005.

[21] P.L. Roe. Characteristic-based schemes for the Euler equations. Annu. Rev. Fluid Mech. 18, 337-365 (1986)., 1986.

[22] N. Villedieu, T. Quintino, M. Ricchiuto, and H. Deconinck. A third order residual distribution scheme for the navier stokes equations. *J. Comput. Phys.*, 2010.

[23] N. Villedieu, T. Quintino, N. Villedieu, and H. Deconinck. A third order residual distribution scheme for the Navier Stokes equations. *J. Comput. Phys*, 2010.

[24] Martin Vymazal, Tiago Quintino, Nadège Villedieu, and Herman Deconinck. High–order upwind residual distribution schemes on isoparametric curved elements. *J. Comput. Phys.*, in press. doi:10.1016/j.jcp.2010.05.027.

A Hyperbolicity of (??).

We consider the matrix

$$M = \begin{pmatrix} A & D \\ \frac{1}{T} & 0 \end{pmatrix}$$

and S symmetric invertible such that SA is symmetric and SQ symmetric definite positive. There exists Q symmetric such that $Q^2 = SD$. The matrix $\begin{pmatrix} S & 0 \\ 0 & TQ^2 \end{pmatrix}$ is symmetric because S and TQ^2 are symmetric ($T \in \mathbb{R}$). Hence,

$$\begin{pmatrix} S & 0 \\ 0 & TQ^2 \end{pmatrix} \times \begin{pmatrix} A & \mathbf{D} \\ \frac{1}{T} & 0 \end{pmatrix} = \begin{pmatrix} SA & SD \\ Q^2 & 0 \end{pmatrix} = \begin{pmatrix} SA & Q^2 \\ Q^2 & 0 \end{pmatrix}$$

which is symmetric because both SA and Q are symmetric.

This shows that M is symetrisable, and the system (**??**)is hyperbolic.

B Implementation remark.

One of the difficulty of the scheme using reconstruction is that one needs to evaluate

$$\int_K \Big(\nabla_u f(u^h)\cdot\nabla v^h - \nabla(\mathbf{D}\nabla v^h)\Big)\Big(\nabla_u f(u^h)\cdot\nabla u^h - \nabla(\mathbf{D}\nabla u^h)\Big)dx$$

This is done using the same trick as in [3] to reduce the number of arithmetic operations. The quadrature points are some of the Lagrange points. The second difficulty is to evaluate $\nabla(\mathbf{D}\nabla u^h)$. To do this, we notice that if $u^h \in \mathbb{P}^k(K)$, then $\nabla u^h \in \mathbb{P}^{k-1}(K)$, so that

$$\nabla u^h = \sum_{\sigma\in K} \nabla u^h(\sigma)\varphi_\sigma$$

where the φ_σ are the Lagrange basis functions. Then $\mathbf{D}\nabla u^h$ is approximated with the right order by

$$\mathbf{D}\nabla u^h \approx \sum_{\sigma\in K} \Big(\mathbf{D}\nabla u^h\Big)(\sigma)\varphi_\sigma.$$

To evaluate pointwise $\nabla(\mathbf{D}\nabla u^h)$, it is enough to apply twice the algorithm to evaluate the gradient.

Centre de recherche INRIA Bordeaux – Sud Ouest
Domaine Universitaire - 351, cours de la Libération - 33405 Talence Cedex (France)

Centre de recherche INRIA Grenoble – Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier
Centre de recherche INRIA Lille – Nord Europe : Parc Scientifique de la Haute Borne - 40, avenue Halley - 59650 Villeneuve d'Ascq
Centre de recherche INRIA Nancy – Grand Est : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex
Centre de recherche INRIA Paris – Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex
Centre de recherche INRIA Rennes – Bretagne Atlantique : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex
Centre de recherche INRIA Saclay – Île-de-France : Parc Orsay Université - ZAC des Vignes : 4, rue Jacques Monod - 91893 Orsay Cedex
Centre de recherche INRIA Sophia Antipolis – Méditerranée : 2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399