A new algorithm for approaching Nash equilibrium and Kalai Smoridinsky solution

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RESEARCH ARTICLE

A new algorithm for approching Nash equilibrium and Kalai Smoridinsky solution

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In the present paper, a new formulation of Nash games is proposed for solving general multi-objective optimization problems. The main idea of this approach is to split the optimization variables which allow us to determine numerically the strategies between two players. The first player minimizes his cost function using the variables of the first table P, the second player, using the second table Q. The original contribution of this work concerns the construction of the two tables of allocations that lead to a Nash equilibrium on the Pareto front. On the other hand, we search P and Q that lead to a solution which is both a Nash equilibrium and a Kalai Smorodinsky solution. For this, we proposed and tried out successfully two algorithms which calculate P, Q and their associated Nash equilibrium, by using some extension of Normal Boundary Intersection approach (NBI).

Keywords: Multiobjective Optimization, Split of territories, Nash equilibrium, NBI approach, Kalai Smorodinsky solution.

1. Introduction

There exists several approaches to solve problems of multicriteria optimization [7], [14]. All these methods, until now, deal with the multidisciplinary problem by considering a kind of implicit weighting of all the disciplinary criteria. Another idea consists in assigning to each discipline its own criterion. This multicriterion problem can be solved by allowing to each criterion a weight [10] (a coefficient of substitution); we get back to a mono criterion problem. This approach has a serious disadvantage. The choice of the weights to be allowed to each criteria is arbitrary; this has an influence on the optimum reached. Another alternative which can be used to solve the multicriterion problems consists to identify the Pareto front [8], [4] which represent the set of not-dominated strategies. This approach is generally expensive since it needs a great number of evaluations of several criterions. The second difficulty is related to the choice of the best point on the Pareto front. The game theory defines another framework to solve the problems of multicriterion optimization. This theory was studied by Périaux [11], [12] and by J. A. Désidéri [6] as a powerful way to solve multidisciplinary optimization problems. In [9] Habbal et al. solved a multidisciplinary optimization problem using a non-cooperative game (Nash game), where the strategy of the players is naturally defined. Several

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multidisciplinary optimization problems arise in the form:

$$(M)\begin{cases}\min\limits_{y\in R^n} f_1(y),\\ \min\limits_{y\in R^n} f_2(y)\end{cases} \tag{1}$$

f_1 and f_2 two convex function.
To solve the problem (M), there is a lot of methods, weighted method, NBI...
In this paper, we propose to solve this problem by using the Nash equalibrium, since it is simple to calculate numerically the solution of problem (M). The split of the variable y amounts to construct two allocation tables P and Q in $\{0,1\}^n$, where $P_i + Q_i = 1$, $1 \le i \le n$. Let $I_{12} = \{1, ..., n\}$ be a set of indices of cardinality n, I_1 is a subset of I_{12} of cardinal $n-p$, and I_2 is its complement of cardinal p, that is to say $I_{12} = I_1 \cup I_2$.
Suppose that:

$$\begin{cases} U = (y_i), & \text{for } i \in I_1,\\ V = (y_i), & \text{for } i \in I_2.\end{cases} \tag{2}$$

We define in this case the integer allocation table P of size n:

$$P_i = 1, \forall i \in I_1,\ \ P_i = 0, \forall i \in I_2,$$

so that

$$y = P.y + (\mathcal{I} - P).y = (U, V) \qquad \text{where } \mathcal{I} = (1, ..., 1), \tag{3}$$

where "." denote the Hadamard product (i.e. $(P.y)_i = P_i y_i,\ P.y \in \mathbb{R}^n$), and (U, V) is defined in (2).
$(U^*, V^*) \in \mathbb{R}^{n-p} \times \mathbb{R}^p$ is a Nash equilibrium if and only if:

$$\begin{cases} f_1(U^*, V^*) = \min\limits_U f_1(U, V^*),\\ f_2(U^*, V^*) = \min\limits_V f_2(U^*, V).\end{cases} \tag{4}$$

Let's consider two positive convex functions f_1 and f_2, and the Nash game (5) which is written in the following form:

$$\begin{cases} \text{Find } y_{EN} \text{ solution of:}\\ \min\limits_U f_1(P.y + (\mathcal{I} - P).y_{EN}),\\ \min\limits_V f_2((\mathcal{I} - P).y + P.y_{EN}).\end{cases} \tag{5}$$

where $y_{EN} = (U^*, V^*)$ Consider the following fixed point problem (6):

$$\begin{cases} \text{Find } y_{EN} \text{ solution of :}\\ \min\limits_y f_1(P.y + (\mathcal{I} - P).y_{EN}) + f_2((\mathcal{I} - P).y + P.y_{EN}) = f_1(y_{EN}) + f_2(y_{EN}).\end{cases} \tag{6}$$

The allocation table P is fixed, and then strategies of each player are the variables corresponding to P and $\mathcal{I} - P$, i.e R^p and R^{n-p}. If y_{EN} is a solution of (6), then y_{EN} is a Nash Equilibrium of (5), and conversely. For the proof, it suffices, write

the optimality condition of problem (5) and (6)
For each choice of P, we find a Nash equilibrium, in this case, we have at most 2^n (where n is the size of y) Nash equilibria. The natural question is, how to choose among all these equilibriums the best Nash equilibrium. That means how to choose the best splitting of territories between the two players that gives an equilibrium belonging to the Pareto front if it exists, which is not always the case. Mixed allocations (the elements of P belonging to $[0, 1]$) are obtained by convexification of the set of pure ones. We also drop the mutual exclusivity constraint, to allow both players to share the same variable. In [1], Aboulaich et al proposes two heuristic algorithmes in order to split the territory. These algorithms allows to compute succesfly the Nash equilibrium, but the obtained equilibriums are not on the pareto front.
In this work we will test in the first part a splitting using P and $Q = \mathcal{I} - P$. In the second part we propose two algorithms NS1 and NS2. The first algorithm calculates the two tables P, Q and the Nash equilibrium associated. In such case, this equilibrium belongs to Pareto front, we use the strategy of Nash games coupled with an extension of the approach "Normal Boundary Intersection" NBI (NBI-Nash).
In the second one, we present a new technique to split the optimization variable y, of such kind the Nash equilibrium associated with this splitting is a solution of Kalai Smorodinsky [13]. The calculation of Kalai Smorodinsky solution, is the intersection between the Pareto front and the line joining the utopian point Ut and the disagreement point D.
In the following we recall brievely the NBI approch [5] and the splitting algorithm proposed in [2].

2. Preliminary result

Let x_i^* and f_i^* denote respectively the minimizer and minimum value of the f_i and let F^* denote the shadow minimum, i.e., the vector whose components are f_i^*. Consider the shifted pay-off matrix Φ whose i^th column is $F(x_i^*) - F^*$. The Convex Hull of Individual Minima or CHIM is defined as the set of points that are convex combinations of the columns of Φ , i.e., $\{\Phi\beta : \beta_i \geq 0, \sum_i \beta_i = 1\}$.
For a two dimensional problem illustrated in Figure 1, CHIM is represented by segment AB.
The idea behind NBI is to pick an even spread of points on the CHIM (for example W in Fig.1), and find the intersection point between the efficient frontier and a set of parallel normals emanating from the chosen set of points on the CHIM (C in Fig.1). Given a convex combination parameter vector β, and a normal direction n pointing towards the origin, the point of intersection between the normal emanating from $\Phi\beta$ and the efficient frontier can be found by solving the following NBI_β subproblem :

$$\begin{cases} \text{Maximize } t \\ \text{Subject to } : x \in A \\ \Phi\beta + tn = F(x) - F^* \end{cases} \tag{7}$$

where A is the set of feasible solution.

By solving subproblem NBI_β 7 for different settings of β, various points on

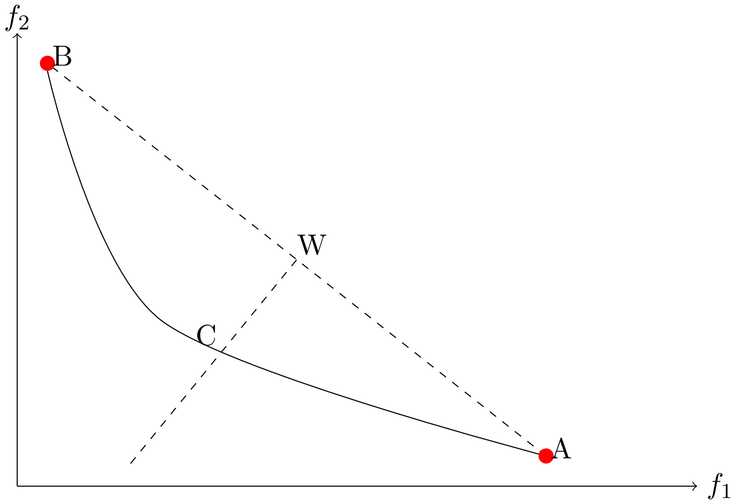

Figure 1. An illustrative integrated design

the efficient frontier can be generated. The advantage of the β parameter is that an even spread of β parameters corresponds to an even sped of points on the CHIM.

Aboulaich et al. [2] demonstrate the equivalence between the research of the Nash equilibrium of the problem (5) and fixed point of the problem (6), for values of P binary. This equivalence is true only if P is binary. In the following we propose an extension of the algorithm introduced in [2] to the non binary case. We search for the Nash equilibrium associated to two given tables of allocation P and $(\mathcal{I} - P)$ which are not necessarily binary, the elements of P belongs to the interval $[0, 1]$. In this case we solve the following problem:

$$\begin{cases} \text{Find } y_{EN} \text{ solution of:} \\ \min\limits_{y} f_1(P.y + (\mathcal{I} - P).y_{EN}^{k-1}), \Rightarrow y_{opt1}^k \\ \min\limits_{y} f_2((\mathcal{I} - P).y + P.y_{EN}^{k-1}), \Rightarrow y_{opt2}^k \\ \text{with the update: } \Rightarrow y_{EN}^k = P.y_{opt1}^k + (\mathcal{I} - P).y_{opt2}^k. \end{cases} \tag{8}$$

Algorithm (NS0):

(1) Initialization: y_{opt1}^0, y_{opt2}^0 and $y_{EN}^0 = P.y_{opt1}^0 + (\mathcal{I} - P).y_{opt2}^0$.
(2) for $k \geq 1$:
 a) solve the problem:

$$\min_{y} f_1(P.y + (\mathcal{I} - P).y_{EN}^{k-1}) \Rightarrow y_{opt1}^k$$

 b) solve the problem:

$$\min_{y} f_2((\mathcal{I} - P).y + P.y_{EN}^{k-1}) \Rightarrow y_{opt2}^k$$

 c) the update

$$y_{EN}^k = P.y_{opt1}^k + (\mathcal{I} - P).y_{opt2}^k, \text{ the update of } y_{EN}$$

$$y_{EN}^{k} = t y_{EN}^{k} + (1-t) y_{EN}^{k-1}, \text{ where } t \in]0,1], \text{ the relaxation of } y_{EN}$$

(3) while $||y_{EN}^{(k)} - y_{EN}^{(k-1)}|| > test$, set $k = k+1$, and repeat 2.

We present in the following, the results obtained by the algorithm $(NS0)$ for some tests, by considering two functions f_1 and f_2 defined by:

$$f_1(y) = \frac{1}{2}||Ay - b||^2 \text{ et } f_2(y) = \frac{1}{2}||Cy - d||^2, \; y \in R^{n\times 1}, \tag{9}$$

where A and C are two $n \times n$ matrices, b and d $n \times 1$ matrices, and $\| \, . \, \|$ is the Euclidean norm.
For arbitrary choice of P we find the following results:

$$b = [1; -2; 0; -1; 2]; d = [1; -3; -1; 3; 5]; A = C = tridiag[1; -2; 1].$$

$P = [0.1174; 0.2967; 0.3188; 0.4242; 0.5079]$ $P = [0.0005; 1; 0.9995; 0.0001; 0.9985]$

Figure 2. Test 1 and 2: The Nash overall loop converged in 21 iterations (left) and in 29 iterations (right)

$$b = rand(50, 1); d = rand(50, 1).$$

$A = C = Id$ $A = C = tridiag[1; -2; 1]$

$0.1 < P = rand(50, 1) < 0.99$ $0.1 < P = rand(50, 1) < 0.99$

Figure 3. Test 3 and 4: The Nash overall loop converged in 51 iterations (left) and in 103 iterations (right)

$$b = [1; -2; 0; -1; 2]; d = [1; -3; -1; 3; 5]; A = tridiag[1; -2; 1]; C = diagsup[1; -1].$$

$P = [0.2760; 0.6797; 0.6551; 0.1626; 0.1190]$ $P = [0.0005; 1; 0.9995; 0.0001; 0.9985]$

Figure 4. Test 5 and 6: The Nash overall loop converged in 205 iterations (left) and in 252 iterations (right)

$$b = 2rand(50, 1); d = 5rand(50, 1); A = tridiag[1; -2; 1]; C = diagsup[1; -1].$$

$0 < P = rand(50, 1) < 1$ $0 \cong P = rand(50, 1) \cong 1$

Figure 5. Test 7 and 8: The Nash overall loop converged in 520 iterations (left) and in 463 iterations (right)

According to the results obtained in the tests that we made, we note that, in the tests 1, 3 and 4 we have $A = C$ and the elements of P are not all close to 0 and 1 then the Nash equilibrium coincides with the Kalai Smorodinsky solution. In test 2, we have $A = C$ and elements of P are not far from 0 and 1, then the Nash equilibrium is not any more a Kalai Smorodinsky solution and it is not in the Pareto front but it's close to the minimum of f_1. In tests 5 and 7, we have $A \neq C$ and the elements of P are not all close to 0 and 1 then the Nash equilibrium is on the line passing through the point of disagreement and the utopia point. And in test 6 and 8, we have $A \neq C$ and elements of P are not far from 0 and 1, then the Nash equilibrium is not in the Pareto front.
In the next part we present two new algorithms in order construct the allocation tables P and Q.

3. Algorithm 1 (NS1): Nash equilibrium and NBI approach

The goal of this algorithm is to search among the Nash equilibria that are on the Pareto front, using an extension of the NBI approach [5]. NBI is a technique that seeks Part space which contains the Pareto optimal points. The idea behind NBI is to pick an even spread of points on the Convex Hull of Individual Minima (CHIM), and to find the intersection point between the efficient front and a set of parallel normals emanating from the chosen set of points on the CHIM. This point belongs to the set of the effective points which are on Pareto front. The pure allocation tables are any elements P and Q from $\{0,1\}^n$ that satisfy $P_i + Q_i = 1$ for $1 \leq i \leq n$. Mixed allocations are obtained by convexification of the set of pure ones. We also drop the mutual exclusivity constraint, to allow both players to share the same variable. To split the optimization variable, we construct two sequences of tables for allocation $P^{(m)}$ and $Q^{(m)}$ in $[0,1]^n$, using the approach proposed in [1] as the initialization step. We build $P^{(0)}$ and $Q^{(0)}$ using the iterations results giving by the iterative minimization of f_1 and f_2, the iteration consists in solving successively two optimization problems (M1) and (M2) by combining NBI and Nash games.
In the first step, we use a heuristic approach to construct the allocations tables. It is based on the observation of preferred directions of descent algorithm to optimize each functional separately. For example, the component P_i is the ratio of the number of times (relative to the total number of optimization iterations) where the direction j was used to reduce the test f_1.

Step1: Let $m = 0$, from an initial point $x^{(0)}$ et $y^{(0)} \in R^n$, we calculate $P^{(0)}$ and $Q^{(0)}$ by :

$$\begin{cases} \min\limits_{x \in R^n} f_1(x), & x^{(k+1)} = x^{(k)} - \rho_k \nabla f_1(x^{(k)}), \quad k \geq 0, \\ & P_j^{(0)} = \dfrac{\sum\limits_k |x_j^{(k+1)} - x_j^{(k)}|}{\sum\limits_k \|x^{(k+1)} - x^{(k)}\|}, \\ \min\limits_{y \in R^n} f_2(y), & y^{(k+1)} = y^{(k)} - \rho_k \nabla f_2(y^{(k)}), \quad k \geq 0, \\ & Q_j^{(0)} = \dfrac{\sum\limits_k |y_j^{(k+1)} - y_j^{(k)}|}{\sum\limits_k \|y^{(k+1)} - y^{(k)}\|}, \end{cases} \tag{10}$$

set,

$$y_{EN}^{(0)} = P^{(0)}.x^* + Q^{(0)}.y^*, \ F(x) = (f_1(x), f_2(x))^T \text{ and } F^* = (f_1(x^*), f_2(y^*))^T \tag{11}$$

where,

$$\begin{cases} x^* = Arg\min\limits_x f_1(x) \\ y^* = Arg\min\limits_y f_2(y). \end{cases} \tag{12}$$

Step2: For $m > 0$, solve,

$$(M1)\begin{cases} \max\limits_{x,t,\beta,P} t, \\ s.c\ F(P.x + Q^{(m-1)}.y_{EN}^{(m-1)}) = F^* + tn + \Phi\beta, \end{cases} \tag{13}$$

and,

$$(M2)\begin{cases} \max\limits_{y,t,\beta,Q} t, \\ s.c\ F(Q.y + P^{(m-1)}.y_{EN}^{(m-1)}) = F^* + tn + \Phi\beta. \end{cases} \tag{14}$$

$$y_{EN}^{(m)} = P^{(m)}.x_{opt}^{(m)} + Q^{(m)}.y_{opt}^{(m)},$$

where $x_{opt}^{(m)}$ (resp $P^{(m)}$) is a solution of the problem (M1)with respect to x (resp P), and $y_{opt}^{(m)}$ (resp $Q^{(m)}$) is a solution of the problem (M2) with respect to y (resp Q).

While $||y_{EN}^{(m)} - y_{EN}^{(m-1)}|| > test$, pose $m = m + 1$, and repeat ***Step2***.

In the following we present the results obtained by the algorithm (NS1) for some tests.

$b = [1; -2; 2; 0; -1]; d = [5; 1; -3; -1; 3]$ $\qquad$ $b = [10; -2; 2; \frac{1}{2}; -1], d = [6; 1; 9; -1; 3]$

$A = C = Id$ $\qquad$ $A = tridiag[1; -2; 1], C = triangsup[-1; 1]$

Figure 6. Test 1 and 2: The Nash overall loop converged in 25 iterations (left) and in 7 iterations (right)

Several other tests have been made ($n = 20$, $n = 50...$), the results show that the algorithm (NS1) numerically converges to a Nash equilibrium on the Pareto Front for functions defined by (9).

4. Algorithm 2 (NS2)

In this section, we present a new technique to split the optimization variable y using the two tables P and $(\mathcal{I} - P)$ and the algorithm of Kalai Smorodinsky [13]. This technique is based on the calculation of the utopian point, the disagreement point and the Nash equilibrium associated to P.

$b = 8rand(10,1); d = 5rand(10,1)$ $\qquad$ $b = eye(10,1), d = rand(10,1)$

$A = tridiag[1;-2;1], C = triangsup[-1;1]$ $\qquad$ $A = C = Id$

Figure 7. Test 3 and 4: The Nash overall loop converged in 17 iterations (left) and in 32 iterations (right)

We are looking at each iteration for the Nash equilibrium associated to the allocation table calculated, while approaching the intersection between the Pareto front and the line joining the utopian point Ut and the disagreement point D. Note,

$$Ut = \begin{pmatrix} f_1(x^*) \\ f_2(y^*) \end{pmatrix}, \; D = \begin{pmatrix} f_1(y^*) \\ f_2(x^*) \end{pmatrix} \text{ and } \tau = \frac{Ut - D}{\| Ut - D \|},$$

where

$$\begin{cases} x^* = Arg \min\limits_{x} f_1(x) \\ y^* = Arg \min\limits_{y} f_2(y). \end{cases} \tag{15}$$

We look for the splitting of the optimization variable y (we search table P) in order that the Nash equilibrium coincide with the Kalai Smorodinsky solution, we propose the following algorithm:

(1) Initialization, $m = 0$: ***Step1***. of (NS1).
(2) For $m > 0$, solve the problem

$$(KS1) \begin{cases} \max\limits_{y,t,P} t, \\ s.c \; F(P.y + (\mathcal{I} - P).y_{EN}^{(m-1)}) = D + t\tau, \end{cases} \tag{16}$$

$$y_{EN}^{(m)} = P^{(m)}.K_{opt}^{(m)} + (\mathcal{I} - P^{(m)}).y_{EN}^{(m-1)},$$

where $K_{opt}^{(m)}$ (resp $P^{(m)}$) is a solution of the problem (KS1) over y (resp P) and F is defined in 11.
(3) While

$$||y_{EN}^{(m)} - y_{EN}^{(m-1)}|| \geq test,$$

pose $m = m + 1$, and repeat 2.

In the following we present the results obtained by the algorithm (NS1) for some tests.

$b = [1; -2; 2; 0; -1]; d = [5; 1; -3; -1; 3]$ $b = [1; -2; 2; 0; -1]; d = [5; 1; -3; -1; 3];$

$A = C = Id$ $A = C = tridiag[1; -2; 1]$

Figure 8. Test 1 and 2: The Kalai Smorodinsky solution overall loop converged in 9 iterations (left) and in 4 iterations (right)

$b = rand(10, 1); d = rand(10, 1)$ $b = rand(10, 1); d = rand(10, 1)$

A=tridiag[1;-2;1] , C=triangsup[-1;1] $A = C = Id$

Figure 9. Test 3 and 4: The Kalai Smorodinsky solution overall loop converged in 6 iterations (left) and in 11 iterations (right)

According to the obtained results, we note that the Kalai Smorodinsky solution (KS) is determined as a Nash equilibrium. The proposed algorithm allows to construct two allocation tables P and Q, and a Nash equilibrium wich that is a Kalai Smorodinsky solution.

5. Conclusion

In this paper a new approach is proposed to solve a multi-criteria optimization problem using a game theory using a new approach for the splitting of territory in the case of concurrent optimization. The first algorithm, permits to compute the Nash equilibria that are on the Pareto front. While the second one determine the Nash equilibrium as a Kalai Smorodinsky solution. The numerical examples confirm that our algorithms has powerful ability to find the Nash equilibrium.

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