# Regular solutions of a problem coupling a compressible fluid and an elastic structure

Muriel Boulakia, Sergio Guerrero

# Regular solutions of a problem coupling a compressible fluid and an elastic structure

M. Boulakia* and S. Guerrero*

**Abstract**

We are interested by the three-dimensional coupling between a compressible viscous fluid and an elastic structure immersed inside the fluid. They are contained in a fixed bounded set. The fluid motion is modelled by the compressible Navier-Stokes equations and the structure motion is described by the linearized elasticity equation.
We establish the local in time existence and the uniqueness of regular solutions for this model. We emphasize that the equations do not contain extra regularizing term. The result is proved by first introducing a problem linearized and by proving that it admits a unique regular solution. The regularity is obtained thanks to successive estimates on the unknowns and their derivatives in time and thanks to elliptic estimates. At last, a fixed point theorem allows to prove the existence and uniqueness of regular solution of the nonlinear problem.

**Résumé**

Nous étudions le couplage entre un fluide visqueux compressible et une structure élastique évoluant à l'intérieur en dimension 3. L'ensemble se trouve dans une cavité fixe bornée. Le fluide est décrit par les équations de Navier-Stokes compressible et la structure est décrite par l'équation d'élasticité linéarisée.
Nous montrons l'existence locale en temps et l'unicité de solutions régulières pour ce modèle. Nous soulignons le fait que les équations ne sont pas régularisées par des termes supplémentaires. Le résultat est prouvé en considérant tout d'abord un problème linéarisé pour lequel on montre l'existence et l'unicité de solutions régulières. La régularité est obtenue grâce à des estimations successives sur les inconnues et leurs dérivées en temps et grâce à des estimations elliptiques. Un argument de point fixe permet ensuite d'avoir l'existence et l'unicité de solution régulière du problème non linéaire.

# 1 Introduction

## 1.1 Statement of problem

We are interested by a coupled system involving an elastic structure and a surrounding viscous compressible fluid. We suppose that the structure and the fluid move in a fixed connected bounded domain $\Omega \subset \mathbb{R}^3$. At time $t$, we denote by $\Omega_S(t)$ the domain occupied by the solid and by $\Omega_F(t) = \Omega \setminus \overline{\Omega_S(t)}$ the domain occupied by the fluid. We suppose that the boundaries of $\Omega_S(0)$ and $\Omega$ are smooth ($C^4$ for instance). The fluid velocity $u$ and the fluid density $\rho$ are described by the compressible Navier-Stokes equations: $\forall\, t \in (0,T)$, $\forall\, x \in \Omega_F(t)$,

$$\left\{\begin{array}{l} \big(\rho_t + \nabla\cdot(\rho u)\big)(t,x) = 0, \\ \big(\rho u_t + \rho(u\cdot\nabla)u\big)(t,x) - \nabla\cdot\big(2\mu\epsilon(u) + \mu'(\nabla\cdot u)\mathrm{Id} - p\mathrm{Id}\big)(t,x) = 0, \end{array}\right. \tag{1}$$

where $(\epsilon(u))_{ij} = \frac{1}{2}(\nabla u + \nabla u^t)_{ij} = \frac{1}{2}(\partial_j u_i + \partial_i u_j)$ denotes the symmetric part of the gradient. The viscosity coefficients $\mu$ and $\mu'$ are real constants which are supposed to satisfy

$$\mu > 0,\ \mu' \geqslant 0. \tag{2}$$

*Université Pierre et Marie Curie-Paris 6, UMR 7598 Laboratoire Jacques-Louis Lions, Paris, F-75005 France. E-mails: boulakia@ann.jussieu.fr (corresponding author), guerrero@ann.jussieu.fr

Moreover, we assume that the pressure $p$ only depends on $\rho$ and is given by

$$p = P(\rho) - P(\overline{\rho}) \tag{3}$$

where $P \in C^{\infty}(\mathbb{R}_{+}^{*})$ and $\overline{\rho} > 0$ is a constant.
Many works have been dedicated to study of the Navier-Stokes compressible equations. We only quote a small part of them and refer to the books [20] and [12] for a complete presentation on the compressible fluids. A result of local in time existence and uniqueness of regular solutions has been proved in [23]. In the case of isentropic fluids (i.e. when $P(\rho) = \rho^{\gamma}$ with $\gamma > 0$), the papers [17] for $\gamma = 1$ and [18] for $\gamma > 1$ show the global existence of a weak solution for small initial data. The first global existence result for large data was proved in [19] with $\gamma \geqslant 9/5$ for dimension $N = 3$ and with $\gamma > N/2$ for $N \geqslant 4$. The conditions on the coefficient $\gamma$ have been relaxed in [10] where it is assumed that $\gamma > N/2$ for $N \geqslant 3$.

The motion of the structure is described by its elastic displacement $\xi$ defined on the initial domain $\Omega_S(0)$. We represent the evolution of $\xi$ by the linearized elasticity equation: $\forall\,(t, y) \in (0, T) \times \Omega_S(0)$,

$$\xi_{tt}(t, y) - \nabla \cdot \big(2\lambda\epsilon(\xi) + \lambda'(\nabla \cdot \xi)\mathrm{Id}\big)(t, y) = 0. \tag{4}$$

Here, we have considered, without lack of generality that the solid density is equal to 1. We assume that Lame's constants satisfy:

$$\lambda > 0, \ \ \lambda' \geqslant 0. \tag{5}$$

In fact, these conditions can be relaxed by $\lambda > 0$ and $3\lambda' + 2\lambda > 0$ (and the same holds for the viscosity coefficients $\mu$ and $\mu'$.)

We now introduce the flow $\chi(t, \cdot) : \Omega_F(0) \to \Omega_F(t)$ on the fluid domain which associates to the lagrangian coordinate of a fluid particle its eulerian coordinate. For all $y \in \Omega_F(0)$, the flow $\chi(\cdot, y)$ is solution of the following ODE:

$$\left\{ \begin{aligned} & \chi_t(t, y) = u(t, \chi(t, y)) \\ & \chi(0, y) = y. \end{aligned} \right. \tag{6}$$

This definition allows to make the link between the lagrangian point of view on the structure and the eulerian point of view on the fluid.
The structure and fluid motions are coupled on the interface. Since the fluid is viscous, the velocity at the interface is supposed to be continuous. Moreover, due to the law of reciprocal actions, the normal component of the stress tensors is also supposed to be continuous. Thus, on $(0, T) \times \partial\Omega_S(0)$, we have

$$\left\{ \begin{aligned} & u \circ \chi = \xi_t \\ & \big(2\mu\epsilon(u) + \mu'(\nabla \cdot u)\mathrm{Id} - (P(\rho) - P(\overline{\rho}))\mathrm{Id}\big) \circ \chi \,\mathrm{cof}\, \nabla\chi\, \mathrm{n} = \big(2\lambda\epsilon(\xi) + \lambda'(\nabla \cdot \xi)\mathrm{Id}\big)\mathrm{n}, \end{aligned} \right. \tag{7}$$

where n is the outward unit normal defined on $\partial\Omega_S(0)$. The system is complemented with Dirichlet condition on the external boundary:

$$u = 0 \text{ on } (0, T) \times \partial\Omega. \tag{8}$$

Observe that $(\overline{\rho}, 0, 0)$ is a stationary solution of system (1), (4) and (7)-(8).
Finally, we introduce the initial conditions

$$\rho(0, \cdot) = \rho_0 \text{ in } \Omega_F(0),\ u(0, \cdot) = u_0 \text{ in } \Omega_F(0) \tag{9}$$

and

$$\xi(0, \cdot) = \xi_0 \text{ in } \Omega_S(0),\ \xi_t(0, \cdot) = \xi_1 \text{ in } \Omega_S(0) \tag{10}$$

which satisfy

$$\rho_0 \in H^3(\Omega_F(0)),\ \ \rho_0 \geqslant \rho_{min} > 0 \text{ in } \Omega_F(0),\ u_0 \in H^4(\Omega_F(0)),\ \xi_0 \in H^3(\Omega_S(0)),\ \xi_1 \in H^2(\Omega_S(0)). \tag{11}$$

together with the following compatibility conditions:

$$
\begin{cases}
u_0 = 0 & \text{on } \partial\Omega, \\
u_0 = \xi_1 & \text{on } \partial\Omega_S(0), \\
(2\mu\epsilon(u_0) + \mu'(\nabla \cdot u_0)\mathrm{Id} - (P(\rho_0) - P(\overline{\rho}))\mathrm{Id})n = (2\lambda\epsilon(\xi_0) + \lambda'(\nabla \cdot \xi_0)\mathrm{Id})n & \text{on } \partial\Omega_S(0), \\
\nabla \cdot (2\mu\epsilon(u_0) + \mu'(\nabla \cdot u_0)\mathrm{Id} - (P(\rho_0) - P(\overline{\rho}))\mathrm{Id}) = \rho_0 \nabla \cdot (2\lambda\epsilon(\xi_0) + \lambda'(\nabla \cdot \xi_0)\mathrm{Id}) & \text{on } \partial\Omega_S(0).
\end{cases}
\tag{12}
$$

Different kinds of fluid-structure interaction problems have been studied in the literature.
A large number of studies deal with an incompressible fluid modelled by the incompressible Navier-Stokes equations. For the coupling of an incompressible fluid with a rigid structure, we mention [14] which shows the local in time existence of weak solutions and papers [6] and[8] (with variable density) which prove the global existence of weak solutions. By 'global existence', we mean that the solution exists until collisions between the structure and the external boundary or between two structures. Paper [21] proves the global existence of weak solutions beyond collisions and [22] proves the existence and uniqueness of strong solutions (global in 2D and local in 3D). At last, [15] and [16] study the lack of collision in 2D or 3D.
For the coupling between an incompressible fluid and an elastic structure, the existence of global weak solutions is proved in [9] when the elastic structure is given by a finite sum of modes and in [3] with a regularizing term in the structure motion. These two results give the existence of solutions defined as long as there is no collision between the structure and the boundary and as long as no interpenetration occurs in the structure. The local existence of regular solutions is proved in [7]. Moreover, the coupling with an elastic plate has also been studied: we quote [1] where the existence of local strong solution is obtained, [5] which proves the existence of global weak solution with a regularizing term in the plate equation and [13] which proves the same result without regularizing term in 2D.
Concerning compressible fluids, the global existence of weak solutions for the interaction with a rigid structure is obtained in [8] (with $\gamma \geqslant 2$) and in [11] (with $\gamma > N/2$). Moreover, in [4], the existence of global regular solutions is proved for small initial data. At last, for the interaction between a compressible fluid and an elastic structure, [2] proves the global existence of a weak solution in $3D$ for $\gamma > 3/2$. The result is obtained for an elastic structure described by a regularized elasticity equation. In our paper, we do not need this regularizing term anymore and we obtain the existence and uniqueness of regular solutions for system (1), (3), (4) (7), (8). More precisely, we will prove the following theorem

**Theorem 1** *Let $(\rho_0, u_0, \xi_0, \xi_1)$ satisfy (11)-(12). Then, there exists $T^* > 0$ such that the system of equations (1), (3), (4) complemented with the boundary conditions (7)-(8) and the initial conditions (9)-(10) admits a unique solution $(\rho, u, \xi)$ defined in $(0, T^*)$ and belonging to the space $Y_{T^*} := Y^1_{T^*} \times Y^2_{T^*} \times Y^3_{T^*}$, where*

$$Y^1_{T^*} := L^2(0, T^*; H^2(\Omega_F(t))) \cap C^0([0, T^*]; H^{7/4}(\Omega_F(t))),$$

$$Y^2_{T^*} := L^2(0, T^*; H^3(\Omega_F(t))) \times H^2(0, T^*; H^1(\Omega_F(t))) \cap C^2([0, T^*]; L^2(\Omega_F(t))) \cap C^0([0, T^*]; H^{11/4}(\Omega_F(t))),$$

$$Y^3_{T^*} := C^0([0, T^*]; H^3(\Omega_S(0))) \cap C^2([0, T^*]; H^1(\Omega_S(0))) \cap C^3([0, T^*]; L^2(\Omega_S(0))).$$

*Moreover, there exists a non-decreasing positive function $f$ depending on $1/\rho_{min}$, $\|\rho_0\|_{\Omega_F(0)}$, $\|u_0\|_{H^4(\Omega_F(0))}$, $\|\xi_0\|_{H^3(\Omega_S(0))}$ and $\|\xi_1\|_{H^2(\Omega_S(0))}$ such that*

$$\|(\rho, u, \xi)\|_{Y_{T^*}} \leqslant f(1/\rho_{min}, \|\rho_0\|_{\Omega_F(0)}, \|u_0\|_{H^4(\Omega_F(0))}, \|\xi_0\|_{H^3(\Omega_S(0))}, \|\xi_1\|_{H^2(\Omega_S(0))}).$$

To prove this result, we will linearize our problem with respect to the fluid velocity, prove a regularity result for this problem and then use a fixed point argument. In the next subsection, we introduce the intermediate problem which is partially linearized with the help of a given fluid velocity.

**Remark 2** *One can prove that*

$$(\rho, u) \in Y^1_{T^*} \times Y^2_{T^*}$$

*is equivalent to*

$$(\rho \circ \chi, u \circ \chi) \in X^1_{T^*} \times X^2_{T^*},$$

*where*
$$X^1_{T^*} := L^2(0,T^*; H^2(\Omega_F(0))) \cap C^0([0,T^*]; H^{7/4}(\Omega_F(0))),$$
*and*
$$X^2_{T^*} := L^2(0,T^*; H^3(\Omega_F(0))) \cap H^2(0,T^*; H^1(\Omega_F(0))) \cap C^2([0,T^*]; L^2(\Omega_F(0))) \cap C^0([0,T^*]; H^{11/4}(\Omega_F(0))).$$
*Moreover, there exists $C > 0$ such that*
$$\|(\rho,u)\|_{Y^1_{T^*}\times Y^2_{T^*}} \leqslant C(\|(\rho\circ\chi, u\circ\chi)\|_{X^1_{T^*}\times X^2_{T^*}} + \|(\rho\circ\chi, u\circ\chi)\|^4_{X^1_{T^*}\times X^2_{T^*}}).$$

## 1.2 A problem linearized with respect to the fluid velocity

Let $(\rho_0, u_0, \xi_0, \xi_1)$ satisfy (11)-(12). We introduce the following notations: for all $t > 0$, we define
$$Q_t = (0,t) \times \Omega_F(0),\ \Sigma_t = (0,t) \times \partial\Omega_S(0).$$
For all $p, r \geqslant 0$ and $q, s \in [1, +\infty]$, we denote by $W^{p,q}(W^{r,s})$ the space $W^{p,q}(0,T; W^{r,s}(\Omega_F(0)))$. Then, we define the following fixed point space, for all $R > 0$
$$X_{T,R} = \{v \in Y_{p,q},\ v = 0 \text{ on } (0,T)\times\partial\Omega,\ v(0) = u_0 \text{ in } \Omega_F(0) \text{ and } N_T(v) \leqslant R\}, \tag{13}$$
where
$$Y_{p,q} := W^{2,p}(H^1) \cap W^{2,q}(L^2) \cap L^q(H^{11/4}) \cap L^p(H^3),$$
for $1 < p < 2$ and $4 < q < \infty$ and $N_T$ is the natural norm associated to $X_{T,R}$:
$$N_T(v) = \|v\|_{W^{2,p}(H^1)} + \|v\|_{W^{2,q}(L^2)} + \|v\|_{L^q(H^{11/4})} + \|v\|_{L^p(H^3)}.$$
Let $0 < T < 1$ and let $\hat{v}$ be a lagrangian velocity given in $X_{T,R}$ with $0 < R < 1$ specified later. We will use this velocity to linearize our problem. Let us now define the flow $\hat{\chi}$ by
$$\hat{\chi}(t,y) = y + \int_0^t \hat{v}(s,y)\,ds \quad \forall y \in \Omega_F(0). \tag{14}$$
We notice that
$$\|\hat{\chi}\|_{W^{3,p}(H^1)} + \|\hat{\chi}\|_{W^{2,\infty}(H^1)} + \|\hat{\chi}\|_{W^{3,q}(L^2)} + \|\hat{\chi}\|_{W^{1,q}(H^{11/4})} + \|\hat{\chi}\|_{L^\infty(H^{11/4})} \leqslant C \tag{15}$$
and
$$\|\nabla\hat{\chi} - \mathrm{Id}\|_{W^{1,q}(H^{7/4})} \leqslant \|\hat{v}\|_{L^q(H^{11/4})} \leqslant R. \tag{16}$$
Here, and in the following, $C$ represents a constant which only depends on the domain $\Omega_F(0)$. Since $W^{1,q}(H^{7/4})$ is an algebra (i.e. $g \in W^{1,q}(H^{7/4})$, $h \in W^{1,q}(H^{7/4}) \Rightarrow gh \in W^{1,q}(H^{7/4})$), we have, for $R$ small enough,
$$\|(\nabla\hat{\chi})^{-1} - \mathrm{Id}\|_{W^{1,q}(H^{7/4})} \leqslant CR. \tag{17}$$
Consequently, for $R$ small enough, $\hat{\chi}(t,\cdot)$ is invertible from $\Omega_F(0)$ onto $\widehat{\Omega}_F(t) = \hat{\chi}(t, \Omega_F(0))$ for all $t \in (0,T)$. Thus, we can define the eulerian velocity $\hat{u}$ by
$$\hat{u}(t,x) := \hat{v}(t, \hat{\chi}^{-1}(t,x)),\ \forall t \in (0,T),\ \forall x \in \widehat{\Omega}_F(t) = \hat{\chi}(t, \Omega_F(0)).$$
We can now linearize partially the system satisfied by $(\rho, u)$. We consider the following problem: for all $t \in (0,T)$
$$\begin{cases} \big(\rho_t + \nabla\cdot(\rho\hat{u})\big)(t,x) = 0,\ \forall\, x \in \widehat{\Omega}_F(t), \\ \big(\rho u_t + \rho(\hat{u}\cdot\nabla)u\big)(t,x) - \nabla\cdot\big(2\mu\epsilon(u) + \mu'(\nabla\cdot u)\mathrm{Id} - (P(\rho) - P(\overline{\rho}))\mathrm{Id}\big)(t,x) = 0,\ \forall\, x \in \widehat{\Omega}_F(t). \end{cases} \tag{18}$$

We complement this system by the equation (4) satisfied by $\xi$ and by the boundary conditions (8) and on $\Sigma_T$,

$$\begin{cases} u \circ \hat{\chi} = \xi_t, \\ \big(2\mu\epsilon(u) + \mu'(\nabla\cdot u)\mathrm{Id} - (P(\rho) - P(\overline{\rho}))\mathrm{Id}\big) \circ \hat{\chi}\,\mathrm{cof}\,\nabla\hat{\chi}\,\mathrm{n} = \big(2\lambda\epsilon(\xi) + \lambda'(\nabla\cdot\xi)\mathrm{Id}\big)\mathrm{n}. \end{cases} \tag{19}$$

The system is still non-linear since it is not linearized with respect to the fluid density. This system can be written on the reference domain $\Omega_F(0)$ with the help of the flow $\hat{\chi}$. Let us define, for all $(t,y) \in Q_T$

$$v(t,y) = u(t,\hat{\chi}(t,y)),\ \gamma(t,y) = \rho(t,\hat{\chi}(t,y)) - \overline{\rho}. \tag{20}$$

The first equation of system (18) becomes

$$\gamma_t + \gamma(\nabla\hat{v}(\nabla\hat{\chi})^{-1} : \mathrm{Id}) + \overline{\rho}(\nabla\hat{v}(\nabla\hat{\chi})^{-1} : \mathrm{Id}) = 0 \text{ in } Q_T. \tag{21}$$

Next, the second equation of system (18) becomes

$$\begin{aligned}(\overline{\rho}+\gamma)\det\nabla\hat{\chi}v_t - \nabla\cdot\Big[\big(\mu(\nabla v(\nabla\hat{\chi})^{-1} + (\nabla\hat{\chi})^{-t}\nabla v^t) + \mu'(\nabla v(\nabla\hat{\chi})^{-1} : \mathrm{Id})\mathrm{Id} \\ - (P(\overline{\rho}+\gamma) - P(\overline{\rho}))\mathrm{Id}\big)\mathrm{cof}\,\nabla\hat{\chi}\Big] = 0 \text{ in } Q_T.\end{aligned} \tag{22}$$

At last, on $\Sigma_T$, we have

$$\begin{cases} v = \xi_t, \\ \big(\mu(\nabla v(\nabla\hat{\chi})^{-1} + (\nabla\hat{\chi})^{-t}\nabla v^t) + \mu'(\nabla v(\nabla\hat{\chi})^{-1} : \mathrm{Id})\mathrm{Id} - (P(\overline{\rho}+\gamma) - P(\overline{\rho}))\mathrm{Id}\big)\mathrm{cof}\,\nabla\hat{\chi}\,\mathrm{n} \\ = \big(2\lambda\epsilon(\xi) + \lambda'(\nabla\cdot\xi)\mathrm{Id}\big)\mathrm{n} \end{cases} \tag{23}$$

and the intial conditions satisfied by $(\gamma, v)$ are

$$\gamma(0,\cdot) = \rho_0 - \overline{\rho} \text{ in } \Omega_F(0),\ v(0,\cdot) = u_0 \text{ in } \Omega_F(0). \tag{24}$$

We introduce the following fixed point mapping:

$$\Lambda : \quad \hat{v} \in X_{T,R} \quad \rightarrow \quad v \tag{25}$$

where $v$, together with $\gamma$ and $\xi$, is solution of system (4), (21), (22) and (23) with the initial conditions (10) and (24). First, we will prove that $\Lambda$ maps from $X_{T,R}$ in $X_{T,R}$ for $T, R > 0$ small enough. More precisely, we will prove that, for $\hat{v} \in X_{T,R}$, $v = \Lambda(\hat{v})$ belongs to

$$\overline{X}_T := C^2([0,T];L^2) \cap H^2(H^1) \cap C^0([0,T];H^{11/4}) \cap L^2(H^3)$$

and satisfies

$$\overline{N}_T(v) = \|v\|_{H^2(H^1)} + \|v\|_{W^{2,\infty}(L^2)} + \|v\|_{L^\infty(H^{11/4})} + \|v\|_{L^2(H^3)} \leqslant M, \tag{26}$$

where $M$ is a constant which depends on the norms of the initial conditions. Since $N_T(v) \leqslant T^\alpha \overline{N}_T(v)$ with $\alpha > 0$, this inequality allows to obtain that, for $T$ small enough, $v$ belongs to $X_{T,R}$.

# 2 Regularity results for the partially linearized problem

## 2.1 Regularity of the density

Since the equation (21) satisfied by $\gamma$ is decoupled from the other variables $v$ and $\xi$, we can obtain a first regularity result independently from the other equations.

**Lemma 3** *Let $1 < p < 2$, $4 < q < \infty$, $0 < T < 1$, $0 < R < 1$ and $\hat{v} \in X_{T,R}$. Then, the solution $\gamma$ of (21) belongs to $W^{1,q}(H^{7/4}) \cap W^{2,q}(L^2) \cap L^2(H^2)$ and*

$$\begin{cases} \|\gamma\|_{W^{1,q}(H^{7/4})} + \|\gamma\|_{W^{2,q}(L^2)} + \|\gamma\|_{L^2(H^2)} + \|\gamma\|_{L^\infty(H^{7/4})} \\ \qquad\qquad \leqslant C(\|\gamma(0)\|_{H^2} + \|u_0\|_{H^1} + \|\gamma(0)\|^2_{H^2} + \|u_0\|^2_{H^1} + R). \end{cases} \tag{27}$$

*Moreover, for $R > 0$ small enough, there exists $\gamma_{min} > -\overline{\rho}$ such that, on $Q_T$*

$$\gamma \geqslant \gamma_{min}. \tag{28}$$

**Proof:** Equation (21) can be written as

$$\gamma_t + \gamma\hat{z} = -\overline{\rho}\hat{z} \text{ in } Q_T \tag{29}$$

where $\hat{z} = \nabla\hat{v}(\nabla\hat{\chi})^{-1} : \mathrm{Id}$. Thus, $\gamma$ is explicitely given by, for all $t \in (0,T)$

$$\gamma(t) = -\overline{\rho}\int_0^t \hat{z}(s)\exp\left(\int_t^s \hat{z}(r)\,dr\right)\,ds + \gamma(0)\exp\left(-\int_0^t \hat{z}(s)\,ds\right) \text{ in } \Omega_F(0). \tag{30}$$

Thus, according to (17) and the definition of $X_{T,R}$ (see (13)), $\hat{z}$ belongs to $L^q(H^{7/4}) \cap L^p(H^2)$ and $\|\hat{z}\|_{L^q(H^{7/4})} + \|\hat{z}\|_{L^p(H^2)} \leqslant CR$. Moreover, since $H^{7/4}$ is an algebra and since $R<1$, we have

$$\left\|\int_0^t \hat{z}(s)\exp\left(\int_t^s \hat{z}(r)\,dr\right)\,ds\right\|_{L^\infty(H^{7/4})\cap L^2(H^2)} \leqslant CR\exp(CR) \leqslant CR$$

and

$$\left\|\gamma(0)\exp\left(-\int_0^t \hat{z}(s)\,ds\right)\right\|_{L^\infty(H^{7/4})\cap L^2(H^2)} \leqslant C\|\gamma(0)\|_{H^2}\exp(CR) \leqslant C\|\gamma(0)\|_{H^2}.$$

This implies that

$$\|\gamma\|_{L^\infty(H^{7/4})\cap L^2(H^2)} \leqslant C(\|\gamma(0)\|_{H^2} + R). \tag{31}$$

Next, the regularity of $\gamma_t$ in $L^q(H^{7/4})$ is readily obtained from equation (29):

$$\|\gamma_t\|_{L^q(H^{7/4})} \leqslant C(\|\gamma(0)\|_{H^2} + R). \tag{32}$$

To prove that $\gamma_{tt}$ belongs to $L^q(L^2)$, we differentiate this equation with respect to time and we obtain:

$$\gamma_{tt} = -\gamma_t\hat{z} - \gamma\hat{z}_t - \overline{\rho}\hat{z}_t.$$

We have the following estimates

$$\|\hat{z}\|_{L^\infty(L^2)} \leqslant C\|\hat{v}\|_{L^\infty(H^1)} \leqslant C(\|u_0\|_{H^1} + \|\hat{v}\|_{W^{1,q}(H^1)}) \leqslant C(\|u_0\|_{H^1} + R) \tag{33}$$

and

$$\|\hat{z}_t\|_{L^q(L^2)} \leqslant C(\|\hat{v}\|_{W^{1,q}(H^1)} + \|\hat{v}\|_{L^\infty(H^1)}\|\hat{v}\|_{L^q(W^{1,\infty})}) \leqslant CR(1+\|u_0\|_{H^1}). \tag{34}$$

Thus, using (31)-(34), we get

$$\begin{aligned}
\|\gamma_{tt}\|_{L^q(L^2)} &\leqslant C\|\gamma_t\|_{L^q(L^\infty)}\|\hat{z}\|_{L^\infty(L^2)} + C\|\hat{z}_t\|_{L^q(L^2)}(1+\|\gamma\|_{L^\infty(L^\infty)}) \\
&\leqslant C(\|\gamma(0)\|_{H^2} + R)(\|u_0\|_{H^1} + R) + CR(1+\|u_0\|_{H^1})(1+\|\gamma(0)\|_{H^{7/4}}).
\end{aligned}$$

Thus, (27) holds.
To prove (28), we remark (by considering the equation satisfied by $\overline{\rho}+\gamma$) that the formula satisfied by $\gamma$ can also be written as follows:

$$\gamma(t) = \rho_0\exp\left(-\int_0^t \hat{z}(s)\,ds\right) - \overline{\rho} \text{ in } \Omega_F(0),$$

where $\rho_0$ was introduced in (9). We have

$$\exp\left(-\int_0^t \hat{z}(s)\,ds\right) \geqslant \exp\left(-\|\nabla\hat{v}\|_{L^1(L^\infty)}\|(\nabla\hat{\chi})^{-1}\|_{L^\infty(L^\infty)}\right) \geqslant \exp(-CR).$$

Thus, if $R>0$ is such that $\exp(-CR) \geqslant \frac{1}{2}$, using (11) we deduce that

$$\gamma(t) \geqslant \frac{\rho_{min}}{2} - \overline{\rho} \text{ in } \Omega_F(0).$$

This inequality proves (28).

## 2.2 Regularity of the fluid velocity and the elastic displacement

We will prove the following regularity result:

**Lemma 4** *There exists a function $h : (\mathbf{R}_*^+)^2 \times (\mathbf{R}^+)^3 \to \mathbf{R}$ increasing with respect to all five variables and there exists $R_0 \in (0,1)$ such that for every $0 < R < R_0$ and every $(\rho_0, u_0, \xi_0, \xi_1)$ satisfying (11)-(12), there exists $T_0 > 0$ such that $\forall T < T_0$ and every $\hat{v} \in X_{T,R}$, system (4), (8), (10) and (22)-(24) admits a unique solution $(v, \xi)$ in $\overline{X}_T \times C^3([0,T]; L^2(\Omega_S(0))) \cap C^0([0,T]; H^3(\Omega_S(0))))$. This solution satisfies the following estimate:*

$$\begin{aligned}\overline{N}_T(v) + \|\xi\|_{W^{3,\infty}(L^2(\Omega_S(0)))} + \|\xi\|_{L^\infty(H^3(\Omega_S(0)))} \\ \leqslant h(1/\rho_{min}, \|\rho_0\|_{H^3}, \|u_0\|_{H^4}, \|\xi_0\|_{H^3(\Omega_S(0))}, \|\xi_1\|_{H^2(\Omega_S(0))}).\end{aligned} \tag{35}$$

*Consequently, there exists $T_1 > 0$ such that for every $T < T_1$ we have $v \in X_{T,R}$, that is to say,*

$$N_T(v) \leqslant R. \tag{36}$$

**Proof:**

1) Existence and uniqueness.

Let $\{w_i\}_{i\in\mathbf{N}} \in H_0^1(\Omega_F(0))$ and $\{z_i\}_{i\in\mathbf{N}} \in H^1(\Omega_S(0))$ two orthogonal basis in $L^2$ and $\{\tilde{z}_i\}_{i\in\mathbf{N}}$ the extension on $\Omega$ of $\{w_i\}_{i\in\mathbf{N}}$ by 0. The initial conditions $\xi_0$, $\xi_1$ and $u_0$ can be decomposed on these basis:

$$\xi_0 = \sum_{i=0}^{\infty} \alpha_i^0 z_i,\ \xi_1 = \sum_{i=0}^{\infty} \alpha_i^1 z_i \text{ and } u_0 = \sum_{i=0}^{\infty} \alpha_i^1 \tilde{z}_i + \sum_{i=0}^{\infty} \beta_i^0 w_i$$

We try to find $(v^n, \xi^n)$ satisfying

$$\left\{\begin{aligned} &\int_{\Omega_F(0)} (\mu(\nabla v^n (\nabla\hat{\chi})^{-1} + (\nabla\hat{\chi})^{-t}(\nabla\hat{v})^t) + \mu'(\nabla v^n(\nabla\hat{\chi})^{-1} : \mathrm{Id})\mathrm{Id} - (P(\overline{\rho}+\gamma) - P(\overline{\rho})\mathrm{Id}))\mathrm{cof}\,(\nabla\hat{\chi}) : \nabla w^n\, dy \\ &+ \int_{\Omega_S(0)} \xi_{tt}^n\, z_t^n\, dy + \int_{\Omega_S(0)} (2\lambda\epsilon(\xi^n) + \lambda'(\nabla\cdot\xi^n)\mathrm{Id}) : \nabla z_t^n\, dy + \int_{\Omega_F(0)} (\overline{\rho}+\gamma)\det(\nabla\hat{\chi}) v_t^n w^n\, dy = 0,\end{aligned}\right. \tag{37}$$

for $t \in (0,T)$, where

$$w^n(t,y) = \sum_{i=0}^{n} \chi_{i,t}(t)\tilde{z}_i(y) + \sum_{i=0}^{n} \kappa_i(t) w_i(y), \quad t \in (0,T),\ y \in \Omega_F(0)$$

and

$$z^n(t,y) = \sum_{i=0}^{n} \chi_i(t) z_i(y), \quad t \in (0,T),\ y \in \Omega_S(0)$$

for $\chi_i$, $\kappa_i \in C^\infty([0,T])\,(1 \leqslant i \leqslant n)$.

We look for $(v^n, \xi^n)$ in the form

$$v^n(t,y) = \sum_{i=0}^{n} \alpha_{i,t}(t)\tilde{z}_i(y) + \sum_{i=0}^{n} \beta_i(t) w_i(y) \quad (t,y) \in (0,T)\times\Omega_F(0)$$

and

$$\xi^n(t,y) = \sum_{i=0}^{n} \alpha_i(t)\tilde{z}_i(y) \quad (t,y) \in (0,T)\times\Omega_S(0)$$

This yields the system

$$A(t)\frac{d}{dt}\begin{pmatrix}\alpha_i\\ \alpha_{i,t}\\ \beta_i\end{pmatrix} = M(t)\begin{pmatrix}\alpha_i\\ \alpha_{i,t}\\ \beta_i\end{pmatrix} + B(t), \quad t \in (0,T),$$

complemented by the initial conditions:

$$\begin{pmatrix} \alpha_i \\ \alpha_{i,t} \\ \beta_i \end{pmatrix}(0) = \begin{pmatrix} \alpha_i^0 \\ \alpha_i^1 \\ \beta_i^0 \end{pmatrix}.$$

The matrix $A(t) := (A_{ij}(t))_{1\leqslant i,j\leqslant 3}$ for $A_{ij} \in \mathcal{M}_{n+1}$ is given by $A_{11} := \mathrm{Id}$, $A_{1j} \equiv 0$ for $j = 2,3$, $A_{i1} \equiv 0$ for $i = 2,3$,

$$A_{22}(t) := \left(\int_{\Omega_S(0)} z_i\, z_j\, dy + \int_{\Omega_F(0)} (\overline{\rho}+\gamma)\det(\nabla\hat{\chi})\tilde{z}_i\,\tilde{z}_j\, dy\right)_{1\leqslant i,j\leqslant n+1},$$

$$A_{23}(t) := \left(\int_{\Omega_F(0)} (\overline{\rho}+\gamma)\det(\nabla\hat{\chi})\tilde{z}_i\, w_j\, dy\right)_{1\leqslant i,j\leqslant n+1},$$

$A_{3,2} := A_{2,3}^t$ and

$$A_{33}(t) := \left(\int_{\Omega_F(0)} (\overline{\rho}+\gamma)\det(\nabla\hat{\chi}) w_i\, w_j\, dy\right)_{1\leqslant i,j\leqslant n+1}.$$

Next, $M(t) := (M_{ij}(t))_{1\leqslant i,j\leqslant 3}$, where $M_{ij} \in \mathcal{M}_{n+1}$ are given by $M_{1j} \equiv 0$ for $j = 1,3$, $M_{12} := \mathrm{Id}$,

$$M_{21}(t) := -\left(\int_{\Omega_S(0)} (2\lambda\epsilon(z_j) + \lambda'(\nabla\cdot z_j)\mathrm{Id}) : \nabla z_i\, dy\right)_{1\leqslant i,j\leqslant n+1},$$

$$M_{22}(t) := -\left(\int_{\Omega_F(0)} (\mu(\nabla\tilde{z}_j(\nabla\hat{\chi})^{-1} + (\nabla\hat{\chi})^{-t}(\nabla\tilde{z}_j)^t) + \mu'(\nabla\tilde{z}_j(\nabla\hat{\chi})^{-1} : \mathrm{Id})\mathrm{Id})\mathrm{cof}\,(\nabla\hat{\chi}) : \nabla\tilde{z}_i\, dy\right)_{1\leqslant i,j\leqslant n+1},$$

$$M_{23}(t) := -\left(\int_{\Omega_F(0)} (\mu(\nabla w_j(\nabla\hat{\chi})^{-1} + (\nabla\hat{\chi})^{-t}(\nabla w_j)^t) + \mu'(\nabla w_j(\nabla\hat{\chi})^{-1} : \mathrm{Id})\mathrm{Id})\mathrm{cof}\,(\nabla\hat{\chi}) : \nabla\tilde{z}_i\, dy\right)_{1\leqslant i,j\leqslant n+1},$$

$M_{31} \equiv 0$,

$$M_{32}(t) := -\left(\int_{\Omega_F(0)} (\mu(\tilde{z}_j(\nabla\hat{\chi})^{-1} + (\nabla\hat{\chi})^{-t}(\nabla\tilde{z}_j)^t) + \mu'(\nabla\tilde{z}_j(\nabla\hat{\chi})^{-1} : \mathrm{Id})\mathrm{Id})\mathrm{cof}\,(\nabla\hat{\chi}) : \nabla w_i\, dy\right)_{1\leqslant i,j\leqslant n+1},$$

and

$$M_{33}(t) := -\left(\int_{\Omega_F(0)} (\mu(w_j(\nabla\hat{\chi})^{-1} + (\nabla\hat{\chi})^{-t}(\nabla w_j)^t) + \mu'(\nabla w_j(\nabla\hat{\chi})^{-1} : \mathrm{Id})\mathrm{Id})\mathrm{cof}\,(\nabla\hat{\chi}) : \nabla w_i\, dy\right)_{1\leqslant i,j\leqslant n+1}.$$

On the other hand, $B(t) := (B_i(t))_{1\leqslant i\leqslant 3}$ with $B_i(t) \in \mathbf{R}^{n+1}$ given by $B_1(t) \equiv 0$,

$$B_2(t) = \left(\int_{\Omega_F(0)} (P(\overline{\rho}+\gamma) - P(\overline{\rho}))\mathrm{cof}\,(\nabla\hat{\chi}) : \nabla\tilde{z}_j\, dy\right)_{1\leqslant j\leqslant n+1}$$

and

$$B_3(t) = \left(\int_{\Omega_F(0)} (P(\overline{\rho}+\gamma) - P(\overline{\rho}))\mathrm{cof}\,(\nabla\hat{\chi}) : \nabla w_j\, dy\right)_{1\leqslant j\leqslant n+1}$$

One can easily see that $A(t)$ is positive definite thanks to $\overline{\rho}+\gamma > \overline{\rho}+\gamma_{min} > 0$ (see (28)) and $\det(\nabla\hat{\chi})(t) \geqslant C > 0$, since $\hat{v} \in X_{T,R}$ and $R$ is small enough. Moreover, $A^{-1}, M, B \in L^\infty(0,T)$. This gives the existence of a unique solution

$$(v^n, \xi^n) \in W^{1,\infty}(0,T; H^1(\Omega_F(0))) \times W^{2,\infty}(0,T; H^1(\Omega_S(0))).$$

Finally, from (37) we can obtain an energy estimate of the form

$$\|v^n\|_{L^\infty(L^2)} + \|v^n\|_{L^2(H^1)} + \|\xi^n\|_{L^\infty(H^1(\Omega_S(0)))} + \|\xi^n\|_{W^{1,\infty}(L^2(\Omega_S(0)))} \leqslant C$$

(see the details below (38)-(47)). Thanks to this estimate, one can pass to the limit as $n \to \infty$ in (37) and show the existence and uniqueness of $(v, \xi) \in (L^\infty(L^2) \cap L^2(H^1)) \times (L^\infty(H^1(\Omega_S(0))) \cap W^{1,\infty}(L^2(\Omega_S(0))))$ a weak solution of (8), (10) and (22)-(23), (4).

Now, in the remaining of the proof, we will establish (35).

2) Regularity in time

• First, we multiply the equation (22) by $v$, we integrate on $Q_t$ for any $t \in (0,T)$. After a spatial integration by parts and according to the boundary conditions (23), we have

$$\begin{aligned}
&\frac{1}{2}\int_{\Omega_F(0)} (\overline{\rho}+\gamma)(t)|v(t)|^2 \det \nabla\hat{\chi}(t)\, dy - \frac{1}{2}\int_{\Omega_F(0)} \rho_0 |u_0|^2\, dy - \frac{1}{2}\iint_{Q_t} |v|^2((\overline{\rho}+\gamma)\det\nabla\hat{\chi})_t\, dy\, ds \\
&+\iint_{Q_t} \Big(\frac{\mu}{2}|\nabla v(\nabla\hat{\chi})^{-1} + (\nabla\hat{\chi})^{-t}\nabla v^t|^2 + \mu'|\nabla v(\nabla\hat{\chi})^{-1} : \mathrm{Id}|^2\Big) \det\nabla\hat{\chi}\, dy\, ds \qquad (38)\\
&-\iint_{Q_t} (P(\overline{\rho}+\gamma) - P(\overline{\rho}))\,\mathrm{cof}\,\nabla\hat{\chi} : \nabla v\, dy\, ds + \iint_{\Sigma_t} \big(2\lambda\epsilon(\xi) + \lambda'(\nabla\cdot\xi)\mathrm{Id}\big)\mathrm{n}\xi_t\, d\gamma\, ds = 0.
\end{aligned}$$

For the first term, according to (28) and (16)

$$\int_{\Omega_F(0)} (\overline{\rho}+\gamma)(t)|v(t)|^2 \det\nabla\hat{\chi}(t)\, dy \geqslant (\overline{\rho}+\gamma_{min})(1-CR)\int_{\Omega_F(0)} |v(t)|^2\, dy. \qquad (39)$$

The second term is bounded by

$$\frac{1}{2}\int_{\Omega_F(0)} \rho_0|u_0|^2\, dy \leqslant C(\|\gamma(0)\|^2_{L^\infty} + \|u_0\|^2_{L^2} + \|u_0\|^4_{L^2}). \qquad (40)$$

The third term is estimated by

$$\begin{aligned}
\iint_{Q_t} |v|^2|((\overline{\rho}+\gamma)\det\nabla\hat{\chi})_t|\, dy\, ds &\leqslant \iint_{Q_t} |v|^2(|\gamma_t|\,|\det\nabla\hat{\chi}| + |\overline{\rho}+\gamma|\,|(\det\nabla\hat{\chi})_t|)\, dy\, ds \\
&\leqslant CT^{1/2}\|v\|^2_{L^\infty(L^2)}(\|\gamma\|_{H^1(L^\infty)} + (\overline{\rho} + \|\gamma\|_{L^4(L^\infty)})\|\nabla\hat{v}\|_{L^4(L^\infty)}).
\end{aligned}$$

Here we have used (15) and the fact that $(\det\nabla\hat{\chi})_t = \det\nabla\hat{\chi}\,\mathrm{tr}(\nabla\hat{\chi}^{-1}\nabla\hat{v})$. Thus, according to (27) and the fact that $\hat{v}$ belongs to $X_{T,R}$,

$$\iint_{Q_t} |v|^2|((\overline{\rho}+\gamma)\det\nabla\hat{\chi})_t|\, dy\, ds \leqslant CT^{1/2}\|v\|^2_{L^\infty(L^2)}(M_1 + R) \qquad (41)$$

where $M_1$ is defined by (it corresponds to the initial conditions term in the right-hand side of inequality (27))

$$M_1 = \|\gamma(0)\|_{H^2} + \|u_0\|_{H^1} + \|\gamma(0)\|^2_{H^2} + \|u_0\|^2_{H^1}. \qquad (42)$$

We consider now the viscosity term corresponding to the second line of (38). The term in $\mu'$ is estimated in the following way

$$\begin{aligned}
\mu'\iint_{Q_t} |\nabla v(\nabla\hat{\chi})^{-1} : \mathrm{Id}|^2 \det\nabla\hat{\chi}\, dy\, ds &\geqslant \frac{\mu'}{2}\iint_{Q_t} |\nabla\cdot v|^2\, dy\, ds - \mu'\iint_{Q_t} |\nabla v((\nabla\hat{\chi})^{-1} - \mathrm{Id}) : \mathrm{Id}|^2\, dy\, ds \\
+\mu'\iint_{Q_t} |\nabla v(\nabla\hat{\chi})^{-1} : \mathrm{Id}|^2(\det\nabla\hat{\chi} - 1)\, dy\, ds &\geqslant \frac{\mu'}{2}\iint_{Q_t} |\nabla\cdot v|^2\, dy\, ds - CR\|v\|^2_{L^2(H^1)}.
\end{aligned} \qquad (43)$$

In the same way,

$$\frac{\mu}{2}\iint_{Q_t} |\nabla v(\nabla\hat{\chi})^{-1} + (\nabla\hat{\chi})^{-t}\nabla v^t|^2 \det\nabla\hat{\chi}\,dy\,ds \geqslant \mu\iint_{Q_t} |\epsilon(v)|^2\,dy\,ds - CR\|v\|^2_{L^2(H^1)}. \tag{44}$$

For the first term in the third line of (38), we notice that, for any $\delta > 0$, there exists a positive constant $C$ such that

$$\left|\iint_{Q_t} (P(\overline{\rho}+\gamma) - P(\overline{\rho}))\operatorname{cof}\nabla\hat{\chi} : \nabla v\,dy\,ds\right| \leqslant \delta\|\nabla v\|^2_{L^2(L^2)} + C\iint_{Q_t} |P(\overline{\rho}+\gamma) - P(\overline{\rho})|^2\,dy\,ds.$$

According to Lemma 3

$$0 < a = \overline{\rho} + \gamma_{min} \leqslant \overline{\rho} + \gamma \leqslant C(M_1 + 1) = b.$$

Thus, there exists an interval $I \subset \mathbb{R}^*_+$ such that $\overline{\rho} \in I$ and $[a,b] \subset I$. Then, we obtain

$$\iint_{Q_t} |P(\overline{\rho}+\gamma) - P(\overline{\rho})|^2\,dy\,ds \leqslant \|P'\|^2_{L^\infty(I)}\|\gamma\|^2_{L^2(L^2)} \leqslant C_{M_1}(M_1 + R)^2,$$

where $C_{M_1}$ depends on $M_1$. This implies that

$$\left|\iint_{Q_t} (P(\overline{\rho}+\gamma) - P(\overline{\rho}))\operatorname{cof}\nabla\hat{\chi} : \nabla v\,dy\,ds\right| \leqslant \delta\|\nabla v\|^2_{L^2(L^2)} + C_{M_1}(M_1 + R)^2. \tag{45}$$

At last, we have to estimate the last term in the left-hand side of (38). Let us multiply the equation (4) by $\xi_t$ and integrate on $(0,t)\times\Omega_S(0)$. We obtain

$$\begin{aligned}\iint_{\Sigma_t} \big(2\lambda\epsilon(\xi) + \lambda'(\nabla\cdot\xi)\mathrm{Id}\big)\mathrm{n}\xi_t\,d\gamma\,ds &= \frac{1}{2}\int_{\Omega_S(0)} |\xi_t(t)|^2\,dy - \frac{1}{2}\int_{\Omega_S(0)} |\xi_1|^2\,dy + \lambda\int_{\Omega_S(0)} |\epsilon(\xi)(t)|^2\,dy \\ -\lambda\int_{\Omega_S(0)} |\epsilon(\xi_0)|^2\,dy &+ \frac{\lambda'}{2}\int_{\Omega_S(0)} |\nabla\cdot\xi(t)|^2\,dy - \frac{\lambda'}{2}\int_{\Omega_S(0)} |\nabla\cdot\xi_0|^2\,dy.\end{aligned} \tag{46}$$

Thus, we can reassemble inequalities (39) to (46) to estimate the terms in (38). Taking the supremum of (38) in $t$ in $(0,T)$ and using Korn's inequality, we obtain, for $\delta$ and $R$ small enough,

$$\begin{aligned}&\|v\|_{L^\infty(L^2)} + \|v\|_{L^2(H^1)} + \|\xi\|_{W^{1,\infty}(L^2(\Omega_S(0)))} + \|\xi\|_{L^\infty(H^1(\Omega_S(0)))} \\ &\leqslant \max\{1, 1/\rho_{min}\}\Big(C(\|\xi_0\|_{H^1(\Omega_S(0))} + \|\xi_1\|_{L^2(\Omega_S(0))}) + C_{M_1} + CT^{1/4}\overline{N}_T(v)(M_1+R)^{1/2}\Big) \\ &= C_0 + C_0T^{1/4}\overline{N}_T(v),\end{aligned} \tag{47}$$

where $\overline{N}_T(v)$ is defined by (26). Here and in the sequel, $C_0$ denotes a generic constant depending increasingly on $1/\rho_{min}$, $\|\rho_0\|_{H^3}$, $\|u_0\|_{H^4}$, $\|\xi_0\|_{H^3(\Omega_S(0))}$ and $\|\xi_1\|_{H^2(\Omega_S(0))}$.

• Now, we differentiate the equation (22) satisfied by $v$ with respect to time. We obtain

$$\begin{aligned}(\overline{\rho}+\gamma)\det\nabla\hat{\chi}v_{tt} + ((\overline{\rho}+\gamma)\det\nabla\hat{\chi})_t v_t - \nabla\cdot\big[\big(\mu(\nabla v(\nabla\hat{\chi})^{-1} + (\nabla\hat{\chi})^{-t}\nabla v^t) + \mu'(\nabla v(\nabla\hat{\chi})^{-1} : \mathrm{Id})\mathrm{Id} \\ - (P(\overline{\rho}+\gamma) - P(\overline{\rho}))\mathrm{Id}\big)\operatorname{cof}\nabla\hat{\chi}\big]_t = 0 \text{ in } Q_T.\end{aligned} \tag{48}$$

Next, we multiply this equation by $v_t$ and we integrate on $Q_t$ for any $t \in (0,T)$. For the first two terms of (48), we have:

$$\begin{aligned}&\iint_{Q_t} \big((\overline{\rho}+\gamma)\det\nabla\hat{\chi}v_{tt} + ((\overline{\rho}+\gamma)\det\nabla\hat{\chi})_t v_t\big)v_t\,dy\,ds \\ &= \frac{1}{2}\int_{\Omega_F(0)} (\overline{\rho}+\gamma)(t)\det\nabla\hat{\chi}(t)|v_t(t)|^2\,dy - \frac{1}{2}\int_{\Omega_F(0)} \rho_0|v_t(0)|^2\,dy + \frac{1}{2}\iint_{Q_t} |v_t|^2((\overline{\rho}+\gamma)\det\nabla\hat{\chi})_t\,dy\,ds.\end{aligned}$$

Thus, arguing exactly as in the previous step, we have

$$\begin{aligned}
&\iint_{Q_t} \big((\overline{\rho}+\gamma)\det\nabla\hat{\chi} v_{tt} + ((\overline{\rho}+\gamma)\det\nabla\hat{\chi})_t v_t\big) v_t \, dy\, ds \\
&\geqslant \rho_{min} C \int_{\Omega_F(0)} |v_t(t)|^2\, dy - \frac{1}{2}\int_{\Omega_F(0)} \rho_0 |v_t(0)|^2 \, dy - CT^{1/2}\|v\|^2_{W^{1,\infty}(L^2)}(M_1+R) \\
&\geqslant \rho_{min} C \int_{\Omega_F(0)} |v_t(t)|^2 \, dy - C_0 - C_0 T^{1/2}\overline{N}_T(v)^2.
\end{aligned} \tag{49}$$

Now, the remaining terms of (48) are

$$\begin{aligned}
&\iint_{Q_t} \Big[\big(\mu(\nabla v(\nabla\hat{\chi})^{-1} + (\nabla\hat{\chi})^{-t}\nabla v^t) + \mu'(\nabla v(\nabla\hat{\chi})^{-1} : \mathrm{Id})\mathrm{Id} - (P(\overline{\rho}+\gamma) - P(\overline{\rho}))\mathrm{Id}\big)\mathrm{cof}\,\nabla\hat{\chi}\Big]_t : \nabla v_t \, dy\, ds \\
&+ \iint_{\Sigma_t} \big(2\lambda\epsilon(\xi_t) + \lambda'(\nabla\cdot\xi_t)\mathrm{Id}\big)\mathrm{n}\xi_{tt}\, d\gamma\, ds.
\end{aligned} \tag{50}$$

We notice that

$$\begin{aligned}
&\mu' \iint_{Q_t} [(\nabla v(\nabla\hat{\chi})^{-1} : \mathrm{Id})\mathrm{cof}\,\nabla\hat{\chi}]_t : \nabla v_t \, dy\, ds = \mu' \iint_{Q_t} |\nabla v_t(\nabla\hat{\chi})^{-1} : \mathrm{Id}|^2 \det\nabla\hat{\chi}\, dy\, ds \\
&+\mu' \iint_{Q_t} (\nabla v((\nabla\hat{\chi})^{-1})_t : \mathrm{Id})\mathrm{cof}\,\nabla\hat{\chi} : \nabla v_t\, dy\, ds + \mu' \iint_{Q_t} (\nabla v(\nabla\hat{\chi})^{-1} : \mathrm{Id})(\mathrm{cof}\,\nabla\hat{\chi})_t : \nabla v_t\, dy\, ds.
\end{aligned} \tag{51}$$

Arguing again as in the previous step, the first term in the right-hand side is estimated by

$$\mu' \iint_{Q_t} |\nabla v_t(\nabla\hat{\chi})^{-1} : \mathrm{Id}|^2 \det\nabla\hat{\chi}\, dy\, ds \geqslant \frac{\mu'}{2}\iint_{Q_t} |\nabla\cdot v_t|^2\, dy\, ds - CR\|v\|^2_{H^1(H^1)}.$$

To bound the second line of (51), we use that $\|((\nabla\hat{\chi})^{-1})_t\|_{L^2(L^\infty)} \leqslant C\|\nabla\hat{v}\|_{L^2(L^\infty)} \leqslant CR$ and $\|(\mathrm{cof}\,\nabla\hat{\chi})_t\|_{L^2(L^\infty)} \leqslant C\|\nabla\hat{v}\|_{L^2(L^\infty)} \leqslant CR$. Thus, for the term in $\mu'$ in (50), we have

$$\begin{aligned}
&\mu' \iint_{Q_t} [(\nabla v(\nabla\hat{\chi})^{-1} : \mathrm{Id})\mathrm{cof}\,\nabla\hat{\chi}]_t : \nabla v_t\, dy\, ds \geqslant \frac{\mu'}{2}\iint_{Q_t} |\nabla\cdot v_t|^2\, dy\, ds - CR(\|v\|^2_{H^1(H^1)} + \|v\|^2_{L^\infty(H^1)}) \\
&\geqslant \frac{\mu'}{2}\iint_{Q_t} |\nabla\cdot v_t|^2\, dy\, ds - CR(\|v\|^2_{H^1(H^1)} + \|v_0\|^2_{H^1}).
\end{aligned} \tag{52}$$

The term in $\mu$ in (50) can be estimated in the same way by

$$\begin{aligned}
&\mu \iint_{Q_t} \big[\big((\nabla v(\nabla\hat{\chi})^{-1} + (\nabla\hat{\chi})^{-t}\nabla v^t)\big)\mathrm{cof}\,\nabla\hat{\chi}\big]_t : \nabla v_t\, dy\, ds \geqslant \mu \iint_{Q_t} |\epsilon(v_t)|^2\, dy\, ds \\
&-CR(\|v\|^2_{H^1(H^1)} + \|v\|^2_{L^\infty(H^1)}) \geqslant \mu \iint_{Q_t} |\epsilon(v_t)|^2\, dy\, ds - CR(\|v\|^2_{H^1(H^1)} + \|v_0\|^2_{H^1}).
\end{aligned} \tag{53}$$

Next, for the pressure term in (50), we see that, for any $\delta > 0$, there exists a positive constant $C$ such that

$$\begin{aligned}
\iint_{Q_t} \big|[(P(\overline{\rho}+\gamma) - P(\overline{\rho}))\mathrm{cof}\,\nabla\hat{\chi}]_t\big|\, |\nabla v_t|\, dy\, ds \leqslant \delta\|\nabla v_t\|^2_{L^2(L^2)} + C\iint_{Q_t} |P'(\overline{\rho}+\gamma)|^2|\gamma_t|^2\, dy\, ds \\
+C\iint_{Q_t} |P(\overline{\rho}+\gamma) - P(\overline{\rho}))|^2 |(\mathrm{cof}\,\nabla\hat{\chi})_t|^2\, dy\, ds.
\end{aligned}$$

Since

$$\iint_{Q_t} |P(\overline{\rho}+\gamma) - P(\overline{\rho}))|^2|(\mathrm{cof}\,\nabla\hat{\chi})_t|^2\, dy\, ds \leqslant C_{M_1}\|\gamma\|^2_{L^\infty(L^\infty)}\|\nabla\hat{v}\|^2_{L^2(L^2)} \leqslant C_0 R^2\|\gamma\|^2_{L^\infty(L^\infty)},$$

we have, according to (27),

$$\iint_{Q_t} \left|[(P(\overline{\rho}+\gamma)-P(\overline{\rho}))\mathrm{cof}\,\nabla\hat{\chi}]_t\right| |\nabla v_t|\,dy\,ds \leqslant \delta\|\nabla v_t\|^2_{L^2(L^2)}+C_0. \tag{54}$$

At last, let us consider the boundary term in (50). If we differentiate with respect to time equation (4), we multiply by $\xi_{tt}$ and integrate on $(0,t)\times\Omega_S(0)$, we obtain

$$\begin{aligned}&\iint_{\Sigma_t}\big(2\lambda\epsilon(\xi_t)+\lambda'(\nabla\cdot\xi_t)\mathrm{Id}\big)\mathrm{n}\xi_{tt}\,d\gamma\,ds=\frac{1}{2}\int_{\Omega_S(0)}|\xi_{tt}(t)|^2\,dy-\frac{1}{2}\int_{\Omega_S(0)}|\xi_{tt}(0)|^2\,dy\\&+\lambda\int_{\Omega_S(0)}|\epsilon(\xi_t)(t)|^2\,dy-\lambda\int_{\Omega_S(0)}|\epsilon(\xi_1)|^2\,dy+\frac{\lambda'}{2}\int_{\Omega_S(0)}|\nabla\cdot\xi_t(t)|^2\,dy-\frac{\lambda'}{2}\int_{\Omega_S(0)}|\nabla\cdot\xi_1|^2\,dy.\end{aligned} \tag{55}$$

Thus, thanks to estimates (49) and (52) to (55), we obtain, for $\delta$ and $R$ small enough

$$\|v\|_{W^{1,\infty}(L^2)}+\|v\|_{H^1(H^1)}+\|\xi\|_{W^{2,\infty}(L^2(\Omega_S(0)))}+\|\xi\|_{W^{1,\infty}(H^1(\Omega_S(0)))}\leqslant C_0+C_0T^{1/4}\overline{N}_T(v). \tag{56}$$

• Let us differentiate two times with respect to time the equation (22) satisfied by $v$. We obtain in $Q_T$

$$\begin{aligned}&(\overline{\rho}+\gamma)\det\nabla\hat{\chi}v_{ttt}+2((\overline{\rho}+\gamma)\det\nabla\hat{\chi})_t v_{tt}+((\overline{\rho}+\gamma)\det\nabla\hat{\chi})_{tt}v_t\\&-\nabla\cdot\left[\big(\mu(\nabla v(\nabla\hat{\chi})^{-1}+(\nabla\hat{\chi})^{-t}\nabla v^t)+\mu'(\nabla v(\nabla\hat{\chi})^{-1}:\mathrm{Id})\mathrm{Id}-(P(\overline{\rho}+\gamma)-P(\overline{\rho}))\mathrm{Id}\big)\mathrm{cof}\,\nabla\hat{\chi}\right]_{tt}=0.\end{aligned} \tag{57}$$

Then, we multiply this equation by $v_{tt}$ and integrate on $Q_t$. For the first two terms, we argue as in the first and second steps:

$$\begin{aligned}&\iint_{Q_t}\big((\overline{\rho}+\gamma)\det\nabla\hat{\chi}v_{ttt}+2((\overline{\rho}+\gamma)\det\nabla\hat{\chi})_t v_{tt}\big)v_{tt}\,dy\,ds\\&=\frac{1}{2}\int_{\Omega_F(0)}(\overline{\rho}+\gamma)(t)\det\nabla\hat{\chi}(t)|v_{tt}(t)|^2\,dy-\frac{1}{2}\int_{\Omega_F(0)}\rho_0|v_{tt}(0)|^2\,dy+\frac{3}{2}\iint_{Q_t}|v_{tt}|^2((\overline{\rho}+\gamma)\det\nabla\hat{\chi})_t\,dy\,ds.\end{aligned}$$

This gives

$$\begin{aligned}&\iint_{Q_t}\big((\overline{\rho}+\gamma)\det\nabla\hat{\chi}v_{ttt}+2((\overline{\rho}+\gamma)\det\nabla\hat{\chi})_t v_{tt}\big)v_{tt}\,dy\,ds\\&\geqslant\rho_{min}C\int_{\Omega_F(0)}|v_{tt}(t)|^2\,dy-\frac{1}{2}\int_{\Omega_F(0)}\rho_0|v_{tt}(0)|^2\,dy-CT^{1/2}\|v\|^2_{W^{2,\infty}(L^2)}(M_1+R)\\&\geqslant\rho_{min}C\int_{\Omega_F(0)}|v_{tt}(t)|^2\,dy-C_0-C_0T^{1/2}\overline{N}_T(v)^2.\end{aligned} \tag{58}$$

For the last term in the first line of (57), we have

$$\left|\iint_{Q_t}((\overline{\rho}+\gamma)\det\nabla\hat{\chi})_{tt}v_t v_{tt}\,dy\,ds\right|\leqslant\|((\overline{\rho}+\gamma)\det\nabla\hat{\chi})_{tt}\|_{L^2(L^2)}\|v_t\|_{L^\infty(L^6)}\|v_{tt}\|_{L^2(L^6)}.$$

We have

$$((\overline{\rho}+\gamma)\det\nabla\hat{\chi})_{tt}=\gamma_{tt}\det\nabla\hat{\chi}+2\gamma_t(\det\nabla\hat{\chi})_t+(\overline{\rho}+\gamma)(\det\nabla\hat{\chi})_{tt}$$

Thus, we can estimate the $L^2(L^2)$-norm of $((\overline{\rho}+\gamma)\det\nabla\hat{\chi})_{tt}$ in the following way

$$\begin{aligned}&\|\gamma_{tt}\det\nabla\hat{\chi}\|_{L^2(L^2)}\leqslant CT^{1/2-1/q}\|\gamma_{tt}\|_{L^q(L^2)}\leqslant CT^{1/2-1/q}(M_1+R)=C_0T^{1/2-1/q},\\&\|\gamma_t(\det\nabla\hat{\chi})_t\|_{L^2(L^2)}\leqslant CT^{1/2-1/q}\|\gamma_t\|_{L^\infty(L^2)}\|\nabla\hat{v}\|_{L^q(L^\infty)}\leqslant CT^{1/2-1/q}R(M_1+R)=C_0T^{1/2-1/q},\end{aligned}$$

since $\|\gamma_t\|_{L^\infty(L^2)}\leqslant\|\gamma_t(0)\|_{L^2}+C\|\gamma_t\|_{W^{1,q}(L^2)}$. At last,

$$\begin{aligned}\|(\overline{\rho}+\gamma)(\det\nabla\hat{\chi})_{tt}\|_{L^2(L^2)}&\leqslant\|\overline{\rho}+\gamma\|_{L^\infty(L^\infty)}(T^{1/2-1/q}\|\nabla\hat{v}_t\|_{L^q(L^2)}+T^{1/2-2/q}\|\nabla\hat{v}\|^2_{L^q(L^4)})\\&\leqslant CT^{1/2-2/q}R(1+M_1)=C_0T^{1/2-2/q}.\end{aligned}$$

These estimates lead to the following inequality

$$\left|\iint_{Q_t} ((\overline{\rho}+\gamma)\det\nabla\hat{\chi})_{tt} v_t v_{tt}\,dy\,ds\right| \leqslant C_0 T^{1/2-2/q}\overline{N}_T(v)^2. \tag{59}$$

For the terms coming from the second line of (57), we have

$$\begin{aligned}
&-\iint_{Q_t} \nabla\cdot\big[\big(\mu(\nabla v(\nabla\hat{\chi})^{-1}+(\nabla\hat{\chi})^{-t}\nabla v^t)+\mu'(\nabla v(\nabla\hat{\chi})^{-1}:\mathrm{Id})\mathrm{Id}-(P(\overline{\rho}+\gamma)-P(\overline{\rho}))\mathrm{Id}\big)\mathrm{cof}\,\nabla\hat{\chi}\big]_{tt} v_{tt}\,dy\,ds\\
&=\iint_{Q_t} \big[\big(\mu(\nabla v(\nabla\hat{\chi})^{-1}+(\nabla\hat{\chi})^{-t}\nabla v^t)+\mu'(\nabla v(\nabla\hat{\chi})^{-1}:\mathrm{Id})\mathrm{Id}-(P(\overline{\rho}+\gamma)-P(\overline{\rho}))\mathrm{Id}\big)\mathrm{cof}\,\nabla\hat{\chi}\big]_{tt}:\nabla v_{tt}\,dy\,ds\\
&+\iint_{\Sigma_t}\big(2\lambda\epsilon(\xi_{tt})+\lambda'(\nabla\cdot\xi_{tt})\mathrm{Id}\big)\mathrm{n}\xi_{ttt}\,d\gamma\,ds.
\end{aligned} \tag{60}$$

We detail the estimates for the term in $\mu'$ in the right-hand side of (60).

$$\begin{aligned}
&\mu'\iint_{Q_t}\big[(\nabla v(\nabla\hat{\chi})^{-1}:\mathrm{Id})\mathrm{cof}\,\nabla\hat{\chi}\big]_{tt}:\nabla v_{tt}\,dy\,ds=\mu'\iint_{Q_t}|\nabla v_{tt}(\nabla\hat{\chi})^{-1}:\mathrm{Id}|^2\det\nabla\hat{\chi}\,dy\,ds\\
&+2\mu'\iint_{Q_t}\nabla v_t[((\nabla\hat{\chi})^{-1}:\mathrm{Id})\mathrm{cof}\,\nabla\hat{\chi}]_t:\nabla v_{tt}\,dy\,ds+\mu'\iint_{Q_t}\nabla v[((\nabla\hat{\chi})^{-1}:\mathrm{Id})\mathrm{cof}\,\nabla\hat{\chi}]_{tt}:\nabla v_{tt}\,dy\,ds.
\end{aligned} \tag{61}$$

For the first and second terms in the right-hand side, arguing as in the previous step, we have

$$\mu'\iint_{Q_t}|\nabla v_{tt}(\nabla\hat{\chi})^{-1}:\mathrm{Id}|^2\det\nabla\hat{\chi}\,dy\,ds\geqslant\frac{\mu'}{2}\iint_{Q_t}|\nabla\cdot v_{tt}|^2\,dy\,ds-CR\|v\|^2_{H^2(H^1)}.$$

and

$$\begin{aligned}
&\mu'\left|\iint_{Q_t}\nabla v_t[((\nabla\hat{\chi})^{-1}:\mathrm{Id})\mathrm{cof}\,\nabla\hat{\chi}]_t:\nabla v_{tt}\,dy\,ds\right|\leqslant CR(\|v\|^2_{H^2(H^1)}+\|v\|^2_{W^{1,\infty}(H^1)})\\
&\leqslant CR(\|v\|^2_{H^2(H^1)}+\|v_0\|^2_{H^3}+\|\rho_0\|^2_{H^2}).
\end{aligned}$$

At last, for the third term in the right-hand side of (61), we notice that

$$\mu'\left|\iint_{Q_t}\nabla v[((\nabla\hat{\chi})^{-1}:\mathrm{Id})\mathrm{cof}\,\nabla\hat{\chi}]_{tt}:\nabla v_{tt}\,dy\,ds\right|\leqslant C\|v\|_{L^\infty(W^{1,\infty})}\|[((\nabla\hat{\chi})^{-1}:\mathrm{Id})\mathrm{cof}\,\nabla\hat{\chi}]_{tt}\|_{L^2(L^2)}\|v\|_{H^2(H^1)}.$$

Since $\|[((\nabla\hat{\chi})^{-1}:\mathrm{Id})\mathrm{cof}\,\nabla\hat{\chi}]_{tt}\|_{L^2(L^2)}\leqslant C(\|\nabla\hat{v}_t\|_{L^2(L^2)}+\|\nabla\hat{v}\|^2_{L^4(L^4)})\leqslant CR$, using the previous estimates, we conclude that

$$\begin{aligned}
&\mu'\iint_{Q_t}\big[(\nabla v(\nabla\hat{\chi})^{-1}:\mathrm{Id})\mathrm{cof}\,\nabla\hat{\chi}\big]_{tt}:\nabla v_{tt}\,dy\,ds\\
&\geqslant\frac{\mu'}{2}\iint_{Q_t}|\nabla\cdot v_{tt}|^2\,dy\,ds-CR(\|v\|^2_{H^2(H^1)}+\|v\|^2_{W^{1,\infty}(H^1)}+\|v\|^2_{L^\infty(W^{1,\infty})})\\
&\geqslant\frac{\mu'}{2}\iint_{Q_t}|\nabla\cdot v_{tt}|^2\,dy\,ds-CR(\|v\|^2_{H^2(H^1)}+\|v_0\|^2_{H^3}+\|\rho_0\|^2_{H^2}+\|v\|^2_{L^\infty(W^{1,\infty})}).
\end{aligned} \tag{62}$$

With the same arguments, we obtain that

$$\begin{aligned}
\mu\iint_{Q_t}\big[\big((\nabla v(\nabla\hat{\chi})^{-1}+(\nabla\hat{\chi})^{-t}\nabla v^t)\big)\mathrm{cof}\,\nabla\hat{\chi}\big]_{tt}:\nabla v_{tt}\,dy\,ds\geqslant\mu\iint_{Q_t}|\epsilon(v_{tt})|^2\,dy\,ds&\\
-CR(\|v\|^2_{H^2(H^1)}+\|v\|^2_{W^{1,\infty}(H^1)}+\|v\|^2_{L^\infty(W^{1,\infty})}).&
\end{aligned} \tag{63}$$

Next, for the pressure term in (60), we have

$$\begin{aligned}
\big|[(P(\overline{\rho}+\gamma)-P(\overline{\rho}))\mathrm{cof}\,\nabla\hat{\chi}]_{tt}\big|\leqslant C\big(&|P(\overline{\rho}+\gamma)-P(\overline{\rho})|\,(|\nabla\hat{v}_t|+|\nabla\hat{v}|^2)+|P'(\overline{\rho}+\gamma)|\,|\gamma_t|\,|\nabla\hat{v}|\\
&+|P'(\overline{\rho}+\gamma)|\,|\gamma_{tt}|+|P''(\overline{\rho}+\gamma)|\,|\gamma_t|^2\big)
\end{aligned}$$

Thus, using (27), we get

$$\left\|[(P(\overline{\rho}+\gamma)-P(\overline{\rho}))\mathrm{cof}\,\nabla\hat{\chi}]_{tt}\right\|_{L^2(L^2)} \leqslant C_{M_1}\big(R\|\gamma\|_{L^\infty(L^\infty)}+R\|\gamma_t\|_{L^4(L^\infty)}+\|\gamma_{tt}\|_{L^2(L^2)}+\|\gamma_t\|^2_{L^4(L^4)}\big)\leqslant C_0.$$

This allows to assert that, for any $\delta>0$, there exists a positive constant $C$ such that

$$\iint_{Q_t}\left|[(P(\overline{\rho}+\gamma)-P(\overline{\rho}))\mathrm{cof}\,\nabla\hat{\chi}]_{tt}\right||\nabla v_{tt}|\,dy\,ds\leqslant\delta\|\nabla v_{tt}\|^2_{L^2(L^2)}+C_0. \tag{64}$$

We finish this step with the estimate of the boundary term in (60). Again, this term is directly estimated from the elasticity equation. Let us differentiate two times (4) and multiply the equation by $\xi_{ttt}$. We obtain

$$\begin{aligned}&\iint_{\Sigma_t}\big(2\lambda\epsilon(\xi_{tt})+\lambda'(\nabla\cdot\xi_{tt})\mathrm{Id}\big)\mathrm{n}\xi_{ttt}\,d\gamma\,ds=\frac{1}{2}\int_{\Omega_S(0)}|\xi_{ttt}(t)|^2\,dy-\frac{1}{2}\int_{\Omega_S(0)}|\xi_{ttt}(0)|^2\,dy\\&+\lambda\int_{\Omega_S(0)}|\epsilon(\xi_{tt})(t)|^2\,dy-\lambda\int_{\Omega_S(0)}|\epsilon(\xi_{tt}(0))|^2\,dy+\frac{\lambda'}{2}\int_{\Omega_S(0)}|\nabla\cdot\xi_{tt}(t)|^2\,dy-\frac{\lambda'}{2}\int_{\Omega_S(0)}|\nabla\cdot\xi_{tt}(0)|^2\,dy.\end{aligned} \tag{65}$$

Thus, using (58), (59) and estimates (62) to (65), we conclude that

$$\begin{aligned}&\|v\|_{W^{2,\infty}(L^2)}+\|v\|_{H^2(H^1)}+\|\xi\|_{W^{3,\infty}(L^2(\Omega_S(0)))}+\|\xi\|_{W^{2,\infty}(H^1(\Omega_S(0)))}\\&\leqslant C_0+C_0T^{1/4-1/(2q)}\overline{N}_T(v)+CR^{1/2}\|v\|_{L^\infty(W^{1,\infty})}.\end{aligned}$$

Since $R<1$ and taking $T$ small enough (depending on the initial conditions), we deduce that

$$\begin{aligned}&\|v\|_{W^{2,\infty}(L^2)}+\|v\|_{H^2(H^1)}+\|\xi\|_{W^{3,\infty}(L^2(\Omega_S(0)))}+\|\xi\|_{W^{2,\infty}(H^1(\Omega_S(0)))}\\&\leqslant C_0+C_0T^{1/4-1/(2q)}\overline{N}_T(v)+CR^{1/2}\|v\|_{L^\infty(W^{1,\infty})}.\end{aligned} \tag{66}$$

Here and in the sequel, $\alpha$ stands for a generic positive constant depending on $q$.

2) Regularity in space

We regard system (22)-(23) like a stationary elliptic problem:

$$-\nabla\cdot(\mu(\nabla v+\nabla v^t)+\mu'(\nabla\cdot v)Id)=F\ \text{ in }\Omega_F(0), \tag{67}$$

with

$$\begin{aligned}&F:=-(\gamma+\overline{\rho})\det(\nabla\hat{\chi})v_t+\nabla\cdot((P(\gamma+\overline{\rho})-P(\overline{\rho}))\mathrm{cof}\,(\nabla\hat{\chi}))\\&+\mu\nabla\cdot((\nabla v((\nabla\hat{\chi})^{-1}-\mathrm{Id})+((\nabla\hat{\chi})^{-t}-\mathrm{Id})(\nabla v)^t)\mathrm{cof}\,(\nabla\hat{\chi}))+\mu\nabla\cdot((\nabla v+\nabla v^t)(\mathrm{cof}\,(\nabla\hat{\chi})-\mathrm{Id}))\\&+\mu'\nabla\cdot(\nabla v(((\nabla\hat{\chi})^{-1}-\mathrm{Id}):\mathrm{Id})\mathrm{cof}\,(\nabla\hat{\chi}))+\mu'\nabla\cdot(\nabla v(\mathrm{cof}\,(\nabla\hat{\chi})-\mathrm{Id}))\end{aligned}$$

and

$$(\mu(\nabla v+\nabla v^t)+\mu'\nabla\cdot v)n=G\ \text{ on }\partial\Omega_S(0),\quad v=0\text{ on }\partial\Omega, \tag{68}$$

with

$$\begin{aligned}&G:=-\mu(\nabla v((\nabla\hat{\chi})^{-1}-\mathrm{Id})+((\nabla\hat{\chi})^{-t}-\mathrm{Id})(\nabla v)^t)\mathrm{cof}\,(\nabla\hat{\chi})n-\mu(\nabla v+(\nabla v)^t)(\mathrm{cof}\,(\nabla\hat{\chi})-\mathrm{Id})n\\&-\mu'(\nabla v:\mathrm{Id})(\mathrm{cof}\,(\nabla\hat{\chi}):\mathrm{Id})n-\mu'(\nabla v((\nabla\hat{\chi})^{-1}-\mathrm{Id}):\mathrm{Id})\mathrm{cof}\,(\nabla\hat{\chi})n\\&+(P(\overline{\rho}+\gamma)-P(\overline{\rho}))\mathrm{cof}\,(\nabla\hat{\chi})n+(\lambda(\nabla\xi+\nabla\xi^t)+\lambda'\nabla\cdot\xi)n.\end{aligned}$$

• We will first prove that $(v,\xi)\in L^\infty(H^2)\times L^\infty(H^2(\Omega_S(0)))$. In order to prove this for $v$, we will show that $F\in L^\infty(L^2)$ and $G\in L^\infty(H^{1/2}(\partial\Omega_S(0)))$ with suitable estimates.

We observe that, thanks to (15) and (16),

$$\|\det(\nabla\hat{\chi})\|_{L^\infty(L^2)}+\|\mathrm{cof}\,(\nabla\hat{\chi})\|_{L^\infty(L^\infty)}\leqslant C\quad\text{and}\quad\|\mathrm{cof}\,(\nabla\hat{\chi})-\mathrm{Id}\|_{L^\infty(L^\infty)}\leqslant CR.$$

Combining this with (27) and (56), we find

$$\|F\|_{L^\infty(L^2)} \leqslant C_0 + T^\alpha \overline{N}_T(v) + CR\|v\|_{L^\infty(H^2)}.$$

Analogously, we get

$$\|G\|_{L^\infty(H^{1/2}(\partial\Omega_S(0)))} \leqslant C_0 + C(R\|v\|_{L^\infty(H^2)} + \|\xi\|_{L^\infty(H^2(\Omega_S(0)))}).$$

Using the elliptic regularity for (67)-(68), we obtain $v \in L^\infty(H^2)$ and

$$\|v\|_{L^\infty(H^2)} \leqslant C_0 + T^\alpha \overline{N}_T(v) + C(R\|v\|_{L^\infty(H^2)} + \|\xi\|_{L^\infty(H^2(\Omega_S(0)))}).$$

Taking $R$ small enough, we find that

$$\|v\|_{L^\infty(H^2)} \leqslant C_0 + T^\alpha \overline{N}_T(v) + C\|\xi\|_{L^\infty(H^2(\Omega_S(0)))}. \tag{69}$$

Let us take a look now at the equation satisfied by $\xi$:

$$\begin{cases} -\nabla\cdot(2\lambda\epsilon(\xi) + \lambda'(\nabla\cdot\xi)\mathrm{Id}) = -\xi_{tt} & \text{in } \Omega_S(0), \\ \xi(t,\cdot) = \xi_0 + \int_0^t v & \text{on } \partial\Omega_S(0). \end{cases} \tag{70}$$

Again classical elliptic estimates give

$$\begin{aligned} \|\xi\|_{L^\infty(H^2(\Omega_S(0)))} &\leqslant C(\|\xi_{tt}\|_{L^\infty(L^2(\Omega_S(0)))} + \|\xi_0\|_{H^2(\Omega_S(0))} + \|\int_0^t v\|_{L^\infty(H^2)}), \\ &\leqslant C(T\|\xi_{ttt}\|_{L^\infty(L^2(\Omega_S(0)))} + \|\xi_0\|_{H^2(\Omega_S(0))} + T\|v\|_{L^\infty(H^2)}). \end{aligned}$$

Using (66) to estimate the first term, we obtain

$$\begin{aligned} \|\xi\|_{L^\infty(H^2(\Omega_S(0)))} &\leqslant C_0 + T^\alpha \overline{N}_T(v) + CTR^{1/2}\|v\|_{L^\infty(W^{1,\infty})} + C(\|\xi_0\|_{H^2(\Omega_S(0))} + T\|v\|_{L^\infty(H^2)}) \\ &= C_0 + T^\alpha \overline{N}_T(v). \end{aligned} \tag{71}$$

Recall that $\alpha > 0$ stands for a generic constant which may depend on $q$. Combining (69) and (71), we find

$$\|v\|_{L^\infty(H^2)} + \|\xi\|_{L^\infty(H^2(\Omega_S(0)))} \leqslant C_0 + T^\alpha \overline{N}_T(v). \tag{72}$$

• Let us now prove that $(v,\xi) \in L^\infty(H^{11/4}) \times L^\infty(H^{11/4}(\Omega_S(0)))$. Thus, we have to prove that $F \in L^\infty(H^{3/4})$ and $G \in L^\infty(H^{5/4}(\partial\Omega_S(0)))$. First, using (27), we have

$$\|(\gamma+\overline{\rho})\det(\nabla\hat{\chi})v_t\|_{L^\infty(H^1)} \leqslant C\|\gamma+\overline{\rho}\|_{L^\infty(H^{7/4})}\|v_t\|_{L^\infty(H^1)} \leqslant C_0\|v_t\|_{L^\infty(H^1)}.$$

Observe now that $W^{2,\infty}(L^2) \cap L^\infty(H^2) \subset W^{1,\infty}(H^1)$. Hence, (66) and (72) provide

$$\begin{aligned} \|(\gamma+\overline{\rho})\det(\nabla\hat{\chi})v_t\|_{L^\infty(H^1)} &\leqslant C_0\|v\|_{W^{1,\infty}(H^1)} \leqslant C_0\|v\|^{1/2}_{L^\infty(H^2)}\|v\|^{1/2}_{W^{2,\infty}(L^2)} \\ &\leqslant C_0 + T^\alpha \overline{N}_T(v) + CR^{1/2}\|v\|_{L^\infty(W^{1,\infty})}. \end{aligned}$$

Next,

$$\|(P(\gamma+\overline{\rho}) - P(\overline{\rho}))\mathrm{cof}\,(\nabla\hat{\chi})\|_{L^\infty(H^{7/4})} \leqslant C_{M_1}\|\gamma\|_{L^\infty(H^{7/4})} \leqslant C_0$$

and

$$\begin{aligned} &\|\nabla v((\nabla\hat{\chi})^{-1} - \mathrm{Id})\mathrm{cof}\,(\nabla\hat{\chi})\|_{L^\infty(H^{7/4})} \leqslant C\|\nabla v\|_{L^\infty(H^{7/4})}\|(\nabla\hat{\chi})^{-1} - \mathrm{Id}\|_{L^\infty(H^{7/4})}\|\mathrm{cof}\,(\nabla\hat{\chi})\|_{L^\infty(H^{7/4})} \\ &\leqslant CR\|v\|_{L^\infty(H^{11/4})}. \end{aligned}$$

The other terms in the definition of $F$ are bounded similarly.

Hence, we have

$$\|F\|_{L^\infty(H^{3/4})} \leqslant C_0 + T^\alpha \overline{N}_T(v) + CR\|v\|_{L^\infty(H^{11/4})} + CR^{1/2}\|v\|_{L^\infty(W^{1,\infty})}.$$

Then, we get

$$\|G\|_{L^\infty(H^{5/4}(\partial\Omega_S(0)))} \leqslant C_0 + C(R\|v\|_{L^\infty(H^{11/4})} + \|\xi\|_{L^\infty(H^{11/4}(\Omega_S(0)))}),$$

so that

$$\|v\|_{L^\infty(H^{11/4})} \leqslant C_0 + T^\alpha \overline{N}_T(v) + C(R\|v\|_{L^\infty(H^{11/4})} + \|\xi\|_{L^\infty(H^{11/4}(\Omega_S(0)))}).$$

Taking $R$ small enough, we find

$$\|v\|_{L^\infty(H^{11/4})} \leqslant C_0 + T^\alpha \overline{N}_T(v) + C\|\xi\|_{L^\infty(H^{11/4}(\Omega_S(0)))}. \tag{73}$$

Now, the solution $\xi$ of the elliptic problem (70) satisfies $\xi \in L^\infty(H^{11/4})$ and

$$\begin{aligned}\|\xi\|_{L^\infty(H^{11/4}(\Omega_S(0)))} &\leqslant C(\|\xi_{tt}\|_{L^\infty(H^{3/4}(\Omega_S(0)))} + \|\xi_0\|_{H^{11/4}(\Omega_S(0))} + \|\textstyle\int_0^t v\|_{L^\infty(H^{11/4})})\\ &\leqslant C(\|\xi_{tt}\|_{L^\infty(H^{3/4}(\Omega_S(0)))} + \|\xi_0\|_{H^{11/4}(\Omega_S(0))} + T\|v\|_{L^\infty(H^{11/4})}).\end{aligned}$$

Then, using (66) we deduce that

$$\|\xi\|_{L^\infty(H^{11/4}(\Omega_S(0)))} \leqslant C_0 + CT^\alpha \overline{N}_T(v) + CR^{1/2}\|v\|_{L^\infty(W^{1,\infty})} + C(\|\xi_0\|_{H^{11/4}(\Omega_S(0))} + T\|v\|_{L^\infty(H^{11/4})}).$$

Combining this with (73) and taking $T$ and $R$ small enough yields

$$\|v\|_{L^\infty(H^{11/4})} + \|\xi\|_{L^\infty(H^{11/4}(\Omega_S(0)))} \leqslant C_0 + T^\alpha \overline{N}_T(v). \tag{74}$$

• Let us finally show that $(v,\xi) \in L^2(H^3) \times L^\infty(H^3(\Omega_S(0)))$. In order to do this, we will estimate $F$ in $L^2(H^1)$ and $G$ in $L^2(H^{3/2}(\partial\Omega_S(0)))$.

For the first term in the expression of $F$, we use that

$$\|\gamma\|_{L^\infty(H^{7/4})} \leqslant C_0, \quad \|\det(\nabla\hat{\chi})\|_{L^\infty(H^2)} \leqslant C$$

(see (27)) and

$$\|v_t\|_{L^2(H^1)} \leqslant C_0 + T^\alpha \overline{N}_T(v)$$

(see (56)), so that

$$\|(\gamma+\overline{\rho})\det(\nabla\hat{\chi})v_t\|_{L^2(H^1)} \leqslant C_0 + T^\alpha \overline{N}_T(v).$$

For the second term, we combine $\|\gamma\|_{L^2(H^2)} \leqslant C_0$ (see (27)) with $\|\mathrm{cof}\,(\nabla\hat{\chi})\|_{L^\infty(H^2)} \leqslant C$ and we get

$$\|\nabla\cdot((P(\gamma+\overline{\rho}) - P(\overline{\rho}))\mathrm{cof}\,(\nabla\hat{\chi}))\|_{L^2(H^1)} \leqslant C_0.$$

Finally, the $L^2(H^1)$-norm of the last four terms is bounded by $R\|v\|_{L^2(H^3)}$.

On the other hand, one can estimate $G$ analogously as follows:

$$\|G\|_{L^2(H^{3/2}(\partial\Omega_S(0)))} \leqslant C_0 + C(R\|v\|_{L^2(H^3)} + \|\xi\|_{L^2(H^3(\Omega_S(0)))}).$$

Consequently,

$$\|v\|_{L^2(H^3)} \leqslant C_0 + C\|\xi\|_{L^2(H^3(\Omega_S(0)))} + C_0T^\alpha \overline{N}_T(v). \tag{75}$$

Regarding now (70), we get

$$\begin{aligned}\|\xi\|_{L^\infty(H^3(\Omega_S(0)))} &\leqslant C(\|\xi_{tt}\|_{L^\infty(H^1(\Omega_S(0)))} + \|\xi_0\|_{H^{11/4}(\Omega_S(0))} + \|\textstyle\int_0^t v\|_{L^\infty(H^3)})\\ &\leqslant C(\|\xi\|_{W^{2,\infty}(H^1(\Omega_S(0)))} + \|\xi_0\|_{H^{11/4}(\Omega_S(0))} + T^{1/2}\|v\|_{L^2(H^3)}).\end{aligned}$$

Using again (66), we have

$$\begin{aligned}\|\xi\|_{L^\infty(H^3(\Omega_S(0)))} &\leqslant C_0 + T^\alpha \overline{N}_T(v) + CR^{1/2}\|v\|_{L^\infty(W^{1,\infty})} + C(\|\xi_0\|_{H^{11/4}(\Omega_S(0))} + T^{1/2}\|v\|_{L^2(H^3)})\\ &\leqslant C_0 + T^\alpha \overline{N}_T(v),\end{aligned}$$

where we have used (74) to estimate the third term.

Combining this with (75), we find that

$$\|v\|_{L^2(H^3)} + \|\xi\|_{L^\infty(H^3(\Omega_S(0)))} \leqslant C_0 + T^\alpha \overline{N}_T(v). \tag{76}$$

As a conclusion, (76), (74) and (66) provide the desired estimate (35).

# 3 Fixed point argument

We apply the following contraction fixed-point theorem:

**Theorem 5** *Let $M$ be a nonempty, compact, convex subset of a Banach space $Z$ and suppose that $\Lambda : M \to M$ satisfies*

$$\|\Lambda(\hat{v}_1) - \Lambda(\hat{v}_2)\|_Z \leqslant \|\hat{v}_1 - \hat{v}_2\|_Z \quad \forall \hat{v}_1, \hat{v}_2 \in M. \tag{77}$$

*Then, $\Lambda$ has a unique fixed point.*

Let $R_0 > 0$ and $T_1 > 0$ provided by Lemma 4. We apply Theorem 5 to $\Lambda$ introduced in (25), $Z_T := L^2(0,T;H^2) \cap H^1(0,T;L^2)$ and $M := X_{T,R}$, where $T < T_1$ and $R < R_0$.

$\Lambda$ is well-defined from $X_{T,R}$ to $X_{T,R}$ thanks to Lemma 4 (see (36)).

It remains to prove (77). Indeed, we will prove that there exists $T_2 > 0$ such that for any $T < T_2$, the following inequality holds:

$$\|\Lambda(\hat{v}_1) - \Lambda(\hat{v}_2)\|_{Z_T} \leqslant T^\kappa \|\hat{v}_1 - \hat{v}_2\|_{Z_T} \quad \forall \hat{v}_1, \hat{v}_2 \in M$$

for some $\kappa > 0$. Then, estimate (77) readily follows.

Let $\hat{v}_1, \hat{v}_2 \in X_{T,R}$ and $(\gamma_1, v_1, \xi_1)$ (resp. $(\gamma_2, v_2, \xi_2)$) the corresponding solution of (4), (21)-(24) associated to $\hat{v}_1$ (resp. $\hat{v}_2$). Then, $\alpha := \gamma_1 - \gamma_2$ satisfies

$$\begin{cases} \alpha_t + (\nabla\hat{v}_2(\nabla\hat{\chi}_2)^{-1} : \mathrm{Id})\alpha = H & \text{in } \Omega_S(0), \\ \alpha_{|t=0} = 0 & \text{in } \Omega_S(0), \end{cases}$$

with

$$H := -\overline{\rho}\nabla\hat{v}_1(\nabla\hat{\chi}_1)^{-1} : \mathrm{Id} + \overline{\rho}\nabla\hat{v}_2(\nabla\hat{\chi}_2)^{-1} : \mathrm{Id} + (\nabla\hat{v}_2(\nabla\hat{\chi}_2)^{-1} : \mathrm{Id} - \nabla\hat{v}_1(\nabla\hat{\chi}_1)^{-1} : \mathrm{Id})\gamma_1.$$

Since $\hat{v}_1, \hat{v}_2 \in X_{T,R}$ and $\gamma_1$ satisfies (27), we have that

$$\|H\|_{L^2(H^1)} \leqslant C_0 \|\hat{v}_1 - \hat{v}_2\|_{Z_T}.$$

This proves that $\|\alpha\|_{L^\infty(H^1)} \leqslant T^{1/2} C_0 \|\hat{v}_1 - \hat{v}_2\|_{Z_T}$ (see (30) for instance). From the equation of $\alpha$, we also have that

$$\|\alpha_t\|_{L^2(H^1)} \leqslant C_0 \|\hat{v}_1 - \hat{v}_2\|_{Z_T}. \tag{78}$$

Let now $w := v_1 - v_2$ and $\beta := \xi_1 - \xi_2$. They satisfy $w(0,\cdot) = 0$ in $\Omega_F(0)$, $w = 0$ on $\partial\Omega$, $\beta(0,\cdot) = 0$ in $\Omega_S(0)$ and the following equations:

$$\begin{cases} (\gamma_2 + \overline{\rho})\det(\nabla\hat{\chi}_2)w_t - \nabla\cdot[(\mu(\nabla w(\nabla\hat{\chi}_2)^{-1} + (\nabla\hat{\chi}_2)^{-t}(\nabla w)^t) + \mu'(\nabla w(\nabla\hat{\chi}_2)^{-1} : \mathrm{Id})\mathrm{Id})\mathrm{cof}\,(\nabla\hat{\chi}_2)] \\ \qquad -\nabla\cdot[(P(\gamma_2+\overline{\rho}) - P(\gamma_1+\overline{\rho}))\mathrm{cof}\,(\nabla\hat{\chi}_2)] = g \quad \text{in } \Omega_F(0), \end{cases} \tag{79}$$

$$\beta_{tt} - \nabla\cdot(2\lambda\epsilon(\beta) + \lambda'(\nabla\cdot\beta)\mathrm{Id}) = 0 \quad \text{in } \Omega_S(0), \tag{80}$$

$$\begin{cases} w = \beta_t \quad \text{on } \partial\Omega_S(0), \\ \big(\mu(\nabla w(\nabla\hat{\chi}_2)^{-1} + (\nabla\hat{\chi}_2)^{-t}(\nabla w)^t) + \mu'(\nabla w(\nabla\hat{\chi}_2)^{-1} : \mathrm{Id})\mathrm{Id} + (P(\overline{\rho}+\gamma_2) - P(\overline{\rho}+\gamma_1))\big)\mathrm{cof}\,\nabla\hat{\chi}_2\,\mathrm{n} \\ = \big(2\lambda\epsilon(\beta) + \lambda'(\nabla\cdot\beta)\mathrm{Id}\big)\mathrm{n} + h \quad \text{on } \partial\Omega_S(0), \end{cases} \tag{81}$$

where the right hand sides are given by

$$\begin{aligned} g := {} & ((\gamma_2+\overline{\rho})\det(\nabla\hat{\chi}_2) - (\gamma_1+\overline{\rho})\det(\nabla\hat{\chi}_1))v_{1,t} - \mu\nabla\cdot[\nabla v_1((\nabla\hat{\chi}_2)^{-1}\mathrm{cof}\,(\nabla\hat{\chi}_2) - (\nabla\hat{\chi}_1)^{-1}\mathrm{cof}\,(\nabla\hat{\chi}_1))] \\ & -\mu\nabla\cdot((\nabla\hat{\chi}_2)^{-t}(\nabla v_1)^t\mathrm{cof}\,(\nabla\hat{\chi}_2) - (\nabla\hat{\chi}_1)^{-t}(\nabla v_1)^t\mathrm{cof}\,(\nabla\hat{\chi}_1)) \\ & -\mu'\nabla\cdot[(\nabla v_1(\nabla\hat{\chi}_2)^{-1} : \mathrm{Id})\mathrm{cof}\,(\nabla\hat{\chi}_2) - (\nabla v_1(\nabla\hat{\chi}_1)^{-1} : \mathrm{Id})\mathrm{cof}\,(\nabla\hat{\chi}_1)] \\ & -\nabla\cdot[(P(\gamma_1+\overline{\rho}) - P(\overline{\rho}))(\mathrm{cof}\,(\nabla\hat{\chi}_1) - \mathrm{cof}\,(\nabla\hat{\chi}_2))] \end{aligned}$$

and

$$\begin{aligned}
h :=\ & \mu\nabla v_1((\nabla\hat\chi_2)^{-1}\mathrm{cof}\,(\nabla\hat\chi_2) - (\nabla\hat\chi_1)^{-1}\mathrm{cof}\,(\nabla\hat\chi_1))n\\
&+\mu((\nabla\hat\chi_2)^{-t}(\nabla v_1)^t\mathrm{cof}\,(\nabla\hat\chi_2) - (\nabla\hat\chi_1)^{-t}(\nabla v_1)^t\mathrm{cof}\,(\nabla\hat\chi_1))n\\
&+\mu'((\nabla v_1(\nabla\hat\chi_2)^{-1}:\mathrm{Id})\mathrm{cof}\,(\nabla\hat\chi_2) - (\nabla v_1(\nabla\hat\chi_1)^{-1}:\mathrm{Id})\mathrm{cof}\,(\nabla\hat\chi_1))n\\
&-(P(\gamma_1+\overline\rho)-P(\overline\rho))(\mathrm{cof}\,(\nabla\hat\chi_1)-\mathrm{cof}\,(\nabla\hat\chi_2))n.
\end{aligned}$$

1) Estimate of $w$ in $L^\infty(L^2)$

Using the estimates

$$\|(\det(\nabla\hat\chi_1)-\det(\nabla\hat\chi_2), (\mathrm{cof}\,(\nabla\hat\chi_1)-\mathrm{cof}\,(\nabla\hat\chi_2)), ((\nabla\hat\chi_1)^{-1}-(\nabla\hat\chi_2)^{-1})\|_{L^\infty(H^1)} \leqslant C\|\hat v_1-\hat v_2\|_{Z_T},$$

$\|\gamma_1-\gamma_2\|_{L^\infty(H^1)} \leqslant C\|\hat v_1-\hat v_2\|_{Z_T}$ and $\hat v_1, \hat v_2 \in X_{T,R}$ we have that

$$\|g\|_{L^2(L^2)} + \|h\|_{L^2(L^2(\partial\Omega_S(0)))} \leqslant T^\kappa\|\hat v_1-\hat v_2\|_{Z_T} \tag{82}$$

for some $\kappa>0$. Consequently, multiplying the equation of $w$ by $w$ and the equation of $\beta$ by $\beta_t$, we find that

$$\|w\|_{L^\infty(L^2)} + \|\beta\|_{L^\infty(H^1)\cap W^{1,\infty}(L^2)} \leqslant T^\kappa\|\hat v_1-\hat v_2\|_{Z_T}.$$

Here and in what follows, $\kappa>0$ is a constant which may change from line to line.

2) Estimate of $w$ in $H^1(H^1)\cap W^{1,\infty}(L^2)$

Let us now differentiate the equation of $w$ with respect to $t$. This yields

$$\left\{\begin{array}{l}
(\gamma_2+\overline\rho)\det(\nabla\hat\chi_2)w_{tt} - \nabla\cdot[(\mu(\nabla w(\nabla\hat\chi_2)^{-1} + (\nabla\hat\chi_2)^{-t}(\nabla w)^t) + \mu'(\nabla w(\nabla\hat\chi_2)^{-1}:\mathrm{Id})\mathrm{Id})\mathrm{cof}\,(\nabla\hat\chi_2)]_t\\
\qquad -\nabla\cdot[(P(\gamma_2+\overline\rho)-P(\gamma_1+\overline\rho))\mathrm{cof}\,(\nabla\hat\chi_2)]_t = \tilde g \quad \text{in } \Omega_F(0),
\end{array}\right. \tag{83}$$

$$\beta_{ttt} - \nabla\cdot(2\lambda\epsilon(\beta_t)+\lambda'(\nabla\cdot\beta_t)\mathrm{Id}) = 0 \quad \text{in } \Omega_S(0). \tag{84}$$

and

$$\left\{\begin{array}{l}
w_t = \beta_{tt} \quad \text{on } \partial\Omega_S(0),\\
[\big(\mu(\nabla w(\nabla\hat\chi_2)^{-1} + (\nabla\hat\chi_2)^{-t}(\nabla w)^t) + \mu'(\nabla w(\nabla\hat\chi_2)^{-1}:\mathrm{Id})\mathrm{Id} + (P(\overline\rho+\gamma_2)-P(\overline\rho+\gamma_1))\big)\mathrm{cof}\,\nabla\hat\chi_2]_t\mathrm{n}\\
= \big(2\lambda\epsilon(\beta_t)+\lambda'(\nabla\cdot\beta_t)\mathrm{Id}\big)\mathrm{n} + h_t \quad \text{on } \partial\Omega_S(0),
\end{array}\right. \tag{85}$$

where

$$\tilde g := g_t - \gamma_{2,t}\det(\nabla\hat\chi_2)w_t - (\gamma_2+\overline\rho)(\det(\nabla\hat\chi_2))_t w_t.$$

Arguing as we did for (48), we multiply equation (83) by $w_t$ and equation (84) by $\beta_{tt}$. This gives an estimate of

$$\|w_t\|^2_{L^2(H^1)} + \|w_t\|^2_{L^\infty(L^2)} + \|\beta_{tt}\|^2_{L^\infty(L^2)} + \|\beta_t\|^2_{L^\infty(H^1)}$$

in terms of

$$C\left(\left|\iint_{Q_T} w_t\cdot g_t\,dy\,dt\right| + \left|\iint_{\Sigma_T} w_t\cdot h_t\,d\sigma\,dt\right|\right). \tag{86}$$

a) Estimate of $g_t$.

Observe that $g_t = g_1 + \nabla\cdot(g_2)$ with

$$g_1 := [((\gamma_2+\overline\rho)\det(\nabla\hat\chi_2) - (\gamma_1+\overline\rho)\det(\nabla\hat\chi_1))v_{1,t}]_t$$

$$\begin{aligned}
g_2 :=\ & -\mu[\nabla v_1((\nabla\hat\chi_2)^{-1}\mathrm{cof}\,(\nabla\hat\chi_2) - (\nabla\hat\chi_1)^{-1}\mathrm{cof}\,(\nabla\hat\chi_1))]_t\\
&-\mu[((\nabla\hat\chi_2)^{-t}\mathrm{cof}\,(\nabla\hat\chi_2) - (\nabla\hat\chi_1)^{-t}\mathrm{cof}\,(\nabla\hat\chi_1))(\nabla v_1)^t]_t\\
&-\mu'[(\nabla v_1(\nabla\hat\chi_2)^{-1}:\mathrm{Id})\mathrm{cof}\,(\nabla\hat\chi_2) - (\nabla v_1(\nabla\hat\chi_1)^{-1}:\mathrm{Id})\mathrm{cof}\,(\nabla\hat\chi_1)]_t - [P(\gamma_1+\overline\rho)(\mathrm{cof}\,(\nabla\hat\chi_1)-\mathrm{cof}\,(\nabla\hat\chi_2))]_t
\end{aligned}$$

We have

$$\left|\iint_{Q_T} w_t \cdot g_1 \, dy \, dt\right| \leqslant \varepsilon \|w_t\|^2_{L^\infty(L^2)} + C\|g_1\|^2_{L^1(L^2)},$$

$$\begin{aligned}\left|\iint_{Q_T} w_t \cdot (\nabla \cdot g_2) \, dy \, dt\right| &= \left| - \iint_{Q_T} \nabla w_t : g_2 \, dy \, dt - \iint_{\Sigma_T} w_t \cdot (g_2 \cdot n) \, d\sigma \, dt\right| \\ &\leqslant \varepsilon \|w_t\|^2_{L^2(H^1)} + C\|g_2\|^2_{L^2(W^{s,4/3})},\end{aligned} \tag{87}$$

for any $s > 3/4$. Here, we have used that $H^{1/2}(\partial\Omega_S(0)) \hookrightarrow L^4(\partial\Omega_S(0))$, $W^{3/4,4/3}(\Omega_F(0)) \hookrightarrow L^2(\Omega_F(0))$ and that the trace operator is continuous from $W^{s,4/3}(\Omega_F(0))$ to $L^{4/3}(\partial\Omega_S(0))$ for any $s > 3/4$. Thus, it remains to show an estimate of $\|g_1\|_{L^1(L^2)}$ and $\|g_2\|_{L^2(W^{s,4/3})}$ for some $s > 3/4$.

• Estimate of $g_1$ in $L^1(L^2)$.

We are going to prove that

$$\|((\gamma_2 + \overline{\rho}) \det(\nabla\hat{\chi}_2) - (\gamma_1 + \overline{\rho}) \det(\nabla\hat{\chi}_1))v_{1,t}\|_{W^{1,1}(L^2)} \leqslant T^\kappa \|\hat{v}_1 - \hat{v}_2\|_{Z_T}. \tag{88}$$

This expression can be rewritten as follows:

$$((\gamma_2 - \gamma_1) \det(\nabla\hat{\chi}_2) - (\gamma_1 + \overline{\rho})(\det(\nabla\hat{\chi}_1) - \det(\nabla\hat{\chi}_2)))v_{1,t}.$$

We have that $\|v_{1,t}\|_{W^{1,p}(H^1)} \leqslant R$ since $v_{1,t} \in X_{T,R}$, $\|\gamma_1 - \gamma_2\|_{W^{1,1}(H^1)} \leqslant T^{1/2}C_0\|\hat{v}_1 - \hat{v}_2\|_{Z_T}$ from (78) and $\|\nabla\hat{\chi}_1 - \nabla\hat{\chi}_2\|_{H^1(H^1)} \leqslant \|\hat{v}_1 - \hat{v}_2\|_{Z_T}$ from (14). Finally, $\|\gamma_1\|_{W^{1,q}(H^{7/4})} \leqslant C$ thanks to (27) and $\|\nabla\hat{\chi}_1\|_{W^{1,1}(H^2)} \leqslant R$. Consequently, (88) is true for any $\kappa < 1 - 1/p$.

• Estimate of $g_2$ in $L^2(W^{4/5,4/3})$.

Due to the similar structure of the three first terms in the expression of $g_2$, we only deal with the first one. Let us prove that

$$\|\nabla v_1((\nabla\hat{\chi}_2)^{-1}\mathrm{cof}\,(\nabla\hat{\chi}_2) - (\nabla\hat{\chi}_1)^{-1}\mathrm{cof}\,(\nabla\hat{\chi}_1))\|_{H^1(W^{4/5,4/3})} \leqslant T^\kappa \|\hat{v}_1 - \hat{v}_2\|_{Z_T}. \tag{89}$$

This term can be rewritten as follows:

$$\nabla v_1(((\nabla\hat{\chi}_2)^{-1} - (\nabla\hat{\chi}_1)^{-1})\mathrm{cof}\,(\nabla\hat{\chi}_2) - (\nabla\hat{\chi}_1)^{-1}(\mathrm{cof}\,(\nabla\hat{\chi}_1) - \mathrm{cof}\,(\nabla\hat{\chi}_2))).$$

First, we have that $\|(\nabla\hat{\chi}_1)^{-1} - (\nabla\hat{\chi}_2)^{-1}\|_{H^{11/10}(H^{4/5})} + \|\mathrm{cof}\,(\nabla\hat{\chi}_1) - \mathrm{cof}\,(\nabla\hat{\chi}_2)\|_{H^{9/8}(H^{3/4})} \leqslant \|\hat{v}_1 - \hat{v}_2\|_{Z_T}$. Now, from (35), we get

$$\|v_1\|_{H^{11/10}(H^{19/10})} \leqslant \|v_1\|_{L^2(H^3)\cap H^2(H^1)} \leqslant C_0.$$

Hence, $\|\nabla v_1\|_{H^{11/10}(H^{9/10})} \leqslant C_0$. Since $(H^{9/10} \cdot H^{4/5})(\Omega_F(0)) \subset W^{4/5,10/7}(\Omega_F(0)) \subset W^{4/5,4/3}(\Omega_F(0))$ and $\|\nabla\hat{\chi}_1\|_{H^{11/10}(H^{19/10})} \leqslant C_0$, (89) holds for any $\kappa < 1/10$.

For the fourth and last term in the expression of $g_2$, we are going to establish that

$$\|P(\gamma_1 + \overline{\rho})(\mathrm{cof}\,(\nabla\hat{\chi}_1) - \mathrm{cof}\,(\nabla\hat{\chi}_2))\|_{H^1(W^{4/5,4/3})} \leqslant T^\kappa \|\hat{v}_1 - \hat{v}_2\|_{Z_T}. \tag{90}$$

This comes from the estimates

$$\|\mathrm{cof}\,(\nabla\hat{\chi}_1) - \mathrm{cof}\,(\nabla\hat{\chi}_2)\|_{H^{11/10}(H^{4/5})} + \|\mathrm{cof}\,(\nabla\hat{\chi}_1) - \mathrm{cof}\,(\nabla\hat{\chi}_2)\|_{L^\infty(H^1)} \leqslant \|\hat{v}_1 - \hat{v}_2\|_{Z_T},$$

$\|P(\gamma_1 + \overline{\rho})\|_{W^{1,q}(H^{7/4})} + \|P(\gamma_1 + \overline{\rho})\|_{L^\infty(H^{7/4})} \leqslant C_0$ (see (27)) and $H^{7/4} \cdot H^{4/5} \subset H^{4/5} \subset W^{4/5,4/3}$. Since $q > 4$, we can also take any $\kappa > 1/10$ in (90).

b) Estimate of $h_t$

In view of estimates (86)-(87), it suffices to prove that

$$\|h\|_{H^1(L^{4/3}(\partial\Omega_S(0)))} \leqslant T^\kappa \|\hat{v}_1 - \hat{v}_2\|_{Z_T} \tag{91}$$

For the three first terms we use that

$$\|(\mathrm{cof}\,(\nabla\hat{\chi}_1), \mathrm{cof}\,(\nabla\hat{\chi}_2), (\nabla\hat{\chi}_1)^{-1}, (\nabla\hat{\chi}_2)^{-1}\|_{H^1(L^\infty(\partial\Omega_S(0)))} \leqslant C_0,$$

$$\|\nabla v_1\|_{H^1(H^{1/2}(\partial\Omega_S(0)))} \leqslant C_0, \quad \|\mathrm{cof}\,(\nabla\hat\chi_1) - \mathrm{cof}\,(\nabla\hat\chi_2)\|_{H^1(L^2(\partial\Omega_S(0)))} \leqslant T^\kappa \|\hat v_1 - \hat v_2\|_{Z_T}$$

for any $\kappa > 1/5$. For the last term, we observe that

$$\|(\nabla\hat\chi_1)^{-1} - (\nabla\hat\chi_2)^{-1}\|_{H^1(H^{1/4}(\partial\Omega_S(0)))} \leqslant C_0 T^{1/8} \|\hat v_1 - \hat v_2\|_{Z_T}$$
$$\|P(\gamma_1 + \overline{\rho})\|_{W^{1,q}(H^{5/4}(\partial\Omega_S(0)))} \leqslant C_0.$$

Since $(H^{1/2} \cdot L^2)(\partial\Omega_S(0)) \subset L^{4/3}(\partial\Omega_S(0))$ and $H^{5/4}(\partial\Omega_S(0)) \subset L^\infty(\partial\Omega_S(0))$, (91) holds for any $\kappa < 1/8$.

So, for the moment we have that

$$\|w\|_{H^1(H^1)\cap W^{1,\infty}(L^2)} + \|\beta\|_{W^{2,\infty}(L^2)\cap W^{1,\infty}(H^1)} \leqslant T^\kappa \|\hat v_1 - \hat v_2\|_{Z_T}, \tag{92}$$

for some $\kappa > 0$.

3) Estimate of $w$ in $L^2(H^2)$

Let us consider the elliptic equation satisfied by $w$ (see (79), (81)):

$$-\nabla\cdot(\mu(\nabla w + (\nabla w)^t) + \mu'\nabla\cdot w) = g - (\gamma_2 + \overline{\rho})\det(\nabla\hat\chi_2) w_t + \nabla\cdot((P(\gamma_2+\overline{\rho}) - P(\gamma_1+\overline{\rho}))\mathrm{cof}\,(\nabla\hat\chi_2))$$
$$-\mu\nabla\cdot(\nabla w(\mathrm{Id} - (\nabla\hat\chi_2)^{-1}\mathrm{cof}\,(\nabla\hat\chi_2)) + ((\nabla w)^t - (\nabla\hat\chi_2)^{-1}(\nabla w)^t\mathrm{cof}\,(\nabla\hat\chi_2)))$$
$$-\mu'\nabla\cdot(\nabla w(\mathrm{Id} - ((\nabla\hat\chi_2)^{-1} : \mathrm{Id})\mathrm{cof}\,(\nabla\hat\chi_2))) := g + \sum_{i=1}^4 g^i \quad \text{in } \Omega_F(0),$$
$$\mu(\nabla w + (\nabla w)^t)n + \mu'(\nabla\cdot w)n = h + (2\lambda\epsilon(\beta) + \lambda'(\nabla\cdot\beta)\mathrm{Id})n - (P(\gamma_2+\overline{\rho}) - P(\gamma_1+\overline{\rho}))\mathrm{cof}\,(\nabla\hat\chi_2)n$$
$$+\mu(\nabla w(\mathrm{Id} - (\nabla\hat\chi_2)^{-1}\mathrm{cof}\,(\nabla\hat\chi_2)) + (\nabla w)^t - (\nabla\hat\chi_2)^{-t}(\nabla w)^t\mathrm{cof}\,(\nabla\hat\chi_2))n$$
$$+\mu'\nabla w(\mathrm{Id} - ((\nabla\hat\chi_2)^{-1} : \mathrm{Id})\mathrm{cof}\,(\nabla\hat\chi_2))n := h + \sum_{i=1}^4 h^i \quad \text{on } \partial\Omega_S(0).$$

As we showed before, $\|g\|_{L^2(L^2)} \leqslant T^\kappa\|\hat v_1 - \hat v_2\|_{Z_T}$ (see (82)). From (92), we have that $\|g^1\|_{L^2(L^2)} \leqslant T^\kappa\|\hat v_1 - \hat v_2\|_{Z_T}$. Since $\|\gamma_1 - \gamma_2\|_{L^\infty(H^1)} \leqslant C_0 T^{1/2}\|\hat v_1 - \hat v_2\|_{Z_T}$, we find that $\|g^2\|_{L^\infty(L^2)} \leqslant C_0 T^{1/2}\|\hat v_1 - \hat v_2\|_{Z_T}$. At last, the last two terms satisfy

$$\sum_{i=3}^4 \|g^i\|_{L^2(L^2)} \leqslant CR\|w\|_{L^2(H^2)}.$$

As long as the boundary term is concerned, we have that $\|h\|_{L^\infty(H^{1/2}(\partial\Omega_S(0)))} \leqslant C\|\hat v_1 - \hat v_2\|_{Z_T}$ since $\|(\mathrm{cof}\,(\nabla\hat\chi_2) - \mathrm{cof}\,(\nabla\hat\chi_1))\|_{L^\infty(H^{1/2}(\partial\Omega_S(0)))} \leqslant C\|\hat v_1 - \hat v_2\|_{Z_T}$, $\|\gamma_1\|_{L^\infty(H^{5/4}(\partial\Omega_S(0)))} \leqslant C_0$ and $\|\nabla v_1\|_{L^\infty(H^{5/4}(\partial\Omega_S(0)))} \leqslant C_0$ (see (35)). Next,

$$\|h^2\|_{L^\infty(H^{1/2}(\partial\Omega_S(0)))} \leqslant C\|\gamma_1 - \gamma_2\|_{L^\infty(H^{1/2}(\partial\Omega_S(0)))} \leqslant C_0 T^{1/2}\|\hat v_1 - \hat v_2\|_{Z_T}.$$

Finally,

$$\sum_{i=3}^4 \|h^i\|_{L^2(H^{1/2}(\partial\Omega_S(0)))} \leqslant CR\|w\|_{L^2(H^2)}.$$

Consequently, we obtain $w \in L^2(H^2)$ and

$$\|w\|_{L^2(H^2)} \leqslant C(T^\kappa\|\hat v_1 - \hat v_2\|_{Z_T} + \|\beta\|_{L^2(H^2(\Omega_S(0)))}), \tag{93}$$

for some $\kappa > 0$. On the other hand, $\beta$ satisfies the following elliptic problem:

$$\begin{cases} -\nabla\cdot(2\lambda\epsilon(\beta) + \lambda'(\nabla\cdot\beta)\mathrm{Id}) = -\beta_{tt} & \text{in } \Omega_S(0), \\ \beta = \int_0^t w & \text{on } \partial\Omega_S(0). \end{cases}$$

This provides

$$\|\beta\|_{L^2(H^2(\Omega_S(0)))} \leqslant C(\|\beta_{tt}\|_{L^2(L^2)} + \|\textstyle\int_0^t w\|_{L^2(H^2)}) \leqslant C(\|\beta\|_{H^2(L^2)} + T\|w\|_{L^2(H^2)}).$$

Using (92) and combining this with (93), we conclude that $\|w\|_{L^2(H^2)} \leqslant T^\kappa\|\hat v_1 - \hat v_2\|_{Z_T}$ (we also have $\|\beta\|_{L^2(H^2(\Omega_S(0)))} \leqslant T^\kappa\|\hat v_1 - \hat v_2\|_{Z_T}$).

# References


[1] H. Beirão da Veiga, *On the existence of strong solutions to a coupled fluid-structure evolution problem*, J. Math. Fluid Mech., **6** (2004), no. 1, 21–52.

[2] M. Boulakia, *Existence of weak solutions for an interaction problem between an elastic structure and a compressible viscous fluid*, J. Math. Pures et Appliquées, **84** (2005), no. 11, 1515–1554.

[3] M. Boulakia, *Existence of weak solutions for the three dimensional motion of an elastic structure in an incompressible fluid*, J. Math. Fluid Mech., **9** (2007), no. 2, 262–294.

[4] M. Boulakia, S. Guerrero, *A regularity result for a solid-fluid system associated to the compressible Navier-Stokes equations*, to appear in Ann. Inst. H. Poincaré Anal. Non Linéaire

[5] A. Chambolle, B. Desjardins, M.J. Esteban, C. Grandmont, *Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate*, J. Math. Fluid Mech., **7** (2005), no. 3, 368–404.

[6] C. Conca, J. San Martin, M. Tucsnak, *Existence of solutions for the equations modelling the motion of a rigid body in a viscous fluid*, Comm. Partial Differential Equations, **25** (2000), no. 5-6, 1019–1042.

[7] D. Coutand, S. Shkoller, *Motion of an elastic solid inside an incompressible viscous fluid*, Arch. Ration. Mech. Anal., **176** (2005), no. 1, 25–102.

[8] B. Desjardins, M. J. Esteban, *On weak solutions for fluid-rigid structure interaction: compressible and incompressible models*, Comm. Partial Differential Equations, **25** (2000), no. 7-8, 1399–1413.

[9] B. Desjardins, M. J. Esteban, C. Grandmont, P. Le Tallec, *Weak solutions for a fluid-elastic structure interaction model*, Rev. Mat. Complut. **14** (2001), no. 2, 523–538.

[10] E. Feireisl, A. Novotný, H. Petzeltová, *On the existence of globally defined weak solutions to the Navier-Stokes equations*, J. Math. Fluid Mech. **3** (2001), no. 4, 358–392.

[11] E. Feireisl, *On the motion of rigid bodies in a viscous compressible fluid*, Arch. Ration. Mech. Anal. **167** (2003), no. 4, 281–308.

[12] E. Feireisl, *Dynamics of viscous compressible fluids*, Oxford Science Publications, Oxford (2004).

[13] C. Grandmont, *Existence of weak solutions for the unsteady interaction of a viscous fluid with an elastic plate*, SIAM J. Math. Anal., **40** (2008), no. 2, 716–737.

[14] C. Grandmont, Y. Maday, *Existence for an unsteady fluid-structure interaction problem*, M2AN Math. Model. Numer. Anal., **34** (2000), no. 3, 609–636.

[15] M. Hillairet, *Lack of collision between solid bodies in a 2D incompressible viscous flow*, Comm. Partial Differential Equations, **32** (2007), no. 7-9, 1345–1371.

[16] M. Hillairet, T. Takahashi, *Collisions in three-dimensional fluid structure interaction problems*, SIAM J. Math. Anal., **40** (2009), no. 6, 2451–2477.

[17] D. Hoff, *Global solutions of the Navier-Stokes equations for multidimensional compressible flow with discontinuous initial data*, J. Differential Equations **120** (1995), no. 1, 215–254.

[18] D. Hoff, *Strong convergence to global solutions for multidimensional flows of compressible, viscous fluids with polytropic equations of state and discontinuous initial data*, Arch. Rational Mech. Anal. **132** (1995), no. 1, 1–14.

[19] P.-L. Lions, *Existence globale de solutions pour les équations de Navier-Stokes compressibles isentropiques*, C. R. Acad. Sci. Paris Sr. I Math. **316** (1993), no. 12, 1335–1340.

[20] P.L. Lions, *Mathematical Topics in Fluid Mechanics*, Oxford Science Publications, Oxford (1996).

[21] J. San Martin, V. Starovoitov, M. Tucsnak, *Global weak solutions for the two dimensional motion of several rigid bodies in an incompressible viscous fluid*, Arch. Rational Mech. Anal., **161** (2002), no. 2, 93–112.

[22] T. Takahashi, *Analysis of strong solutions for the equations modeling the motion of a rigid-fluid system in a bounded domain*, Adv. Differential Equations, **8** (2003), no. 12, 1499–1532.

[23] A. Tani, *On the first initial-boundary value problem of compressible viscous fluid motion*, Publ. RIMS, Kyoto Univ. **13** (1977), 193–253.