Budan Tables of Real Univariate Polynomials

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Abstract

The Budan table of f collects the signs of the iterated derivatives of f. We revisit the classical Budan-Fourier theorem for a univariate real polynomial f and establish a new connectivity property of its Budan table. We use this property to characterize the virtual roots of f (introduced by Gonzalez-Vega, Lombardi, Mahé in 1998); they are continuous functions of the coefficients of f. We also consider a property $(\mathcal{P})$ of a polynomial f, which is generically satisfied, it eases the topological-combinatorial description and study of Budan tables. A natural extension of the information collected by the virtual roots provides alternative representations of $(\mathcal{P})$-polynomials; while an attached tree structure allows a finite stratification of the space of polynomials with fixed degree.

Key words: Budan-Fourier theorem; virtual roots; roots of real univariate polynomial; Budan table; genericity; representation of univariate polynomials; Hermite-Birkhoff interpolation; B-tree; stratification; incidence relations.

1. Introduction

Counting and localizing roots of univariate polynomials is a basic task in Algebra and Numerical analysis. Many known techniques and interesting historical notes can be found e.g. in the book (24) of Rahman and Schmeisser. Established two centuries ago the Budan-Fourier theorem connects the number of real roots of a polynomial in an interval with the sign variations of a sequence and inspired the invention of Sturm sequences. It also gave rise to efficient and certified subdivision algorithms see (7) and (1).

Subdivision methods exploit the ordered structure of the real numbers, and are widely applied for calculating good approximations of solutions of polynomial equations or intersections of surfaces in many applied sciences. The analysis of their complexity and

efficiency relies on the the geometric and algebraic nature of the inputs. The dictionary between invariants readable on equations and features of varieties, in complex algebraic geometry, is ultimately based on the fact that a polynomial of degree n admits n roots. This is not the case for real roots. A natural strategy for studying properties of real algebraic varieties is to consider simultaneously roots of iterated derivatives of the input. A progress was achieved by Gonzalez-Vega, Lombardi, Mahé in (11) when they introduced the concept of the n virtual roots of a degree n polynomial f (which depend continuously on the coefficients of f), to provide a good substitute to its n complex roots. Their motivation was an attempt to prove Pierce-Birkhoff conjecture on the representation of C^0-polynomial spline functions. Virtual roots were then studied by Coste, Lajous, Lombardi, Roy in (9), and recently by Bembé in his PhD thesis (5).

The table containing the signs of all the derivatives of a polynomial f is called, in this paper, Budan table. The table is an infinite rectangle contained in the real plane, formed by positive and negative rectangular blocks, separated by segments corresponding to roots of derivatives. This structure was used by various mathematicians for separating and labeling the different real roots of a polynomial, see (7). However up to now, authors considered properties of either rows or columns of a table, or their sign variations but not the table "itself". Here, we adopt a 2D point of view, re-interpret known results and present new ones. We analyze the different admissible configurations of successive rows in such a table and deduce the 2D same-sign-connectivity of the positive (resp. negative) blocks components, then characterize the virtual roots using this data.
We define a simple property, we call $(\mathcal{P})$, on the roots of the iterative derivatives of a monic polynomial f, which is generically satisfied. Then, for a $(\mathcal{P})$-polynomial, i.e. assuming $(\mathcal{P})$, we study and characterize the topological and combinatorial properties of the corresponding Budan table and virtual roots. We specialize results of Hermite-Birkhoff interpolation (see e.g. (21)) to our setting, to better analyze the relation between a $(\mathcal{P})$-polynomial f and its virtual roots. We also introduce a finite discrete invariant, the B-tree of a $(\mathcal{P})$-polynomial f; and initiate the study of the attached stratification by semi-algebraic sets of the space of $(\mathcal{P})$-polynomials. Then we extend most of the results to the general case when $(\mathcal{P})$ is not necessarily satisfied.

The same-sign-connected components of the Budan table can be interpreted as plane surfaces delimited by discretized curves which can be approximated by continuous ones. This point of view was initiated in (6) and in (12) using derivatives with non-integer orders, called fractional derivatives. We plan to develop it further in a future work (13).

The paper is organized as follows. Section 2 provides basic definitions and properties presented with our point of view. Section 3 introduces our generic assumption $(\mathcal{P})$, then presents our results in this generic case and define our new invariant the B-tree; they are illustrated with simple examples; the proofs are collected in Section 4. Section 5 examines admissible configurations for Budan tables and other invariants in the non generic case and establishes continuity results. Section 6 concludes with some extensions and potential applications.

A partial and preliminary version of our approach was presented in the first part of our talk at the ISSAC'2011 conference, (and in the corresponding proceeding paper (6)).

Acknowledgements

Notations:

We denote by $\mathbb{R}$ the field of real numbers, by $\mathbb{R}[x]$ the ring of real univariate polynomial, by $\mathbb{R}[x]_n$ the corresponding vector space when the degree is at most n; it is equipped with the L_2 norm on the coefficients in the monomial basis. We denote by f a monic polynomial of degree n, $f := x^n + \sum_{i=0}^{i=n-1} a_i x^i$, the set of coefficients (a_i) belongs to $\mathbb{R}^n$. The affine space $\mathbb{E}_n$ of monic polynomials of degree n is identified with $\mathbb{R}^n$.
For an integer m, we denote by $f^{(m)}$ the $m-$th derivative of f, with $f^{(0)} := f$.

2.   Basics on Budan tables

In this section, after the definition of a Budan table, we list classical results around Budan-Fourier theorem, the proofs are elementary and can be found e.g. in (24) chapter 10, section 1. We also recall results of (11; 9).

The simplest observation that we will exploit in our approach is the following elementary lemma.

Lemma 1. *Assume that a is a root of f, then in a small enough interval $]a, a + h[$, $h > 0$, f and its derivative f' have the same sign.*

Proof. Express $f(x)$ as a definite integral of $f'(x)$ on a small interval $]a, a + h[$ where f' keeps the same sign.   □

Definition 2. Let f be a monic univariate polynomial of degree n. The Budan table of f is a subset of the real plane equal to the union of $n + 1$ infinite rectangles of height one $L_i := \mathbb{R} \times [i - 1/2, i + 1/2[$ for i from 0 to n, called rows.
For i from 0 to n, each row L_i is the union of a set of open rectangles (infinite on the left and right sides of the row) having the property that the sign of the $(n-i)-$th derivative of f is constant on each of them, they are separated by vertical segments, corresponding to roots of this derivative. We color in black the rectangles corresponding to negative values of the $(n-i)$-th derivative $f^{(n-i)}$ of f, and we color in gray the rectangles corresponding to positive values of $f^{(n-i)}$.

Remark 3.   (1) Since f is assumed monic, every infinite right rectangle of each row is gray.
  (2) Since $f^{(n)}$ is a positive constant, the row L_0 is a gray infinite rectangle.
  (3) The first (infinite) rectangle of each row L_i is alternatively gray or black, depending on the parity of i: it is gray if $n - i$ is even.

3

(4) We are interested by the connected components of the union of the closures of the gray rectangles; and respectively for the black rectangles.

It is clear that there is a gray connected component containing the infinite right rectangles of all rows. The other connected components (gray or black) are said bounded on the right.

Proposition 4 (multiple roots)**.** *Let f be a monic real polynomial of degree n. If b is a root of multiplicity k of f with $k \leq n$ then for sufficiently small positive h, denoting by s the sign of $f^{(k)}(b)$, the columns of the Budan table of f near b are shown in the left part of the following figure.*

If c is a root of multiplicity k of $f^{(m)}$ with $0 < m < m + k \leq n$ then, for sufficiently small positive h, denoting by s_1 the sign of $f^{(m-1)}(c)$, and by s_2 the sign of $f^{(m+k)}(c)$, the columns of the Budan table of f near c are shown in the right part of the following figure.

	b-h	b	b+h			c-h	c	c+h
$sgn(f)$	$(-1)^k s$	0	s		$sgn(f^{(m-1)})$	s_1	s_1	s_1
$sgn(f')$	$(-1)^{k+1} s$	0	s		$sgn(f^{(m)})$	$(-1)^k s_2$	0	s_2
$\ldots$	$\ldots$	0	s		$\ldots$	$\ldots$	0	s_2
$sgn(f^{(k-1)})$	$-s$	0	s		$sgn(f^{(m+k-1)})$	$-s_2$	0	s_2
$sgn(f^{(k)})$	s	s	s		$sgn(f^{(m+k)})$	s_2	s_2	s_2

See (24) for a proof.

A classical "descriptor" attached to a Budan table is the function $V_f(x)$ of the real indeterminate x with values in the set of integers $\mathbf{N}$, it counts the number of sign changes in the sequence formed by f and its derivatives evaluated at x.

Definition 5. For a sequence $(a_0, \ldots, a_n) \in (\mathbb{R} \setminus \{0\})^{n+1}$ the *number of sign changes* $\mathbf{V}(a_0, \ldots, a_n)$ is defined inductively in the following way:

$$\mathbf{V}(a_0) := 0;$$

$$\mathbf{V}(a_0, \ldots, a_i) := \begin{cases} \mathbf{V}(a_0, \ldots, a_{i-1}) & \text{if } a_{i-1} a_i > 0, \\ \mathbf{V}(a_0, \ldots, a_{i-1}) + 1 & \text{if } a_{i-1} a_i < 0. \end{cases}$$

To determine the number of sign changes of a sequence $(a_0, \ldots, a_n) \in \mathbb{R}^{n+1}$, delete the zeros in $(a_0, \ldots, a_n)$ and apply the previous rule. ($\mathbf{V}$ of the empty sequence equals 0).

We notice that the function V_f is computed from the Budan table of f, but two different tables (of two polynomials f and g) may have the same function $V_f = V_g$. Therefore the Budan table is a finer invariant than V_f attached to the polynomial f. We will study other invariants attached to f.

Proposition 6 (Budan-Fourier theorem)**.** *Let $f \in \mathbb{R}[X]$ be monic of degree n. Then,*
- $V_f(-\infty) = n$, $V_f(\infty) = 0$.

4

- *Near a real root c of multiplicity k of f, which is not a root of another derivative $f^{(l)}$ of f with $l > k$, V_f decreases by k when x moves from $c - h$ to $c + h$, for sufficiently small positive h .*
- *Near a real root c of multiplicity k of $f^{(m)}$, which is not a root of another derivative $f^{(l)}$ of f with $l > m + k$ or $l < m$, the following happens:*
  If k is even, V_f decreases by k.
  If k is odd, V_f decreases by the even integer $k + s_1 s_2$, with the notations of Proposition 4.
- *Near c, a real root of several non successive derivatives of f, V_f decreases by the sum of the quantities corresponding to each of them.*
- *Near the other points of $\mathbb{R}$, V_f is constant.*
  The function V_f is decreasing on $\mathbb{R}$.
- *For $a, b \in \mathbb{R}$ with $a < b$, the number of real roots of f in the interval $]a, b]$ counted with multiplicities is at most $V_f(a) - V_f(b)$. Moreover the difference between the number of roots and this bound is an even integer.*

See (24) for a proof.

Definition 7. Let f be a monic real polynomial of degree n. We call virtual root of f any real value a where the function $V_f(x)$ jumps, and we call virtual multiplicity at a the integer value of the jump.

The previous proposition implies that there are n virtual roots of f counted with multiplicity. Each real roots of f is also a virtual root, the others virtual roots are called virtual non real root of f.

Moreover the following result obviously holds.

Proposition 8. *Let $f \in \mathbb{R}[X]$ be monic of degree n. Let $y_1 \leq \cdots \leq y_n$ the ordered virtual roots of f, repeated according to their multiplicities, and $y_0 = -\infty$, $y_{n+1} = \infty$. Then we have for $1 \leq r \leq n + 1$, with $y_{r-1} \neq y_r$,*

$$x \in [y_{r-1}, y_r[\Longleftrightarrow$$
$$\mathbf{V}(f(x), f'(x), \ldots, f^{(n)}(x)) = n + 1 - r$$

(resp. for $r = 1$ the interval $x \in] -\infty, y_1[$).

The interest of the concept of virtual root introduced (with an equivalent definition) in (11) comes from the following important result.

Theorem 9 ((11)). *The virtual roots of a monic polynomial f depend continuously on the coefficients of f.*

See also another equivalent definition in (9).
In the sequel, we will see a geometric interpretation of virtual roots of f in terms of the same-color-connected components of the Budan table of f.

3. A generic case

In this section and the next one, we assume a condition $(\mathcal{P})$, which is generically satisfied. We introduce and study several data, continuous or discrete, attached to f and its Budan table.

The proofs of the results are collected in Section 4.

3.1. $(\mathcal{P})$-polynomials

Definition 10. A polynomial g in $\mathbb{R}[x]$ satisfies condition $(\mathcal{P})$ if and only if: each derivative of g has simple roots, and all these roots are pairwise distinct. A monic polynomial satisfying this condition will be called a $(\mathcal{P})$-polynomial.

Obviously, the parities of the number of real roots and degree of a $(\mathcal{P})$-polynomial are equal.

We now see that the property $(\mathcal{P})$ is generically satisfied.

Proposition 11. *Let f be a monic real polynomial of degree n. Let 1 and $(a_i), 0 \le i \le n-1$ be the ordered set of its coefficients in the monomial basis. For any positive ϵ, there exists g, another monic real polynomial of degree n, such that:*
- *g satisfies condition $(\mathcal{P})$.*
- *$|a_i - b_i| < \epsilon$, $0 \le i \le n-1$,*
  where 1 and $(b_i), 0 \le i \le n$ form the ordered set of the coefficients of g.
- *the number of real roots (counted with multiplicities) of f and g are equal.*
Moreover, the set $\mathbb{H}_n$ of monic polynomials in $\mathbb{E}_n$, identified with $\mathbb{R}^n$, satisfying $(\mathcal{P})$ form a semi-algebraic set of $\mathbb{E}_n$.

3.2. Generic Budan tables

Now, we determine the features of the Budan table B of a generic monic polynomial f with degree n.

Definition 12. We say that a table B with $(n+1)$ rows L_i, $0 \le i \le n$ numbered bottom-up, formed by rectangles of alternating colors, gray and black, separated by vertical segments, is a $\mathcal{B}$ table of degree n if it satisfies the following properties.
- The row L_0 is a gray infinite rectangle. The infinite rightmost rectangle of each row is gray. The first (infinite) rectangle of each row L_i is alternatively gray or black, depending on the parity of i: it is gray if $n - i$ is even.
- If i is even (resp. odd) the number of rectangles on the row L_i is odd (resp. even).
- Let $(l+1)$ be the number of rectangles of the top row L_n, then $l \le n$ and $n-l$ is an even number $2p$. There are $l + p + 1$ same-color-connected components of B. Each non first rectangle of L_i, $i > 0$ is connected on the left to a rectangle of the same color of the row L_{i-1}. The l first rectangles of L_n are in separated same-color-connected components. There are p other connected components, bounded on the right, and surrounded by connected components of the opposite color. There is also a gray component unbounded on the right.
- The previous item is true, replacing n by any m, $0 < m < n$, and B by the table formed by the lower $m + 1$ rows.
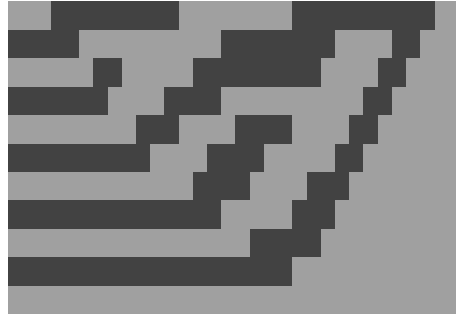
Fig. 1. A $\mathcal{GB}$ table of degree 6

Fig. 2. A $\mathcal{GB}$ of of degree 10

Moreover we say that a $\mathcal{B}$ table is a $\mathcal{GB}$ table if all the x-values of the end points of all the rectangles are pairwise distinct.

Theorem 13. *Let f be a $(\mathcal{P})$-polynomial of degree n, and let $m \leq n$ be the number or real (simple) roots of f. Then m and n have the same parity, $n = m + 2p$ and the Budan table of f is a $\mathcal{GB}$ table of degree n.*

Example 1. Figures 1 and 2 show two $\mathcal{GB}$ tables: one of degree 6 with $m = 0, p = 3$ and the other of degree 10 with $m = 4, p = 3$. The first one corresponds to the polynomial $f := int(F, x) + 200$, where "int" denotes the integral and $F := x^5 - 12x^4 + 42x^3 - 53x^2 - 28x - 2$.

Proposition 14. Let f be a $(\mathcal{P})$-polynomial of degree n. Consider a gray (resp. black) connected component bounded on the right of the Budan table of f. The x value of the rightmost (i.e top-right) corner of this component is a virtual root of f.
Let $m \leq n$ be the number or real (simple) roots of f, and let $n - m = 2p$.
There are p virtual non real roots of f, their virtual multiplicity is two; each of them is a root of some derivative of f of positive order.

We recover that f admits n virtual roots, counted with multiplicities.

Definition 15. We call augmented virtual root of f the pair (y, k) formed by a virtual root of f and n minus the order of the derivative of f which vanishes at y, i.e. $f^{(n-k)}(y) = 0$. Notice that k is the degree of that derivative.

Remark 16. The augmented virtual roots of f depend only on the Budan table B of f. Virtual roots of a $\mathcal{GB}$ table are well defined.

Proposition 17. *Let f be a $(\mathcal{P})$-polynomial of degree n.*
By Rolle's theorem between two successive roots $a < b$ of some derivative $f^{(m)}$ with $0 \leq m \leq n - 2$, (or in $\mathbb{R}$ if $f^{(m)}$ has no root), there is an odd number $2r + 1$ of roots $(X_1 < ... < X_{2r+1})$ of the next derivative $f^{(m+1)}$. Then the r roots with an even index

$(X_2, ... X_{2r})$ are virtual non real roots of f.

Similarly if a is the smallest (resp. the largest) root of $f^{(m)}$, in the infinite interval $]-\infty, a[$ (resp. $]a, \infty[$) there is an even number $2r$ of roots $(X_1 < ... < X_{2r})$ of the next derivative $f^{(m+1)}$. Then the r roots with an odd index $(X_1, ... X_{2r-1})$, (resp. the k roots with an even index $(X_2, ... X_{2r})$) are virtual non real roots of f.

For each augmented virtual non real root (y, k) of f, we have

$$f^{(k-1)}(y) f^{(k+1)}(y) > 0.$$

The following corollary explicits the relation between the virtual roots of f and f'.

Corollary 18. *With the same hypothesis and notations.*

The Budan table of the derivative f' of f is obtained by deleting the first row of the Budan table of f.

Denote by $(x_i); 1 \le i \le u$ the ordered set of real roots of f and by $(y_j); 1 \le j \le v$ the ordered set of virtual roots of f with order 1; call U the ordered union of these 2 sets. Then f' has $u + 2v - 1$ real roots: the $(y_j); 1 \le j \le v$ and a set $(z_r); 1 \le r \le u + v - 1$ of values in each of the $u + v - 1$ bounded intervals limited by the successive elements of U. The virtual roots of order k of f' for $k > 0$ are the virtual roots of order $k + 1$ of f.

Hence, the virtual roots of f and f' satisfy the "classical" interlacing property. This was already noticed in (9).

3.3. Interpolation

Since it is monic, a $(\mathcal{P})$-polynomial f of degree n depends on n real coefficients, denoted by $a_0, ..., a_{n-1}$. However it has "only" m real roots and p virtual non real roots, with $n = m + 2p$. In order to get the same number of parameters and constraints, we must attach to each of the p virtual non real roots another real value.

We choose to attach to each augmented virtual root (y, k) with $k > 0$ the value $f^{(n-k-1)}(y)$, and so define a balanced system $\mathcal{S}$ of constraints.

Definition 19. We denote by $\mathcal{S}(f)$ the system of $n = m + 2p$ data formed by the m real roots $(x_i), 1 \le i \le m$, of f; the p augmented virtual root (y_j, k_j), $1 \le j \le p$ with $k_j > 0$ and the p corresponding values $w_j := f^{(n-k_j-1)}(y_j), 1 \le j \le p$.

Notice that $\mathcal{S}(f)$ defines a system of affine equations in the coefficients of f.

Two natural questions arise:

Is it true that two different $(\mathcal{P})$-polynomials f and g in $\mathbb{H}_n$, define different systems $\mathcal{S}(f) \ne \mathcal{S}(g)$?

Given a system $\mathcal{S}$ as above, does there always exists a $(\mathcal{P})$-polynomial f which satisfies these data ?

We will see that the answer to the first question is "Yes" while the answer to the second one is in general "No".

The first question is a so-called homogeneous Hermite-Birkhoff interpolation problem, Hermite-Birkhoff interpolation, (HB) in short, is an old subject comprehensively reviewed till 1983 in the book (21). In contrast with Lagrange or Hermite interpolation, it does not always have a unique solution. In this theory, an important tool is the incidence matrix.

3.4. Incidence matrix

In HB theory, the list of constraints are written line by line, hence the rows of the incidence matrix correspond to real values where the functions are evaluated, and the columns correspond to degrees (or orders of differentiation). Since in our definition of Budan tables the roles of rows and columns are reversed, we will consider here the transposed of the usual HB incidence matrix. Yet the rows of the matrix are indexed (as usual) top-down.

Let h be the difference between two $(\mathcal{P})$-polynomials f and g having same augmented virtual roots and set of values $w_j : 1 \leq j \leq p$, as above. Hence h has degree at most $n-1$, and its coefficients form n unknowns. It should satisfy $n = m+2p$ linear vanishing conditions: m corresponding to the m real roots of f and $2p$ corresponding to the p double virtual non real roots of f. We order the $m+p$ distinct virtual roots $y_1 < ... < y_{m+p}$ of f.

Definition 20. The incidence matrix E of a $(\mathcal{P})$-polynomial f only depends on the augmented virtual roots (y_i, k_i) of f. It is the $(n, m+p)-$matrix $E = (e_{j,i})$ such that n of its entries are 1, and the others are 0 according to the following rule:
(1) If $n - k_i = 0$ then $e_{1,i} = 1$.
(2) If $n - k_i > 0$ then $e_{n-k_i+1,i} = 1$ and $e_{n-k_i,i} = 1$.
(3) Otherwise $e_{j,i} = 0$.

Example 2. For a $(\mathcal{P})$-polynomial of degree 5, having one real root y_2 and two augmented virtual roots $(y_1, 2)$ and $(y_3, 4)$, with $y_1 < y_2 < y_3$; the matrix E is:

$$\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

In HB theory the following properties of the incidence matrix are extensively used.

Definition 21. E satisfies Polya condition if and only if $M_j > j$ for all $j = 1...n$; where M_j denotes the sum of the entries in the first $j+1$ rows of the incidence matrix E.
A maximal sequence of 1 in a column of E is called even (resp. odd) if the number of its elements is even (resp. odd).
E is said conservative, if and only if every odd sequence of E is non supported.
A maximal sequence of 1 in a column i of E starting at (j, i) is non supported if at least one of the two following sub-matrices of E (which can be void) does not contain any 1:
- The upper left sub-matrix of E formed by the rows $1...j$ and the columns $1...i-1$
- The upper right sub-matrix of E formed by the rows $1...j$ and the columns $i+1...m+p$.

In (3) it is proved that a homogeneous HB problem whose incidence matrix is both conservative and satisfies Polya condition, has only the trivial solution. See also (20) for simpler proofs.

Proposition 22. *The previously defined matrix E (attached to $m+p$ augmented virtual roots) is conservative and satisfies Polya condition.*
The corresponding homogeneous HB problem has only the trivial solution.

Hence the answer to the first previous question is "Yes".

Proposition 23. *Given a system S formed by pairs of real numbers and integers: $(x_i, 0)$ with $1 \leq i \leq m$, (y_i, k_i) with $1 \leq i \leq p$, $k_i \leq n$, the set of values w_i with $1 \leq i \leq p$, $n = m + p$; there exist a unique monic polynomial F satisfying the conditions :*
$$F(x_i) = 0, 1 \leq j \leq m+, \ F^{(n-k_i)}(y_i) = 0, 1 \leq j \leq p \ , \ w_l := F^{(n-k_i-1)}(y_i), 1 \leq i \leq p.$$

Remark 24. • The (y_i, k_i) with $1 \leq i \leq m + p$, are not necessarily the augmented virtual roots of F, as illustrated by the following Example 3.
• The invertible matrix of the linear transformation is a generalization of a Vandermonde matrix, see the following Example 4.

Example 3. Consider the general degree 3 monic polynomial $F := x^3 + a_2 x^2 + a_1 x + a_0$. Require that $y_1 := 0$ is a root of F, $y_2 := 1$ is a root of F', and the extra condition $F(1) = w$.
The 3 constraints are: $a_0 = 0$, $3 + 2a_2 + a_1 = 0$, $1 + a_2 + a_1 = w$.
The solution is: $a_0 = 0$, $a_1 = 2w + 1$, $a_2 = -w - 2$.
Then $y_2 := 1$ will be a virtual non real root of F, if and only if $f(1)f''(1) > 0$. This means $w(w - 1) < 0$, hence $0 < w < 1$.
Notice that for $w = 0$, f admits a double real root at 1 and that for $w = 1$, f' admits a double real root at 1.

Hence the answer to the second question is "No".

Example 4. The generalized Vandermonde matrix corresponding to Example 2 is a $(5, 5)$-matrix with entries monomials in (y_1, y_2, y_3). Its determinant $D(y_1, y_2, y_3)$ is a polynomial of degree 5 which does not vanish if the indeterminate are real and pairwise distinct.
$$D(y_1, y_2, y_3) = 12(y_1 - y_2)^2((y_1 + y_2 - 2y_3)^2 + 2(y_1 - y_3)^2).$$

Fix a $\mathcal{GB}$ table of degree n, through the set of its $m + p$ augmented virtual roots, it defines an incidence matrix E. Hence with a set of values $(w_l) \in \mathbb{R}^p$, it defines a system of constraints S.

Proposition 25. *The set W of values $(w_i) \in \mathbb{R}^p$ such that the system S represents a $(\mathcal{P})$-polynomial, form a semi algebraic set (possibly empty) in $\mathbb{R}^p$. More precisely, if we fix the signs σ of the $(w_i) \in \mathbb{R}^p$ such that the system S represents a $(\mathcal{P})$-polynomial, then the corresponding set W_σ is a (possibly empty) open convex polytope in $\mathbb{R}^p$.*

A question similar to interpolation is to ask if any $\mathcal{GB}$ table B of degree n is realizable, i.e. does there exists a $(\mathcal{P})$-polynomial f of degree n, which admits B as its Budan table?

The answer is "No" because there are quantitative inequality between the roots of a polynomial and its derivative; see (23). So we need to weaken the requirements.

V. Kostov, in (18), proved that any sequence of numbers of roots of the derivatives, compatible with Rolle's theorem, is realizable. Then he investigated in a series of articles, ((15), (16), (19)) arrangements of roots, compatible with Rolle's theorem. He proved that if we fix an order of derivation k, then all arrangements of roots of f and f^k, compatible with Rolle's theorem are realizable. But he showed that in degree 4 any arrangement of the roots of all the derivatives, compatible with Rolle's theorem are not realizable; more precisely only 10 out of 12 are realized. There is also a related work by B. and M. Shapiro (25).

An arrangement of the roots of all the derivatives, compatible with Rolle's theorem is equivalent to what we will call a qualitative $\mathcal{B}$ table.

Definition 26. Given a $\mathcal{B}$ table B of degree n as above, we call qualitative version of B the table obtained by ordering the roots of f and all its derivatives, say z_i with $1 \leq i \leq N$; then replacing z_i by i. There are a finite number of such qualitative $\mathcal{B}$ tables with fixed degree.

Since in Figures 1 and 2, we did not mention the values z_i, they are indeed examples of qualitative $\mathcal{GB}$ tables.

Hence we need to extract from a qualitative $\mathcal{GB}$ table discrete invariants in order to expect, with this weaker requirement, that all configurations are realizable.

A candidate can be the previous incidence matrix but we prefer the B-tree, defined in the next subsection, which keeps some of the information on the same-color-connected components.

3.6. B-tree

in this subsection we define our new invariant the B-tree. With the previous notations it is obtained, from a qualitative Budan table, by contacting the $m + p$ same-color-connected components to obtain a bi-color tree with $m + p$ nodes and $n + 1$ leaves. Figures 3 and 4 illustrate our construction with the trees corresponding to example 1.

Definition 27. The B-tree extracted from a qualitative $\mathcal{B}$ table B of degree n is a bi-color tree constructed on a grid $[0..m + p + 3, 0..n]$ as follows.
We position the $n + 1$ leaves at the points $(0, i)$, $0 \leq i \leq n$ and color them alternatively in gray and black; $(0, 0)$ is gray. The root of the tree is positioned at $(m + p + 3, 0)$. The root of the tree, the nodes and the leaves are connected by colored edges which are the contraction of the the same-color-connected components of B.
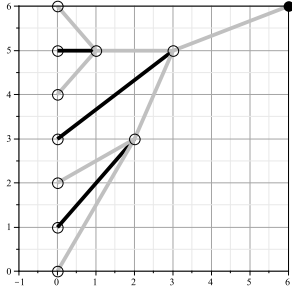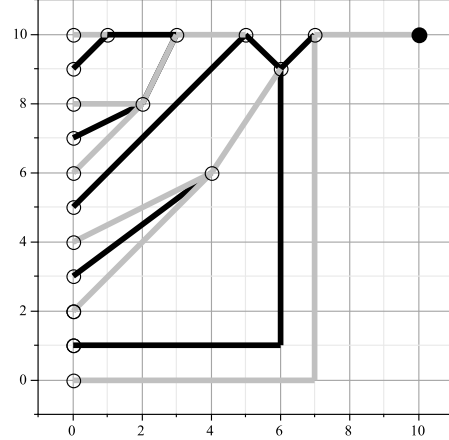
Fig. 3. A B-tree of degree 6

Fig. 4. A B-tree of degree 10

We say that the B-tree is decorated if the coordinates (i, k_i), $1 \leq i \leq m + p$ of its nodes are given.

The obtained tree is rooted on the right of the picture and can be drawn (as usual) with short branches.
From a node (i, n) corresponding to a simple root of f start two branches of different colors. From a node (i, k_i) with $n - k_i > 0$ corresponding to a virtual non real root of f, start three branches of different colors: the two extremal ones have the same color as the one of its stem and the middle one is of opposite color.

All the degree n strict hyperbolic polynomials (i.e. having n distinct real roots) have the same decorated B-tree. Since they have several qualitative Budan table, we see that the decorated B-tree is a strictly weaker invariant.

So we may ask the two following questions about realizable configurations.

Question 1. Let B be a $\mathcal{GB}$ table of degree n, does there exist a $(\mathcal{P})$-polynomial f of degree n, which admits the same decorated B-tree than B ? respectively the same B-tree as B ?

We conjecture that they are both true. Notice that the first question is stronger than prescribing the numbers of real roots in each row.

Definition 28. Let $\mathcal{T}(f)$ and $T(f)$ denote the B-tree and the decorated B-tree associated to a $(\mathcal{P})$-polynomial f. For a fixed decorated B-tree T, respectively B-tree $\mathcal{T}$, corresponding to a $\mathcal{GB}$ table of degree n, we consider the strata Σ_T, respectively $\Theta_{\mathcal{T}}$, of $\mathbb{H}_n$ formed by all the $(\mathcal{P})$-polynomials f such that $T(f) = T$, $\mathcal{T}(f) = \mathcal{T}$. We denote by Σ and Θ these two stratification of $\mathbb{H}_n$.

Proposition 29. *Each stratum Σ_T, respectively $\Theta_{\mathcal{T}}$, of $\mathbb{H}_n$ is a semi-algebraic set of $\mathbb{E}_n$.*

Proposition 30. *The virtual roots of a the polynomials f in a stratum Σ_T of $\mathbb{H}_n$ depend analytically on the coefficients of f.*

Question 2. Are the strata Σ_T connected in $\mathbb{H}_n$?

4. Proofs

This section provides the proofs of the statements of the previous section. In the proofs of each statement we keep the notations of that statement.

4.0.1. Proposition 11
It is precisely Lemma 10.2.2 of (24), its proof relies on a continuous representation of f by conjugated complex roots and Ostrowski continuity theorem, plus the use of discriminants.

4.0.2. Theorem 13
We consider the different conditions of Definition 12. Items 1 and 2 and the last condition are obvious. Let's prove item 3.

By Lemma 1, each (non first) rectangle of row L_i is connected to a rectangle of the same color on row L_{i-1} located on its left. So, by induction, any rectangle is same-color-connected to one of the $(n+1)$ first infinite rectangles on the left side of the table.

Each black connected component is bounded on its right; so it has a rightmost upper segment (call x its abscissa): if it is on L_n, then x is a real root of f, else if it is on L_i, $i < n$, it is surrounded below above and on the right by gray rectangles. In this last case, the black component stops at (x, i) and we also consider that the gray component above the segment coming from the left stops at $(x, i-1)$ since it is in an affluent of the gray component coming from the right which continues upward. Therefore two components (a black one and a gray one) stop at the abscissa x. The same reasoning works for the gray components bounded on the right, exchanging the roles of gray and black.

Let l denote the number of real roots of f and p denote the number of rightmost upper segment of a same-color-connected component bounded on the right. Since the gray component containing L_0 is unbounded, there are n other tails moving upward from the left side. Hence $n = l + 2p$.

4.0.3. Proposition 14
When a same-color-connected component stops, the jump in the Budan-Fourier count is 2 if the component is surrounded below above and on the right by rectangles of opposite color, while it is 1 if it stops on the top row.

4.0.4. Proposition 17
Here is the proof of the first assertion, the second is very similar.

Assume without loss of generality that between a and b, $f^{(m)}$ is positive. Then by Lemma 1, on the left of x_1, $f^{(m+1)}$ is also positive and changes sign at each of its successive root, so $f^{(m+1)}$ is positive at the right of X_2. Between X_1 and X_2, there are an odd number $2u+1$ of roots of $f^{(m+2)}$, denoted by $Y_1, ..., Y_{2u+1}$. Then $f^{(m+2)}$ is positive at the right of Y_{2u+1}. Hence the rectangle $]X_1..X_2[$ on L_{n-m} is black and we deduce that it is surrounded (above, on its right and below) by gray rectangles.

4.0.5. Corollary 18

Apply Rolle's theorem.

4.0.6. Proposition 22

E is conservative because the only columns with one 1, have the 1 situated on the first row; the other columns have two 1.

Polya condition is clearly satisfied for the first two rows, because if a polynomial f has no real root, then its derivative has a at least one real root, hence f has at least a virtual root of order 1.

Let ρ_j denote the number of real roots of $f^{(j)}$, $j \geq 0$. Then by Proposition 17, the number λ_j, $j > 0$, of virtual roots of order j of f is: $\lambda_j = \frac{1}{2}(\rho_j - \rho_{j-1} + 1)$. Each of them gives rise to two 1 in E. The count of 1 in the first $j+1$ rows of E gives: $M_j = j + \frac{1}{2}(\rho_j + \rho_{j-1} + 1)$. Hence $M_j > j$.

4.0.7. Proposition 23

Indeed the corresponding generalized Vandermonde matrix is invertible, by Proposition 22.

4.0.8. Proposition 25

Consider p other extra constraints $f^{(k_i+1)}(y_i) = z_i$, at the augmented non real virtual roots. The new system $\mathcal{S}'$ is solvable if and only if the $2p$ parameters (z_i, w_i) satisfy a set of affine relations. And we also require for each $i, 1 \leq i \leq p$ the non linear inequality $z_i.w_i > 0$. The solution is a union of convex polytopes whose vertices depend rationally on the y_i, then we consider its projection on the (w_i) space $\mathbb{R}^p$. If we fix the signs σ of the (w_i), hence those of (z_i), the solutions correspond to the intersection of an affine space with an hyper-octant in $\mathbb{R}^{2p}$, that we project on $\mathbb{R}^p$ to get an open polytope.

4.0.9. Proposition 29

The proof of Proposition 25 gives semi-algebraicity by elimination of the (w_i). $\Theta_{\mathcal{T}}$ is a finite union of strata Σ_T.

4.0.10. Proposition 30

The complex roots of f as well as the virtual roots of f depend continuously on the coefficients of f. But this dependency is analytic as far as the roots do not collide and create multiple roots. But this event cannot occur when f moves in a stratum Σ_T, since it is formed by $(\mathcal{P})$-polynomials.

5. Without condition $(\mathcal{P})$

In this section we consider the case where the condition $(\mathcal{P})$ is not necessarily satisfied. In order to generalize the concepts and results presented in Section 3, we need to slightly adapt the definition of a Budan table.

5.1. Multiplicities and simplified Budan table

Definition 31. We call simplified Budan table of f, the table obtained from the Budan table of f by skipping the segments between any two adjacent rectangles of the same color on a same row. In other words, for any integer k the roots of $f^{(k)}$ with even multiplicity are no longer indicated.

Theorem 32. *Let f be a monic polynomial of degree n, and let $m \leq n$ be the number or real roots of odd multiplicity of f. Then m and n have the same parity, $n = m + 2p$ and the simplified Budan table of f is a $\mathcal{B}$ table of degree n with $m + p$ same-color-connected components.*

Proof. Lemma 1 also applies when f or one of its derivatives admits multiple roots, and since we suppressed the segments corresponding to roots with even multiplicity, the endpoints of the rectangles only correspond to roots of $f^{(i)}$, $0 \leq i \leq n - 1$ with odd multiplicities. So we can repeat the argument of the proof of Theorem 13 replacing the word "root" by the expression "root with odd multiplicity". $\square$

Proposition 33. Let f be a monic real polynomial of degree n. The x value of the rightmost upper segment of a connected component (either gray or black) of the simplified Budan table of f is a virtual root of f. The virtual multiplicities are counted as follows:
- the multiplicities of events appearing along a same $x-$value are added,
- the multiplicity of a simple root of f counts 1,
- the multiplicity of a simple virtual non real root (i.e. it is not a multiple root of a derivative of f) counts 2,
- the multiplicity of a multiple root of f of order k counts k,
- the multiplicity of a multiple virtual non real root which is a multiple root of order k of a derivative of f counts k if k is even, and otherwise $k + s_1 s_2$ with the notations of the second table of Proposition 4.

Definition 34. We call augmented virtual root of f a pair (y, k) formed by a virtual root of f and an integer $0 \leq k \leq n-1$ which form the coordinates of the rightmost upper segment of a connected component (either gray or black) of the simplified Budan table of f. So there are m augmented virtual roots of type (y, n), and p augmented virtual roots of type (y, k), with $k < n$, such that $n = m + 2p$.
Different augmented virtual roots of f can correspond to the same virtual root of f.

5.2. Incidence matrix and B-tree

The definitions of B-tree, decorated B-tree, qualitative version of a Budan table, and of the stratifications given in Section 3, apply verbatim for the simplified Budan table of a monic polynomial f, even if it (or one of its derivatives) admits a multiple root. Similarly we have:

Definition 35. To a simplified Budan table B of degree n of a monic polynomial f corresponds an incidence matrix $E = (e_{i,j})$. Let m be the number of real roots of odd multiplicity of f and $m + p$ be the number of augmented virtual roots, such that $n = m + 2p$. Then E is a $(n, m + p)$ matrix with n entries equal to 1 and the others to 0,

15

Fig. 5. Budan table of example 5

according to the following rule.

Let (i, k_i) be a node of the decorated B-tree then:

If $n - k_i = 0$ then $e_{1,i} = 1$. If $n - k_i > 0$ then $e_{n-k_i,i} = 1$ and $e_{n-k_i+1,i} = 1$. Otherwise $e_{i,j} = 0$.

Again we have, with the same arguments:

Proposition 36. *The matrix E is conservative and satisfies Polya condition. The corresponding homogeneous HB problem has only the trivial solution.*

Example 5. We consider the monic polynomial $f := (x-1)^4 g/20$ of degree 7 such that $g := 20x^3 + 10x^2 + 4x + 1$ chosen such that we have at $x = 0$: $f(0) > 0$, $f'(0) = 0$, $f^{(2)}(0) = 0$, $f^{(3)}(0) = 0$, $f^{(4)}(0) > 0$. The polynomial f also has a real root of order 4 at $x = 1$ and another simple real negative root.

The simplified Budan table of f is shown in Figures 5.

5.3. Deformations

Conjecture 1. Let $f \in \mathbb{R}[X]$ be monic of degree n and T be its decorated B-tree. Then, there exists a $\mathcal{P}$-polynomial g of degree n with the same B-tree.

To prove that we need to control the deformations of f at the multiple roots of f and of its derivatives. The requirement is to deform f into F such that if x_0 is a root of $f^{(k)}$ of multiplicity r it gives rise near-by x_0 to no root of $F^{(k)}$ if r is even and to only one root of $F(k)$ if r is odd.

Example 6. For f as in example 5, the deformation $F := x^7 + 10.5x^6 + 46.2x^5 + 110.x^4 + 154.x^3 + 126.x^2 + 55.6x + 10.4$ of f gives the desired result. We constructed it by a sequence of deformations and integrations starting from the third derivative of f. Figures 6 and 7 show the two families of graphs up to the third derivative.
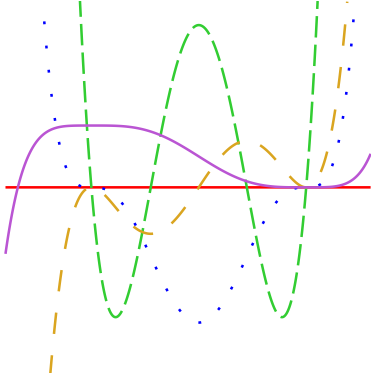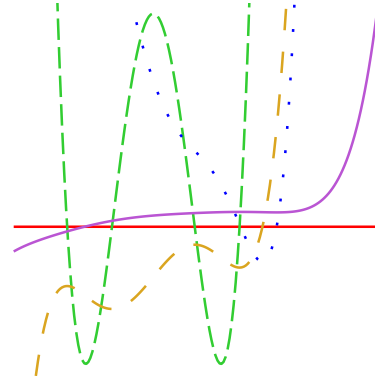
16

Fig. 6. Graphs of the derivatives of example 5

Fig. 7. Graphs corresponding to the deformation

The stratifications Σ and Θ of $\mathbb{H}_n$ extends to $\mathbb{E}_n$. The study of incidence relations among the strata of Θ (or Σ) is an interesting question, but out of the scope of the present paper. A first task could be to answer the following question.

Question 3. Are the strata $\Theta_\mathcal{T}$ connected in $\mathbb{E}_n$?

6. Conclusion

6.1. Extensions

In this paper, we have studied the same-sign-connected components of the Budan table of a univariate polynomial f, and extracted two types of information: the virtual roots which depend continuously on the coefficients of f and its Budan tree which is a finite discrete invariant attached to f. We related this construction to realizable configurations studied by V. Kostov and to Hermite-Birkhoff interpolation, which has received recently a renewed interest in Computer algebra, see (4). Then we defined a stratification of the space of polynomials and started its study. We left open several questions on the properties of the strata and their incidence relations. They provide directions for further investigations and experimentation.

There are other possibilities of extensions of the previous approach.

6.1.1. Algorithms for virtual roots isolation

In (14), we exploited the intersection of Budan tables with grids to develop algorithms for virtual roots isolation, and consequently a new method for real roots isolation of a univariate polynomial.

6.1.2. Fewnomials

In (9), the Budan-Fourier theorem and the continuity property of the virtual roots, were generalized to the case of Fewnomials, with a modified set of differentiations depending on a sequence of functions $\mathbf{f}$.
More precisely, with our notations, our Lemma 2.1 and Proposition 2.4 are valid in that setting. Therefore our constructions can also be extended to that setting. In a joint paper

17

with Mariemi Alonso Garcia (2),for Issac 2012, we applied the method presented in this article to develop real roots isolation algorithms for sparse polynomials.

6.1.3. "Circular" differentiation

Another extension could be to replace the input polynomial $f(x)$ of degree n by its homogenization $F(X, Y)$ in degree n, and then set $X = cos(t)$, $Y = sin(t)$ to get a trigonometric polynomial $G(t)$ depending on n coefficients, hence G is periodic and takes bounded values in $\mathbb{R}$. In that case, the derivative $G'(t)$ of $G(t)$ with respect to t can be obtained similarly from another homogeneous polynomial.

Then we can define a periodic generalization of the Budan table, bounded on the columns and unbounded on the rows, but we can truncate it. There is again a generic case and one can study the corresponding "virtual roots" and B-trees.

6.2. Two more directions of research

- Study the relationship between virtual roots in an interval and pairs of conjugate complex roots, which lie in a sector close to this interval as initiated in Obreschkoff's theorem, see (24), chapter 10.
- Investigate what happens if the coefficients of the input polynomials follow Gaussian random distributions. See (13).

References

[1] Akritas Alkiviadis G.: Reflections on an pair of theorems by Budan and Fourier, Mathematics Magazine, Vol. 55, No. 5, 292-298, (1982).

[2] Alonso-Garcia, M and Galligo, A : Real roots isolation for sparse polynomials. Proceedings ISSAC'2012, ACM; 35-42, (2012).

[3] Atkinson, K and Sharma, A: A partial characterization of poised Hermite-Birkhoff interpolation problems. SIAM J. Numer. Anal. vol 6 pp 230-236, (1969).

[4] Butcher, J.C and Corless, R.M and Gonzlez-Vega, L and Shakoori, A: Polynomial algebra for Birkhoff interpolants. Numerical Algorithms 56(3): 319-347 (2011).

[5] Bembé, D: Algebraic certificates for Budan's theorem. PhD thesis (2011).

[6] Bembé, D and Galligo, A: Virtual Roots of Real Polynomials and Fractional Derivatives. Proceedings of ISSAC'2011, ACM (2011).

[7] Bochnack, J. and Coste, M. and Roy, M-F.: Real Algebraic Geometry. Springer (1998).

[8] Budan de Boislaurent: Nouvelle méthode pour la résolution des équations numériques d'un degré quelconque. Paris (1822). Contains in the appendix a proof of Budan's theorem presented at the Académie des Sciences (1811).

[9] Coste, M and Lajous, T and Lombardi, H and Roy, M-F : Generalized Budan-Fourier theorem and virtual roots. Journal of Complexity, 21, 478-486 (2005).

[10] Dyn, N., Lorentz, G.G., Riemenschneider, S.D.: Continuity of the Birkhoff interpolation. SIAM J. Numer. Anal. 19(3), 507-509 (1982).

[11] Gonzalez-Vega, L and Lombardi, H and Mahé, L : Virtual roots of real polynomials. J. Pure Appl. Algebra,124, pp 147-166, (1998).

[12] Galligo, A: Roots of the Derivatives of some Random Polynomials. Proceedings SNC'2011, ACM (2011).

[13] Galligo, A: Deformation of Roots of Polynomials via Fractional Derivatives. Accepted for publication in JSC, 2012.

[14] Galligo, A: Improved Budan-Fourier Count for Root Finding. Preprint (2011).

[15] Kostov, V.P.: Root arrangements of hyperbolic polynomial-like functions. Rev. Mat. Complut. 19 (2006) no. 1, 197-225.

[16] Kostov, V.P.: On root arrangements of polynomial-like functions and their derivatives. Serdica Math. J. 31 (2005) no. 3, 201-216.

[17] Kostov, V.P.: On root arrangements for hyperbolic polynomial-like functions and their derivatives. Bull. Sci. Math. 131 (2007), no. 5, 477-492.

[18] Kostov, V.P.: A realization theorem about D-sequences. Comptes Rendus de l'Academie Bulgare Des Sciences, 60:12 (2007).

[19] Kostov, V.P.: On hyperbolic polynomial-like functions and their derivatives. proc. Roy. Soc. Edimburugh, Sect. A 137 819-8452 (2007).

[20] Lorentz, G.G., Zeller, K.: Birkhoff interpolation. SIAM J. Numer. Anal.vol8, 1, pp43-48, (1971).

[21] Lorentz, G.G., Jetter, K., Riemenschneider, S.D.: Birkhoff Interpolation, vol. 19. Addison-Wesley, Reading, MA; Don Mills, ON (1983).

[22] Muhlbach, G.: An algorithmic approach to Hermite-Birkhoff interpolation. Numer. Math. 37, 339-347 (1981).

[23] Peyser, G: On the Roots of the Derivative of a Polynomial with Real Roots. The American Mathematical Monthly , Vol. 74, No. 9 (Nov., 1967), pp. 1102-1104.

[24] Rahman, Q.I and Schmeisser, G: Analytic theory of polynomials, Oxford Univ. press (2002).

[25] Shapiro,B and Shapiro, M.: This strange and mysterious Rolle's theorem. submitted to Amer. math. Monthly, arXiv: math CA/0302215.

[26] Vincent M,: Sur la résolution des équations numériques, Journal de mathématiques pures et appliquées **44** (1836) 235–372.