Deformation of Roots of Polynomials via Fractional Derivatives

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1. Introduction

Fractional calculus has attracted interest of engineers and researchers in several fundamental or applied fields, but it is often ignored by the researchers in Polynomial algebra and too seldom encountered in Computer algebra. In this paper we emphasize that Fractional calculus is based on simple rules which can be exploited in Computer algebra, then we build a novel geometric object associated to a real univariate polynomial. Fractional derivatives define a deformation of a univariate polynomial: the order of differentiation is the continuous parameter of deformation. This construction appeared naturally in our study of Budan tables (a 2 D diagram where the vertical axis represents the integer orders of differentiation) of polynomials with large degrees when we replace a discretized curve by a continuous one, see (1), (6), (7), and Section 4.

For a fixed polynomial $f(x)$, we consider the order of differentiation q as a new indeterminate then factor the fractional derivative to get an associated bivariate polynomial that we denote by $P_f(x, q)$. For $0 \leq q \leq 1$, $P_f(x, q)$ defines an homotopy between the polynomials $f(x)$ and $xf'(x)$.

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We associate to a degree n real polynomial f a plane spline curve, the stem of f. It is the union of $n-1$ real zero sets defined by $P_{f^{(m)}}(x,q) = 0,\ 0 \leq q \leq 1,\ 0 \leq m \leq n-2$ we called the layers of the stem. We classify the admissible configurations of layers into 4 types and proved that they are realizable by providing examples; they complement Rolle's theorem on the respective location of the real roots of a polynomial and of its derivative. Type 1 is the simplest and more common one while type 4 is more unusual. Stems of classical random polynomials exhibit regular patterns. Relying on many computational experiments we proposed conjectures to describe their geometry. In particular, we conjecture that in the layers of the considered random polynomials, almost surely types 3 and 4 never happen. The analysis of these patterns led us to extend the deformation process to the complex values of x. We observed experimentally that in the complex plane, $P_f(x,q)$ creates a correspondence between the complex roots and the critical points of our random polynomials $f(x)$. It seems that this kind of pairing has not been reported before. In order to explain it, we consider a classical electrostatic interpretation: a critical point of f is viewed as an equilibrium of a particle submitted to a field of forces; and we suggest a mean field approach based on limit symmetries of the roots sets. However the analysis becomes complicated since there are two conflicting limit symmetries and we were not able to prove our claim, so we left it as a conjecture. In the conclusion we indicate several potential extensions of our constructions. We relied on the computer algebra system Maple to compute examples, and create pictures which illustrate our presentation.

The paper is organized as follows. Section 2 provides a brief introduction to Fractional calculus: history, definitions and properties. Sections 3 and 4 present our polynomial factor $P_f(x,q)$ of the q-th fractional derivative of a monic polynomial f, then describe two kinds of plane curves (FD-curve and stem) attached to f. Section 5 applies our constructions to some classical random polynomials, and set conjectures tested on many computational experiments. Section 6 concentrates on the homotopy between $f(x)$ and $xf'(x)$ defined when $0 \leq q \leq 1$; and on the correspondence between the complex roots of $f(x)$ and $xf'(x)$. Then we sketch our mean field approach relying on limit symmetries of roots sets.

Notations:

We denote by $\mathbf{R}$ the field of real numbers, by f a monic polynomial of degree n, $f := x^n + \sum_{i=0}^{i=n-1} a_i x^i$; and by $f^{(m)}$ the $m-$th derivative of f for an integer m, with $f^{(0)} := f$.

2. Fractional derivatives

The attempt to introduce and compute with derivatives or anti derivatives of non integer orders goes back to the 17-th century. The traditional adjective "fractional", corresponding to the order of differentiation, is misleading since it need not be rational. An historical progression of the concept from 1695 to 1975, through a hundred citations of mathematicians, is provided in (14). More recently, fractional derivatives has attracted a renewed interest of engeniers and researchers in several fundamental or applied fields: applied mathematics, viscoelasticity, system theory, physics, chemistry, and even finance and social sciences. The paper (13) reports hundreds of specialized conferences, monographes, articles and softwares in the area of Fractional calculus during the last 40 years.

Let us emphasize that nowadays in Mathematics, fractional derivatives are mostly used in Functional analysis for the study of PDEs. A classical example is the formal factorization of the heat equation using the semi differentiation S such that $SS = \frac{\partial}{\partial t}$. Fractional derivatives are presented via Fourier or Laplace transforms, which are well adapted to the functions exponentials, sinus and their infinite sums. Moreover Fractional calculus can be considered as a laboratory for creating new representations of special functions and integral transforms. Unfortunately, Fractional derivatives are often ignored by the researchers in Polynomial algebra and too seldom encountered in Computer algebra.

An important property which remains true in Fractional calculus is that two fractional differentiations commute. However, for non-integer orders of differentiation, the fractional derivative at a point x of a function f does not only depend on the graph of f very near x; fractional differentiations do not commute with the translations on the variable x. Moreover, like for integrals a fractional derivative depends on the choice of an initial point.

For applications in engineering and science, the order of differentiation of fractional derivatives is generally either fixed or limited to few values; it will not be the case here since the order of differentiation will serve as a continuous parameter of deformation. This interpretation arrived naturally in our study of Budan tables of univariate polynomials with large degrees: The vertical axis represented the integer orders of differentiation, and we wanted to replace a discretized curve by a continuous one, see (1), (6), (7), and our Section 4. Notice that the idea of deforming a function via its fractional derivatives is not original since it already appeared in (10), and (12); the authors were motivated by mathematics education.

Several definitions

Fractional calculus can be introduced "pedagogically" by giving 3 easy formulae which apply respectively to the finite sums of monomials, exponentials and sinus. Each of them corresponds to the extension to the real orders of a usual formula of differentiation with integral orders.

Diff^q denotes the fractional differentiation of order $q \geq 0$. We have $\mathrm{Diff}^0(f) = f$, $\mathrm{Diff}^q(\mathrm{Diff}^r(f)) = \mathrm{Diff}^{q+r}(f)$ and $\mathrm{Diff}^1(f) = f'$.

In the next sections, we will concentrate on the first one, attributed to Lacroix (1815) or to Peacock (1833), since our interest is in polynomial algebra.

(1) For monomials:

$$\mathrm{Diff}^q(x^n) := \frac{n!}{(n-q)!} x^{n-q};\ for\, q \geq 0,\ n\,\text{integer}.$$

Where $(n-q)! = \Gamma(n-q+1)$; Γ is the gamma function, so:

$$Diff^{1/2}(x^2 - 2x + 3) = (\frac{8}{3}x^2 - 4x + 3)x^{-1/2}\frac{1}{\sqrt{\pi}}.$$

(2) For exponentials:

$$\mathrm{Diff}^q(e^{sx}) := s^q e^{sx};\ \text{for } q \geq 0,\ s \text{ real positive.}$$

(3) For sinus:

$$\mathrm{Diff}^q(sin(sx)) := s^q sin(sx + q\frac{\pi}{2});\ \text{for } q \geq 0,\ s \text{ real positive.}$$

An important task of Fractional calculus was finding a unique framework to interpret all special formulae or procedures. It was achieved by considering generalized functions (such as distributions) together with Laplace or Fourier transforms, and then rely on integrations (which correspond to fractional derivatives with negative orders) to extend the domain of validity of the proposed formulae. The theory now applies widely in Functional analysis and for numerical approximate computations. The general definition expresses clearly the fact that a fractional derivative is a non local concept. Hence Fractional calculus is necessarily more complicated than usual calculus.

The integral formulation shows that a good definition of fractional derivative of a function requires to remove one indetermination, e.g by choosing a lower limit (or origin) $a \in \mathbf{R}$; the corresponding notation is Diff_a^q. For instance in our first formula this origin is $a = 0$, in the second it is $a = -\infty$, and in the third it has a more complicated expression since the function is periodic.

An unfortunate consequence of this generalization process is that fractional derivatives of polynomials which is elementary using the first above formula, seems very complicated through the computation of Fourier or other transforms using indefinite integrals.

In the sequel, we will only consider fractional derivatives with $q \geq 0$ of polynomials and use the above formula.

3. Bivariate polynomial associated to f

3.1. A simple construction

We will consider a polynomial factor of the fractional derivatives of the polynomial f. We rely on Lacroix-Peacock rule for monomials, see above.

Lemma 1. Let $f(x)$ be a polynomial of degree n, then

$$P_f(x,q) := \frac{1}{n!} x^q \Gamma(n-q) \mathrm{Diff}^q(f)$$

is a polynomial in x and q, of total degree n.

Notice that, for an initial value $a \in \mathbf{R}$, the polynomial can be written in the basis formed by the powers of $(x-a)$, then one considers the fractional derivative operator Diff_a^q and the corresponding polynomial $P_{f,a}(x,q)$. Here we set $a = 0$.

We decided that, since we focus on the polynomial roots of a fixed univariate polynomial f, it is convenient to choose for a the mean of the complex roots of f. Equivalently, thanks to the famous Tchirnaus transformation, after a real translation on the abscissa x we can assume that the coefficient of x^{n-1} vanishes, hence the mean of the roots of f is 0. This last property is also true for all derivatives with integer orders of f.
To simplify the presentation of our study we will also assume that the other coefficients of f are not 0, a property generically satisfied. So we will consider only $q \geq 0$ and concentrate on the following set.

Definition 1. • We denote by $\mathcal{E}_n$, the subclass of monic polynomials f in $\mathbf{R}[x]$ of degree n such that:

$$f = x^n + \sum_{i=0}^{n-2} a_i x^i \ , \ a_i \in \mathbf{R}, \ a_i \neq 0.$$

• Let $f \in \mathcal{E}_n$. We call bivariate polynomial associated to f, the polynomial $P_f(x,q) := x^q \frac{(n-q)!}{n!} \mathrm{Diff}^q(f,x))$. It is a polynomial of total degree n in x and q, which may be written:

$$P_f(x,q) := x^n \ + \ \sum_{i=0}^{n-2} (\prod_{j=i+1}^{n} (1 - \frac{q}{j})) a_i x^i.$$

The polynomial $P_f(x,q)$ has the following property.

Lemma 2. Let $f \in \mathcal{E}_n$. Then, $P_f(0,q) = 0$ if and only if q is an integer between 1 and n.

Hence it makes sense to consider the deformations of the roots of f (or of its derivatives) defined by $P_f(x,q) = 0$ separately in the two half (x,q)-planes, $x \geq 0$ and $x \leq 0$.

For $x > 0$ and $q > 0$, this amounts to consider as in (12) the deformation of (the graph of) f by fractional derivatives but skip fractional powers of x.

Remark 1. $P_f(x,q)$, considered as a polynomial in x, is an Hadamard product of two univariate polynomials, $f(x)$ and a fixed polynomial $S(q,x)$ defined by:

$$S(q,x) := x^n \; + \; \sum_{i=0}^{n-1} (\prod_{j=i+1}^{n} 1 - \frac{q}{j}) \binom{n}{i} x^i.$$

Roots of Hadamard products have nice properties, see (15) section 3.5 and chapter 5.

3.2. FD-curves

Definition 2. We call FD-curve of a polynomial $f \in \mathcal{E}_n$ the real algebraic plane curve defined by the bivariate associated equation $P_f(x,q) = 0$.

Figure 1 shows an example with $f := (x-1)(x-2)(x-3)(x+0.5)(x+2.5)(x+3)$, $n = 6$, an hyperbolic polynomial. The roots of f and its derivatives are represented by small green disks. Figure 2 shows another simple example with $f := (x-2)(x-3)(x+0.5)(x+2.5)(x^2+1)$, $n = 6$.

FD-curves provide an interesting family of algebraic plane curves: each one is "naturally" attached to a univariate polynomial f and interpolates the real roots of the derivatives of

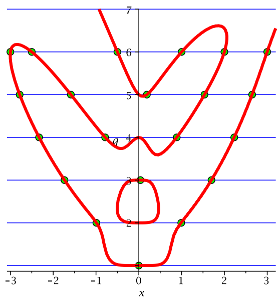

Fig. 1. A FD-curve

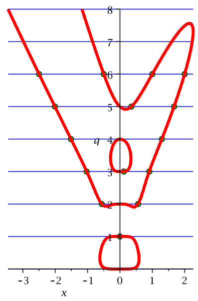

Fig. 2. Another FD-curve

f, hence gives geometric insights on the correlations between these roots. However we found that these correlations appear more neatly on another curve associated to f, the stem of f.

4. Budan tables and stems

4.1. Definitions

For a polynomial $f(x)$ in $\mathcal{E}_n$, the real roots and the signs of all its derivatives with integer order are collected in a 2D diagram we called, in (1) and (6), a Budan table. The second coordinate indicates the degree of the derivative. Notice that if all roots are simple then in each row, the signs change at each root and the representation can be sketched to only indicate pairs formed by a root of a derivative and the degree of that derivative.

Example:

$$f := (x-5).(x^2-x+4)\ ;\ f' = 3(x-1).(x-3)\ ;\ f'' = 6(x-2).$$

These polynomials have respectively 1, 2, 1 real roots, the Budan table BT can be represented by:

$$BT = \{[5,3],[1,2],[3,2],[2,1]\}.$$

Figures 3 and 4 show Budan tables of two polynomials of degrees 64.

Definition 3. Let f be a polynomial in $\mathcal{E}_n$. The stem of f is a

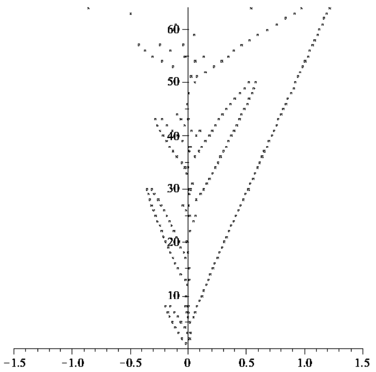

Fig. 3. Budan table of a Kac polynomial of degree 64

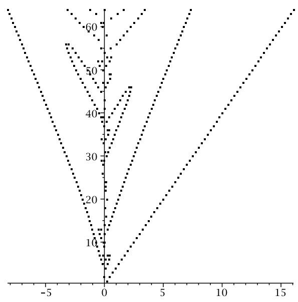

Fig. 4. Budan table of a Weyl polynomial of degree 64

C^0 spline formed by the union of $n-1$ portions of real curves, we called layers: For i from 0 to $n-2$ each layer is the zero set, of degree $n-i$, defined by $P_{f^{(i)}}(x,q)=0$ and $0 \le q \le 1$.

Stems were designed to study the roots of the derivatives of random polynomials of high degrees and exploit their symmetries.

To illustrate the differences between FD-curves and stems, Figures 5, 6 show the stems corresponding to the two previous FD-curves: stems are much less curved and there is a C^0 continuity between the adjacent layers of a stem. However stems provide a good approximation of a Budan table when n is large and f is a random polynomial, compare Figures 7 and 8 which correspond to polynomials of degree 128.

4.2. Configurations for $0 \le q \le 1$

An important point in the study of stems is the analysis of the configurations of segments of curves (and ovals) appearing in one layer (i.e. between two integer values of the vertical coordinate); without loss of generality, in the deformations of the roots of $f(x)$ into the roots of $xf'(x)$ when the parameter q in P_f takes all the values between 0 and 1.

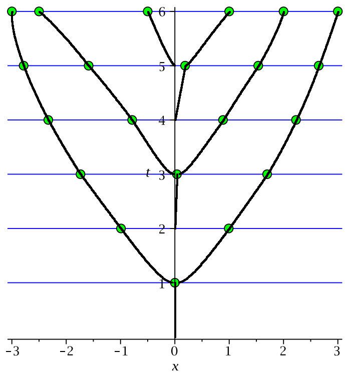

Fig. 5. Stem of Figure 1

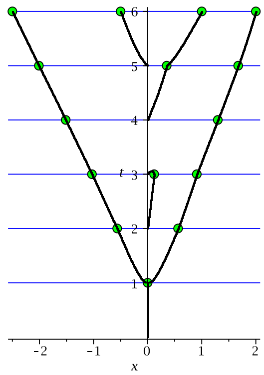

Fig. 6. Stem of Figure 2

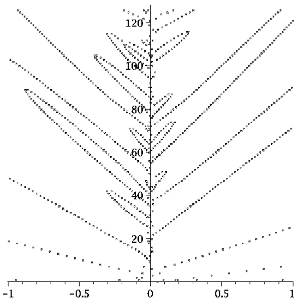

Fig. 7. Budan table of a SO(2) polynomial (detail)

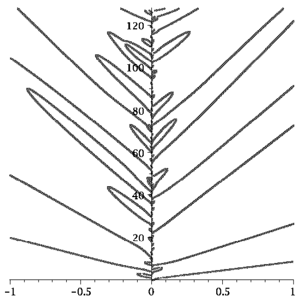

Fig. 8. Stem of that SO(2) polynomial

To simplify the description of the configurations and since we will apply it to random polynomials, we will assume that $f(x)$ and $xf'(x)$ have simple roots and that the curve C (the layer) defined by $P_f(x,q) = 0,\ 0 \leq q \leq 1$ is smooth. This condition is generically true since it is implied by the non vanishing of the bivariate resultant $Res_{(x,q)}(P_f, (P_f)'_x, (P_f)'_q)$.

Rolle's theorem implies that between two roots of f there are an odd number of roots of f'. But we do not know a priori how they are connected through C.

Proposition 1. *With these assumptions, there are 4 types of configurations which happen in a layer C of the stem of f. They* *are enumerated here after.*

(1) f has r real roots, $X_1 < ... < X_r$ and xf' has also r real roots $Y_1 < ... < Y_r$ and each X_i is connected, through C by a segment of curve, to Y_i, for $1 \leq i \leq r$.

(2) f has r real roots, $X_1 < ... < X_r$ and xf' has $r + 2u$ real roots $Y_1 < ... < Y_{r+2u}$ and each X_i is connected, through C by a segment of curve, to some $Y_{\theta(i)}$, for $1 \leq i \leq r$, with $1 \leq \theta(i) < \theta(i+1) \leq r + u$. The others Y_j, not equal to any $Y_{\theta(i)}$, are connected pairwise through C by segments of curves which do not touch the line $q = 0$.

(3) f has $r+2v$ real roots, $X_1 < ... < X_{r+2v}$ and xf' has $r+2u$ real roots $Y_1 < ... < Y_{r+2u}$; r roots of f, $X_{j_1} < ... < X_{j_r}$, are connected, through C by a segment of curve, to some $Y_{\theta(i)}$, for $1 \leq i \leq r$, with $1 \leq \theta(i) < \theta(i+1) \leq r + u$. The other roots of $f(x)$, not equal to any X_{j_i}, are connected pairwise through C by segments of curves which do not touch the line $q = 1$. And the others roots of $xf'(x)$, not equal to any $Y_{\theta(i)}$, are connected pairwise through C by segments of curves which do not touch the line $q = 0$.

(4) There is a loop in C which does not touch the line $q = 0$ nor the line $q = 1$.

It is enough to show that these configurations are effectively realized.

Example 2. (1) All layers in Figures 5 and 6 are of Type 1.

(2) Type 2 happens for all layers in Figures 8, where the curve admits an horizontal tangent. Another example is given in

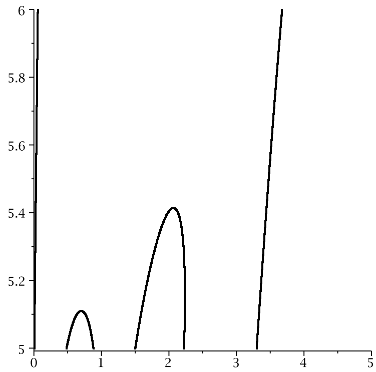

Fig. 9. a type 2 layer

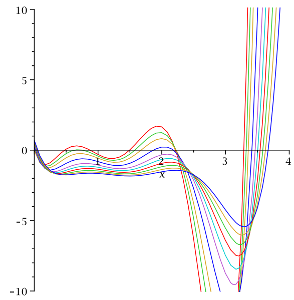

Fig. 10. a type 2 deformation between f and xf'

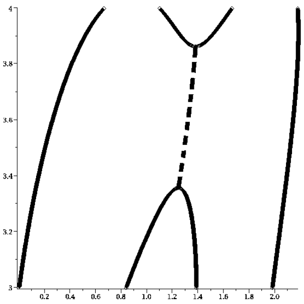

Fig. 11. a type 3 layer

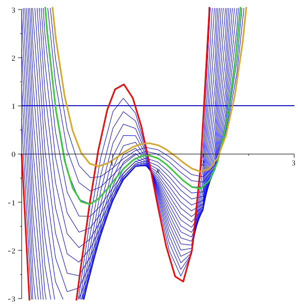

Fig. 12. a type 3 deformation between f and xf'

Figure 9 , the corresponding deformation of the graphs of f and xf' is shown in Figure 10.

(3) Type 3 happens in the layer shown in Figure 11, the corresponding deformation of the graphs of f and xf' is shown in Figure 12.

(4) A type 4 example is more difficult to find: we obtained one of degree 10 by first computing an example with two prescribed singular points (in the considered range) and then deform it to get an oval.

5. Gaussian Random polynomials

Random polynomials is a classic and active subject in Mathematics and Statistics. It is at the core of extensive recent research and has also many applications in Physics and Economics; two books (2) and (5) are dedicated to it. Already in 1943, Mark Kac (9) gave an explicit formula for the expectation of the number of roots of a polynomial in a class that now bears his name (see below). The subject is naturally related to the study of eigenvalues of random matrices with its applications in Physics, see e.g. (4).

Here, for a fixed degree n, we consider a polynomial f in $\mathcal{E}_n$ given in the monomial basis:

$$f := x^n + \sum_{i=0}^{n-2} a_i x^i$$

where the coefficients a_i are instances of $n-1$ independent centered normal distributions $N(0,\sigma)$. We are concerned with averaged asymptotic behaviors when n tends to infinity.

We consider 3 classes of random polynomials, specify their names and the corresponding variances $\sigma^2(i)$ of the coefficients a_i.

- Kac polynomials: $\sigma^2(i) = 1$.
- $SO(2)$-polynomials: $\sigma^2(i) = \binom{n}{i}$.
- Weyl-polynomials: $\sigma^2(i) = \frac{1}{i!}$.

The limit distributions when n tends to infinity, of the complex roots of these random polynomials is almost uniform in angles around the origin, this property is completed by an axial symmetry over the real axis due to complex conjugation. This observation can be quantified: (16) computed for Kac polynomials the density function $h_n(x,y)$ of the number of complex roots near a complex point $x+iy$, completing Kac's computation of the density function of the number of real roots near a real point x.

Our first experiments with Kac polynomials found that their

Budan tables admit unexpected structured patterns, see Figure 3. The roots of the successive derivatives present almost dotted curves and alignments. To our best knowledge, this phenomenon has not been explored before.

We made more experiments with different instances of Kac polynomials and got very similar patterns, then we repeated the experiments with the different random polynomials defined above, see Figures 5 and 6.

Based on our experiments, we propose the following conjectures on probabilistic behaviors when n tends to infinity.

Conjecture 1. Let f be a Kac, $SO(2)$ or Weyl random polynomial, as above. If the complex root with maximal module of f is real positive, then almost surely all the maximal real roots of $f^{(m)}$ for $0 \leq m \leq n-k$ with $k << n$ tend to be aligned.

A similar statement for negative real roots is obtained replacing $f(x)$ by $(-1)^n f(-x)$.

Conjecture 2. Let f be a Kac, $SO(2)$ or Weyl random polynomial, as above. In the stem of f almost surely all the layers are of types 1 or 2 (described in Lemma 4.1).

Figures 3, 5 and 6 illustrate these 2 conjectures.

The dotted line in Figure 11 suggests to examine the corresponding deformation of complex roots.

6. Complex critical points

There is an important bibliography on the location of the critical points of a polynomial (i.e. the roots of its derivative f') with respect to the location of its roots, going back to Gauss with Gauss-Lucas theorem. Several recent works concentrate on the following conjecture of Sendov, which has been proved for small degrees and in several special cases. Their proofs rely either on the implicit function theorem or on extremal polynomials, or on refinements of Gauss-Lucas theorem. See the book (15), chapter 7.

Sendov Conjecture: Let f be a polynomial having all its roots in the disk D. If z is a root of f, then the disk $z + D$ contains a root of f'.

6.1. Observations

We performed experiments with the classes of random polynomials described in section 5, they exhibited interesting behaviors:

- for almost each root of f smaller disks $z + \epsilon D$, with $\epsilon << 1$, contain a critical point; in such a way that they suggest a correspondence between the roots of f and the roots of xf',

- one can restrict these disks to small sectors, which indicates a direction towards the real axis or towards the origin.

Figures 13 to 14 illustrate the relation between roots and critical points for an $SO(2)$ random polynomial of degree 32. Fractional derivatives are used to construct (as for real roots) a deformation between the zero sets of f and xf'. The color chart is: the roots of f are blue, the roots of f' are red and the roots of the fractional derivatives are green. Figure 15 shows the "top" part of a complex analog of the Budan table, notice the regular alignments towards the origin (it is slightly perturbed near the real axis).

6.2. Complex stem

By analogy with section 4.1 we attach to a polynomial $f \in \mathcal{E}_n$ the C^0 spline formed by the $n-1$ layers γ_i for $0 \leq i \leq n-2$

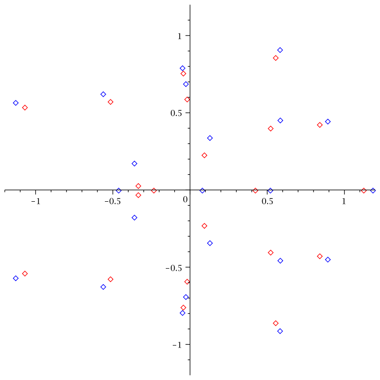

Fig. 13. Roots of f and f'

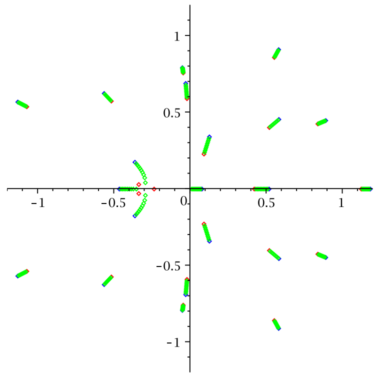

Fig. 14. Pairing via fractional derivatives

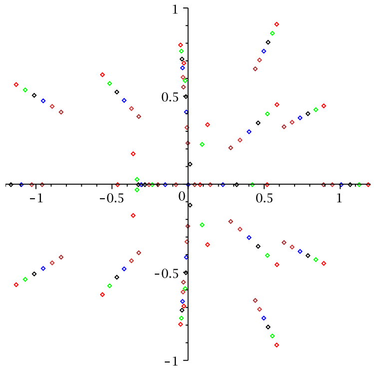

Fig. 15. Truncated $SO(2)$ complex Budan table

in the (x, q)-space $\mathbb{C} \times [0, 1]$. Each layer γ_i is a subset of a 3D algebraic curve of degree $n - i$, its "real" part C_i is one of the layer of the (real) stem defined in section 4.1.

Since f has real coefficients, generically, the only double points of each layer γ_i appear on C_i, otherwise due to complex conjugation they would correspond to pairs of double points with the same value of q. These double points correspond to points of C_i with an horizontal tangent.

Hence, generically, the complex stem of f defines a continuation process which allows to distinguish the paths of each complex non real root of f until it arrives on the real axis.

We view the non integer differentiation order q as a time parameter. For random polynomials all generically true conditions are almost surely satisfied.

Consider a random polynomial f as above and assume Conjecture 2 of section 5. Then when q varies, almost surely the deformation process in the complex plane is obtained when the roots of the derivatives of f move according to the following rules.

- The real roots of $f(x)$ are deformed to real roots of $xf'(x)$ as described by a layer of type 1.
- Some pairs of conjugated complex roots of f are deformed to real double roots of a fractional derivative and then to pairs of real roots of $xf'(x)$ as described by a layer of type 2.
- The other complex non real roots of $f(x)$ are deformed to complex non real roots of $f'(x)$.
- The process is repeated inductively replacing f by each derivative $f^{(m)}$ for $m = 0, ..., m-2$.

6.4. Electrostatic attraction

The interpretation of the location of each critical point of f as an equilibrium of a logarithmic potential, where the roots of f are viewed as positively charged particles (or rods), goes back to F. Gauss. As reported in (15), the following equality to zero provides a quick proof of Gauss-Lucas theorem.

Denote by x_j the complex roots of the polynomial f assumed distinct from each other and distinct from z, another complex number. Then by logarithmic differentiation and conjugation we deduce:

$$f'(z) = 0 \;\; \Rightarrow \;\; \sum \frac{z - x_j}{|z - x_j|^2} \; = \; 0.$$

Thus a critical point z is the barycenter of the roots of f with positive weights $\frac{1}{|z-x_j|^2}$, hence is in the convex hull of the set of roots.

We now sketch a mean field approach to explain the observed phenomena.

The vector in the complex plane $\frac{-z+x_j}{|z-x_j|^2}$ is viewed as a force applied to z directed towards x_j proportional to the reciprocal of the distance. Summing these forces, the point z (viewed as an electron) is attracted by the roots system of f (viewed as positively charged particles). When the limit distribution of the roots of f is uniform in angles, i.e. only depends on the radius, the resulting electrostatic force on a point z inherits a limit symmetry, and tends to be directed towards the origin.

Let us denote by L_1 the real line joining the origin to a root x_k and by L_2 the real line orthogonal to L_1 through the origin. The number and distribution of roots below and above L_1 (respectively L_2) are asymptotically "almost" balanced. So, with a good probability, an equilibrium z_k can be found "near" L_1 with the vector $x_k z_k$ oriented towards the origin. One can expect as well that x_k the farther is from the origin, the smaller the vector $x_k z_k$ should be. This is what we observed in our experiments as illustrated with Figures 13.

The previous balanced count of forces is perturbed when we approach the real axis, because there is another axial symmetry due to complex conjugation, and a positive probability of real roots. This breaks the rotational symmetry, consequently the resulting electrostatic force is now also directed towards the real axis.

Conjecture 3. For f a Kac, $SO(2)$ or Weyl random polynomial, the presented continuation process realizes a bijection such that each non real critical point z_k of f is attached to a root x_k. Moreover in the limit distribution of (x_k, z_k) when n tends to infinity, almost surely the vectors $x_k z_k$ point towards the origin.

Remark 2. Conjecture 3implies that the zero set of f' resembles the zero set of f but "shrunk". Hence the extremal module root of f' will also be positive real. Then a simple computation will give the deplacement of the extremal root and inductively prove Conjecture 1.

Conclusion

In this article we considered deformations of the roots of a univariate polynomial f via a "natural" homotopy computed with a bivariate polynomial obtained as a factor of the fractional derivatives of f. We introduced geometric objects related to that construction, and initiated their study. We reported computational experiments performed with f in some classes of Gaussian random polynomials, then proposed 3 conjectures. Although we also suggested some insight and a mean field approach to better understand the underlying geometry, the proof of these conjectures will require advanced developments in Probability theory.
We now list 3 other directions of research.

- Adapt our approach to fractional derivatives of trigonometric polynomials and study their roots.
- Extend in 2 variables and experiment with deformations of algebraic plane curves.

- In this paper we put some limitations on the considered polynomial f, e.g. having distinct roots, study the more general case.

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