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A METRIC PROOF OF THE CONVERSE LYAPUNOV THEOREM FOR SEMICONTINUOUS MULTIVALUED DYNAMICS

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Abstract. We extend the metric proof of the inverse Lyapunov Theorem, given in [13] for continuous multivalued dynamics, by means of tools issued from weak KAM theory, to the case where the set–valued vector field is just upper semicontinuous. This generality is justified especially in view of application to discontinuous ordinary differential equations. The more relevant new point is that we introduce, to compensate the lack of continuity, a family of perturbed dynamics, obtained through internal approximation of the original one, and perform some stability analysis of it.

1. Introduction

This is the prosecution of our previous work [13], where we have proposed a proof of the inverse Lyapunov Theorem for continuous multivalued dynamics, enjoying suitable stability and attractiveness properties with respect to an attractor $\mathcal{A}$, based on the introduction of a Lipschitz–continuous intrinsic metric suitably related to the set–valued vector field and the exploitation of some ideas borrowed from weak KAM theory. We refer to the Introduction of [13] for a detailed explanation of the main features of our method.

As declared in the title, the step forward here is that we consider dynamics just upper semicontinuous. This is indeed the natural regularity to assume, especially in view of applications to ordinary differential equations with discontinuous right–hand side. Even if the general lines of the proof stay unaffected, such a generalization requires major adjustments to the arguments employed in the continuous setting that we try to explain in what follows.

There are basically two points in [13] where the continuity assumptions on the set–valued vector field giving the dynamics is essentially used.

First, to prove that the intrinsic distance from any compact subset of $\mathbb{R}^N$, see Sections 3 and 4, satisfies almost everywhere an infinitesimal nonincrease condition with respect to F, namely that the pairing of its differential at x with every vector of $F(x)$ is nonpositive for a.e. x.

Secondly, to show that a Lipschitz–continuous Lyapunov function for F, say v, constructed using the intrinsic distance, can be uniformly approximated locally around any point $y \notin \mathcal{A}$ by a sequence of C^∞ functions enjoying the infinitesimal (strict) decrease property with respect to F in some neighborhood of y. Actually this is a crucial step for proving the final regularization theorem.

In the continuous setting this is performed by directly smoothing v through mollifiers with suitably small support. However, in doing it and in checking the validity of the decrease property for the regularized functions, one has to cope, after also using Jensen inequality, with pairings between vectors of $F(y)$, and differentials of v at points close to y. But these quantities, in the present setting, cannot be estimated for the upper semicontinuity of the multivalued field, now assumed, does not prevent from explosions at the limit, so that the size of $F(y)$ can be considerably larger than that of F at approximating points.

To overcome the first difficulty we resort to a theory developed in [4] to define weak solutions for measurable Hamilton–Jacobi equations, and use measure theoretic tools, as the notion of Lebesgue point and Fubini Theorem, to establish the infinitesimal nonincrease condition for the distance without continuity of F, see Lemma 4.2 and Proposition 4.1. However, we cannot derive from this, in contrast with the continuous case, that such a property can be extended to the generalized gradients of the distance at any point of $\mathbb{R}^N$, see Remark 4.3.

The second point is more delicate; to tackle it we consider a family of perturbed dynamics F_δ, for $\delta > 0$, obtained through internal approximation of F, namely

$$F_\delta(x) = \mathrm{co}F(x + \delta\, B) \quad \text{for any } x \in \mathbb{R}^N,$$

where co indicates the convex hull and B is the unit ball about 0. In [8] the authors use more involved perturbations of the multivalued dynamics to finally produce a Lipschitz–continuous multivalued vector field containing the initial one and enjoying suitable properties. Our approach is different and our aim is not to give a more regular approximating dynamics, note that F_δ is just u.s.c. as F.

We first show that the F_δ inherit the stability and attractiveness properties of F in bounded regions, increasing as δ goes to 0 and such that their union equals $\mathbb{R}^N \setminus \mathcal{A}$, which stay at a positive distance from the attractor $\mathcal{A}$, see Section 3. Then, exploiting this, we are able to construct, see Theorem 5.6, a Lipschitz–continuous Lyapunov function v for F with the additional property of being locally uniformly approximated, for $\delta \to 0$, by Lipschitz–continuous functions $\bar{v}_\delta$ satisfying an infinitesimal decrease condition with respect to F_δ, a.e. in any compact set disjoint from $\mathcal{A}$, for δ suitably small.

The trick is now to observe that by regularizing these approximating functions $\bar{v}_\delta$ through an appropriate mollifier, locally around any $y \notin \mathcal{A}$, one obtains a smooth function enjoying the infinitesimal decrease property *with respect to* F in a suitable neighborhood of y. For this, it is enough to take the ball supporting the mollifier of ray less that δ, the lack of continuity being actually compensated by the fact that $F_\delta(y)$ contains the values of F for arguments lying in the ball of ray δ centered at y. This allows to estimate the pairings between the vectors of $F(y)$ and the differential of $\bar{v}_\delta$ at points closer than δ to x.

Since $\bar{v}_\delta$ locally uniformly converges to v, we approximate in the end v by a sequence of smooth functions locally enjoying the infinitesimal decrease property with respect to F, which is enough, as already pointed out, for constructing a smooth Lyapunov function for F, through adaptation of the arguments already used in [13].

The paper is organized as follows: In Section 2 we introduce some preliminary material and detail the assumptions; the statement of the main result is written down as well. Section 3 is devoted to establish the relevant properties of the perturbed dynamics F_δ and to define an intrinsic length ℓ^δ related to it. In Section 4 we introduce the path metric associated to ℓ^δ and show that it enjoys an infinitesimal nonincrease property with respect to F_δ. The information gathered is exploited in Section 5 to construct a Lipschitz continuous Lyapunov function for F which is the local uniform limit of functions locally satisfying an infinitesimal decrease condition with respect to F_δ. This is used in Section 6 to perform the final regularization argument.

2. Preliminaries and statement of the main result

Notation and terminology. For every $x \in \mathbb{R}^N$, $r > 0$, we denote by $B(x,r)$ the Euclidean open ball with radius r, centered at x; we simply write B for the unit Euclidean open ball about 0. Given a subset C of $\mathbb{R}^N$, we indicate by ∂C, $\mathrm{Int} C$, $\overline{C}$ its boundary, interior and closure, respectively. We write $d(\cdot, C)$ for the Euclidean distance from it, and define the *signed distance* $d^\#$ via the formula $d^\#(\cdot, C) = 2d(\cdot, C) - d(\cdot, \partial C)$. We recall that it is convex when C is itself convex. In the case where C is, in addition, a convex cone, $d^\#(\cdot, C)$ is positively homogeneous. For any $\varepsilon > 0$ we denote by $C + \varepsilon B$ the set $\{x \in \mathbb{R}^N : d(x, C) < \varepsilon\}$. If C is convex and closed we set for every $q \in \mathbb{R}^N$

$$\sigma_C(p) = \sup\{p\, q : q \in C\},$$

where $p\, q$ indicates the Euclidean scalar product. It is called the *support function* of C and it is convex positively homogeneous. We furthermore denote by C^- the *polar cone* of C, i.e.

$$C^- = \{p : \sigma_C(p) \leq 0\}.$$

We say that a set–valued map G from $\mathbb{R}^N$ to the space of nonempty closed subsets of $\mathbb{R}^N$ is *upper semicontinuous* (u.s.c. for short) if for any x, any neighborhood W of $G(x)$, there is a neighborhood U of x with $G(U) \subset W$. This readily implies that G is locally bounded and that the graph of G is closed in $\mathbb{R}^N \times \mathbb{R}^N$. We call G *lower semicontinuous* (l.s.c.) if for any x, $y \in G(x)$ and any neighborhood W of y there exists a neighborhood U of x with

$$G(z) \cap W \neq \emptyset \qquad \text{for any } z \in U.$$

This definition can be equivalently rephrased using sequences as follows: for any $y \in G(x)$, any sequence x_n converging to x there is a sequence $y_n \in G(x_n)$ converging to y.

In the case where G is compact valued the notion of upper semicontinuity can be given requiring that for any x and any $\varepsilon > 0$, there exists $\delta > 0$ such that $G(x + \delta B) \subset G(x) + \varepsilon B$. See [1] for these and other continuity definitions for set–valued maps.

For a locally Lipschitz continuous function w we define the (Clarke) *generalized gradient* at a point x by

$$\partial w(x) = \mathrm{co}\,\{\lim_n Dw(x_n) : x_n \in \mathrm{dom}(Dw)\ ,\ x = \lim_n x_n\},$$

where co indicates the convex hull and $\mathrm{dom}(Dw)$ is the set of points where w is differentiable.

All the curves considered throughout the paper are absolutely continuous. Given such a curve ξ, $\ell(\xi)$ indicates its Euclidean length.

The Dynamics. We consider a differential inclusion

$$\dot{\xi}(t) \in F(\xi(t)), \tag{1}$$

and we assume that

$$F \text{ is a compact convex valued upper semicontinuous map.} \tag{2}$$

By solution or integral curve of (1) we intend a curve defined in some interval I satisfying the differential inclusion for a.e. $t \in I$. It is well known, see e.g. [1], that, under the condition (2), local solutions of (1) do exist for any initial point. We furthermore require that

$$\text{the differential inclusion (1) is forward complete.} \tag{3}$$

This means that any solution can be extended on the whole $\mathbb{R}^+ := [0, +\infty)$, or equivalently no solution ξ can exhibit a finite time blow-up, i.e. the relation $\lim\limits_{t\to T^-} |\xi(t)| = +\infty$ is impossible, whenever T is finite.

We assume that there exists a compact set $\mathcal{A}$, called *attractor*, satisfying

Stability

$$\begin{gathered}\text{For any } \varepsilon > 0 \text{ there is } \delta > 0 \text{ such that}\\ \text{any solution } \xi \text{ of (1) with } d(\xi(0), \mathcal{A}) < \delta \text{ satisfies}\\ d(\xi(t), \mathcal{A}) < \varepsilon, \text{ for all } t > 0,\end{gathered} \tag{4}$$

Attractiveness

$$\begin{gathered}\lim_{t\to+\infty} d(\xi(t), \mathcal{A}) = 0\\ \text{for any integral curve } \xi \text{ of (1).}\end{gathered} \tag{5}$$

Being the attractor compact, the previous conditions can be rephrased, as established in [8, Proposition 2.2], displaying some uniformity. The set of equivalent properties can be written as follows:

Uniform Attraction

$$\begin{gathered}\text{For any } r > 0 \text{ and } L > 0 \text{ there exists } T = T(r, L) \text{ such that}\\ \text{any solution } \xi \text{ of (1) with } d(\xi(0), \mathcal{A}) \le L \text{ satisfies}\\ d(\xi(t), \mathcal{A}) \le r, \quad \forall t \ge T.\end{gathered} \tag{6}$$

Uniform Boundedness

$$\begin{gathered}\text{There exists a continuous nondecreasing function } m : \mathbb{R}^+ \to \mathbb{R}^+ \text{ such that}\\ \text{for any solution } \xi \text{ of (1) with } d(\xi(0), \mathcal{A}) \le L \text{ one has}\\ d(\xi(t), \mathcal{A}) \le m(L), \quad \forall t \ge 0.\end{gathered} \tag{7}$$

Stability

$$\lim_{L\to 0^+} m(L) = 0. \tag{8}$$

In the present discontinuous framework we essentially need such uniformity, and so we exploit the above mentioned equivalence result of [8]. On the contrary, we recall that in our previous work [13], under continuity assumption on F, just (4) and (5) have been used.

Note that the stability condition implies that $\mathcal{A}$ is *forward invariant*, in the sense that any solution of (1) starting at some point of $\mathcal{A}$ lies in $\mathcal{A}$ for all $t > 0$.

Statement of the main result. A Lipschitz–continuous *Lyapunov pair* is a pair of locally Lipschitz–continuous functions (V, Ψ) satisfying:

Positive Definiteness

$$V \text{ and } \Psi \text{ are nonnegative and } V^{-1}(0) = \Psi^{-1}(0) = \mathcal{A}. \tag{9}$$

Coercivity

$$\begin{gathered}V(x) \ge \beta(d(x, \mathcal{A})) \quad \text{for any } x\\ \text{with } \beta \in C(\mathbb{R}^+) \cap C^\infty((0, +\infty)) \text{ strictly increasing}\\ \text{and satisfying } \beta(0) = 0,\ \lim_{s\to+\infty} \beta(s) = +\infty.\end{gathered} \tag{10}$$

Infinitesimal Decrease

$$\sigma_{F(x)}(DV(x)) \le -\Psi(x) \quad \text{for a.e. } x. \tag{11}$$

A Lyapunov pair (V, Ψ) is qualified as *smooth* if, in addition, V and Ψ belong to $C^\infty(\mathbb{R}^N)$. Our main result is the following:

Theorem 2.1. *Given a forward complete differential inclusion $\dot{\xi}(t) \in F(\xi(t))$ on $\mathbb{R}^N$, with F compact convex valued upper semicontinuous map, we assume that there is a compact subset $\mathcal{A}$ of $\mathbb{R}^N$ fulfilling the stability condition (4) and the attractiveness condition (5). Then there exists a smooth Lyapunov pair (V, Ψ) with*

$$V(x) \ge k d(x, \mathcal{A}) \quad \textit{for some } k > 0, \textit{ any } x \textit{ with } d(x, \mathcal{A}) \ge 2.$$

The proof will be given in Section 6.

The perturbed dynamics. It is crucial to develop our arguments to introduce perturbed dynamics given by the set valued vector fields

$$F_\delta(x) = \overline{\mathrm{co}}\, F(x + \delta B), \qquad \delta > 0,$$

where co indicates the convex hull.

Remark 2.2. For any δ the multifunction $F_\delta(x)$ is convex compact valued since F is locally bounded. In addition it is upper semicontinuous. In fact, the set-valued function $x \mapsto F(x + \delta B)$ is apparently upper semicontinuous, since $F(x)$ is so; consequently, its support function, which coincides with $x \mapsto \sigma_{F_\delta(x)}(q)$ because the support function is not affected by passing to the convex hull, is upper semicontinuous. We then derive from [1, Theorem 2, Corollary 2 in Section 1.4]), that F_δ, being convex conpact valued, is upper semicontinuous, as claimed.

We will exploit in what follows the following stability property.

Proposition 2.3. *Given positive constants L, r, T there is $\delta > 0$ such that any integral curve ξ of*

$$\dot{\xi}(t) \in F_\delta(\xi(t)) \tag{12}$$

with $d(\xi(0), \mathcal{A}) \le L$ is defined in $[0, T]$, and

$$|\gamma(t) - \xi(t)| < r \quad \textit{for } t \in [0, T], \tag{13}$$

for some solution γ to (1).

The statement is a particular case of [8, Lemma 3.2]. The proof exploits a compactness theorem for solutions of differential inclusions (see [8, Lemma 2.3] and references therein), and the uniform boundedness property of the integral curves of (1).

3. Properties of the perturbed dynamics.

The basic idea underlying the results we are going to present in this Section is that the perturbed dynamics inherit some crucial properties of F in bounded domains lying at a positive distance from the attractor.

For any $R > 1$, we select r_R satisfying $m(6r_R) < \frac{1}{2R}$, where m is the function defined in (7); this implies $6r_R < \frac{1}{2R} < R$). We set

$$L_R := m(2R) + R$$

and

$$\Theta_R = \{x \,:\, 1/R < d(x,\mathcal{A}) < R\}\ ,\ \Theta'_R = \{x \,:\, 3r_R < d(x,\mathcal{A}) < L_R\}.$$

Observe that Θ_R is contained in Θ'_R, for any R. We denote by δ_R a constant provided by Proposition 2.3 in correspondence with r_R, L_R and $T_R := T(r, 2L_R)$, where $T(\cdot,\cdot)$ is the function appearing in (6).

The following two basic properties will be repeatedly exploited in what follows.

Lemma 3.1. *Let $R > 1$ and ξ an integral curve of F_{δ_R},*

i) *if $\xi(0) \in \overline{\Theta'_R}$, then $d(\xi(T_R),\mathcal{A}) < 2r_R$;*
ii) *if, in addition, $\xi(0) \in \Theta_R$ then $d(\xi(t),\mathcal{A}) < L_R$ for $t \in [0, T_R]$.*

Proof. We omit the subscript R to ease notations. By Proposition 2.3, ξ is defined in $[0,T]$ and there is an integral curve γ of F with

$$|\gamma(t) - \xi(t)| < r \quad \text{for } t \in [0,T]. \tag{14}$$

This implies $d(\gamma(0),\mathcal{A}) \le L + r < 2L$, and so $d(\gamma(T),\mathcal{A}) \le r$ by the very definition of T. This gives i) in view of (14). If $\xi(0) \in \Theta$ then

$$d(\gamma(0),\mathcal{A}) < R + r < 2R,$$

from which the item ii) follows, taking into account (14) and the definition of $m(\cdot)$ given in (7).

□

We proceed proving a continuity property for the perturbed dynamics.

Lemma 3.2. *Let $R > 1$, then*

$$0 \notin F_{\delta_R}(x) \quad \textit{for any } x \in \overline{\Theta'_R}. \tag{15}$$

This implies that $\mathrm{Int}F_{\delta_R}(x)^- \neq \emptyset$ and $F^-_{\delta_R}$ is l.s.c. in $\overline{\Theta'_R}$.

Proof. We omit the subscript R. The argument to show (15) is by contradiction. If $0 \in F_\delta(x)$ for some $x \in \overline{\Theta'}$, then $\xi(t) \equiv x$ is a stationary solution of F_δ, and so, by Lemma 3.1 i)

$$d(\xi(T),\mathcal{A}) = d(x,\mathcal{A}) < 2r,$$

which is impossible.

Since $0 \notin F_\delta(x)$, the set

$$\{q : q = \lambda q_0 \ \text{ with } q_0 \in F_\delta(x)\,,\, \lambda \ge 0\}$$

is closed and coincides with the polar cone of $F_\delta(x)^-$, it cannot moreover contain linear subspaces. Therefore $F_\delta(x)^-$ has dimension N and its interior is nonempty.

For proving the last part of the assertion, we take $p_0 \in \mathrm{Int}F_\delta(x)^-$ and preliminarily claim that, as a consequence of (15)

$$p_0 q < 0 \quad \text{for any } q \in F_\delta(x). \tag{16}$$

If, in fact, $p_0 q_0 = 0$ for some $q_0 \in F_\delta(x)$, then

$$(p_0 + \varepsilon q_0) q_0 = \varepsilon |q_0|^2 > 0 \quad \text{for any } \varepsilon > 0,$$

which shows that $p_0 + \varepsilon q_0 \notin F_\delta(x)^-$, in contrast with p_0 being in the interior of $F_\delta^-(x)$. The strict inequality (16) implies that $p_0 \in F_\delta(y)^-$, for y sufficiently close to x, otherwise there exists a sequence x_n converging to x and $q_n \in F_\delta(x_n)$ with

$$p_0 q_n > 0 \quad \text{for any } n.$$

This, in turn, entails, by the u.s.c. of F_δ

$$p_0 q_0 \geq 0 \quad \text{for some } q_0 \in F_\delta(x),$$

which indeed contradicts (16). The asserted l.s.c. of F_δ then follows since $F_\delta^-(x)$, being convex, is the closure of its interior.

□

We set

$$Z^\delta(x) = F_\delta(x)^- \cap \overline{B} \quad \text{for any } x$$

and define a length functional ℓ^δ through the formula

$$\ell^\delta(\xi) = \int_I \sigma^\delta(\xi, \dot{\xi}) \mathrm{d}t \tag{17}$$

for any interval I, and any curve ξ defined in I, where $\sigma^\delta(x,q)$ stands for the support function of $Z^\delta(x)$ at q, for all x, q. We have $\ell^\delta(\xi) \leq \ell(\xi)$ for any ξ, any $\delta > 0$. The function $\sigma^\delta(x,\cdot)$ is convex positively homogeneous, for every x. Moreover, for every $R > 1$, as a consequence of Lemma 3.2, $\sigma^{\delta_R}(\cdot, q)$ is lower semicontinuous in $\overline{\Theta'_R}$, for every q.

The intrinsic length ℓ^δ is invariant under change of parameter, being the support function positively homogeneous, and satisfies the following lower semicontinuity property: if ξ_n is a sequence of equiLipschitz–continuous curves contained in a compact set disjoint from $\mathcal{A}$, defined in some compact interval I and uniformly converging in I to a curve ξ then $\liminf_n \ell^\delta(\xi_n) \geq \ell^\delta(\xi)$, see [5].

It is clear that ℓ^δ is nonnegative, but can possibly be zero for some nonconstant curves. The next three Propositions are actually about curves with vanishing intrinsic length.

Proposition 3.3. *Let $R > 1$, and ξ a curve with $\ell(\xi) < +\infty$, defined in $\mathbb{R}^+$ and contained in $\overline{\Theta'_R}$. Then ξ cannot be the local uniform limit in $\mathbb{R}^+$ of a sequence of curves ξ_n defined in $\mathbb{R}^+$, parametrized by the arc–length in $[0, T_n]$, for some T_n positively diverging, and such that $\lim_n \ell^{\delta_R}(\xi_n\big|_I) = 0$, for any compact interval I.*

Proof. We omit, as usual, the subscript R. Since $\ell(\xi)$ is finite, we have

$$\lim_{t \to +\infty} \xi(t) = y_0 \quad \text{for some } y_0 \in \overline{\Theta'}.$$

We know, thanks to Lemma 3.2, that the interior of $Z^\delta(y_0)$ is nonempty, and so we see, arguing as in Lemma 3.2, that there are p_0, $\mu > 0$ and $\frac{\mu}{4} > \nu > 0$ such that

$$d^\#(p_0, Z^\delta(x)) < -\mu \quad \text{for } x \in B(y_0, \nu). \tag{18}$$

There thus exists a compact interval $I = I(\nu)$, of length greater than 1, such that $\xi(t) \in B(y_0, \nu/2)$ for $t \in I$. If, by contradiction, ξ_n were a sequence of curves as indicated in the statement, then for n large enough and satisfying $I \subset [0, T_n]$

$$\xi_n(t) \in B(y_0, \nu) \quad \text{for } t \in I, \tag{19}$$

and

$$\ell(\xi_n\big|_I) \geq 1. \tag{20}$$

Taking into account (18), (19), (20), and that $|p_0| < 1$ we find, for n large enough

$$\begin{aligned} \ell^\delta(\xi_n\big|_I) &= \int_I \sigma^\delta(\xi_n, \dot{\xi}_n)\,\mathrm{d}t \geq \int_I p_0 \dot{\xi}_n\,\mathrm{d}t + \mu \int_I |\dot{\xi}_n|\,\mathrm{d}t \geq \\ &\geq -2\nu + \mu \geq \frac{\mu}{2}, \end{aligned}$$

which is in contrast with the assumptions.

□

Proposition 3.4. *Let $R > 1$ and ξ be a curve with positive natural length and zero ℓ^{δ_R} intrinsic length contained in $\overline{\Theta'_R}$, then ξ is an integral curve of F_{δ_R}, up to change of parameter.*

Proof. We omit the subscript R. The proof goes as in [13, Proposition 3.2]. We exploit that $0 \notin F_\delta(x)$ when $x \in \overline{\Theta'}$, see Lemma 3.2, the upper semicontinuity and the local boundedness of F_δ. If we assume that ξ is parametrized by the arc–length in some interval $[0, t_0]$, then we define the reparametrized curve γ via the formula $\gamma(s) := \xi(\Lambda(s))$ where Λ is the inverse, with respect of the composition, of

$$t \mapsto \int_0^t \frac{1}{\mu(r)}\,\mathrm{d}r,$$

and

$$\mu(t) = \min\{\lambda \geq 0 : \lambda\,\dot{\xi}(t) \in F_\delta(\xi(t))\} \quad \text{for a.e. } t.$$

□

Proposition 3.5. *Let $R > 1$ and ξ be a curve defined in $[0, t_0)$, for some $t_0 \in \mathbb{R}^+ \cup \{+\infty\}$, with $\xi(0) \in \Theta_R$, possessing zero intrinsic length ℓ^{δ_R}. Then there is an alternative:*
either
i) ξ has finite natural length and is contained in $\overline{\Theta'_R}$,
or
ii) the first exit time of ξ from $\overline{\Theta'_R}$, denoted by t_1, satisfies $d(\xi(t_1), \mathcal{A}) = 3r$.

Proof. Assume that ξ is not contained in $\overline{\Theta'}$, then, by Proposition 3.4, $\xi\big|_{[0,t_1)}$ is an integral curve of F_δ, up to change of parameter. Let us denote by γ the reparametrized curve and by $[0, s_1)$ the interval corresponding to $[0, t_1)$. By Lemma 3.1 i) $s_1 < T$, and by Lemma 3.1 ii) $d(\gamma(s_1), \mathcal{A}) = 3r$, which shows that the item ii) holds true.

If, instead, ξ does not go out from $\overline{\Theta'}$ then a change of parameter can be provided on the whole $[0, t_0)$ in order to obtain an integral curve of F_δ, say γ, defined on $[0, s_0)$, for some $s_0 > 0$. Then, exploiting again Lemma 3.1 i), we see that $s_0 < T$ and, since F_δ is locally bounded, the natural length of ξ is finite. Then i) is satisfied.

□

4. Intrinsic length and intrinsic distance

We consider the path distance S^δ associated to ℓ^δ, and defined, for any x, y, as

$$S^\delta(x,y) := \inf\{\ell^\delta(\xi) : \xi \text{ is a curve joining } x \text{ to } y\}. \tag{21}$$

S^δ is more precisely a nonsymmetric semidistance, i.e. it is nonnegative with $S^\delta(x,x) = 0$ for every x, and satisfies the triangle inequality. Moreover $S^\delta(x,y) \le |x-y|$ for every x, y. The function $(x,y) \mapsto S^\delta(x,y)$ is consequently Lipschitz–continuous.

The metric S^δ is related to F_δ by the following crucial result.

Proposition 4.1. *Let $\delta > 0$ be fixed. For any compact subset C of $\mathbb{R}^N$, the function*

$$u := \min\{S^\delta(z,\cdot) : z \in C\}$$

is Lipschitz–continuous and satisfies

$$|Du|_\infty \le 1 \quad , \quad \sigma_{F_\delta(y)}(Du(y)) \le 0 \quad \text{for a.e. } y \in \mathbb{R}^N.$$

To prove the Proposition 4.1 we need a preliminary Lemma.

Lemma 4.2. *Let $\delta > 0$ be fixed. There exists a set E of vanishing N–dimensional Lebesgue measure such that*

$$\limsup_{h\to 0^+} \frac{S^\delta(y, y+hq)}{h} \le \sigma^\delta(x,q),$$

for any $y \in \mathbb{R}^N \setminus E$ and any $q \in \mathbb{R}^N$.

Proof. We proceed along the same line as the proof of [4, Proposition 2.9]. We take a sequence of unit vectors q_n dense in the boundary of the unit ball. We fix n, y orthogonal to q_n, and consider the line $\gamma_y(s) := y + sq_n$, $s \in \mathbb{R}$. For any Lebesgue point t of the function $s \mapsto \sigma^\delta(\gamma_y(s), \dot\gamma_y(s))$ one has

$$\sigma^\delta(\gamma_y(t), q_n) = \lim_{h\to 0^+} \frac{1}{h}\int_t^{t+h} \sigma^\delta(\gamma_y(s), q_n)\mathrm{d}s \ge \limsup_{h\to 0^+} \frac{S^\delta(\gamma_y(t), \gamma_y(t)+hq_n)}{h}. \tag{22}$$

Since a.e. $t \in \mathbb{R}$ is a Lebesgue point of $\sigma^\delta(\gamma_y, \dot\gamma_y)$, (22) holds for a.e. t. Then, taking into account that y has been arbitrarily chosen in $q_n^\perp$, Fubini's Theorem implies

$$\limsup_{h\to 0^+} \frac{S^\delta(x, x+hq_n)}{h} \le \sigma^\delta(x, q_n), \quad \text{for a.e. } x \in \mathbb{R}^N.$$

Since the functions $q \mapsto \limsup_{h\to 0^+} \frac{S^\delta(x,x+hq)}{h}$, $q \mapsto \sigma^\delta(x,q)$ are both continuous, we reach the conclusion by density.

□

Proof of Proposition 4.1. By the triangle inequality we immediately get

$$|u(x) - u(y)| \le |x-y| \quad \text{for any } x,\, y \text{ in } \mathbb{R}^N,$$

then u is Lipschitz continuous with $|Du|_\infty \le 1$, and so differentiable outside a set $\mathcal{N}$ with vanishing N–dimensional Lebesgue measure. Let E be the set provided by the preceding Lemma. The set $\Omega := \mathbb{R}^N \setminus (E \cup \mathcal{N})$ has full Lebesgue measure.

For $y \in \Omega$, $q \in \mathbb{R}^N$ one has

$$Du(y)\, q = \lim_{h\to 0^+} \frac{u(y+hq) - u(y)}{h} \le \limsup_{h\to 0^+} \frac{S^\delta(y, y+hq)}{h} \le \sigma^\delta(y,q).$$

Then, by the Hahn–Banach Theorem $Du(y) \in Z^\delta(y)$ for *a.e.* $y \in \mathbb{R}^N$, which gives the assertion.

$\square$

Remark 4.3. Notice that in contrast to what happens in the case where F is continuous, see [13], we cannot deduce from Proposition 4.1 any information on the sign of $\sigma_{F_\delta(y)}(p)$ when $p \in \partial u(y)$ and y is any point of $\mathbb{R}^N$. This will have some consequences, for instance, in the proof of the forthcoming Theorem 5.6.

We exploit the results obtained in Section 3 to establish some properties pertaining to points with vanishing intrinsic distance from some compact set or from the attractor.

Proposition 4.4. *Let K, R be a compact subset of $\mathbb{R}^N$ and a constant greater than 1, respectively, with $\max\{d(x,\mathcal{A}) : x \in K\} < R$. Then*

$$\{y \,:\, \min_{x \in K} S^{\delta_R}(x,y) = 0\}$$

is compact.

Proof. Arguing by contradiction, we assume the assertion not true for some K, R satisfying the statement. We can moreover assume, without loosing generality, that K contains $\{x : d(x,\mathcal{A}) \le 2/R\}$. There thus exists a sequence ξ_n, parametrized by the arc–length in the interval $[0,T_n]$ for some T_n positive, with

$$\begin{array}{ll} d(\xi_n(0),\mathcal{A}) = \max\{d(x,\mathcal{A}) : x \in K\} \in (\frac{2}{R}, R), & \xi_n(t) \in \Theta'_R \quad \text{for } t \in [0,T_n), \\ d(\xi_n(T_n),\mathcal{A}) = L_R, & \lim_{n \to +\infty} \ell^{\delta_R}(\xi_n) = 0. \end{array}$$

If $T := \sup_n T_n$ is finite, we extend the ξ_n to $[0,T]$ by keeping them constant in $[T_n,T]$, and pass to the uniform limit in $[0,T]$, up to a subsequence, through Ascoli Theorem. We get in this way a limit curve ξ with support contained in $\overline{\Theta'_R}$ and

$$\xi(0) \in \Theta_R \,,\; d(\xi(T),\mathcal{A}) = L_R \,,\; \ell^{\delta_R}(\xi) = 0.$$

We deduce, by Proposition 3.4, that ξ is an integral curve of F^{δ_R}, which contradicts Lemma 3.1 ii). If instead $\sup_n T_n = +\infty$, we extend, as above, the ξ_n in $\mathbb{R}^+$ and pass to the local uniform limit, up to a subsequence, obtaining a curve ξ with support contained in $\overline{\Theta'_R}$. Then ξ must have finite Euclidean length in force of Proposition 3.5, and this is in contrast with Proposition 3.3.

$\square$

Proposition 4.5. *Let $R > 1$ and $x_0 \in \Theta_R$, then*

$$\min\{S^{\delta_R}(y,x_0) \,:\, y \in \mathcal{A}\} > 0.$$

Proof. We omit the index R to ease notations. If the assertion is not true there exist a sequence z_n, with $d(z_n,\mathcal{A}) = 5\,r$ for any n, and curves γ_n, with support contained in Θ', joining z_n to x_0 and satisfying $\ell^\delta(\gamma_n) < \frac{1}{n}$.

If $T := \sup_n \ell(\gamma_n)$ is finite, we obtain at the limit, arguing along subsequences as in Proposition 4.4, an integral curve γ of F_δ, up to change of parameter, with support contained in $\overline{\Theta}'$. If the reparametrized curve, say $\bar\gamma$, is defined in $[0,T']$, for some $T' > 0$, then $T' < T_R$ according to Lemma 3.1 i).

There thus exists, thanks to Proposition 2.3, an integral curve ξ of F with

$$|\bar{\gamma}(t) - \xi(t)| < r \quad \text{for } t \in [0, T'].$$

This implies $d(\xi(0), \mathcal{A}) < 6\, r$ and consequently

$$d(x_0, \mathcal{A}) = d(\bar{\gamma}(T'), \mathcal{A}) \leq m(6\, r) + r$$

by (7). This is impossible because $m(6\, r) < \frac{1}{2\, R}$ and

$$m(6\, r) + r < \frac{1}{2\, R} + \frac{1}{12\, R} < \frac{1}{R},$$

while $x_0 \in \Theta_R$.

If, on the contrary, $\sup_n \ell(\gamma_n) = +\infty$, we argue as in Proposition 4.4 to reach a contradiction with Proposition 3.3.

□

We readily derive from the previous result:

Corollary 4.6. *Let R a positive constant greater than 1. For any $x_0 \in \Theta_R$ there exists $\varepsilon > 0$ such that*

$$S^{\delta_R}(x_0, \cdot) = \min\{S^{\delta_R}(z, \cdot) : z \in \mathcal{A} \cup \{x_0\}\} \quad \text{in } B(x_0, \varepsilon). \tag{23}$$

5. Lyapunov pairs

We start by showing the existence of local Lipschitz–continuous Lyapunov pairs (u_x, ϕ_x) for any $x \notin \mathcal{A}$. Actually u_x fulfills a stronger condition, namely it possess derivatives having negative scalar product with all the vectors of F_δ, for suitable $\delta = \delta(x) > 0$. This point will be crucial in what follows.

Proposition 5.1. *For any $R > 1$, $x \in \Theta_R$ there is an open neighborhood $B_x \subset \Theta_R$, and nonnegative Lipschitz–continuous functions u_x, ϕ_x, with Lipschitz constant less than or equal to 1, such that*

$$\phi_x, u_x > 0 \quad \text{in } B_x, \quad \phi_x \equiv 0 \quad \text{in } \mathbb{R}^N \setminus B_x, \quad u_x \equiv 0 \quad \text{in } \mathcal{A} \tag{24}$$

and

$$\sigma_{F_{\delta(y)}}(Du_x(y)) \leq -\phi_x(y) \quad \text{for a.e. } y, \tag{25}$$

where $\delta = \delta_R$.

The proof of the Proposition is based upon a localization Lemma for the intrinsic distances related to the perturbed dynamics. The demonstration of it is obtained by suitably adjusting that of [13, Proposition 4.4] and exploiting the results of Section 3.

Lemma 5.2. *For any $R > 1$, $x_0 \in \Theta_R$, $\varepsilon > 0$ there exists $\rho = \rho(\varepsilon, x_0, R) \in (0, \varepsilon)$ such that*

$$S^{\delta_R}(x_0, y) = \inf\{\ell^{\delta_R}(\xi) : \xi \text{ connects } x_0 \text{ to } y \text{ and } \ell(\xi) < \varepsilon\} \quad \text{for } y \in B(x_0, \rho). \tag{26}$$

Proof. We omit the subscript R. Arguing by contradiction as in [13, Proposition 4.4] we construct, starting from any $x_0 \in \Theta$, a sequence of cycles ξ_n, parametrized by the arc–length in $\mathbb{R}^+$, with period T_n, satisfying $\xi_n(0) = x_0$ and

$$\lim_n \ell^\delta(\xi_n\big|_I) = 0 \quad \text{for any compact interval } I. \tag{27}$$

We also show that ξ_n possess local uniform limit, denoted by ξ, in $\mathbb{R}^+$, up to subsequences, and $\ell^\delta(\xi) = 0$ because of (27).

By exploiting Proposition 3.5 we know that either $\inf_{\mathbb{R}^+} d(\xi(t), \mathcal{A}) \leq 3r$ or ξ has finite length and is contained in $\overline{\Theta'}$, but the latter option is impossible by Proposition 3.3 since the Euclidean length of the approximating curves is infinite.

We then define for n sufficiently large

$$T'_n = \max\{t \in [0, T_n) \,:\, d(\xi_n(t), \mathcal{A}) \leq 4r\},$$

and set

$$\gamma_n = \xi_n(\cdot + T'_n).$$

Arguing, as in the first part of the proof, along subsequences, we see that γ_n has a local uniform limit γ on $\mathbb{R}^+$ with $\ell^\delta(\gamma) = 0$, and the first exit time of γ from $\overline{\Theta'}$, say t_1, satisfies by Proposition 3.5

$$d(\gamma(t_1), \mathcal{A}) = 3r.$$

We proceed to show that x_0 belongs to the support of γ. We deduce from the very definition of T'_n that $T'_n + t_1 > T_n$ for n large, consequently the sequence $T_n - T'_n$ is bounded and so convergent, up to a subsequence. For any limit point T_0, we actually have

$$\gamma(T_0) = \lim_n \gamma_n(T_n - T'_n) = \lim_n \xi_n(T_n) = x_0.$$

The curve $\gamma\big|_{[0,T_0]}$ is contained in $\overline{\Theta'}$, since $T_0 \leq t_1$, and so it is an integral curve of F_δ, up to a change of parameter, by Proposition 3.4. We denote by η the reparametrized curve and by $[0, s_1]$ the interval corresponding to $[0, T_0]$. By Lemma 3.1 $s_1 < T_R$, therefore there exists, in force of Proposition 2.3, an integral curve of F, denoted by $\tilde{\eta}$, with

$$|\tilde{\eta}(t) - \eta(t)| < r \quad \text{in } [0, s_1].$$

We then have

$$d(\tilde{\eta}(0), \mathcal{A}) \leq d(\gamma(0), \mathcal{A}) + r \leq 4r + r = 5r$$

and

$$m(5r) \leq m(6r) < \frac{1}{2R},$$

which contradicts (7) since

$$d(\tilde{\eta}(s_1), \mathcal{A}) \geq d(x_0, \mathcal{A}) - r \geq \frac{1}{R} - r > \frac{1}{2R}.$$

□

Proof of Proposition 5.1. We set $\delta = \delta_R$. Thanks to Corollary 4.6 and Lemma 3.2, we can find a neighborhood $U_x \subset \Theta_R$ and a p_x such that

$$S^\delta(x, \cdot) = \min\{S^\delta(z, \cdot) : z \in \mathcal{A} \cup \{x\}\} \quad \text{in } U_x, \tag{28}$$

$$p_x \in \operatorname{int} Z^\delta(y) \quad \text{for any } y \in \overline{U_x}, \tag{29}$$

which implies $\sigma_{F_\delta(y)}(p_x) < 0$ for $y \in \overline{U_x}$. By exploiting (29), we discover, through Lemma 5.2, that the function

$$y \mapsto p_x(y-x)$$

is a strict subtangent to $S^\delta(x,\cdot)$ at x. By pulling up such test function around x, we consequently find, thanks to (28), that

$$u_x(y) := \begin{cases} \mu_x + p_x(y-x) & y \in B_x \\ \min\{S^\delta(z,y) : z \in \mathcal{A} \cup \{x\}\} & \text{otherwise} \end{cases} \tag{30}$$

satisfies, for a suitable choice of the neighborhood $B_x \subset U_x$ and the positive constant μ_x

$$|Du_x|_\infty \le 1, \quad \sigma_{F_\delta(y)}(Du_x(y)) \le 0 \text{ for a.e. } y, \quad u_x > 0 \text{ in } B_x, \quad u_x \equiv 0 \text{ in } \mathcal{A}.$$

Finally we select a Lipschitz–continuous function ϕ_x, with $0 \le \phi_x \le \varepsilon$, where

$$\varepsilon = -\max_{\overline{U_x}} \sigma_{F_\delta(y)}(p_x),$$

and, in addition, $\phi_x > 0$ in B_x, $\phi_x \equiv 0$ in $\mathbb{R}^N \setminus B_x$, and $|D\phi_x|_\infty \le 1$. This concludes the proof.

□

To prove that the first element of the global Lyapunov pair we are going to construct satisfies a coercivity property, we need to generalize the construction provided in the previous Section to define S^δ, for δ positive, and introduce a family of distances given through the support function of the dual cone of F_δ intersected with balls of continuously varying radii.

Given a continuous positive function g, we set

$$Z^\delta_g(x) = F_\delta(x)^- \cap \overline{B(0,g(x))} \quad \text{for any } x,$$

and define the length functional ℓ^δ_g as in (17) with σ^δ replaced by the support function of Z^δ_g. We furthermore denote by S^δ_g the related length distance. All the results given in the previous Section for S^δ, apply to S^δ_g, as well, with some obvious adaptations in the proofs. We restate the Proposition 4.1 for the distance S^δ_g:

Proposition 5.3. *Given a positive constant δ, a continuous positive function g, and compact subsets C, K of $\mathbb{R}^N$, the function*

$$u := \min\{S^\delta_g(z,\cdot) : z \in C\}$$

is locally Lipschitz–continuous and satisfies

$$|Du|_{\infty,K} \le |g|_{\infty,K}, \quad \sigma_{F_\delta(y)}(Du(y)) \le 0 \quad \textit{for a.e. } y \in \mathbb{R}^N.$$

In the next result we show that the values taken by g in a region suitably far from some point x_0 do not influence S^δ_g in a neighborhood of x_0 small enough, for appropriate $\delta > 0$.

Proposition 5.4. *Given a closed subset C of $\mathbb{R}^N$ containing $\mathcal{A}$, $R > 1$, and x with*

$$\inf_{z \in C} S^{\delta_R}(z,x) > 0,$$

there is a constant $\rho = \rho(x,C,R)$ such that $B(x,\rho) \cap C = \emptyset$ and

$$S^{\delta_R}(x,\cdot) = \min\{S^{\delta_R}_g(z,\cdot) : z \in \mathcal{A} \cup \{x\}\} \quad \textit{in } B(x,\rho),$$

whenever g is a continuous positive function with $g \equiv 1$ in $\mathbb{R}^N \setminus C$.

Proof. We choose $l \in (0, \inf_{z \in C} S^{\delta_R}(z,x))$, and set

$$K = \{y : \inf_{z \in C} S^{\delta_R}(z,y) \leq l\}.$$

If ξ is a curve connecting a point of C to one belonging to $\mathbb{R}^N \setminus K$, parametrized in $[0,1]$, and t_1 is its last exit time from C, then

$$\ell_g^{\delta_R}(\xi\big|_{[t_1,1]}) = \int_{t_1}^1 \sigma_g^{\delta_R}(\xi,\dot{\xi})\,\mathrm{d}t \geq \int_{t_1}^1 \sigma^{\delta_{R_0}}(\xi,\dot{\xi})\,\mathrm{d}t > l. \tag{31}$$

We now fix

$$\rho < \min\{l/2, d(x,K)\}, \tag{32}$$

and keeping in mind that the lengths $\ell^{\delta_R}(\xi)$ and $\ell_g^{\delta_R}(\xi)$ of a curve ξ lying in $B(x,\rho)$ coincide, and are both less than the natural length, we see that

$$S_g^{\delta_R}(x,y) \leq \frac{l}{2} \quad \text{when } y \in B(x,\rho). \tag{33}$$

Formulae (31), (33) give the assertion, with ρ chosen according to (32).

□

We define for each $n \in \mathbb{N}$, $n \geq 2$

$$\mathcal{K}_n = \{y : S^{\delta_n}(x,y) = 0 \text{ for some } x \text{ with } d(x,\mathcal{A}) \leq n-1\}. \tag{34}$$

Note that the sets $\mathcal{K}_n$ are compact, for any n, thanks to Proposition 4.4. We set for any $x \notin \mathcal{K}_2$

$$\mathcal{G}(x) = \{y : d(y,\mathcal{A}) \leq j(x) - 1\}, \tag{35}$$

where $j(x) = \max\{j : x \notin \mathcal{K}_j\}$, we finally select a radius $\rho(x)$ satisfying the statement of Proposition 5.4, with $C = \mathcal{G}(x)$. We directly derive from Proposition 5.4:

Corollary 5.5. *Let $\mathcal{G}(\cdot)$, $\rho(\cdot)$ be defined as above. For any $x \notin \mathcal{K}_2$, any positive function g with $g \equiv 1$ outside $\mathcal{G}(x)$, one has*

$$S^{\delta_R}(x,\cdot) = \min\{S_g^{\delta_R}(z,\cdot) : z \in \mathcal{A} \cup \{x\}\} \quad \textit{for } R \geq j(x), \textit{ in } B(x,\rho(x)).$$

Moreover $B(x,\rho(x)) \cap \mathcal{G}(x) = \emptyset$.

We proceed by stating and proving the main result of the Section. The new crucial point here with respect to [13, Theorem 5.1] is that not only we get a Lipschitz–continuous Lyapunov pair (v,ψ), but we show that v is locally uniformly approximated by functions $\bar{v}_n$ which possess derivatives having negative scalar product with all the vectors of $F_{\bar{\delta}_n}$, for suitable $\bar{\delta}_n > 0$. This is indeed the main outcome of the previous analysis on the perturbed dynamics and will be essentially exploited in the forthcoming Lemma 6.1.

Theorem 5.6. *There exists a Lipschitz–continuous Lyapunov pair (v,ψ) satisfying*

i) *(v,ψ) is the local uniform limit of a sequence $(\bar{v}_n,\bar{\psi}_n)$ with $\bar{\psi}_n \geq 0$, $\bar{\psi}_n \equiv 0$ on $\mathcal{A}$, and $\sigma_{F_{\bar{\delta}_n}}(D\bar{v}_n) \leq -\bar{\psi}_n(x)$ for a.e. x and some $\bar{\delta}_n > 0$;*

ii) *$v(x) \geq d(x,\mathcal{A})$ for any x.*

Proof. We use the same notations of Proposition 5.1. We extract from $\{B_x\}$ a countable locally finite cover $\{B_i\} = \{B_{x_i}\}$ of $\mathbb{R}^N \setminus \mathcal{A}$, we fix, for any i, $R_i > 1$ such that $x_i \in \Theta_{R_i}$, and write u_i, ϕ_i, δ_i instead of u_{x_i}, ϕ_{x_i}, δ_{R_i}; we set moreover

$$\varepsilon_i = -\operatorname*{ess\,sup}_{B_i} d^{\#}(Du_i(x), F_{\delta_i}(x)^-) < 0 \quad \text{for any } i \in \mathbb{N},$$

and

$$I = \{i \in \mathbb{N} \,:\, x_i \notin \mathcal{K}_2\}.$$

Note that indices not belonging to I are finite, being $\mathcal{K}_2$ compact and the cover $\{B_i\}$ locally finite. We can assume, without loosing generality,

$$R_i > j(x_i) \quad \text{and} \quad B_i \subset B(x_i, \rho(x_i)) \quad \text{for any } i \in I. \tag{36}$$

Finally we select a positive sequence λ_i with $\sum_i \lambda_i = 1$.

We perturb the functions u_i, for $i \in I$, by carrying out two steps. First we define new distances $S^{\delta_i}_{g_i}$ by setting

$$g_i = \begin{cases} \varepsilon_i \lambda_i & \text{in } \widehat{\mathcal{G}}(x_i) \\ 1 & \text{in } \mathbb{R}^N \setminus \mathcal{G}(x_i), \end{cases}$$

where $\mathcal{G}$ has is defined as in (35), and

$$\widehat{\mathcal{G}}(x_i) = \left\{ x \,:\, d^{\#}(x, \mathcal{G}(x_i)) < -\frac{1}{2\,i} \right\}.$$

Exploiting Proposition 5.4 and Corollary 5.5, we construct $\widehat{u}_i$ as indicated in Proposition 5.1, with $S^{\delta_i}_{g_i}$ in place of S^{δ_i}. It is again a consequence of Corollary 5.5 and (36) that

$$\widehat{u}_i = u_i \quad \text{in } B_i \quad \text{for } i \in I. \tag{37}$$

We then proceed to define

$$w_i = \begin{cases} \frac{1}{\varepsilon_i \lambda_i}\, \widehat{u}_i & \text{for } i \in I \\ \frac{1}{\varepsilon_i \lambda_i}\, u_i & \text{for } i \notin I \end{cases}$$

and

$$\psi_i = \frac{1}{\varepsilon_i \lambda_i}\, \phi_i \quad \text{for } i \in \mathbb{N}.$$

We have for any $i \in \mathbb{N}$, by using Proposition 5.1

$$\psi_i, w_i > 0 \quad \text{in } B_i, \quad \psi_i \equiv 0 \quad \text{in } \mathbb{R}^N \setminus B_i, \quad w_i \equiv 0 \quad \text{in } \mathcal{A}$$

and

$$\sigma_{F_{\delta_i}(y)}(Dw_i(y)) \leq -\psi_i(y) \quad \text{for a.e. } y.$$

Loosely speaking, thanks to the previous construction, the w_i possess derivatives of large norm in B_i, and this will be crucial for proving the estimate in item ii); on the contrary, such norms are equibounded in $\mathcal{G}(x_i)$, at least for $i \in I$. This will allow to get a convergence result. More precisely we have, by (37) and the definition of ε_i

$$d^{\#}(Dw_i(x), F_{\delta_i}(x)^-) = \frac{1}{\varepsilon_i \lambda_i} d^{\#}(Du_i(x), F_{\delta_i}(x)^-) \leq -\frac{1}{\lambda_i} \quad \text{for } i \in \mathbb{N}, \text{ a.e. } x \in B_i \tag{38}$$

and

$$|Dw_i|_{\infty, \widehat{\mathcal{G}}(x_i)} \leq \frac{1}{\varepsilon_i \lambda_i} |g_i|_{\infty, \widehat{\mathcal{G}}(x_i)} = 1 \quad \text{for } i \in I. \tag{39}$$

To show that the sequence w_i is locally equiLipschitz–continuous, we consider a bounded open set Ω containing $\mathcal{A}$. We take into account that Ω, being bounded, is contained in $\widehat{\mathcal{G}}(x_i)$ for $i \in I$ sufficiently large, (39), and that the indices not belonging to I are finite, to get

$$|Dw_i|_\Omega \leq 1, \quad \text{except for a finite number of } i.$$

From this we also deduce that the w_i are equibounded in Ω, since they all vanish in $\mathcal{A}$, and we see that the series $\sum_1^\infty \lambda_i w_i$, uniformly converge in Ω to a function v with $Dv = \sum_1^\infty \lambda_i Dw_i$ a.e. We have

$$\sigma_{F(x)}(Dv(x)) \leq -\psi(x) := -\sum_1^\infty \lambda_i \psi_i(x) \quad \text{for a.e. } x.$$

We moreover set

$$\bar{v}_n = \sum_1^n \lambda_i w_i, \qquad \bar{\psi}_n = \sum_1^n \lambda_i \psi_i$$

and

$$\bar{\delta}_n = \min\{\delta_i \,:\, i = 1 \ldots, n\}.$$

We have

$$\sigma_{F_{\bar{\delta}_n}(x)}(D\bar{v}_n(x)) \leq -\bar{\psi}_n(x) \quad \text{for a.e. } x,$$

which establishes the item i) in the statement.

The proof of ii) is complicated with respect to the corresponding point in [13, Theorem 5.1] because the relations between Dv and $F(\cdot)^-$ hold a.e. and cannot be extended to the generalized gradients at any point, due to the lack of continuity of F, see Remark 4.3 . For this reason we have to perform a preliminary regularization of $\bar{v}_n$, for suitable n.

Let x_0, ξ, ε be a point not in $\mathcal{A}$, an integral curve of (1) issued from x_0 and a positive constant with $d(x_0, \mathcal{A}) > \varepsilon$, respectively. By the attractiveness condition we can find a time $T > 0$ such that $d(\xi(T), \mathcal{A}) = \varepsilon$ and $\xi(t) \notin \mathcal{A}$ for $t \in [0, T]$. We denote by U a bounded neighborhood of $\xi\big|_{[0,T]}$ with $U + \delta_0 B$ disjoint from $\mathcal{A}$ for some fixed $\delta_0 > 0$. We pick n_0 such that

$$U + \delta_0 B \subset \cup_{i=1}^{n_0} B_i$$

and fix $n > n_0$. Let $y \in U + \delta_0 B$ be a differentiability point for w_i, $i = 1, \cdots n$, then $y \in B_k$ for some $1 \leq k \leq n_0$, and by the convex character of the signed distance from a convex set and (38)

$$\begin{aligned} d^\#(D\bar{v}_n(y), F_{\bar{\delta}_n}(y)^-) &\leq \sum_{i=1}^n \lambda_i d^\#(Dw_i(y), F_{\bar{\delta}_n}(y)^-) \\ &\leq \lambda_k d^\#(Dw_k(y), F_{\bar{\delta}_n}(y)^-) \leq -1. \end{aligned} \tag{40}$$

We regularize $\bar{v}_n$ in U through a mollifier ζ_δ supported in $B(0, \delta)$, for a $\delta < \min\{\delta_0, \bar{\delta}_n\}$; we get

$$\bar{v}_n^\delta(x) := \int \zeta_\delta(y - x)\, \bar{v}_n(y)\, \mathrm{d}y \qquad \text{for } x \in U$$

and, via Jensen's inequality

$$d^\#(\bar{v}_n^\delta(x), F(x)^-) \leq \int \zeta_\delta(y - x)\, d^\#(D\bar{v}_n^\delta(y), F(x)^-)\, \mathrm{d}y$$

for any $x \in U$. We know that $F(x) \subset F_{\bar{\delta}_n}(y)$ for any $y \in B(x, \delta)$, so that

$$d^{\#}(D\bar{v}_n^{\delta}(y), F(x)^{-}) \leq d^{\#}(D\bar{v}_n^{\delta}(y), F_{\bar{\delta}_n}(y)^{-}),$$

which finally gives by (40)

$$d^{\#}(D\bar{v}_n^{\delta}(x), F(x)^{-}) \leq \int \zeta_{\delta}(y-x)\, d^{\#}(D\bar{v}_n^{\delta}(y), F_{\bar{\delta}_n}(y)^{-})\, \mathrm{d}y \leq -1.$$

We deduce

$$\left(D\bar{v}_n^{\delta}(\xi(t)) + \frac{\dot{\xi}(t)}{|\dot{\xi}(t)|} \right) \dot{\xi}(t) \leq 0 \quad \text{for a.e. } t \in [0, T]$$

and

$$\begin{aligned} \bar{v}_n^{\delta}(x_0) &\geq \bar{v}_n^{\delta}(x_0) - \bar{v}_n^{\delta}(\xi(T)) = -\int_0^T D\bar{v}_n^{\delta}(\xi(t))\, \dot{\xi}(t)\, \mathrm{d}t \\ &\geq \ell(\xi\big|_{[0,T]}) \geq |x_0 - \xi(T)| \geq d(x_0, \mathcal{A}) - \varepsilon. \end{aligned}$$

Since $\bar{v}_n^{\delta}(x_0) \to \bar{v}_n(x_0)$ as $\delta \to 0$ and $\bar{v}_n(x_0) \to v(x_0)$ as $n \to +\infty$, we obtain the estimate in ii) taking into account that ε has been arbitrarily chosen.

□

6. Main result

Here we finally prove Theorem 2.1, our main result, stated in Section 2. In this final step becomes clear our choice of introducing auxiliary larger dynamics. As already pointed out in the Introduction, because of the lack of continuity, the regularization procedure adopted in our previous paper [13] cannot be applied here. We essentially use that the Lipschitz–continuous Lyapunov function for F constructed in the previous Section enjoys the additional property of being approximated by functions satisfying an infinitesimal decrease condition with respect to F_{δ}.

Lemma 6.1. *Let (v, ψ) the Lipschitz–continuous Lyapunov pair provided by Theorem 5.6. For any bounded open set Ω with closure disjoint from $\mathcal{A}$ there are sequences u_n^{Ω}, φ_n^{Ω} of smooth functions uniformly converging in Ω to v and ψ, respectively, such that*

$$\sigma_{F(x)}(Du_n^{\Omega}(x)) \leq -\varphi_n^{\Omega}(x),$$
$$u_n^{\Omega}(x) > \frac{d(x, \mathcal{A})}{2}, \quad \varphi_n^{\Omega}(x) > 0,$$

for any $x \in \Omega$, $n \in \mathbb{N}$.

Proof. We choose $\delta_0 > 0$ in such a way that $\Omega + \delta_0 B =: \Omega'$ still has closure disjoint from the attractor. We know that (v, ψ) is the uniform limit in Ω' of a sequence $(\bar{v}_n, \bar{\psi}_n)$ with

$$\bar{v}_n(x) > \frac{d(x, \mathcal{A})}{2} \quad , \quad \sigma_{F_{\bar{\delta}_n}(x)}(D\bar{v}_n(x)) \leq -\bar{\psi}_n(x),$$

where the first relation is satisfied for any $x \in \Omega'$ and the latter for a.e. $x \in \Omega'$ and some $\bar{\delta}_n > 0$. Since $\psi > 0$ in Ω', we can assume, without loosing generality, that

$\bar{\psi}_n(x) > 0$ for any $x \in \Omega'$, any $n \in \mathbb{N}$, and, in addition, that $\bar{\delta}_n$ is infinitesimal as $n \to +\infty$, $\bar{\delta}_n < \delta_0$ for any n. We set

$$\begin{aligned} u_n^\Omega(x) &= \int \zeta_{\rho_n}(z-y)\,\bar{v}_n(z)\,\mathrm{d}z \\ \varphi_n^\Omega(x) &= \int \zeta_{\rho_n}(z-y)\,\bar{\psi}_n(z)\,\mathrm{d}z, \end{aligned}$$

where the mollifier ζ_{ρ_n} is supported in $B(0,\rho_n)$ and ρ_n is any positive constant satisfying $\rho_n < \bar{\delta}_n$, $u_n(x) > \frac{d(x,\mathcal{A})}{2}$ for $x \in \Omega$. Exploiting the Jensen's inequality and observing that

$$F(y) \subset F_{\bar{\delta}_n}(z) \quad \text{for any } y \in \Omega,\, z \in B(y,\rho_n),$$

we get for any $y \in \Omega$

$$\begin{aligned} \sigma_{F(y)}(Du_n^\Omega(y)) &= \sigma_{F(y)}\left(\int \zeta_{\rho_n}(z-y) D\bar{v}_n(z)\,\mathrm{d}z\right) \\ &\leq \int \zeta_{\rho_n}(z-y)\sigma_{F(y)}(D\bar{v}_n(z))\,\mathrm{d}z \\ &\leq \int \zeta_{\rho_n}(z-y)\sigma_{F_{\bar{\delta}_n}(z)}(D\bar{v}_n(z))\,\mathrm{d}z \\ &\leq -\int \zeta_{\rho_n}(z-y)\bar{\psi}_n(z)\,\mathrm{d}z = -\varphi_n^\Omega(y). \end{aligned}$$

It is then apparent that u_n^Ω, φ_n^Ω, defined by the previous formulae, satisfy the statement.

□

Proof of Theorem 2.1. Starting from the previous Lemma, the proof goes as in [13, Theorem 2.2]. We just sketch the main points for reader's convenience. We consider a countable locally finite cover of $\mathbb{R}^N \setminus \mathcal{A}$ made up by bounded open sets Ω_i whose closure is disjoint from $\mathcal{A}$, a C^∞ partition of unity β_i subordinated to it, and a sequence of positive numbers λ_i with $\sum_i \lambda_i = 1$.

We select, for any $i \in \mathbb{N}$, indices n_i such that $u_i := u_{n_i}^{\Omega_i}$, $\varphi_i := \varphi_{n_i}^i$ are suitably close to v, ψ respectively, in $L^\infty(\Omega_i)$. We then define

$$\widehat{V} = \begin{cases} \sum_i \beta_i u_i & \text{in } \mathbb{R}^N \setminus \mathcal{A} \\ 0 & \text{in } \mathcal{A}, \end{cases}$$

and

$$\widehat{\Psi} = \begin{cases} \sum_i \beta_i \varphi_i & \text{in } \mathbb{R}^N \setminus \mathcal{A} \\ 0 & \text{in } \mathcal{A}. \end{cases}$$

We show that $(\widehat{V}, \widehat{\Psi}/2)$ is a Lipschitz–continuous Lyapunov pair with $\widehat{V}$, $\widehat{\Psi}$ belonging to $C^\infty(\mathbb{R}^N \setminus \mathcal{A})$ and, in addition, $\widehat{V}(x) \geq d(x,\mathcal{A})/2$ for any x. We finally use [13, Lemma 6.1] to construct from $\widehat{V}$, $\widehat{\Psi}$ a smooth Lyapunov pair (V,Ψ) satisfying the statement.

□

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