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Abstract—[1] **We develop a framework to model and study cellular networks with randomly wandering users. Our model captures user displacement resulting from random mobility patterns and arrival positions, which in turn influences the transmission rates. We model the user movements by a random walk with exponential wandering times. Each cell is composed of disjoint (transmission) rate regions and we model each rate region as an equivalent step in the random walk model. We further use spatial queuing theoretic tools to obtain explicit expressions for the expected service time (total time duration for which the user derives service from the cell), call busy and drop probabilities. We obtain approximate closed form expressions for optimal cell size (for some asymptotic speed cases and small cell radii) and validate them via numerical simulations. We show that the optimal cell size increases with the increase in the speed of the users or in the power budget and decreases with increase in path loss factor.**

Index Terms—**Small cells; Random walk; Wandering users;**

I. Introduction

Recent trends in mobile broadband access and services is paving the way for dense deployment of base stations, popularly known as small cell networks (SCNs) [1]. Typically SCNs comprise of portable pico and femto base stations (BSs) that serve dense urban areas, commercial and office spaces, hot-spots, etc. The design and deployment of such networks pose many a new challenges. One of the key challenges with mobile users in SCNs is managing frequent handovers (HO). Each HO can potentially result in a call drop and also requires some amount of information exchange with the new BS. These two factors can degrade the system performance. Virtual cells (resources reserved across multiple cells) and fast base station switching (reducing the exchange information) are some of the ideas proposed (see [3]) to reduce HO losses. However, they cannot completely eliminate the same. As the cell size decreases, on one hand the frequency of the HOs increase resulting in losses and on the other hand the cell edge users obtain services at better communication rates. As the HOs increase the calls get dropped before completing the service with higher probabilities, while with better communication rates the amount of time taken for the same service reduces. Thus the performance of the system depends upon these contrasting phenomenon and one needs to address this trade-off while designing optimal systems. That is, one needs to first answer the question, *how small must be a small cell network?*, and this should be answered considering the above trade-off.

In a recent work [2], we used spatial queuing theory to study this contrast in the case of high speed unidirectional users. Key system performance metrics (that capture the above mentioned trade-off) like expected waiting time, service time, call busy and drop probabilities for various traffic types were derived and the cell sizes, which optimize these metrics for a given user velocity profile are computed. In [2], the users move in a fixed direction at random speed which remains unchanged for the entire duration of the call. Users traversing on well structured streets in urban areas and deriving service from BSs located on street infrastructure fall under this category. However, there are many scenarios in which the users move randomly. Typically, this happens in commercial centers, hot-spots and office spaces. This is an equally important case study and the analysis of such systems with randomly wandering users would be way different from that of the fixed direction users. In this paper, we study, analyze and obtain optimal dimensioning rules when users move in a random manner, using random walk model techniques.

Further, in [2], we considered *maximal* rate of service, i.e., we assume that the service rates can change continually based on the distance between the user and the serving BS and also is maximum possible (i.e., capacity). We now consider a more practical scenario. We assume that the system can support one of the finitely many transmission (or service) rates and that a user derives his service at one of these rates based on his distance from the serving BS. The finite set of all possible transmission rates can further depend upon the cell dimension. A cell is partitioned into as many disjoint regions as the number of possible transmission rates and we assume to know/estimate (e.g., via Signal to Noise ratio estimation) the current rate region in which the user is located. Hence *we model user movements by a random walk model, in which each step represents a rate region.* The wandering times in each region can depend upon the region itself as well as the cell dimension. We obtain performance measures like expected service time, call busy and drop probabilities, etc, further using queuing theoretic tools. We then obtain analytical expressions for optimal cell sizes for some (speed) asymptotic cases, while the same is obtained for general cases using numerical examples. Finite choice of transmission rates has significant implications: larger cells work efficiently only if the total power in the system is boosted, when one can't

[1]The work of the first and the second author was carried out while they were at INRIA, Sophia Antipolis, France.

expand the number of choices of the transmission rates. *We found that this requires the transmit power to be scaled by a factor greater than β, the path-loss factor, and called it as β^+ scaling.*

Mathematical models that capture the dynamics of stocks, animals, humans, traffic, etc., have been a well studied subject over the past decades. Random walk models serve as a fundamental model that can capture the observed stochastic dynamics in many such cases. The theory of random walks has a long history which goes back to the beginning of the last century by Karl Pearson. Feller's and Spitzer's books [4], [5] contain preliminary material on this topic.

Random walk and other mobility models are often used to study user movement in cellular networks in various contexts. In the following we list a few. An excellent survey of mobility models used in the simulations of wireless networks, is provided in [6] and the references therein. Some such simulation models for wireless networks are developed in [8], [9] etc, using random walk techniques.

Random walk models are also used to obtain performance analysis, like in our paper. For example, in [7], the authors develop a two dimensional random walk model to study mobility in wireless networks and derive the cumulative distribution function of the dwell time in a circular or a rectangular cell. These papers exploit directly the results pertaining to random walk pattern while in our paper, the random walk models aid in capturing the instantaneous (random) user position which in turn determines the transmission rate. The expected service time (time taken either to transmit the entire file or to reach the boundary, whichever occurs first) is then calculated using the random walk techniques. Assuming that at each step (in the random walk model), the user is served by one of the transmission rates available at the BS for a duration that is exponential, we derive important system performance metrics using queuing theoretic tools and further use them to derive dimensioning rules in such networks.

The paper is organized as follows. Section II describes the system model, while section III, presents the theoretical analysis for a general case. Section IV studies the case of random walk with exponential wandering times.

II. System Model

We consider a cellular network, where each cell is of dimension L. In the case of one dimensional networks (see Figure 1), the entire network spans over a line segment, say $[-M, M]$, which is divided into cells of length $2L$, while in the case of two dimensional networks (see Figure 2), each cell is a circle of radius L. Our aim is to find optimal L^*, while the network caters to moving users. Let $\eta := \mathbf{1}_{\{1D\}} + 2 \ \ \mathbf{1}_{\{2D\}}$ represent the dimension. We assume no interference from the other cells.

Rate Regions: The cell is divided into $2N$ (or N for 2D) disjoint segments such that the users in a segment are served with the same transmission rate. Let $\{\mathbb{A}_n\}_{n\in\mathbb{N}}$ represent these rate regions (see figures 1 and 2). We assume in this paper (because of small cell radii) that the signal attenuations are dominated by the distance based path loss and hence the rate regions depend completely upon the distance between the user and the serving BS. Thus, with $|.|$ representing the norm or the absolute value:

$$\mathbb{A}_n := \begin{cases} \left[\frac{(n-1)L}{N}, \frac{nL}{N}\right] 1_{\{n>0\}} + \left[\frac{nL}{N}, \frac{(n+1)L}{N}\right] 1_{\{n<0\}} & \text{if 1D} \\ \left\{\mathbf{x} \in \mathcal{R}^2 : \frac{(n-1)L}{N} \le |\mathbf{x}| \le \frac{nL}{N}\right\} & \text{if 2D} \end{cases}$$

$$\text{and } \mathbb{N} \ := \ \begin{cases} \{-N, \cdots, -1, 1, \cdots, N\} & \text{if 1D} \\ \{1, \cdots, N\} & \text{if 2D.} \end{cases} \quad (1)$$

User in region n receives service at rate $r_{|n|}$. Let $\mathbb{R} := \{r_1, \ldots, r_N\}$ represent the ensemble of all possible transmission rates. Note that this set is arranged in the decreasing order. For example in a two dimensional (circular) cell of Figure 2, each annular ring is served with a common rate, which decreases as the distance from the center (where BS is located) increases. The rate at which the service is offered changes once the user switches from one region to another.

Embedded (Rate) Markov Chain: Users can be in any one of the rate regions $\{\mathbb{A}_n\}_n$. We represent the user location at time k by Φ_k . When $\Phi_k = n$, it implies that the user at time k is wandering in segment $\mathbb{A}_n$ and is receiving service at rate $r_{|n|}$. Let W_n represent the time for which the user remains in n^{th} region, $\mathbb{A}_n$. This represents (for any k), the actual time for which the k^{th} step lasts, given that $\Phi_k = n$. Note here, we are inherently assuming that the consequent times, the user spends in the same rate region, are identically and independently distributed (IID). However the characteristics of these times can depend upon the region in which the user is wandering. After wandering in $\mathbb{A}_n$ for time W_n user moves either to $\mathbb{A}_{n+1}$ or to $\mathbb{A}_{n-1}$ respectively with probabilities p_n, $1 - p_n$. *Note that $p_1 = 1$ always for 2D.* That is, $\{p_n\}$ represent the transition probabilities of the embedded Markov chain $\{\Phi_k\}$.

All the quantities $\{p_n\}$, $\{W_n\}$, $\{r_n\}$ can depend upon the dimension L (which we are trying to optimize) and the dependence is shown explicitly, only if required, by adding L as a parameter, e.g., as in $p_{n;L}$.

Arrivals: There are two types of arrivals: 1) arrivals from external world (represented by subscript e, whenever there is an ambiguity) modeled as Poisson arrivals with parameter λ; 2) HO arrivals (always indicated either using subscript h or ho) modeled again as Poisson arrivals[2] and this stream is derived from a fraction of external arrivals whose service is not completed at a cross over. Rate of arrivals into cell of interest depends upon the cell dimension L and this is shown by either λ_L (for external arrivals) or $\lambda_{h;L}$ (for HO arrivals). For external arrivals, we assume[3] $\lambda_L = \lambda L^\eta$, while $\lambda_{h;L}$ will be calculated in later sections.

Every arrival, brings with it marks (Φ, S): $\Phi \in \mathbb{N}$ is position of arrival with distribution $\Pi := \{\pi_n\}$ and S is number of bytes to be transmitted with $S \sim \mu \ \ exp^{-\mu t} dt$ ($\mu > 0$).

[2]This is a commonly made assumption, for example see [10], [11].

[3]If the arrivals in the entire line segment $[-M, M]$ occur at rate λ', those in segment $[-L, L]$ occur at a smaller rate $\lambda_L = \lambda' Prob$(arrival in $[-L, L]$). For the special case of uniform arrivals (i.e., arrivals landing uniformly in $[-M, M]$), $\lambda_L = \lambda L$ for some $\lambda > 0$. Similarly for 2D, $\lambda_L = \lambda L^2$ for some $\lambda \geq 0$.

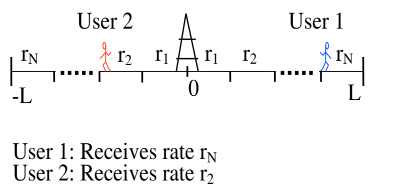

Fig. 1. One dimensional cell, rate partitioning and user's movement

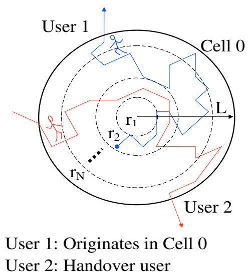

Fig. 2. Two dimensional cell, rate partitioning and user's movement

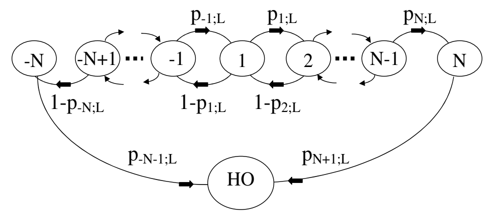

Fig. 3. Transitions of the embedded Markov chain $\{\Phi_k\}$ for 1D.

Resources: A cell can attend K parallel calls. The power per transmission, P_L, depends upon L the cell dimension and this dependency will be discussed later.

An example of $\mathbb{R}$**:** One can *choose the set of possible transmission rates,* $\mathbb{R}$ *and* N *based on the practical channel coding schemes* that would be used in the network design. *The analysis presented can be utilized to study a system with any given* $\mathbb{R}$ *and* N. However, in this paper, we consider a specific example. This specific $\mathbb{R}$ is obtained using low SNR approximation of the following theoretical (capacity) rate[4]:

$$r(d) := P_L \left(1_{\{d \le d_0\}} + r_0 \, |d|^{-\beta} \, 1_{\{d > d_0\}} \right) \text{ with } r_0 = d_0^{\beta}.$$

Here $r(d)$ is the rate at distance d, d_0 is a small lossless distance[5] while β is the propagation co-efficient. We consider a specific system which supports transmission at the maximum possible rate for the entire region. For example in $\mathbb{A}_n$ the farthest user will be at distance $|n|L/N$ and hence maximal transmission rate, that can be allocated equals

$$r_n = r(|n|L/N) = r_0 P_L N^{\beta} L^{-\beta} |n|^{-\beta}. \tag{2}$$

Remark: *Alternatively, if the system under consideration can design modulation and or channel coding schemes so as to achieve (almost)* ν *percent of the theoretical rates where* $\nu < 1$ *is a fixed coefficient, then again the above rate structure is applicable (after absorbing* ν *into* r_0 *of (2)).*

Handovers: Whenever the user reaches the boundary $\{|\mathbf{x}| = L\}$ the call is handed over to the neighboring cell (if not completed) and each HO requires extra s_h bytes to be exchanged. *We assume* $s_h \ll S$ *and that* s_h *bytes are exchanged w.p.1 (with probability one), while user is in the last region (*r_N *or* r_{-N}*).*

Notations: Let the flag, η, represent 1 for 1D and 2 for 2D. We denote the transpose by ^{t}. Calligraphic letters represent matrices. Mathbb letters represent sets (e.g., $\mathbb{N}$ - set of segment numbers, $\mathbb{R}$ - set of all possible transmission rates, $\mathbb{A}_n$ - rate region n). The contents inside two flower brackets represent either a set or an ordered tuple (as according to convenience): for example $\{r_n\}$ represents the set $\mathbb{R}$ while $\{\pi_n\}$ represents the ordered tuple Π. Lowercase letters represent time index (k) or the segment index (n). Lowercase bold letters represent the vectors.

Uppercase letters either represent system parameters (e.g., M - dimension of Macrocell, L - dimension of small cell, P - Power per transmission, K - Number of servers, N - Total number of possible transmission rates (number of elements in $\mathbb{R}$) , $\Pi = \{\pi_n\}_n = \{Prob(\text{Arrival in segment } n)\}_n$ - Vector of arrival probabilities, etc.) or represent random variables (W - wandering time, S - number of bytes to be transferred, Φ - the segment in which the user is wandering, etc.). When any of the above have to be indexed by n or k and further the dependency on parameter L has to be shown, then we use notation like $\pi_{n;L}$, $W_{n;L}$, P_L, $\Phi_{k;L}$ etc.

III. Analysis

In this section, using queuing theoretic tools, we obtain relevant performance metrics like, the expected service time, call busy and or drop probability etc. We also derive *capacity per cell*: a notion that gives the maximum number of bytes that can be transfered while the user traverses in a cell, normalized by the cell size. We obtain optimal cell dimension in the later sections utilizing the performance metrics derived. We begin with the analysis of the embedded Markov chain $\{\Phi_k\}$, whose transitions are as in Fig. 3. We obtain most of the analysis using conditional expectation techniques and the transition properties.

1) Expected service time: Let B_e represent the total amount of time for which an user derives service from the cell (in which the call originated): either the time to complete the call or the time until HO to a neighbor cell (earlier of the two). Let $\bar{b}_e$ represent its average, b_n the average given the call originated in region n (which happens with probability π_n). Then $\bar{b}_e = \sum_n \pi_n b_n$.

Time taken to transfer S bytes (at rate r_n) equals S/r_n and so a user wandering in $\mathbb{A}_n$ completes his service if $W_n > S/r_n$. Thus, the probability of completing the service while the user is in region n equals,

$$q_n \quad = \quad E[W_n > S/r_n] = 1 - E\left[exp^{-\mu W_n r_n}\right], \tag{3}$$

[4]For unit noise variance, capacity equals $log(1 + SNR)$, where signal to noise ratio $SNR = P_L A$ and attenuation $A = 1_{\{d \le d_0\}} + (d/d_0)^{-\beta} 1_{\{d > d_0\}}$. For low SNRs, $log(1 + SNR) \approx SNR$ and hence capacity equals $P_L A$.

[5]Typically d_0 is negligibly small and so **we consider optimizing over cell sizes** $L > d_0 N$ so that $r(d) = r_0 P_L d^{-\beta}$ always.

and the expected time for which the user receives the service, while moving in region n will be

$$t_n = E\left[\min\left\{W_n, \ \frac{S}{r_n}\right\}\right] = E\left[\frac{1-exp^{-\mu W_n r_n}}{\mu r_n}\right] = \frac{q_n}{\mu r_n}. \quad (4)$$

By memoryless nature of S, the bytes remaining after receiving the service in a rate region will again be exponentially distributed with the same parameter. Now, $\{b_n\}$ can be computed by conditioning on appropriate events. While in region n, it derives service on an average for time t_n and then it moves to either region $n+1$ with probability p_n or to region $n-1$ with probability $1-p_{n-1}$. If the service is not completed in region n (which happens with probability $1-q_n$) then the remaining bytes (note S is exponential) will be served in a similar manner in the new region entered albeit with new rate. This repeats either till the service is completed or till the user exits the cell. Thus, $\{b_n\}$ satisfies the linear equations (note $p_1 = 1$ and negative indices are not applicable for 2D):

$$\begin{aligned} b_n &= t_n + (q_n 0 + (1-q_n)p_n b_{n+1} + (1-q_n)(1-p_n)b_{n-1}), \\ b_n &= 0 \text{ when } n = N+1 \text{ or } -(N+1). \end{aligned} \quad (5)$$

The last equation indicates when the user moves out of the last rate region(s) (e.g., N) it no more derives service from the cell under consideration. That is, $\mathcal{Z}\mathbf{b} = \mathbf{t}$ where (for 1D),

$$\text{with } z_n := -(1-q_n)p_n \text{ and } \bar{z}_n := -(1-q_n)(1-p_n), \quad (6)$$

$$\mathcal{Z} := \begin{bmatrix} 1 & z_{-N} & 0 & & & & \cdots & 0 \\ \bar{z}_{-(N-1)} & 1 & z_{-(N-1)} & & & & \cdots & 0 \\ & & \vdots & & & & & \\ 0 & 0 & \cdots & \bar{z}_1 & 1 & z_1 & \cdots & 0 \\ & & \vdots & & & & & \\ 0 & 0 & \cdots & & & \cdots & \bar{z}_N & 1 \end{bmatrix}$$

$$\begin{aligned} \mathbf{b} &= [b_{-N}, b_{-N-1}, \cdots, b_{-1}, b_1, \cdots, b_N]^t \\ \mathbf{t} &= [t_{-N}, t_{-N-1}, \cdots, t_{-1}, t_1, \cdots, t_N]^t. \end{aligned}$$

The above definitions correspond to the case of one dimension (1D). The corresponding definitions for 2D are:

$$\mathcal{Z} := \begin{bmatrix} \bar{z}_1 & 1 & z_1 & 0 & \cdots & 0 \\ & & \vdots & & & \\ 0 & 0 & \cdots & & \bar{z}_N & 1 \end{bmatrix}$$

$$\mathbf{b} = [b_1, b_2, \cdots, b_N]^t, \qquad \mathbf{t} = [t_1, t_2, \cdots, t_N]^t.$$

Matrix $\mathcal{Z}$ is invertible in either case and hence, one can solve for $\{b_n\}$. The expected service time equals

$$\bar{b}_e = \Pi^t \mathcal{Z}^{-1}\mathbf{t} \quad \text{with} \quad \Pi := [\pi_{-N}, \pi_{-N+1} \cdots, \pi_N]^t. \quad (7)$$

2) Handovers (HO): A call that crossed over to a neighboring cell before completing its service is termed as a HO call. We assume that HOs can also be modeled by Poisson arrivals (see for e.g., [10], [11]). These HOs, just like the new calls, are picked up (continued) only when the new cell has free servers and require s_h additional bytes of information to be exchanged for initiating the call.

Stochastic equivalence, HO-SE: Due to stationarity, *the HOs into the cell of interest (cell 0, $[-L, L]$) are stochastically same as those that go out of the same cell*, because, for example in 1D: 1) by symmetry, the HOs from cell 0 ($[-L, L]$) to cell 1 ($[L, 3L]$) are stochastically same as those from cell -1 ($[-3L, -L]$) to cell 0; 2) the same is true for HOs when an user travels from right to left. The same is true even for 2D networks. Using this stochastic equivalence (which we will henceforth refer as HO-SE), we calculate all the quantities related to handovers (that are required for further analysis) via fixed point equations.

HO arrival positions: Let $\pi_{h,n}$ represent the probability that a HO call arrives at n. For 2D the HO can occur only at N and hence, $\pi_{h,n} = 0$ for all $1 \le n < N$ and $\pi_{h,N} = 1$.

For 1D HO can occur either at N or at $-N$ and hence

$$\pi_{h,n} = 0 \text{ for all } -Nn < N \text{ and } \pi_{h,N} \neq 0, \pi_{h,-N} \neq 0.$$

For 1D networks, *we assume symmetry in either direction. That is we assume that $p_n = 1 - p_{-n}$ and that $W_n \stackrel{d}{=} W_{-n}$ (stochastically equivalent).* Thus, $\pi_{h,N} = \pi_{h,-N} = 1/2$.

$$\text{Let } \Pi_h := \begin{cases} [0.5, \ 0, \ \cdots, \ 0, \ 0.5]^t & \text{for 1D} \\ [0, \ 0 \ \cdots, \ 0, \ 1]^t & \text{for 2D.} \end{cases} \quad (8)$$

HO Arrival rate: The probability of a (possible) HO is one minus the probability of service being completed within the cell and this has to be calculated by solving linear equations as in the case of $\bar{b}_e$. Let h_n represent the overall probability of completing the service in cell 0, given the call is originated in the region n. Then $\{h_n\}$ solves (by conditioning as explained for $\{b_n\}$, see equation (5))

$$\begin{aligned} h_n &= q_n + (1-q_n)p_n h_{n+1} + (1-q_n)(1-p_n)h_{n-1} \text{ and} \\ h_n &= 0 \text{ when } n = N+1 \text{ or } -(N+1). \end{aligned}$$

That is, $\mathcal{Z}\mathbf{h} = \mathbf{q}$ where $\mathbf{h} := [h_{-N}, \cdots, h_N]^t$ and $\mathbf{q} := [q_{-N}, \cdots, q_N]^t$. Again for 2D, the vectors $\mathbf{h}$ and $\mathbf{q}$ have only the lower N elements as in the case of the vectors $\mathbf{b}$, $\mathbf{t}$.

The probability of a new arrival not completing service before moving out of the current cell (which results in a HO) equals

$$P_{e,ho} = 1 - \sum_n \pi_n h_n = 1 - \Pi^t \mathcal{Z}^{-1}\mathbf{q}. \quad (9)$$

In other words, out of all the new or external arrivals that arrived in cell 0, $P_{e,ho}$ portion of them get handed over to a neighboring cell. Some of these HOs get converted to a HO again. The probability of this event can be calculated in a similar way and it equals (see equation (8) and see footnote 6 to understand why this does not depend upon s_h, extra HO bytes),

$$P_{h,ho} = 1 - {\Pi_h}^t \mathcal{Z}^{-1}\mathbf{q}.$$

Again by memory less property (as S is exponential) there is no difference in this probability (or any other quantity that we calculate further) for the first HO and for the subsequent HOs. A HO can result in further HOs and so on. Thus (by conditioning on appropriate events) the handover rate $\lambda_{h;L}$ by stochastic equivalence (HO-SE), satisfies:

$$\lambda_{h;L} = \lambda_L P_{e,ho} + \lambda_{h;L} P_{h,ho} \quad \text{and hence} \quad \lambda_{h;L} = \frac{\lambda_L P_{e,ho}}{1 - P_{h,ho}}.$$

3) Overall service time and stability factor: Let $\bar{b}$ represent the average of the service times demanded by external as well as HO arrivals. The expected time for which a HO call utilizes the cell's resources (by memoryless property, irrespective of the number of times the call is already handed over), can be calculated in a similar way as done while obtaining (7) and equals:

$$\Pi_h^t \mathcal{Z}^{-1} \mathbf{t}.$$

Let t_{ho} represent the time taken to serve the HO bytes s_h and the overall service also includes this. By our assumption, these bytes are exchanged in the outermost rate region and hence $t_{ho} := s_h / r_N$. These bytes are exchanged only during HO and the HO call utilizes the system resources even during this period. And so the expected service time of a HO call equals[6]:

$$\bar{b}_h = \Pi_h^t \mathcal{Z}^{-1} \mathbf{t} + t_{ho}.$$

With this, the overall service time equals:

$$\bar{b} = \frac{(\lambda_L \Pi^t + \lambda_{h;L} \Pi_h^t) \mathcal{Z}^{-1} \mathbf{t} + \lambda_{h;L} t_{ho}}{\lambda_L + \lambda_{h;L}}.$$

Then the stability factor is,

$$\rho_L = \frac{\bar{b}(\lambda_L + \lambda_{h;L})}{K} = \frac{(\lambda_L \Pi^t + \lambda_{h;L} \Pi_h^t) \mathcal{Z}^{-1} \mathbf{t} + \lambda_{h;L} t_{ho}}{K}. \quad (10)$$

4) Busy and Drop Probability for Non elastic traffic: Non elastic traffic comprises of users demanding immediate service. These users (e.g., voice calls) drop the call if it is not picked up immediately, i.e., if all the servers are busy. The probability that a call is not picked up immediately is called the Busy probability and the probability that a call that was picked up is ever dropped before completing its service is called the Drop probability. We compute both these quantities. A small cell catering to non elastic traffic can be modeled by an M/G/K/K queue (as we have done in [2]). Then using Erlang's loss formula the busy probability can be calculated:

$$P_{Busy}(L) := \frac{\rho_L^K / K!}{\sum_{k=0}^K \rho_L^k / k!}.$$

Busy probability, P_{Busy}, depends upon L only via ρ and both are differentiable in L (see [2] for similar details) and by differentiating twice one can immediately obtain the following:

Lemma *1:* Optimizers of ρ and P_{Busy} are same, i.e.,

$$L_\rho^* := \min_L \rho = \min_L P_{Busy}(L) =: L_{P_{Busy}}^*. \quad \square$$

Drop probability (probability that a call that is picked up will ever be dropped) can now be calculated by conditioning.

$$P_{Drop} = P_{e,ho}(P_{Busy} + (1 - P_{Busy}) P_{h,ho} P_{h,Drop})$$

where $P_{h,ho}$ and $P_{e,ho}$ are defined in previous section and where $P_{h,Drop}$ is the drop probability given that the call is a HO call, which satisfies by HO-SE (see [2]):

$$P_{h,Drop} = P_{Busy} + (1 - P_{Busy}) P_{h,ho} P_{h,Drop}$$

and so

$$P_{h,Drop} = \frac{P_{Busy}}{1 - (1 - P_{Busy}) P_{h,ho}}.$$

Substituting,

$$P_{Drop} = \frac{P_{e,ho} P_{Busy}}{1 - (1 - P_{Busy}) P_{h,ho}}. \quad (11)$$

5) Expected waiting time for Elastic Traffic: One can follow the approach as in ([2]) to derive the corresponding performance, the average waiting time of a call. However this is not considered in this paper.

6) Capacity per cell: We define capacity of a cell as the average number of *maximum*[7] bytes that can be transferred in a cell per cell size. Let c_n represent the maximum number of bytes that can be transmitted when a call originates in $\mathbb{A}_n$. While staying in region n a maximum of $W_n r_n$ number of bytes can be transferred and hence c_n can be obtained using the following iteration (by same procedure as used for (5))

$$c_n = E[W_n] r_n + p_n c_{n+1} + (1 - p_n) c_{n-1}.$$

Thus the capacity of the cell and the capacity per cell equals (for 1D the length of a cell $\propto L$ while for 2D, the area of the cell is $\propto L^2$)

$$C_{cap} = \Pi^t \mathcal{P}^{-1} \mathbf{r_w} \text{ and } C_{cell} := \frac{C_{cap}}{L^\eta} = \frac{1}{L^\eta} \Pi^t \mathcal{P}^{-1} \mathbf{r_w} \text{ with} \quad (12)$$

$$\mathcal{P} := \begin{bmatrix} 1 & \hat{p}_N & 0 & & & \cdots & 0 \\ \bar{p}_{-(N-1)} & 1 & \hat{p}_{N-1} & & \cdots & & 0 \\ & & \vdots & & & & \\ 0 & 0\cdots & & \bar{p}_1 & 1 & \hat{p}_1 \quad \cdots & 0 \\ & & \vdots & & & & \\ 0 & 0\cdots & & & \cdots & \bar{p}_N & 1 \end{bmatrix}$$

$$\mathbf{c} := [c_{-N}, c_{-N-1}, \cdots, c_{-1}, c_1, \cdots, c_N]^t$$

$$\mathbf{r_w} := [r_{-N} E[W_{-N}], \cdots, r_{-1} E[W_{-1}], r_1 E[W_1], \cdots, r_N E[W_N]]^t.$$

where $\bar{p}_n := p_n - 1$ and $\hat{p}_n := -p_{-n}$. Again for 2D the quantities are reduced matrices/vectors as explained before, e.g., as in the definition of matrix $\mathcal{Z}$.

7) Time to reach boundary: Expected time to reach boundary can be calculated on similar lines and this equals

$$\begin{aligned} \tau_L &:= E[T_L] = \Pi^t \mathcal{P}^{-1} \mathbf{w}, \text{ with} \quad (13) \\ \mathbf{w} &:= [E[W_{-N}], E[W_{-N-1}], \\ &\quad \cdots, E[W_{-1}], E[W_1], \cdots, E[W_N]]^t. \end{aligned}$$

We summarize all the expressions derived in the Table I. From this table, it is clear that one can analyze and derive performance measures for any system (i.e., given N, K and $\mathbb{R}$, etc.,) for which $\{p_n\}$ (the transition probabilities w.r.t. the rate regions) and $\{q_n\}$ (the Laplace transform of the wandering times $W_{n;L}$) can be calculated. We next consider one example user movement model (random walk model) and apply the analysis of this section to obtain expressions

[6] By exponential nature of the wandering times, the leftover time in the last region will again be exponential and hence the remaining calculations are unchanged.

[7] By "maximum" we mean the bytes of information transfered via (capacity) maximum possible rate, given the rate partitioning. The rates given by (2) exactly represent this *maximum* rates when $\nu = 1$.

for various performance measures. We also obtain the closed form expressions for the optimizers (optimal cell dimension) of these performance measures in some cases.

TABLE I
THE VARIOUS EXPRESSIONS

$q_n = 1 - E\left[exp^{-\mu W_n r_n}\right]$	$t_n = \frac{q_n}{\mu r_n}$
$\bar{b}_e = \Pi^t \mathcal{Z}^{-1}\mathbf{t}$	$\bar{b}_h = \Pi_h^t \mathcal{Z}^{-1}\mathbf{t} + t_{ho}$
$P_{e,ho} = 1 - \Pi^t \mathcal{Z}^{-1}\mathbf{q}$	$P_{h,ho} = 1 - \Pi_h^t \mathcal{Z}^{-1}\mathbf{q}$
$\lambda_{h;L} = \lambda_L \frac{P_{e,ho}}{1-P_{h,ho}}$	$\rho = \frac{(\lambda_L \Pi^t + \lambda_{h;L}\Pi_h^t)\mathcal{Z}^{-1}t + \lambda_{h;L}t_{ho}}{K}$
$P_{Busy}(L) = \frac{\rho_L^K/K!}{\sum_{k=0}^K \rho_L^k/k!}$	$P_{Drop} = \frac{P_{e,ho}P_{Busy}}{1-(1-P_{Busy})P_{h,ho}}$
$C_{cell} = L^{-\eta}\Pi^t \mathcal{P}^{-1}\mathbf{r_w}$	Example, $r_n = r_0 P_L N^\beta L^{-\beta}\lvert n\rvert^{-\beta}$

IV. RANDOM WALK WITH EXPONENTIAL WANDERING TIMES

The users arrive in one of the rate regions n, wander for time $W_{n;L}$ which is exponentially distributed (whose distribution is independent of every other process) and then switches to one of its neighboring rate regions or moves over to a next cell if region n is at the cell edge. The mean of the wandering time $W_{n;L}$ is proportional to the measure (length in case of 1D, area in case of 2D) of the region in which it is moving. The area of the 2D annular ring n equals $\pi(L/N(n+1))^2 - \pi(L/Nn)^2 = \pi L^2/N^2(2n-1)$. That is (recall $\eta = 1$ for 1D and 2 for 2D),

$$E[W_{n;L}] = \frac{1}{\omega_L} = \frac{L^\eta(2n^{\eta-1}-1)}{\omega} \text{ for some } \omega > 0.$$

This dependence upon the cell size L ensures that the mean variations of the mobility model remains (almost) same irrespective of the cell size. In this case (from (3) and definitions of the elements of the matrix $\mathcal{Z}$),

$$\begin{aligned} q_n &= \frac{\mu r_n}{\omega_L + \mu r_n}, \quad t_n = \frac{1}{\omega_L + \mu r_n}, \\ z_n &= -\frac{\omega_L}{\omega_L + \mu r_n} p_n, \text{ and } \bar{z}_n = -\frac{\omega_L}{\omega_L + \mu r_n}(1-p_n). \end{aligned} \tag{14}$$

We assume uniform arrivals, i.e., the arrivals position themselves uniformly across the entire system and hence $\pi_n = 1/((3-\eta)N)$. We further assume that the rates used depend upon the distance from the BS. In particular we choose the theoretical rates (in low SNR regime) as in equation (2), reduced by a ν factor (which is absorbed into r_0) as explained in section II.

1) Capacity per cell: In this case, the capacity per cell (12) simplifies to,

$$C_{cell} = \frac{r_0 P_L L^{-\beta}}{N^{-\beta}\omega} \Pi^t \mathcal{P}^{-1}\mathbf{n}_\beta \text{ with} \tag{15}$$

$$\mathbf{n}_\beta^t := \begin{cases} \left[N^{-\beta}, \cdots, 2^{-\beta}, 1, 1, 2^{-\beta}, \cdots, N^{-\beta}\right] & \text{if 1D} \\ \left[1, 3 * 2^{-\beta}, \cdots, (2N-1)N^{-\beta}\right] & \text{if 2D.} \end{cases}$$

The capacity per cell is a fundamental limit that represents the maximum transferable information per cell size that can be transfered while an user moves in the cell which can support N distinct rates. If the total power in the system has to remain constant[8] then $P_L = PL^\eta$. With P_L scaling as PL^η, we notice from equation (15) that C_{cell} decreases with L (note practical values of $\beta \geq 2$, even $\beta = 2$ is not considered as a very practical value of path loss factor). This implies that the optimal cell size (optimizing the fundamental limit C_{cell}) is Nd_0, which is practically an infeasible cell dimension. In other words, the total power budget has to be increased with L, if cell sizes greater than the trivial Nd_0 has to perform better. The necessary growth rate can easily be read from (15) and hence we have,

Lemma *2 (β^+***-scaling):** Capacity per cell increases with L only if the power P_L scales with L according to

$$P_L = PL^{\beta+\gamma} \text{ for some } \gamma > 0. \qquad \square$$

To obtain this result we used low SNR approximation of the capacity formula $log(1 + P_L r_n) \approx P_L r_n$. However one can easily see that Lemma 2 is true, even without this approximation. This approximation is used only for simplifying further analysis.

The above lemma shows that the fundamental capacity improves monotonically with cell size only when $P_L = PL^{\beta+\gamma}$ and would actually decrease with L if it is not boosted with $\gamma > 0$. Henceforth, we call this as β^+*-scaling*. However, this fundamental limit does not consider the HO losses. Hence, one needs other performance metrics to capture the HO trade offs mentioned at the beginning of this paper, after boosting the power according to β^+-scaling.

2) Drop and Busy probability: HO losses become significant for small cell sizes and performance metrics like drop probability (P_{Drop}) or busy probability (or equivalently ρ (by Lemma 1)) capture these losses. The rest of the section focuses on obtaining the optimal cell size for these metrics, when the power scales as in Lemma 2. We also show in some cases that the optimal cell size (for P_{Busy}) is Nd_0, if this scaling is not done. Towards the end we also consider/propose an optimal cell size that optimizes a cost combining the busy probability and the total power used.

For exponential wandering times , from (2) and (14):

$$q_n = \frac{\mu r_0 P_L L^{\eta-\beta}|n|^{-\beta}}{\mu r_0 P_L L^{\eta-\beta}|n|^{-\beta} + N^{-\beta}(2n^{\eta-1}-1)^{-1}\omega}. \tag{16}$$

Average time to reach the boundary, τ_L, is indicative of the speed of the user and we obtain further analysis in two asymptotic limits of τ_L. Intermediate values are studied via simulations. From (13) , for exponential wandering times,

$$\tau_L = \frac{L}{\omega}\Pi^t \mathcal{P}^{-1}\mathbf{1}, \text{ with } \mathbf{1}^t := [1, \cdots, 1, \cdots, 1]. \tag{17}$$

[8]The number of pico cells of dimension L, is proportional to $L^{-\eta}$ and hence total power would be proportional to $P_L L^{-\eta}$. Thus to maintain the total power constant, $P_L = PL^\eta$ for some constant $P > 0$.

3) Low speeds: As $\omega \to 0$, from (17) user covers a cell with large τ_L, i.e, *the user is moving with low speed.* In this limit, $q_n \approx 1$. This, in turn implies $\mathcal{Z} \approx$ identity matrix and that

$$t_n \approx \frac{1}{\mu r_n} = \frac{N^{-\beta}}{\mu r_0 P_L L^{-\beta} |n|^{-\beta}}, \quad P_{e,ho} \approx 0 \approx P_{h,ho}.$$

When the users wander in the same cell for considerable amount of time, its service gets completed within one cell itself and this is the reason for no HOs (i.e, $P_{e,ho} = P_{h,ho} = \lambda_{h;L} \approx 0$). *With no HOs the drop probability is zero.* Further, with β^+ scaling, one can expect an improvement in busy probability as the cell size increases. Indeed, substituting for the power scaling, $P_L = PL^{\beta+\gamma}$ from Table I:

$$\rho \approx \frac{\lambda L^\eta \left(\Pi^t \check{\mathbf{n}}_\beta\right)}{K\mu r_0 P_L L^{-\beta} N^\beta} = \frac{\lambda L^{\eta-\gamma}\left(\Pi^t \mathbf{n}_\beta^{-1}\right)}{K\mu r_0 P N^\beta} \text{ with} \tag{18}$$
$$\check{\mathbf{n}}_\beta := [N^\beta, \cdots, 1, 1, \cdots, N^\beta]^t \text{ and } P_{Drop} \approx 0.$$

From the above equation it is clear that the stability factor (ρ) improves with L only if $\gamma > \eta$ asserting again the need for (actually more than) β^+-power scaling. With $\gamma > \eta$, ρ decreases monotonically with L. Further, as γ increases, ρ improves for the same L. By Lemma 1 the busy probability, P_{Busy}, also improves with γ. But with $\gamma > \eta$, total power increases with L. Thus one needs to consider joint cost, consisting of power cost and ρ,

$$L^{\eta-\gamma} + aL^{\beta+\gamma}. \tag{19}$$

Lemma *3:* With $\omega \to 0$ the optimizer of the joint cost, consisting of total power and ρ as given in (19), for any $\gamma > \eta$ and weight factor $a > 0$, equals:

$$L^*_\rho(\gamma) = \left(\frac{\beta+\gamma-\eta}{a(\beta+\gamma)}\right)^{1/(\beta+2\gamma-\eta)}. \qquad \square$$

4) High speeds (as $\omega \to \infty$): From (17), as $\omega \to \infty$, τ_L decreases to 0, implying high speed users. With this one can approximate

$$q_n \approx \frac{\mu r_n}{\omega_L} = L^{\eta-\beta}\frac{\mu P_L r_0 |n|^{-\beta}(2n^{\eta-1}-1)}{N^{-\beta}\omega},$$

$(1-q_n)$ with 1, t_n with $1/\omega_L$ and $\mathcal{Z}$ with $\mathcal{P}$. Then (see the expressions in Table I),

$$\begin{aligned}
\bar{b}_e &\approx \frac{L^\eta}{\omega}\Pi^t\mathcal{P}^{-1}\mathbf{n}_\omega, \quad \bar{b}_h \approx \frac{L^\eta}{\omega}\Pi_h^t\mathcal{P}^{-1}\mathbf{n}_\omega + t_{ho} \text{ with}\\
\mathbf{n}_\omega &:= \mathbf{1}1_{\{\eta=1\}} + [1,3,\cdots,2N-1]^t 1_{\{\eta=2\}},\\
P_{e,ho} &\approx 1 - \frac{\mu P_L r_0 L^{\eta-\beta}}{\omega N^{-\beta}}\Pi^t\mathcal{P}^{-1}\mathbf{n}_\beta, \text{ (see (15) and continuing)}\\
\rho_L &= \frac{\lambda N^{-\beta}\left(\check{\rho}_1 L^{\beta+\eta}P_L^{-1} + \check{\rho}_2 L^{2\eta}\right)}{K\Pi_h^t\mathcal{P}^{-1}\mathbf{n}_\beta \mu r_0} + \frac{s_h\lambda_{h;L}}{Kr_N} \quad \text{with}\\
\check{\rho}_1 &= \Pi_h^t\mathcal{P}^{-1}\mathbf{n}_\omega \text{ and}\\
\check{\rho}_2 &= \frac{\mu r_0}{\omega N^{-\beta}}\mathbf{n}_\beta^t\mathcal{P}^{-1}\left(\Pi_h\Pi^t - \Pi\Pi_h^t\right)\mathcal{P}^{-1}\mathbf{n}_\omega.
\end{aligned}$$

With β^+- power scaling we have,

$$\begin{aligned}
\rho_L &= \frac{\lambda N^{-\beta}\left(\check{\rho}_1 L^{\beta+\eta}P_L^{-1} + \check{\rho}_2 L^{2\eta}\right)}{K\Pi_h^t\mathcal{P}^{-1}\mathbf{n}_\beta \mu r_0} + \frac{s_h\lambda_{h;L}}{Kr_N} \qquad (20)\\
&= \frac{\lambda N^{-\beta}\left(\tilde{\rho}_1 L^{\eta-\gamma} + \tilde{\rho}_2 L^{-2\gamma} + \check{\rho}_2 L^{2\eta}\right)}{K\Pi_h^t\mathcal{P}^{-1}\mathbf{n}_\beta \mu r_0}\\
\tilde{\rho}_1 &= P^{-1}\check{\rho}_1 - \frac{s_h\mu\Pi^t\mathcal{P}^{-1}\mathbf{n}_\beta}{N^{-\beta}}, \quad \tilde{\rho}_2 = \frac{s_h\omega}{P^2 r_0}.
\end{aligned}$$

For large values of ω, $\check{\rho}_2$ is small and hence we have

$$\begin{aligned}
\rho_L &\approx \frac{\lambda N^{-\beta}}{K\Pi_h^t\mathcal{P}^{-1}\mathbf{n}_\beta \mu r_0}\left(\tilde{\rho}_1 L^{\eta-\gamma} + \tilde{\rho}_2 L^{-2\gamma}\right) \qquad (21)\\
\tilde{\rho}_1 &= P^{-1}\Pi_h^t\mathcal{P}^{-1}\mathbf{n}_\omega - \frac{s_h\mu\Pi^t\mathcal{P}^{-1}\mathbf{n}_\beta}{N^{-\beta}}, \quad \tilde{\rho}_2 = \frac{s_h\omega}{P^2 r_0}
\end{aligned}$$

with $\mathbf{n}_\omega := \mathbf{1}1_{\{\eta=1\}} + [1,3,\cdots,2N-1]^t 1_{\{\eta=2\}}$. We *see that Lemma 2 is affirmed again, i.e., the optimizer of ρ (and that of P_{Busy}) equals the trivial one Nd_0 if $\gamma \le 0$.* When $\gamma > 0$, by differentiating twice (first derivative is zero and second derivative is positive at minimizer) we obtain:

Lemma *4:* For large ω, cell size optimizing busy probability, $L^*_{P_{Busy}} = Nd_0$ if $\gamma \le 0$. If $0 < \gamma < \eta$,

$$L^*_\rho = L^*_{P_{Busy}} = \left(\frac{2\gamma\tilde{\rho}_2}{(\eta-\gamma)\tilde{\rho}_1}\right)^{1/(\eta+\gamma)}. \qquad \square$$

One can again optimize a joint cost of ρ and the power: $\tilde{\rho}_1 L^{\eta-\gamma} + \tilde{\rho}_2 L^{-2\gamma} + aPL^{\beta+\gamma}$. Cell size optimizing the P_Drop, can be obtained similarly (proof in Appendix A):

Lemma *5:* For large ω, L^* which optimizes the drop probability is (if $K(\eta-\gamma) > \eta+\gamma$)

$$L^*_{P_{Drop}} = \left(\frac{((2K+1)\gamma+\eta)\tilde{\rho}_2}{(K(\eta-\gamma)-(\eta+\gamma))\tilde{\rho}_1}\right)^{1/(\eta+\gamma)}. \qquad \square$$

Properties of the optimizers: We observe from Lemmas 4 and 5 that the optimal cell size: 1) decreases with increase in path loss factor β ($\tilde{\rho}_1 \uparrow$ with $\beta \uparrow$); 2) increases with γ, the power scaling factor; 3) increases with increase in ω (from (17), when $\omega \uparrow$ *speed* of the user $\uparrow$).

5) Numerical examples: We obtain the optimizers for the general case of ω via numerical examples. We estimate the optimizers for the performance metrics given in Table I, after substituting the values of q_n, t_n etc., with equations (14), via grid search method. We compare the estimated optimizers (shown in figures with $\hat{}^*$ symbols) with that of Lemmas 4 and 5 (shown in figures as *). From figure 4, we observe that the computed optimizers are close to the numerically estimated ones for both the values of β (2.2 and 2.8), for both P_{Drop} and P_{Busy} when ω is large. For small values of ω ($\omega < 70$ for $\beta = 2.8$ and $\omega < 25$ for $\beta = 2.2$) we notice that the approximation is no more good.

In figure 5 we plot the high speed approximation for ρ given by (21) and the actual value of ρ as given in Table I after substituting (16). We notice that the approximation is very close to the actual value. However, the approximation error increases with increase in β the path loss factor, which once

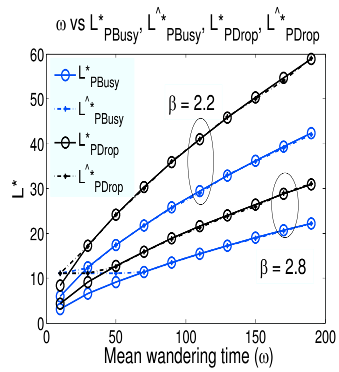

Fig. 4. Mean wandering time ω vs L*

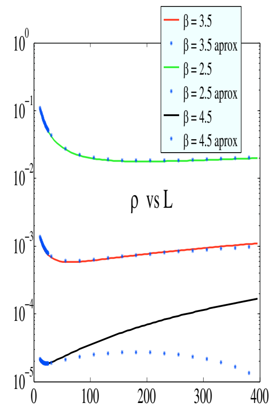

Fig. 5. ρ_L vs L

again confirms the closeness of the two sets of optimizers of Figure 4 for large values of ω.

From these numerical examples, we again observe that, the optimal cell size decreases with increase in path loss factor as well as with decrease in speed of the user given in terms of ω.

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Conclusions

We obtained the performance analysis of cellular networks catering to randomly wandering users. We modeled the user movements by a random walk, in which each step corresponds to a rate region, where the rate regions are obtained by partitioning the cell based on the transmission rates. With exponential wandering times, in each rate region, we obtained key performance measures like service times, busy and drop probabilities, capacity per cell, etc. We showed that the fundamental capacity per cell decreases monotonically with cell size, unless the power budget is increased (by a factor greater than β, the path loss factor) with cell size. We also showed that without β^+ power scaling, the optimal cell size, optimizing the busy probability, would be trivial (equal to the lossless distance). We obtained closed form expressions for optimal cell sizes, with β^+ power scaling, in the two asymptotic regimes of the user speeds (speed tending to zero and infinity). We also obtained the optimizers for intermediate values of speeds via numerical simulations. This procedure gives an (numerical) algorithm to obtain optimal cell size for all speeds. We establish the following: 1) Optimal cell size increases with speed, ω; 2) decreases with path loss factor β and 3) increases with the power scaling factor γ.

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Appendix A

Proof of Lemma 5: By differentiating and simplifying (as $K/\rho >> 1$),

$$\frac{dP_{Busy}}{d\rho} = P_{Busy}\frac{K}{\rho} - P_{Busy}\frac{\sum_{m=0}^{K-1}\frac{\rho^m}{m!}}{\sum_{m=0}^{K}\frac{\rho^m}{m!}} \approx P_{Busy}\frac{K}{\rho}.$$

From table I, since $P_{h,ho}P_{Busy} \ll 1 - P_{h,ho}$ (these probabilities are small usually of the orders 10^{-3} or lesser):

$$P_{Drop} \approx P_{Busy}\frac{P_{e,ho}}{1 - P_{h,ho}}.$$

Hence (for large ω),

$$\begin{aligned}
\frac{dP_{Drop}}{dL} \quad &\approx \quad P_{Busy}\left(\frac{d\left(\frac{P_{e,ho}}{1-P_{h,ho}}\right)}{dL} + \frac{P_{e,ho}}{1-P_{h,ho}}\frac{K}{\rho}\frac{d\rho}{dL}\right) \\
&\approx \frac{P_{Busy}}{\frac{\mu Pr_0}{\omega N^{-\beta}}\Pi_h^t \mathcal{P}^{-1}\mathbf{n}_\beta}\Bigg(- (\eta+\gamma)L^{-\eta-\gamma-1} \\
&\qquad + L^{-\eta-\gamma}K\frac{(\eta-\gamma)\tilde{\rho}_1 L^{\eta-\gamma-1} - 2\gamma\tilde{\rho}_2 L^{-2\gamma-1}}{\tilde{\rho}_1 L^{\eta-\gamma} + \tilde{\rho}_2 L^{-2\gamma}}\Bigg) \\
&= \frac{\lambda P_{Busy} L^{-\eta-\gamma-1}\omega N^{-2\beta}}{\mu^2 Pr_0^2 \left(\Pi_h^t \mathcal{P}^{-1}\mathbf{n}_\beta\right)^2 \rho} \\
&\qquad \left((K(\eta-\gamma) - (\eta+\gamma))\tilde{\rho}_1 L^{\eta-\gamma} - (2K\gamma+\eta+\gamma)\tilde{\rho}_2 L^{-2\gamma}\right).
\end{aligned}$$

The first term in the last equation is always non zero and so the derivative is zero if and only if the second term is zero. Further, the second derivative is positive at that zero. ∎