Nonlinear controller for a modified design of the ball beam system

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Abstract

The global stabilization of the classical ball & beam has been approached in the literature through saturated control which impose restrictions on the reactivity of the closed loop. In this work a modified design for the classical ball & beam system is presented. The beam is driven by two actuators (see figure 1). In comparison to the classical system, this design offers an additional degree of freedom which is the vertical motion of the beam. We show that the new design offers the possibility to get rid of the closed loop low reactivity restriction. We propose two nonlinear controllers to steer the trajectories of the system towards a final desired position. The first controller adapts, to the new design, existing controllers from the literature for the classical ball & beam. The second controller uses the additional degree of freedom to provide a faster stabilization.

1 Introduction

Stabilization of underactuated systems, driven by fewer actuators than degrees of freedom, presents a challenging problem which has attracted a considerable attention in the nonlinear control community (see, for example, [1] [2] [3] [4] and the references therein). The ball & beam system is one of the simplest underactuated systems.

The classical system consists of a beam free to rotate around a fixed axis. The ball is free to translate along the beam. It is underactuated since the position of both the beam and the ball should be controlled through the torque acting on the beam. An obstacle to the stabilization of the system came from the destabilizing centrifugal force. In fact, from the dynamical equations

$$\begin{aligned}\ddot{y} &= y\dot{\theta}^2 - g\sin(\theta)\\ \ddot{\theta} &= \tau_2,\end{aligned}$$

one may naturally consider the gravitational term $-g\sin(\theta)$ as a virtual control input for the y dynamics. τ_2 is then designed to accomplish the desired virtual input θ which stabilize y. This approach works locally since if y is large or $\dot{\theta}$ is high, the system is destabilized by $y\dot{\theta}^2$. The problem arise thus from the centrifugal force $y\dot{\theta}^2$, when both fast stabilization and large domain of attraction are desired. The global stabilization has been addressed through low gain control designs in [5] [6] [7] [8]. The idea is to drive the beam slowly such that $\dot{\theta}$ become very small. A major drawback of low gain design is the slow time response of the closed loop system which induce large transients when the system starts far away from the origin. The present work consider a novel design of the ball & beam system which permit to overcome both mentioned problems. It is described in the following lines.

The beam consist in a double axes system whose inclination is driven through two actuators of different technology, pneumatic and electrical. The upper axis is held up from one side by a prismatic-revolute

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piston rod (PR) and by a revolute piston rod (R) on the other side. The “ball” (actually it is a translating mass) hang on lower axis and can slide along this one. A diagram of the platform can be depicted in figure 2. It is worth mentioning that the new design is also an underactuated system since it has 3 outputs and 2 control inputs. Challenges arising from tight cooperation between both actuators will be addressed and faced using nonlinear control techniques, afterwards validated by means of this prototype in a future work.

This communication proceeds as follows. Section 2 introduces the dynamical model of the system. In section 3 and 4 two nonlinear controllers are developed. The first adapts existing results from literature to the new design. While the second controller take profit of the additional degree of freedom to provide faster time responses in closed loop. Simulation results are presented in section 5 with the aim of comparing the responses of both controllers. Finally, in section 6, conclusions are drawn.

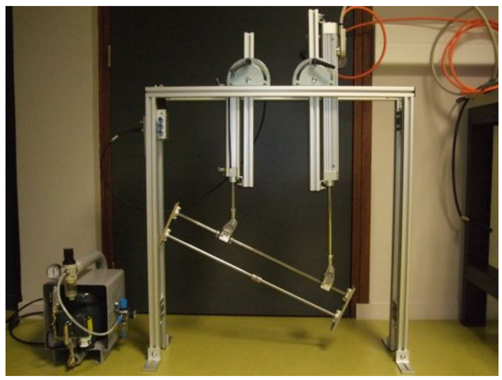


Figure 1: Benchmark picture

2 DYNAMICAL MODEL

Actuators dynamics is not our concern in this work, we assume that a feedback is already designed such that the forces acting on the beam can be considered as control inputs. Nonetheless, actuators dynamics will be one of our main concerns in a future work in order to compare both technologies. After writing the total energy of the system and applying the equations of Euler-Lagrange, the dynamical model writes:

$$\Gamma(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau \tag{1}$$

where $q = [x\ \theta\ y]^t$ is the vector of generalized coordinates. $\Gamma(q) = [\gamma_1(q)\ \gamma_2(q)\ \gamma_3(q)]^T$, $C(q,\dot{q})$, $G(q)$ represent the inertia matrix, the matrix of Coriolis and the term of gravitational forces respectively, are defined as follows

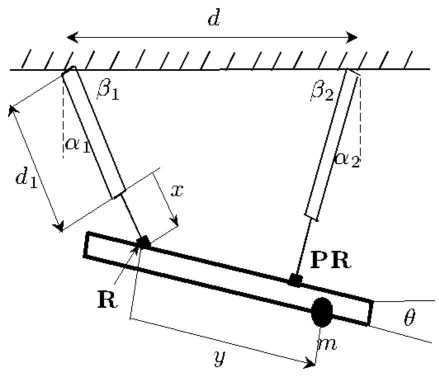
$$\gamma_1(q) = \begin{bmatrix} M+m \\ (Ml+my)\mathfrak{c}_\theta \\ m\mathfrak{s}_\theta \end{bmatrix}, \quad \gamma_3(q) = \begin{bmatrix} m\mathfrak{s}_\theta \\ my(\mathfrak{s}_\theta\mathfrak{c}_\theta - \mathfrak{c}_{\theta+\alpha_1}\mathfrak{s}_{\theta+\alpha_1}) \\ m(\mathfrak{s}_\theta^2 + \mathfrak{c}_{\theta+\alpha_1}^2) \end{bmatrix},$$

$$\gamma_2(q) = \begin{bmatrix} (Ml+my)\mathfrak{c}_\theta \\ J_0 + (Ml^2+my^2)(\mathfrak{c}_\theta^2 + \mathfrak{s}_{\theta+\alpha_1}^2) \\ my(\mathfrak{s}_\theta\mathfrak{c}_\theta - \mathfrak{c}_{\theta+\alpha_1}\mathfrak{s}_{\theta+\alpha_1}) \end{bmatrix},$$

$$G(q) = \begin{bmatrix} -(Mg+mg)\mathfrak{c}_{\alpha_1} \\ -(Ml-my)g\mathfrak{c}_\theta \\ mg\mathfrak{s}_\theta \end{bmatrix} \quad \tau = \begin{bmatrix} F_1 + F_2\mathfrak{c}_{2\alpha_1} \\ \frac{F_2\mathfrak{s}_{\theta+\beta_2}(d-2d_1(t))\mathfrak{c}_{\beta_1}}{\mathfrak{c}_\theta + \mathfrak{s}_\theta\mathfrak{t}_{\alpha_2}} \\ 0 \end{bmatrix},$$

$$C(q,\dot q)=\begin{bmatrix} 0 & -((Ml+my)\mathfrak{s}_\theta\dot\theta+2m\dot y\mathfrak{c}_\theta) & 0\\ 0 & \phi(Ml^2+my^2)\dot\theta+2my\dot y(\mathfrak{c}^2_\theta+\mathfrak{s}^2_{\theta+\alpha_1}) & 0\\ 0 & m\dot y(\mathfrak{s}_{2\theta}-\mathfrak{s}_{2(\theta+\alpha_1)})-my(\mathfrak{c}^2_{\theta+\alpha_1}+\mathfrak{s}^2_\theta)\dot\theta & 0\end{bmatrix},$$

with $\phi=(\mathfrak{s}_{\theta+\alpha_1}\mathfrak{c}_{\theta+\alpha_1}-\mathfrak{c}_\theta\mathfrak{s}_\theta)$. M and m represents the masses of the beam and the "ball" respectively, J the inertia of the beam. The systems variables are represented on the figure 2.



PR : Prismatic & revolute joint

R : Revolute joint

θ : rod inclination

Figure 2: System diagram

For the sake of simplicity, in this preliminary study we assume that both actuators are mounted in a vertical position ($\alpha_1=\alpha_2=0$) reducing the dynamical model to (see fig. 3):

$$\begin{array}{rcl}(M+m)\ddot x & + & (Ml+my)\cos\theta\ddot\theta+m\sin\theta\ddot y-(Ml+my)\sin\theta\dot\theta^2\\ & + & 2m\dot y\cos\theta\dot\theta-(M+m)g=F_1+F_2\\ (J_0+Ml^2 & + & my^2)\ddot\theta+(Ml+my)\cos\theta\ddot x+2my\dot y\dot\theta\\ & - & (Ml+my)g\cos\theta=F_2d\\ \ddot y & + & \sin\theta-y\dot\theta^2+g\sin\theta=0\end{array}\qquad(2)$$

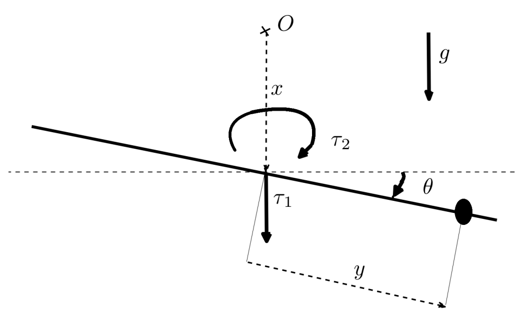


Figure 3: System diagram for $\alpha_1=\alpha_2=0$. τ_1 is a vertical force. τ_2 is the torque acting on the beam.

We apply first a classical partial feedback linearization while taking into account that $(M+m\mathfrak{c}^2_\theta)(J_0+Ml^2+my^2)-(Ml+my)^2$ is strictly positive. This condition can be satisfied by a proper choice of the physical parameters transforming (2) into

$$
\begin{aligned}
\ddot{y} &= y\dot{\theta}^2 - (g+\tau_1)sin(\theta) \\
\ddot{x} &= \tau_1 \\
\ddot{\theta} &= \tau_2
\end{aligned} \tag{3}
$$

3 First nonlinear controller

The domain of interest is constituted by $\mathcal{D} = \mathbb{R}\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\times]-\frac{\pi}{2},\frac{\pi}{2}[\times\mathbb{R}$. We focus here on developing a first nonlinear controller which adapts [5] [6] [7] [8] to the new design in order to globally asymptotically stabilize the origin of system (3).

The controller proceeds in steps. First a homogeneous state feedback from [11] is designed for τ_1. It drives the x dynamics to zero in finite time. Then, a saturated control [9] design drives the y, θ subsystem to the origin asymptotically.

Consider a double integrator

$$\ddot{x} = \tau_1 \tag{4}$$

According to [11], two positive constants K_1 and K_2 exist such that the control:

$$\tau_1 = -K_1 sign(x)|x|^{n_1} - K_2 sign(\dot{x})|\dot{x}|^{n_2} \tag{5}$$

where $n_2 \in [1-\varepsilon, 1]$, stabilizes (4) in finite time. ε is a small positive constant and $n_1 = \frac{n_2}{2-n_2}$. Thus after a finite time denoted T_0 we have $x = \dot{x} = 0$.

Consider the change of variables $z_1 = \frac{y}{g}$, $z_2 = \frac{\dot{y}}{g}$, $z_3 = \theta$, $z_4 = \dot{\theta}$, $z_5 = x$ and $z_6 = \dot{x}$ we have

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -\frac{g+\tau_1}{g}\sin(z_3) + z_1 z_4^2 \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= \tau_2 \\
\dot{z}_5 &= z_6 \\
\dot{z}_6 &= \tau_1.
\end{aligned} \tag{6}
$$

Using the control (5) and taking $\tau_2 = -z_3 - z_4 + u$, we have after a finite time T_0:

$$
\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -\sin(z_3) + z_1 z_4^2 \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= -z_3 - z_4 + u.
\end{aligned} \tag{7}
$$

Consider now the change of variables

$$
\begin{aligned}
y_1 &= z_1 + 2z_2 + 2z_3 + z_4 \\
y_2 &= z_2 + z_3 + z_4 \\
y_3 &= z_3 \\
y_4 &= z_4.
\end{aligned}
$$

It transforms (7) to

$$
\begin{aligned}
\dot{y}_1 &= y_2 + 2P + u \\
\dot{y}_2 &= P + u \\
\dot{y}_3 &= y_4 \\
\dot{y}_4 &= -y_3 - y_4 + u,
\end{aligned} \tag{8}
$$

with $P = sin(y_3) - y_3 + x_1 y_4^2$. Take the control

$$u = -\sigma_2(y_2 + \sigma_1(y_1)), \tag{9}$$

where σ_1 and σ_2 are two state dependent saturation functions defined as follows:

- $\sigma_i(s_i) = s_i$ when $|s_i| \leq \epsilon_i$,
- $\sigma_i(s_i) = \epsilon_i$ when $s_i \geq \epsilon_i$,
- $\sigma_i(s_i) = -\epsilon_i$ when $s_i \leq -\epsilon_i$, $i = 1, 2$.

Take $\epsilon_2 = \min(\bar{\epsilon}_2, \frac{1}{\sqrt{1+Cz_1^2}})$ where $C > 1$ and $\bar{\epsilon}_2$ is chosen sufficiently small such that $|\sin(\epsilon_2) - \epsilon_2| < \epsilon_2^2$, $\epsilon_1 < \frac{\epsilon_2}{2}$ and $\epsilon_2^2 < \frac{\epsilon_1}{2G_2}$. G_2 is a positive constant. By the monotonicity of the sine function and its derivative on the interval $[0, \frac{\pi}{2}]$, it is clear that $|\sin y_3 - y_3| < \frac{y_3^2}{G_1}$ $\forall$ $|y_3| < \epsilon_2$, $G_1 > 1$ is a properly chosen constant. By taking a quadratic Lyapunov function in both y_3 and y_4, it is straightforward to show that in a finite amount of time, say T_1, we have $|y_3| \leq \frac{|u|}{\varrho}$, $|y_4| \leq \frac{|u|}{\varrho'}$ where both ϱ and ϱ' belong to $[1-\delta, 1[$ and δ is a small positive constant. Consider now the y_2 dynamics, the time derivative of the Lyapunov function $V_2 = \frac{1}{2}y_2^2$ is bounded by

$$\begin{aligned} \dot{V}_2 &= y_2(P + u) \leq y_2(\frac{|u|^2}{G_1} + z_1\epsilon_2|u| - u) \\ &\leq y_2(\frac{|u|}{G_1} + |z_1|\epsilon_2 - 1)\sigma_2(y_2 + \sigma_1(y_1)) \end{aligned} \tag{10}$$

where G_1 and ϵ_2 are chosen such that $\frac{|u|}{G_1} + |z_1|\epsilon_2 - 1 < 0$ as a consequence, there exist a finite time, denoted by T_2, and a positive constant $\varrho'' \in [1-\delta, 1[$ such that σ_2 leaves its saturated zone. Consider now the Lyapunov function $V_1 = \frac{1}{2}y_1^2$ associated to the y_1 dynamics which is given by:

$$\dot{y}_1 = y_2 + 2(\sin y_3 - y_3) + 2x_1 y_4^2 + u \tag{11}$$

notice that $u = -y_2 - \sigma_1(y_1)$ from now on. The time derivative of V_1 writes:

$$\dot{V}_1 \leq y_1(2\frac{|u|^2}{G_1} + 2z_1|u|^2 - \sigma_1(y_1)) \tag{12}$$

which ensures by a proper choice of the saturation functions as well as the constants C and G_2 that after a finite amount of time the input u is no more saturated $u = -y_2 - y_1$. The closed loop system reduce to:

$$\begin{aligned} \dot{y}_1 &= -y_1 + 2P \\ \dot{y}_2 &= -y_1 - y_2 + P \\ \dot{y}_3 &= y_4 \\ \dot{y}_4 &= -y_3 - y_4 - y_2 - y_1 \end{aligned} \tag{13}$$

which is locally asymptotically stable.

4 Second nonlinear controller

Suppose that there exist a feedback control which drives the system so that the instantaneous axis of rotation of the beam be the position where the "ball" is located. This consideration, if realized, permit to completely supress the (destabilizing) influence of the centrifugal force. Motivated by this observation, let us consider the following change of variables which is valid on $\mathcal{D}$:

$$\begin{bmatrix} x_m \\ y_m \end{bmatrix} = \begin{bmatrix} 1 & \sin\theta \\ 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{14}$$

It is invertible with $y = \frac{y_m}{\cos\theta}$ and $x = x_m - y_m \tan\theta$. With the new variables the system writes:

$$
\begin{array}{rcl}
\ddot{x}_m & = & -g(\sin\theta)^2 + 2(\dot{y}_m + y_m \tan\theta\dot{\theta})\dot{\theta} + (\cos\theta)^2\tau_1 + y_m\tau_2 \\
\ddot{y}_m & = & -g\sin\theta\cos\theta - 2(\dot{y}_m + y_m \tan\theta\dot{\theta})\tan\theta\dot{\theta} - \sin\theta\cos\theta\tau_1 \\
& - & y_m \tan\theta\tau_2 \\
\ddot{\theta} & = & \tau_2.
\end{array} \qquad (15)
$$

Consider the change of control variables :

$$
\left[\begin{array}{c} u \\ v \end{array}\right] = \left[\begin{array}{cc} (\cos\theta)^2 & y_m \\ 0 & 1 \end{array}\right] \left[\begin{array}{c} \tau_1 \\ \tau_2 \end{array}\right],
$$

with u given by

$$
u = g(\sin\theta)^2 - 2(\dot{y}_m + y_m \tan\theta\dot{\theta})\dot{\theta} - \psi(x_m, \dot{x}_m) \qquad (16)
$$

and $\psi(x_m, \dot{x}_m)$ is a stabilizing controller for the x_m dynamics. Equation (16) transform (15) to

$$
\begin{array}{rcl}
\ddot{x}_m & = & -\psi(x_m, \dot{x}_m) \\
\ddot{y}_m & = & -\tan\theta(g + \ddot{x}_m) \\
\ddot{\theta} & = & v.
\end{array}
$$

We use a saturated control for $\psi(x_m, \dot{x}_m)$ in order to ensure that $g - |\ddot{x}_m| > 0$. Take $z = \tan\theta$ and $v = \frac{1}{(1+z^2)^2}(w - 2z\dot{z}^2)$, the system writes:

$$
\begin{array}{rcl}
\ddot{x}_m & = & -\psi(x_m, \dot{x}_m) \\
\ddot{y}_m & = & -(g + \ddot{x}_m)z \\
\ddot{z} & = & w.
\end{array}
$$

where w is a new control input.

Consider the new variable: $\xi = z - \alpha y_m - \beta\dot{y}_m$. Its second order derivative is given by:

$$
\ddot{\xi} = w + (g + \ddot{x}_m)(\alpha z + \beta\dot{z}) + \beta x_m^{(3)} z
$$

The dynamics $(\xi, \dot{\xi})$ can be stabilized by

$$
w = -(g + \ddot{x}_m)(\alpha z + \beta\dot{z}) - \beta x_m^{(3)} z - \Phi(\xi, \dot{\xi})
$$

where Φ can be taken as in (5) with the purpose of ensuring a finite time stabilization. After a finite time denoted by T_4, we have $\xi = \dot{\xi} = 0$ and subsystem y_m reduced to

$$
\ddot{y}_m = -(g + \ddot{x}_m)(\alpha y_m + \beta\dot{y_m})
$$

which is asymptotically stable due to the choice of ψ which ensure that $g - |\ddot{x}_m| > 0$.

5 Simulation

The performance of both controllers is compared through numerical simulations depicted in figures 4-9. Experimental results will be published in a future work. Two tests were performed.

1. For the first one, the system starts at the initial conditions $y = 1$, $\theta = 0.35rd \approx 20°$ and $x = \dot{x} = \dot{y} = \dot{\theta} = 0$. Simulations results are shown on figure 4 for the first controller, while on figures 6 and 7 for the second one.

2. For the second test, the system starts at the initial conditions $y = 1$, $\theta = 1.4rd \approx 80°$ and $x = \dot{x} = \dot{y} = \dot{\theta} = 0$. Simulations corresponding to the first controller are reported on the figure 5. On the other hand, simulations for the second controller are shown on figures 8 and 9.

By examining the simulations, it can be noticed that the second controller avoids the large transients induced by the first one. Thus, the effectiveness of the new design is confirmed. On the other hand, it is worth mentioning that the first controller provide a saturated input which can be beneficial in some situation.

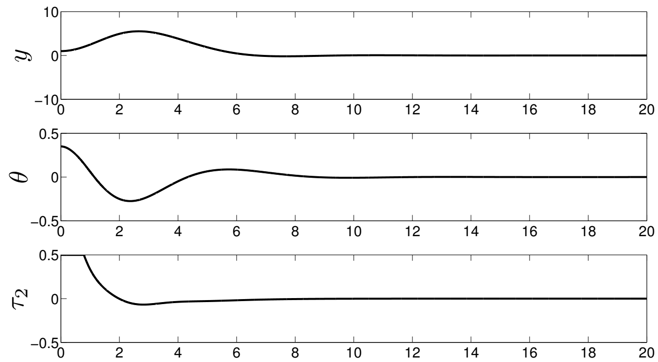


Figure 4: First controller, test 1.

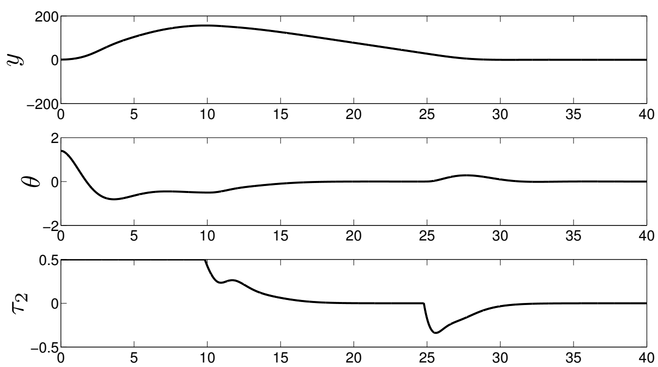


Figure 5: First controller, test 2.

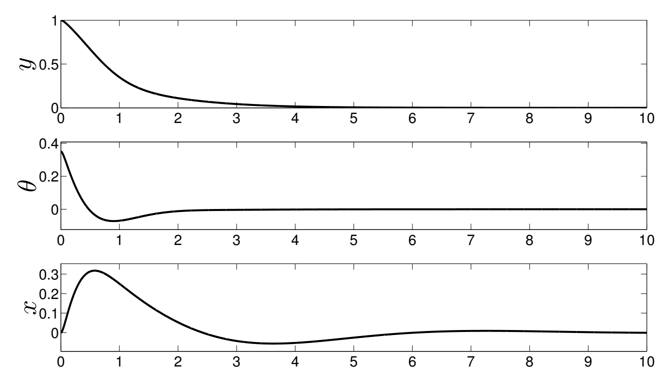


Figure 6: Second controller, test 1, system outputs.

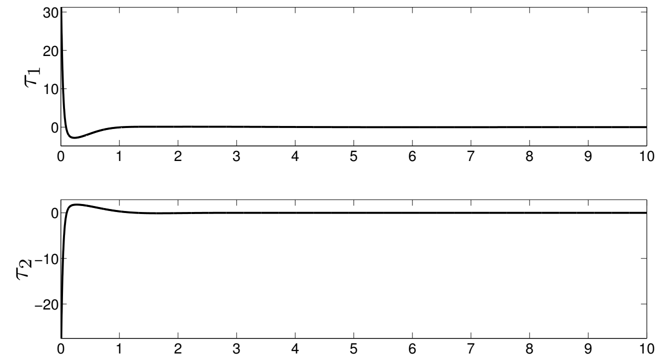


Figure 7: Second controller, test 1, control inputs.

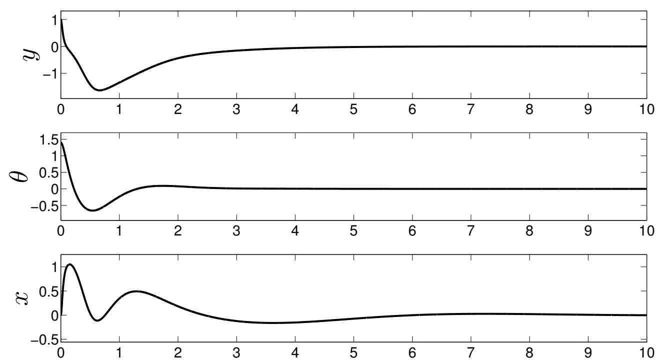


Figure 8: Second controller, test 2, system outputs.

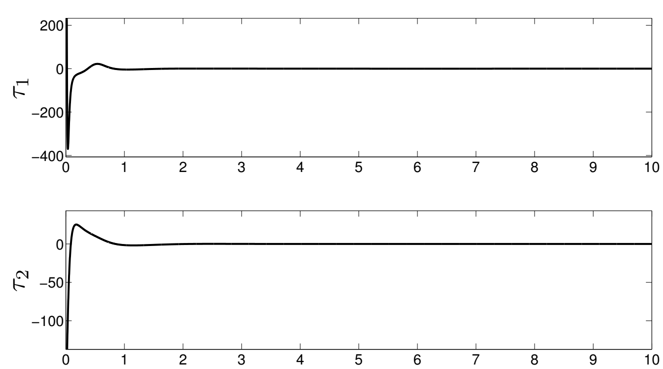


Figure 9: Second controller, test 2, control inputs.

6 Conclusion

This communication presented a new design for the ball and beam system. The idea behind this novel design is to get rid of the centrifugal force which constitute an obstacle for fast and global stabilization of the classical ball & beam system. Two nonlinear controllers have been developed. The first one adapts existing controllers from the literature to the new system. The drawback of this controller is slow reactivity of the closed loop. The second controller takes profit of the additional degree of freedom of the system to provide fast global stabilization. The efficiency of the algorithm was attested by several numerical simulations. The application of the controller to the real life system will be published in a forthcoming work.

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