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# FINDING ECM-FRIENDLY CURVES THROUGH A STUDY OF GALOIS PROPERTIES

RAZVAN BARBULESCU, JOPPE W. BOS, CYRIL BOUVIER, THORSTEN KLEINJUNG,  
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ABSTRACT. In this paper we prove some divisibility properties of the cardinality of elliptic curve groups modulo primes. These proofs explain the good behavior of certain parameters when using Montgomery or Edwards curves in the setting of the elliptic curve method (ECM) for integer factorization. The ideas behind the proofs help us to find new infinite families of elliptic curves with good division properties increasing the success probability of ECM.

## 1. INTRODUCTION

The elliptic curve method (ECM) for integer factorization [16] is the asymptotically fastest known method for finding relatively small factors  $p$  of large integers  $N$ . In practice, ECM is used, on the one hand, to factor large integers. For instance, the 2011 ECM-record is a 241-bit factor of  $2^{1181} - 1$  [9]. On the other hand, ECM is used to factor many small (100 to 200 bits) integers as part of the number field sieve [19, 15, 2], the most efficient general purpose integer factorization method.

Traditionally, the elliptic curve arithmetic used in ECM is implemented using Montgomery curves [17] (e.g., in the widely-used GMP-ECM software [27]). Generalizing the work of Euler and Gauss, Edwards introduced a new normal form for elliptic curves [12] which results in a fast realization of the elliptic curve group operation in practice. These Edwards curves have been generalized by Bernstein and Lange [7] for usage in cryptography. Bernstein et al. explored the possibility to use these curves in the ECM setting [6]. After Hisil et al. [13] published a coordinate system which results in the fastest known realization of curve arithmetic, a follow-up paper by Bernstein et al. discusses the usage of the so-called “ $a = -1$ ” twisted Edwards curves [5] in ECM.

It is common to construct or search for curves which have favorable properties. The success of ECM depends on the smoothness of the cardinality of the curve considered modulo the unknown prime divisor  $p$  of  $N$ . This usually means constructing curves with large torsion group over  $\mathbb{Q}$  or finding curves such that the order of the elliptic curve, when considered modulo a family of primes, is always divisible by an additional factor. Examples are the Suyama construction [25], the curves proposed by Atkin and Morain [1], a translation of these techniques to Edwards curves [6, 5], and a family of curves suitable for Cunningham numbers [10].

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*Key words and phrases.* Elliptic Curve Method (ECM), Edwards curves, Montgomery curves, torsion properties, Galois groups.

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In this paper we study and prove divisibility properties of the cardinality of elliptic curves over prime fields. We do this by studying properties of Galois groups of torsion points using Chebotarev's theorem [18]. Furthermore, we investigate some elliptic curve parameters for which ECM finds exceptionally many primes in practice, but which do not fit in any of the known cases of good torsion properties. We prove this behavior and provide parametrizations for infinite families of elliptic curves with these properties.

## 2. GALOIS PROPERTIES OF TORSION POINTS OF ELLIPTIC CURVES

In this section we give a systematic way to compute the probability that the order of a given elliptic curve reduced by an arbitrary prime is divisible by a certain prime power.

### 2.1. Torsion Properties of Elliptic Curves.

**Definition 2.1.** Let  $K$  be a finite Galois extension of  $\mathbb{Q}$ ,  $p$  a prime and  $\mathfrak{p}$  a prime ideal above  $p$  with residue field  $k_{\mathfrak{p}}$ . The decomposition group  $\text{Dec}(\mathfrak{p})$  of  $\mathfrak{p}$  is the subgroup of  $\text{Gal}(K/\mathbb{Q})$  which stabilizes  $\mathfrak{p}$ . Call  $\alpha^{(\mathfrak{p})}$  the canonical morphism from  $\text{Dec}(\mathfrak{p})$  to  $\text{Gal}(k_{\mathfrak{p}}/\mathbb{F}_p)$  and let  $\phi_{\mathfrak{p}}$  be the Frobenius automorphism on the field  $k_{\mathfrak{p}}$ . We define  $\text{Frobenius}(p) = \bigcup_{\mathfrak{p}|p} (\alpha^{(\mathfrak{p})})^{-1}(\phi_{\mathfrak{p}})$ .

In order to state Chebotarev's theorem we say that a set  $S$  of primes admits a natural density equal to  $\delta$  and we write  $\mathbb{P}(S) = \delta$  if  $\lim_{N \rightarrow \infty} \frac{\#(S \cap \Pi(N))}{\#\Pi(N)}$  exists and equals  $\delta$ , where  $\Pi(N)$  is the set of primes up to  $N$ . If  $\text{event}(p)$  is a property which can be defined for all primes except a finite set (thus of null density), when we note  $\mathbb{P}(\text{event}(p))$  we tacitly exclude the primes where  $\text{event}(p)$  cannot be defined.

**Theorem 2.2** (Chebotarev, [18]). *Let  $K$  be a finite Galois extension of  $\mathbb{Q}$ . Let  $H \subset \text{Gal}(K/\mathbb{Q})$  be a conjugacy class. Then*

$$\mathbb{P}(\text{Frobenius}(p) = H) = \frac{\#H}{\#\text{Gal}(K/\mathbb{Q})}.$$

Before applying Chebotarev's theorem to the case of elliptic curves, we introduce some notation. For every elliptic curve  $E$  over a field  $F$  and all  $m \in \mathbb{N}$ ,  $m \geq 2$ , we consider the field  $F(E[m])$  which is the smallest extension of  $F$  containing all the  $m$ -torsion of  $E$ . The next result is classical, but we present its proof for the intuition it brings.

**Proposition 2.3.** *For every integer  $m \geq 2$  and any elliptic curve  $E$  over some field  $F$ , the following hold:*

- (1)  $F(E[m])/F$  is a Galois extension;
- (2) there is an injective morphism  $\iota_m : \text{Gal}(F(E[m])/F) \hookrightarrow \text{Aut}(E(\overline{F})[m])$ .

*Proof.* (1) Since the addition law of  $E$  can be expressed by rational functions over  $F$ , there exist polynomials  $f_m, g_m \in F[X, Y]$  such that the coordinates of the points in  $E(\overline{F})[m]$  are the solutions of the system  $(f_m = 0, g_m = 0)$ . Therefore  $F(E[m])$  is the splitting field of  $\text{Res}_X(f_m, g_m)$  and  $\text{Res}_Y(f_m, g_m)$  and in particular is Galois. (2) For each  $\sigma \in \text{Gal}(F(E[m])/F)$  we call  $\iota_m(\sigma)$  the application which sends  $(x, y) \in E(\overline{F})[m]$  into  $(\sigma(x), \sigma(y))$ . Thanks to the discussion above,  $\iota_m(\sigma)$  sends points of  $E(\overline{F})[m]$  in  $E(\overline{F})[m]$ . Since the addition law can be expressed by rational functions over  $F$ , for each  $\sigma$ ,  $\iota_m(\sigma) \in \text{Aut}(E(\overline{F})[m])$ . One easily checks that  $\iota_m$  is a group morphism and its kernel is the identity.  $\square$

*Notation 2.4.* We fix generators for  $E(\overline{\mathbb{Q}})[m]$ , thereby inducing an isomorphism  $\psi_m : \text{Aut}(E(\overline{\mathbb{Q}})[m]) \rightarrow \text{GL}_2(\mathbb{Z}/m\mathbb{Z})$ . Let  $\iota_m$  be the injection given by Proposition 2.3. We call  $\rho_m : \text{Gal}(\mathbb{Q}(E[m])/\mathbb{Q}) \rightarrow \text{GL}_2(\mathbb{Z}/m\mathbb{Z})$  the injective morphism  $\psi_m \circ \iota_m$ .

Let  $p$  be a prime such that  $E$  has good reduction at  $p$  and  $p \nmid m$ . Let  $\iota_m^{(p)}$  be the injection of  $\text{Gal}(\mathbb{F}_p(E[m])/\mathbb{F}_p)$  into  $\text{Aut}(E(\overline{\mathbb{F}}_p)[m])$  given by Proposition 2.3. By [22, Prop. VII.3.1] there is a canonical isomorphism  $r_m^{(\mathfrak{p})}$  from  $\text{Aut}(E(\overline{\mathbb{Q}})[m])$  to  $\text{Aut}(E(\overline{\mathbb{F}}_p)[m])$  for each prime ideal  $\mathfrak{p}$  over  $p$ .

*Remark 2.5.* Note that  $\#\text{Gal}(\mathbb{Q}(E[m])/\mathbb{Q})$  is bounded by  $\#\text{GL}_2(\mathbb{Z}/m\mathbb{Z})$ . For every prime  $\pi$ , we have  $\#\text{GL}_2(\mathbb{Z}/\pi\mathbb{Z}) = (\pi - 1)^2(\pi + 1)\pi$ , and for every integer  $k \geq 1$ ,  $\#\text{GL}_2(\mathbb{Z}/\pi^{k+1}\mathbb{Z}) = \pi^4 \#\text{GL}_2(\mathbb{Z}/\pi^k\mathbb{Z})$ .

*Notation 2.6.* For all  $g \in \text{GL}_2(\mathbb{Z}/m\mathbb{Z})$  we put  $\text{Fix}(g) = \{v \in (\mathbb{Z}/m\mathbb{Z})^2 \mid g(v) = v\}$ . Conjugation of  $g$  gives an isomorphic group of fixed elements. If we are interested only in the isomorphism class we use the notation  $\text{Fix}(C)$  where  $C$  is a set of conjugated elements. We use analogous notations for  $\text{Aut}(E(\overline{\mathbb{Q}})[m])$  and  $\text{Aut}(E(\overline{\mathbb{F}}_p)[m])$ .

**Theorem 2.7.** *Let  $E$  be an elliptic curve over  $\mathbb{Q}$  and  $m \geq 2$  be an integer. Put  $K = \mathbb{Q}(E[m])$ . Let  $T$  be a subgroup of  $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ . Then,*

- (1)  $\mathbb{P}(E(\mathbb{F}_p)[m] \simeq T) = \frac{\#\{g \in \rho_m(\text{Gal}(K/\mathbb{Q})) \mid \text{Fix}(g) \simeq T\}}{\#\text{Gal}(K/\mathbb{Q})}$ .
- (2) Let  $a, n \in \mathbb{N}$  such that  $a \leq n$  and  $\gcd(a, n) = 1$  and let  $\zeta_n$  be a primitive  $n$ th root of unity. Put  $G_a = \{\sigma \in \text{Gal}(K(\zeta_n)/\mathbb{Q}) \mid \sigma(\zeta_n) = \zeta_n^a\}$ . Then:  

$$\mathbb{P}(E(\mathbb{F}_p)[m] \simeq T \mid p \equiv a \pmod{n}) = \frac{\#\{\sigma \in G_a \mid \text{Fix}(\rho_m(\sigma|_K)) \simeq T\}}{\#G_a}$$

*Proof.* (1) Let  $p \nmid m$  be a prime for which  $E$  has good reduction and let  $\mathfrak{p}$  be a prime ideal of  $K$  over  $p$ . We abbreviate  $H = \{\sigma \in \text{Gal}(K/\mathbb{Q}) \mid \text{Fix}(\iota_m(\sigma)) \simeq T\}$ . First note that  $E(\mathbb{F}_p)[m] = \text{Fix}(\iota_m^{(p)}(\phi_p))$  where  $\phi_p$  is the Frobenius in  $\text{Gal}(\mathbb{F}_p(E[m])/\mathbb{F}_p)$ . Since the diagram

$$\begin{array}{ccccc} \text{Dec}(\mathfrak{p}) & \hookrightarrow & \text{Gal}(\mathbb{Q}(E[m])/\mathbb{Q}) & \xrightarrow{\iota_m} & \text{Aut}(E(\overline{\mathbb{Q}})[m]) \\ \downarrow \alpha^{(\mathfrak{p})} & & & & \downarrow r_m^{(\mathfrak{p})} \\ \text{Gal}(k_{\mathfrak{p}}/\mathbb{F}_p) & \xrightarrow{\sim} & \text{Gal}(\mathbb{F}_p(E[m])/\mathbb{F}_p) & \xrightarrow{\iota_m^{(p)}} & \text{Aut}(E(\overline{\mathbb{F}}_p)[m]) \end{array}$$

is commutative and since  $\text{Frobenius}(p) \subset \text{Gal}(K/\mathbb{Q})$  is the conjugacy class generated by  $(\alpha^{(\mathfrak{p})})^{-1}(\phi_p)$  we have  $E(\mathbb{F}_p)[m] \simeq \text{Fix}(\iota_m(\text{Frobenius}(p)))$ .

Decompose  $H$  into a disjoint union of conjugacy classes  $C_1, \dots, C_N$ . Then  $\text{Fix}(\iota_m(\text{Frobenius}(p))) \simeq T$  is equivalent to  $\text{Frobenius}(p)$  being one of the  $C_i$ . Thanks to Theorem 2.2 we obtain:

$$\mathbb{P}(E(\mathbb{F}_p)[m] \simeq T) = \sum_{i=1}^N \mathbb{P}(\text{Frobenius}(p) = C_i) = \sum_{i=1}^N \frac{\#C_i}{\#\text{Gal}(K/\mathbb{Q})} = \frac{\#H}{\#\text{Gal}(K/\mathbb{Q})}.$$

(2) Using similar arguments as in (1) we have to evaluate

$$\frac{\mathbb{P}(\text{Frobenius}(p) \in \{C_1, \dots, C_N\}, p \equiv a \pmod{n})}{\mathbb{P}(p \equiv a \pmod{n})}.$$

Let  $p$  be a prime and  $\mathfrak{p}$  a prime ideal as in the first part of the proof, and let  $\mathfrak{P}$  be a prime ideal of  $K(\zeta_n)$  lying over  $\mathfrak{p}$ . Furthermore let  $\tilde{C}_1, \dots, \tilde{C}_N$  be the conjugacy classes of  $\text{Gal}(K(\zeta_n)/\mathbb{Q})$  that are in the pre-images of  $C_1, \dots, C_N$  and whose elements  $\sigma$  satisfy  $\sigma(\zeta_n) = \zeta_n^a$ . Since  $\text{Gal}(K(\zeta_n)/\mathbb{Q})$  maps  $\zeta_n$  to primitive  $n$ th roots of unity we have for  $\sigma \in (\alpha^{(\mathfrak{P})})^{-1}(\phi_{\mathfrak{P}})$  that  $\sigma(\zeta_n) = \zeta_n^b$  holds for some  $b$ . Together with  $\sigma(x) \equiv x^p \pmod{\mathfrak{P}}$  we get  $\zeta_n^b \equiv \zeta_n^p \pmod{\mathfrak{P}}$ . If we exclude the finitely many primes dividing the norms of  $\zeta_n^c - 1$  for  $c = 1, \dots, n-1$  we obtain  $b \equiv p \pmod{n}$ . Since Frobenius( $K(\zeta_n), p$ ), the Frobenius conjugacy class for  $K(\zeta_n)$ , is the pre-image of Frobenius( $p$ ), we get with the argument above  $\mathbb{P}(\text{Frobenius}(p) \in \{C_1, \dots, C_N\}, p \equiv a \pmod{n}) = \mathbb{P}(\text{Frobenius}(K(\zeta_n), p) \in \{\tilde{C}_1, \dots, \tilde{C}_N\})$ . A similar consideration for the denominator  $\mathbb{P}(p \equiv a \pmod{n})$  completes the proof.  $\square$

*Remark 2.8.* Put  $K = \mathbb{Q}(E[m])$ . If  $[K(\zeta_n) : \mathbb{Q}(\zeta_n)] = [K : \mathbb{Q}]$ , then one has  $\mathbb{P}(E(\mathbb{F}_p)[m] \simeq T \mid p \equiv a \pmod{n}) = \mathbb{P}(E(\mathbb{F}_p)[m] \simeq T)$  for  $a$  coprime to  $n$ . Indeed, according to Galois theory,  $\text{Gal}(K(\zeta_n)/\mathbb{Q})/\text{Gal}(K(\zeta_n)/K) \simeq \text{Gal}(K/\mathbb{Q})$  through  $\bar{\sigma} \mapsto \sigma|_K$ . Since  $[K(\zeta_n) : \mathbb{Q}(\zeta_n)] = [K : \mathbb{Q}]$ , we have  $[K(\zeta_n) : K] = \varphi(n)$  and therefore each element  $\sigma$  of  $\text{Gal}(K/\mathbb{Q})$  extends in exactly one way to an element of  $\text{Gal}(K(\zeta_n)/\mathbb{Q})$  which satisfies  $\sigma(\zeta_n) = \zeta_n^a$ . Note that for  $n \in \{3, 4\}$  the condition is equivalent to  $\zeta_n \notin K$ .

The families constructed by Brier and Clavier [10], which are dedicated to integers  $N$  such that the  $n$ th cyclotomic polynomial has roots modulo all prime factors of  $N$ , modify  $[K(\zeta_n) : \mathbb{Q}(\zeta_n)]$  by imposing a large torsion subgroup over  $\mathbb{Q}(\zeta_n)$ .

An important particular case of Theorem 2.7 is as follows:

**Corollary 2.9.** *Let  $E$  be an elliptic curve and  $\pi$  be a prime number. Then,*

$$\mathbb{P}(E(\mathbb{F}_p)[\pi] \simeq \mathbb{Z}/\pi\mathbb{Z}) = \frac{\#\{g \in \rho_{\pi}(\text{Gal}(\mathbb{Q}(E[\pi])/\mathbb{Q})) \mid \det(g - \text{Id}) = 0, g \neq \text{Id}\}}{\#\text{Gal}(\mathbb{Q}(E[\pi])/\mathbb{Q})},$$

$$\mathbb{P}(E(\mathbb{F}_p)[\pi] \simeq \mathbb{Z}/\pi\mathbb{Z} \times \mathbb{Z}/\pi\mathbb{Z}) = \frac{1}{\#\text{Gal}(\mathbb{Q}(E[\pi])/\mathbb{Q})}.$$

**Example 2.10.** Let us compute these probabilities for the curves  $E_1 : y^2 = x^3 + 5x + 7$  and  $E_2 : y^2 = x^3 - 11x + 14$  and the primes  $\pi = 3$  and  $\pi = 5$ . Here  $E_1$  illustrates the generic case, whereas  $E_2$  has special Galois groups. One checks with Sage [23] that  $[\mathbb{Q}(E_1[3]) : \mathbb{Q}] = 48$  and  $\#\text{GL}_2(\mathbb{Z}/3\mathbb{Z}) = 48$ . By Proposition 2.3 we deduce that  $\rho_3(\text{Gal}(\mathbb{Q}(E_1[3])/\mathbb{Q})) = \text{GL}_2(\mathbb{Z}/3\mathbb{Z})$ . A simple computation shows that  $\text{GL}_2(\mathbb{Z}/3\mathbb{Z})$  contains 21 elements having 1 as eigenvalue, one of which is  $\text{Id}$ . Corollary 2.9 gives the following probabilities:  $\mathbb{P}(E_1(\mathbb{F}_p)[3] \simeq \mathbb{Z}/3\mathbb{Z}) = \frac{20}{48}$  and  $\mathbb{P}(E_1(\mathbb{F}_p)[3] \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}) = \frac{1}{48}$ . We used the same method for all the probabilities of Table 1, where we compare them to experimental values.

Note that the relative difference between theoretical and experimental values never exceeds 0.4%. It is interesting to observe that reducing the Galois group does not necessarily increase the probabilities, as it is shown for  $\pi = 3$ .

**2.2. Effective Computations of  $\mathbb{Q}(E[m])$  and  $\rho_m(\text{Gal}(\mathbb{Q}(E[m])/\mathbb{Q}))$  for Prime Powers.** The main tools are the division polynomials as defined below.

**Definition 2.11.** Let  $E : y^2 = x^3 + ax + b$  be an elliptic curve over  $\mathbb{Q}$  and  $m \geq 2$  an integer. The  $m$ -division polynomial  $P_m$  is defined as the monic polynomial whose roots are the  $x$ -coordinates of all the  $m$ -torsion affine points.  $P_m^{\text{new}}$  is defined as the monic polynomial whose roots are the  $x$ -coordinates of the affine points of order exactly  $m$ .

		$E_1$	$E_2$
$\# \text{GL}_2(\mathbb{Z}/3\mathbb{Z})$		48	
$\# \text{Gal}(\mathbb{Q}(E[3])/\mathbb{Q})$		48	16
$\mathbb{P}(E(\mathbb{F}_p)[3] \simeq \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z})$	Th.	$\frac{1}{48} \approx 0.02083$	$\frac{1}{16} = 0.06250$
	Exp.	0.02082	0.06245
$\mathbb{P}(E(\mathbb{F}_p)[3] \simeq \mathbb{Z}/3\mathbb{Z})$	Th.	$\frac{20}{48} \approx 0.4167$	$\frac{4}{16} = 0.2500$
	Exp.	0.4165	0.2501
$\# \text{GL}_2(\mathbb{Z}/5\mathbb{Z})$		480	
$\# \text{Gal}(\mathbb{Q}(E[5])/\mathbb{Q})$		480	32
$\mathbb{P}(E(\mathbb{F}_p)[5] \simeq \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z})$	Th.	$\frac{1}{480} \approx 0.002083$	$\frac{1}{32} = 0.03125$
	Exp.	0.002091	0.03123
$\mathbb{P}(E(\mathbb{F}_p)[5] \simeq \mathbb{Z}/5\mathbb{Z})$	Th.	$\frac{114}{480} = 0.2375$	$\frac{10}{32} = 0.3125$
	Exp.	0.2373	0.3125

TABLE 1. Comparison of the theoretical values (Th) of Corollary 2.9 to the experimental results for all primes below  $2^{25}$  (Exp).

**Proposition 2.12.** *For all  $m \geq 2$  we have:*

- (1)  $P_m, P_m^{\text{new}} \in \mathbb{Q}[X]$ ;
- (2)  $\deg(P_m) = \frac{(m^2+2-3\eta)}{2}$ , where  $\eta$  is the remainder of  $m$  modulo 2.

*Proof.* For a proof we refer to [8]. □

Note that one obtains different division polynomials for other shapes of elliptic curves (Weierstrass, Montgomery, Edwards, etc.). Nevertheless, the Galois group  $\text{Gal}(\mathbb{Q}(E[m])/\mathbb{Q})$  is model independent and can be computed with the division polynomials of Definition 2.11 as, in characteristic different from 2 and 3, every curve can be written in short Weierstrass form.

One can compute  $\mathbb{Q}(E[\pi])$  for any prime  $\pi \geq 3$  using the following method:

1. Make a first extension of  $\mathbb{Q}$  through an irreducible factor of  $P_\pi$  and obtain a number field  $F_1$  where  $P_\pi$  has a root  $\alpha_1$ .
2. Let  $f_2(y) = y^2 - (\alpha_1^3 + a\alpha_1 + b) \in F_1[y]$  and  $F_2$  be the extension of  $F_1$  through  $f_2$ .  $F_2$  contains a  $\pi$ -torsion point  $M_1$ . In  $F_2$ ,  $P_\pi$  has  $\frac{\pi-1}{2}$  trivial roots representing the  $x$  coordinates of the multiples of  $M_1$ .
3. Call  $F_3$  the extension of  $F_2$  through an irreducible factor of  $P_\pi \in F_2[x]$  other than those corresponding to the trivial roots.
4. Let  $\alpha_2$  be the new root of  $P_\pi$  in  $F_3$ . Let  $f_4(y) = y^2 - (\alpha_2^3 + a\alpha_2 + b) \in F_3[y]$  and  $F_4$  be the extension of  $F_3$  through  $f_4$ .  $F_4$  contains all the  $\pi$ -torsion.

The case of prime powers  $\pi^k$  with  $k \geq 2$  is handled recursively. Having computed  $\mathbb{Q}(E[\pi^{k-1}])$ , we obtain  $\mathbb{Q}(E[\pi^k])$  by repeating the 4 steps above with  $P_{\pi^k}^{\text{new}}$  instead of  $P_\pi$  and by considering as trivial roots all the  $x$ -coordinates of the points  $\{P + M_1 \mid P \in E[\pi^{k-1}]\}$ .

In practice, we observe that in general  $P_\pi, f_2, P_\pi^{(F_2)}$  and  $f_4$  are irreducible, where  $P_\pi^{(F_2)}$  is  $P_\pi$  divided by the factors corresponding to the trivial roots. If this is the case, as  $\deg(P_\pi) = \frac{\pi^2-1}{2}$  (Proposition 2.12), the absolute degree of  $F_4$  is  $\frac{\pi^2-1}{2} \cdot 2 \cdot \frac{\pi^2-\pi}{2} \cdot 2 = (\pi-1)^2(\pi+1)\pi$ . By Remark 2.5,  $\# \text{GL}_2(\mathbb{Z}/\pi\mathbb{Z}) = (\pi-1)^2(\pi+1)\pi$ ,

therefore in general we expect  $\rho_\pi(\text{Gal}(\mathbb{Q}(E[\pi])/\mathbb{Q})) = \text{GL}_2(\mathbb{Z}/\pi\mathbb{Z})$ . Also, we observed that in general the degree of the extension  $\mathbb{Q}(E[\pi^k])/\mathbb{Q}(E[\pi^{k-1}])$  is  $\pi^4$ .

Serre [20] proved that the observations above are almost always true. The next theorem is a restatement of items (1) and (6) in the introduction of [20].

**Theorem 2.13** (Serre). *Let  $E$  be an elliptic curve without complex multiplication.*

- (1) *For all primes  $\pi$  and  $k \geq 1$  the index  $[\text{GL}_2(\mathbb{Z}/\pi^k\mathbb{Z}) : \rho_{\pi^k}(\text{Gal}(\mathbb{Q}(E[\pi^k])/\mathbb{Q}))]$  is non-decreasing and bounded by a constant depending on  $E$  and  $\pi$ .*
- (2) *For all primes  $\pi$  outside a finite set depending on  $E$  and for all  $k \geq 1$ ,  $\rho_{\pi^k}(\text{Gal}(\mathbb{Q}(E[\pi^k])/\mathbb{Q})) = \text{GL}_2(\mathbb{Z}/\pi^k\mathbb{Z})$ .*

**Definition 2.14.** Put  $I(E, \pi, k) = [\text{GL}_2(\mathbb{Z}/\pi^k\mathbb{Z}) : \rho_{\pi^k}(\text{Gal}(\mathbb{Q}(E[\pi^k])/\mathbb{Q}))]$ . If  $E$  does not admit complex multiplication, we call Serre's exponent the integer  $n(E, \pi) = \min\{n \in \mathbb{N}^* \mid \forall k \geq n, I(E, \pi, k+1) = I(E, \pi, k)\}$ .

In [21] Serre showed that in some cases one can prove that  $I(E, \pi, k) = 1$  for all positive integers  $k$ . Indeed, Serre proved that the surjectivity of  $\rho_{\pi^k}$  (or equivalently  $I(E, \pi, k) = 1$ ) follows from the surjectivity of  $\rho_\pi$  (or equivalently  $I(E, \pi, 1) = 1$ ) for all rational elliptic curves  $E$  without complex multiplication and for all primes  $\pi \geq 5$ . In order to have the same kind of results for  $\pi = 2$  (resp.  $\pi = 3$ ) one has to suppose that  $\rho_2, \rho_4$  and  $\rho_8$  are surjective (resp.  $\rho_3$  and  $\rho_9$  are surjective).

Serre also conjectured that only a finite number of primes, not depending on the curve  $E$ , can occur in the second point of Theorem 2.13. The current conjecture is that for all rational elliptic curves without complex multiplication and all primes  $\pi \geq 37$ ,  $\rho_\pi$  is surjective. In [28] Zywina described an algorithm that computes the primes  $\pi$  for which  $\rho_\pi$  is not surjective and checked the conjecture for all elliptic curves in Magma's database (currently this covers curves with conductor at most 14000).<sup>1</sup>

The method described above allows us to compute  $\mathbb{Q}(E[m])$  as an extension tower. Then it is easy to obtain its absolute degree and a primitive element. Identifying  $\rho_\pi(\text{Gal}(\mathbb{Q}(E[m])/\mathbb{Q}))$  (up to conjugacy) is easy when there is only one subgroup (up to conjugacy) of  $\text{GL}_2(\mathbb{Z}/m\mathbb{Z})$  with the right order. In the other case we check for each  $g \in \text{GL}_2(\mathbb{Z}/m\mathbb{Z})$  using the fixed generators of  $E(\overline{\mathbb{Q}})[m]$  whether  $g$  gives rise to an automorphism on  $\mathbb{Q}(E[m])$ . In practice, the bottleneck of this method is the factorization of polynomials with coefficients over number fields.

An faster probabilistic algorithm for computing  $\text{Gal}(\mathbb{Q}(E[\pi])/\mathbb{Q})$  was proposed by Sutherland [24]. This algorithm was not known by the authors at the time of writing and would have helped to accelerate the computation of the examples.

**2.3. Divisibility by a Prime Power.** It is a common fact that, for a given prime  $\pi$ , the cardinality of an arbitrary elliptic curve over  $\mathbb{F}_p$  has a larger probability to be divisible by  $\pi$  than an arbitrary integer of size  $p$ . In this subsection we shall rigorously compute those probabilities under some hypothesis of generality.

*Notation 2.15.* Let  $\pi$  be a prime and  $i, j, k \in \mathbb{N}$  such that  $i \leq j$ . We put:

$$p_{\pi,k}(i, j) = \mathbb{P}(E(\mathbb{F}_p)[\pi^k] \simeq \mathbb{Z}/\pi^i\mathbb{Z} \times \mathbb{Z}/\pi^j\mathbb{Z}).$$

Let  $\ell \leq m$  be integers. When it is defined we denote:

$$p_{\pi,k}(\ell, m | i, j) = \mathbb{P}(E(\mathbb{F}_p)[\pi^{k+1}] \simeq \mathbb{Z}/\pi^\ell\mathbb{Z} \times \mathbb{Z}/\pi^m\mathbb{Z} \mid E_p[\pi^k] \simeq \mathbb{Z}/\pi^i\mathbb{Z} \times \mathbb{Z}/\pi^j\mathbb{Z}).$$

<sup>1</sup>Thanks to Andrew Sutherland for bringing this article to our attention.

When it is clear from the context,  $\pi$  is omitted.

*Remark 2.16.* Since for every natural number  $m$  and every prime  $p$  coprime to  $m$ ,  $E(\mathbb{F}_p)[m] \subset \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}$ , we have  $p_{\pi,k}(i, j) = 0$  for  $j > k$ . In the case  $j < k$ , if  $p_{\pi,k}(\ell, m \mid i, j)$  is defined, it equals 1 if  $(\ell, m) = (i, j)$  and equals 0 if  $(\ell, m) \neq (i, j)$ . Finally, for  $j = k$ , there are only three conditional probabilities which can be non-zero:  $p_{\pi,k}(i, k \mid i, k)$ ,  $p_{\pi,k}(i, k+1 \mid i, k)$ , and  $p_{\pi,k}(k+1, k+1 \mid k, k)$ .

**Theorem 2.17.** *Let  $\pi$  be a prime and  $E$  an elliptic curve over  $\mathbb{Q}$ . If  $k$  is an integer such that  $I(E, \pi, k+1) = I(E, \pi, k)$ , in particular if  $E$  has no complex multiplication and  $k \geq n(E, \pi)$ , then for all  $0 \leq i < k$  we have:*

- (1)  $p_{\pi,k}(k+1, k+1 \mid k, k) = \frac{1}{\pi^4}$ ;
- (2)  $p_{\pi,k}(k, k+1 \mid k, k) = \frac{(\pi-1)(\pi+1)^2}{\pi^4}$ ;
- (3)  $p_{\pi,k}(i, k+1 \mid i, k) = \frac{1}{\pi}$ .

*Proof.* Let  $M = (\mathbb{Z}/\pi^k\mathbb{Z})^2$ . For all  $g \in \mathrm{GL}_2(\pi M)$ , we consider the set  $\mathrm{Lift}(g) = \{h \in \mathrm{GL}_2(M) \mid h|_{\pi M} = g\} = \{g + \pi^{k-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z}/\pi\mathbb{Z}\}$ , whose cardinality is  $\pi^4$ . Since  $I(E, \pi, k+1) = I(E, \pi, k)$ , we have  $\frac{\#\mathrm{Gal}(\mathbb{Q}(E[\pi^k])/\mathbb{Q})}{\#\mathrm{Gal}(\mathbb{Q}(E[\pi^{k+1}])/\mathbb{Q})} = \frac{\#\mathrm{GL}_2(\mathbb{Z}/\pi^k\mathbb{Z})}{\#\mathrm{GL}_2(\mathbb{Z}/\pi^{k+1}\mathbb{Z})}$ , which equals  $\frac{1}{\pi^4}$  by Remark 2.5. So for all  $g \in \rho_{\pi^k}(\mathrm{Gal}(\mathbb{Q}(E[\pi^k])/\mathbb{Q}))$ ,  $\mathrm{Lift}(g) \subset \rho_{\pi^{k+1}}(\mathrm{Gal}(\mathbb{Q}(E[\pi^{k+1}])/\mathbb{Q}))$ . Thanks to Theorem 2.7, the proof will follow if we count for each  $g$  the number of lifts with a given fixed group.

- (1) For  $g = \mathrm{Id} \in \rho_{\pi^k}(\mathrm{Gal}(\mathbb{Q}(E[\pi^k])/\mathbb{Q}))$ , there is only one element of  $\mathrm{Lift}(g)$  fixing  $(\mathbb{Z}/\pi^{k+1}\mathbb{Z})^2$ , so  $p_{\pi,k}(k+1, k+1 \mid k, k) = \frac{1}{\pi^4}$ .
- (2) The element  $g = \mathrm{Id} \in \rho_{\pi^k}(\mathrm{Gal}(\mathbb{Q}(E[\pi^k])/\mathbb{Q}))$ , can be lifted in exactly  $\pi^4 - 1 - \#\mathrm{GL}_2(\mathbb{Z}/\pi\mathbb{Z})$  ways to elements in  $\mathrm{GL}_2(\mathbb{Z}/\pi^{k+1}\mathbb{Z})$  which fix the  $\pi^k$ -torsion, a point of order  $\pi^{k+1}$ , but not all the  $\pi^{k+1}$ -torsion. Therefore  $p_{\pi,k}(k, k+1 \mid k, k) = \frac{(\pi-1)(\pi+1)^2}{\pi^4}$ .
- (3) Every element of  $\mathrm{GL}_2(\mathbb{Z}/\pi^k\mathbb{Z})$  which fixes a line, but is not the identity, can be lifted in exactly  $\pi^3$  ways to an element of  $\mathrm{GL}_2(\mathbb{Z}/\pi^{k+1}\mathbb{Z})$  which fixes a line of  $(\mathbb{Z}/\pi^{k+1}\mathbb{Z})^2$ . So  $p_{\pi,k}(i, k+1 \mid i, k) = \frac{\pi^3}{\pi^4} = \frac{1}{\pi}$ .

□

The theorem below uses the information on  $\mathrm{Gal}(\mathbb{Q}(E[\pi^{n(E,\pi)}])/\mathbb{Q})$  for a given prime  $\pi$  in order to compute the probabilities of divisibility by any power of  $\pi$ .

*Notation 2.18.* Let  $\pi$  be a prime and  $\gamma_n(h) = \pi^n \sum_{\ell=0}^h \pi^\ell p_n(\ell, n)$ . We also define

$$\delta(k) = \begin{cases} p_{i+1}(i+1, i+1) & \text{if } k = 2i+1 \\ 0 & \text{otherwise} \end{cases}, \quad S_k(h) = \pi^k \left( \sum_{\ell=h}^{\lfloor \frac{k}{2} \rfloor} p_{k-\ell}(\ell, k-\ell) + \delta(k) \right).$$

**Theorem 2.19.** *Let  $\pi$  be a prime,  $E$  an elliptic curve over  $\mathbb{Q}$ , and  $n$  be a positive integer such that  $\forall k \geq n$ ,  $I(E, \pi, k) = I(E, \pi, n)$  (e.g., a curve without complex*



multiplication and  $n \geq n(E, \pi)$ . Then, for any  $k \geq 1$ ,

$$\mathbb{P}(\pi^k \mid \#E(\mathbb{F}_p)) = \begin{cases} \frac{S_k(0)}{\pi^k} & \text{if } 1 \leq k \leq n, \\ \frac{1}{\pi^k}(\gamma_n(k-n-1) + S_k(k-n)) & \text{if } n < k \leq 2n, \\ \frac{1}{\pi^k}(\gamma_n(n) + p_n(n, n)\pi^{2n-1} - \frac{\pi^{4n-1}p_n(n, n)}{\pi^k}) & \text{if } k > 2n. \end{cases}$$

Let  $\bar{v}_\pi$  be the average valuation of  $\pi$  of  $\#E(\mathbb{F}_p)$  for an arbitrary prime  $p$ . Then,

$$\bar{v}_\pi = 2 \sum_{\ell=1}^{n-1} p_\ell(\ell, \ell) + \frac{\pi}{\pi-1} \sum_{\ell=0}^{n-1} p_n(\ell, n) + \sum_{\ell=0}^{n-2} \sum_{i=\ell+1}^{n-1} p_i(\ell, i) + \frac{\pi(2\pi+1)}{(\pi-1)(\pi+1)} p_n(n, n).$$

*Proof.* Let  $k$  be a positive integer. Using Figure 1, one checks that

$$(1) \quad \mathbb{P}(\pi^k \mid \#E(\mathbb{F}_p)) = \sum_{\ell=0}^{\lfloor \frac{k}{2} \rfloor} p_{k-\ell}(\ell, k-\ell) + \delta(k).$$

Let  $c_1 = \frac{1}{\pi^4}$ ,  $c_2 = \frac{(\pi-1)(\pi+1)^2}{\pi^4}$ , and  $c_3 = \frac{1}{\pi}$ . With these notations, the situation can be illustrated by Figure 1. For  $j > n$  and  $\ell < n$ , the probability  $p_j(\ell, j)$  is

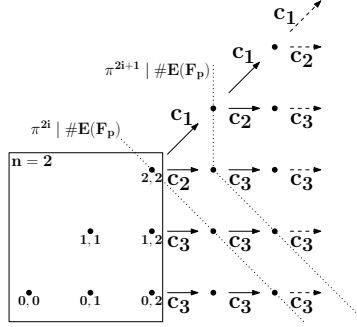


FIGURE 1. Each node of coordinates  $(i, j)$  represents the event  $(E_p[\pi^j] \simeq \mathbb{Z}/\pi^i \mathbb{Z} \times \mathbb{Z}/\pi^j \mathbb{Z})$ . The arrows represent the conditional probabilities of Theorem 2.17.

the product of the conditional probabilities of the unique path from  $(\ell, j)$  to  $(\ell, n)$  in the graph of Figure 1 times the probability  $p_n(\ell, n)$ . For  $j > n$  and  $\ell \geq n$ , the probability  $p_j(\ell, j)$  is the product of the conditional probabilities of the unique path from  $(\ell, j)$  to  $(n, n)$  in the graph of Figure 1 times the probability  $p_n(n, n)$ .

There are three cases that are to be treated separately:  $1 \leq k \leq n$ ,  $n < k \leq 2n$  and  $k > 2n$ . For  $1 \leq k \leq n$ , the result follows from Equation (1). Let us explain

	Average valuation of 2			Average valuation of 3			Average valuation of 5		
	$n$	Th.	Exp.	$n$	Th.	Exp.	$n$	Th.	Exp.
$E_1$	1	$\frac{14}{9} \approx 1.556$	1.555	1	$\frac{87}{128} \approx 0.680$	0.679	1	$\frac{695}{2304} \approx 0.302$	0.301
$E_3$	3	$\frac{895}{576} \approx 1.554$	1.554	1	$\frac{39}{32} \approx 1.219$	1.218	1	$\frac{155}{192} \approx 0.807$	0.807

TABLE 2. Experimental values (Exp.) are obtained with all primes below  $2^{25}$ . Theoretical values (Th.) come from Theorem 2.19.

the case for  $k > 2n$ , with  $k = 2i$ :

$$\begin{aligned}
 \mathbb{P}(\pi^{2i} \mid \#E(\mathbb{F}_p)) &= \sum_{\ell=0}^i p_{2i-\ell}(\ell, 2i-\ell) + \delta(2i) = \sum_{\ell=0}^i p_{2i-\ell}(\ell, 2i-\ell) \\
 &= \sum_{\ell=0}^{n-1} p_{2i-\ell}(\ell, 2i-\ell) + \sum_{\ell=n}^{i-1} p_{2i-\ell}(\ell, 2i-\ell) + p_i(i, i) \\
 &= \sum_{\ell=0}^{n-1} c_3^{2i-\ell-n} p_n(\ell, n) + \sum_{\ell=n}^{i-1} c_3^{2i-2\ell-1} c_2 c_1^{i-n} p_n(n, n) + c_1^{i-n} p_n(n, n).
 \end{aligned}$$

After computations, one obtains the desired formula. The cases  $k > 2n$  odd, and  $n < k \leq 2n$  are treated similarly. The formula for  $\bar{v}_\pi$  is obtained using  $\bar{v}_\pi = \sum_{k \geq 1} \mathbb{P}(\pi^k \mid \#E_p)$ . □

*Remark 2.20.* The theorem proves in particular that there exists a bound  $B$  such that for primes  $\pi > B$ ,  $\mathbb{P}(\pi^2 \mid \#E(\mathbb{F}_p)) < \frac{2}{\pi^2}$ , so the probability that the cardinality is divisible by the square of a prime greater than  $B$  is at most  $\frac{2}{B}$ . This confirms the experimental result that an elliptic curve is close to a cyclic group when reduced modulo an arbitrary prime, regardless on its rank over  $\mathbb{Q}$ .

**Example 2.21.** Let us compare the theoretical and experimental average valuation of  $\pi = 2$ ,  $\pi = 3$  and  $\pi = 5$  for the curve  $E_1 : y^2 = x^3 + 5x + 7$  and  $E_3 : y^2 = x^3 - 10875x + 526250$ . We exclude  $E_2$  in this example since it has complex multiplication. For  $E_1$ , we apply Theorem 2.19 with  $n = 1$  and compute the necessary probabilities with Corollary 2.9 knowing that the Galois groups are isomorphic to  $\text{GL}_2(\mathbb{Z}/\pi\mathbb{Z})$ . For  $E_3$ , we apply Theorem 2.19 with  $n = 3$  for  $\pi = 2$  and  $n = 1$  for  $\pi = 3$  and  $\pi = 5$ , and compute the necessary probabilities with Corollary 2.9 (when  $n = 1$ ) and Theorem 2.7 (when  $n = 3$ ). The results are shown in Table 2.

In order to apply Theorem 2.19, one has to show that  $I(E, \pi, k) = I(E, \pi, n)$  for all  $k \geq n$  (or  $n \geq n(E, \pi)$  since  $E_1$  and  $E_3$  have complex multiplication). For  $E_1$ , we were only able to prove that  $n(E, \pi) = 1$  for  $\pi = 2$ ,  $\pi = 3$  and  $\pi = 5$  by using the remarks at the end of section 2.2. For  $E_3$ , Andrew Sutherland computed for us the Galois groups up to the  $2^5$ -,  $3^3$ - and  $5^2$ -torsion. It is sufficient to compute the probabilities and have some intuition for the values of  $n$ , but we were not able to prove that they are correct. In this case, we have to assume that the values of  $n$  for which we were able to compute the Galois group (and so the probabilities) are correct.

## 3. APPLICATIONS TO SOME FAMILIES OF ELLIPTIC CURVES

As shown in the preceding section, changing the torsion properties is equivalent to modifying the Galois group. One can see the fact of imposing rational torsion points as a way of modifying the Galois group. In this section we change the Galois group either by splitting the division polynomials or by imposing some equations that directly modify the Galois group. With these ideas, we find new infinite ECM-friendly families and we explain the properties of some known curves.

**3.1. Preliminaries on Montgomery and Twisted Edwards Curves.** Let  $K$  be a field whose characteristic is neither 2 nor 3.

**3.1.1. Edwards curves.** For  $a, d \in K$ , with  $ad(a-d) \neq 0$ , the twisted Edwards curve  $ax^2 + y^2 = 1 + dx^2y^2$  is denoted by  $E_{a,d}$ . The “ $a = -1$ ” twisted Edwards curves are denoted by  $E_d$ . In [6] completed twisted Edwards curves are defined by

$$\overline{E}_{a,d} = \{((X : Z), (Y : T)) \in \mathbb{P}^1 \times \mathbb{P}^1 \mid aX^2T^2 + Y^2Z^2 = Z^2T^2 + dX^2Y^2\}.$$

The completed points are the affine  $(x, y)$  embedded into  $\mathbb{P}^1 \times \mathbb{P}^1$  by  $(x, y) \mapsto ((x : 1), (y : 1))$  (see [6] for more information). We denote  $(1 : 0)$  by  $\infty$ .

We give an overview of all the 2- and 4-torsion and some 8-torsion points on  $\overline{E}_{a,d}$ , as specified in [6], in Figure 2.

**3.1.2. Montgomery curves and Suyama family.** Let  $A, B \in K$  be such that  $B(A^2 - 4) \neq 0$ . The Montgomery curve  $By^2 = x^3 + Ax^2 + x$  associated to  $(A, B)$  is denoted by  $M_{A,B}$  (see [17]) and its completion in  $\mathbb{P}^2$  by  $\overline{M}_{A,B}$ .

*Remark 3.1.* If  $a, d, A, B \in K$  are such that  $d = \frac{A-2}{B}$  and  $a = \frac{A+2}{B}$ , then there is a birational map between  $\overline{E}_{a,d}$  and  $\overline{M}_{A,B}$  given by  $((x : z), (y : t)) \mapsto ((t+y)x : (t+y)z : (t-y)x)$  (see [4]). Therefore  $\overline{M}_{A,B}$  and  $\overline{E}_{a,d}$  have the same group structure over any field where defined and in particular the same torsion properties. Any statement in twisted Edwards language can be easily translated into Montgomery coordinates and vice versa.

A Montgomery curve for which there exist  $x_3, y_3, k, x_\infty, y_\infty \in \mathbb{Q}$  such that

$$(2) \quad \begin{cases} P_3(x_3) = 0, & By_3^2 = x_3^3 + Ax_3^2 + x_3 & \text{(3-torsion point)} \\ k = \frac{y_3}{y_\infty}, & k^2 = \frac{x_3^3 + Ax_3^2 + x_3}{x_\infty^3 + Ax_\infty^2 + x_\infty} & \text{(non-torsion point)} \\ x_\infty = x_3^3. & & \text{(Suyama equation)} \end{cases}$$

is called a Suyama curve. As described in [25, 26], the solutions of (2) can be parametrized by a rational value denoted  $\sigma$ . For all  $\sigma \in \mathbb{Q} \setminus \{0, \pm 1, \pm 3, \pm 5, \pm \frac{5}{3}\}$ , the associated Suyama curve has positive rank and a rational point of order 3.

*Remark 3.2.* In the following, when we say that an elliptic curve  $E_{a,d}$  has good reduction modulo a prime  $p$ , we also suppose that we have  $v_p(a) = v_p(d) = v_p(a-d) = 0$  (resp.  $v_p(A-2) = v_p(A+2) = v_p(B) = 0$  for a Montgomery curve). In this case the reduction map is simply given by reducing the coefficients modulo  $p$ . The results below are also true for primes of good reduction which do not satisfy these conditions, by slightly modifying the statements and the proofs. Moreover, in ECM, if the conditions are not satisfied, we immediately find the factor  $p$ .

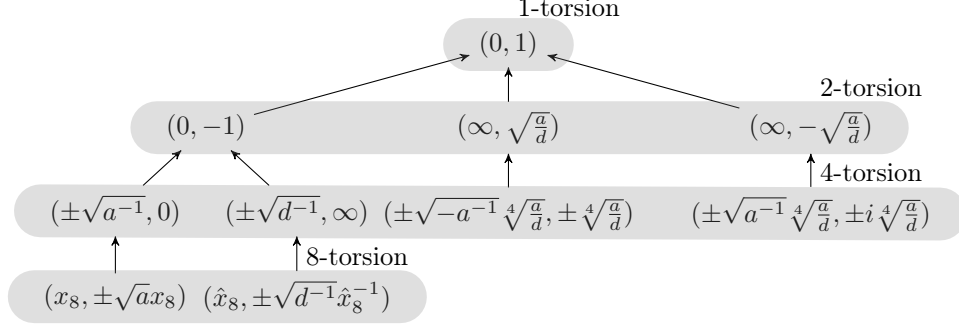


FIGURE 2. An overview of all 1-, 2- and 4-torsion and some 8-torsion points on twisted Edwards curves. The  $x_8$  and  $\hat{x}_8$  in the 8-torsion points are such that  $adx_8^4 - 2ax_8^2 + 1 = 0$  and  $ad\hat{x}_8^4 - 2d\hat{x}_8^2 + 1 = 0$ .

**3.2. Generic Galois Group of a Family of Curves.** In the following, when we talk about the *Galois group of the  $m$ -torsion of a family of curves*, we talk about a group isomorphic to the Galois group of the  $m$ -torsion for all curves of the family except for a sparse set of curves (which can have a smaller Galois group).

For example, let us consider the Galois group of the 2-torsion for the following family  $\{\mathcal{E}_r : y^2 = x^3 + rx^2 + x \mid r \in \mathbb{Q} \setminus \{\pm 2\}\}$ . The Galois group of the 2-torsion of the curve  $\mathcal{E} : y^2 = x^3 + Ax^2 + x$  over  $\mathbb{Q}(A)$  is  $\mathbb{Z}/2\mathbb{Z}$ . Hence, for most values of  $r$  the Galois group is  $\mathbb{Z}/2\mathbb{Z}$  and for a sparse set of values the Galois group is the trivial group. So, we say that the Galois group of the 2-torsion of this family is  $\mathbb{Z}/2\mathbb{Z}$ .

To our best knowledge, there is no implementation of an algorithm computing Galois groups of polynomials with coefficients in a function field. Instead we can compute the Galois group for every curve of the family, so we can guess the Galois group of the family from a finite number of instantiations. In practice, we took a dozen random curves in the family and if all these Galois groups of the  $m$ -torsion were the same, we guessed that it was the Galois group of the  $m$ -torsion of the family of curves.

**3.3. Study of the  $2^k$ -Torsion of Montgomery/Twisted Edwards Curves.** The rational torsion of a Montgomery/twisted Edwards curve is  $\mathbb{Z}/2\mathbb{Z}$  but it is known that 4 divides the order of the curve when reduced modulo any prime  $p$  [25]. The following theorem gives more detail on the  $2^k$ -torsion.

**Theorem 3.3.** *Let  $E = E_{a,d}$  be a twisted Edwards curve (resp. a Montgomery curve  $M_{A,B}$ ) over  $\mathbb{Q}$ . Let  $p$  be a prime such that  $E$  has good reduction at  $p$ .*

- (1) *If  $p \equiv 3 \pmod{4}$  and  $\frac{a}{d}$  (resp.  $A^2 - 4$ ) is a quadratic residue modulo  $p$ , then  $E(\mathbb{F}_p)[4] \simeq \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ ;*
- (2) *If  $p \equiv 1 \pmod{4}$ ,  $a$  (resp.  $\frac{A+2}{B}$ ) is a quadratic residue modulo  $p$  (in particular  $a = \pm 1$ ) and  $\frac{a}{d}$  (resp.  $A^2 - 4$ ) is a quadratic residue modulo  $p$ , then  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z} \subset E(\mathbb{F}_p)[4]$ ;*
- (3) *If  $p \equiv 1 \pmod{4}$ ,  $\frac{a}{d}$  (resp.  $A^2 - 4$ ) is a quadratic non-residue modulo  $p$  and  $a - d$  (resp.  $B$ ) is a quadratic residue modulo  $p$ , then  $E(\mathbb{F}_p)[8] \simeq \mathbb{Z}/8\mathbb{Z}$ .*

*Proof.* Using Remark 3.1, it is enough to prove the results in the Edwards language, which follow by some calculations using Figure 2.  $\square$

Theorem 3.3 suggests that by imposing equations on the parameters  $a$  and  $d$  we can improve the torsion properties. The case where  $\frac{a}{d}$  is a square has been studied in [6] and [5] for the family of Edwards curves having  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$  (when  $a = 1$ ) respectively  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$  (when  $a = -1$ ) rational torsion. Here we focus on two other equations:

$$(3) \quad \exists c \in \mathbb{Q}, a = -c^2 \quad (A + 2 = -Bc^2 \text{ for Montgomery curves}),$$

$$(4) \quad \exists c \in \mathbb{Q}, a - d = c^2 \quad (B = c^2 \text{ for Montgomery curves}).$$

The cardinality of the Galois group of the 4-torsion for generic Montgomery curves is 16 and this is reduced to 8 for the family of curves satisfying (3). Using Theorem 2.7, we can compute the changes of probabilities due to this new Galois group. For all curves satisfying (3) and all primes  $p \equiv 1 \pmod{4}$ , the probability of having  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$  as the 4-torsion group becomes 0 (instead of  $\frac{1}{4}$ ); the probabilities of having  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$  and  $\mathbb{Z}/4\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$  as the 4-torsion group become  $\frac{1}{4}$  (instead of  $\frac{1}{8}$ ).

The Galois group of the 8-torsion of the family of curves satisfying (4) has cardinality 128 instead of 256 for generic Montgomery curves. Using Theorem 2.7, one can see that the probabilities of having an 8-torsion point are improved.

Using Theorem 2.19, one can show that both families of curves, the family satisfying (3) and the one satisfying (4), increase the probability of having the cardinality divisible by 8 from 62.5% to 75% and the average valuation of 2 from  $\frac{10}{3}$  to  $\frac{11}{3}$ .

**3.4. Better Twisted Edwards Curves with Torsion  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$  using Division Polynomials.** In this section we search for curves such that some of the factors of the division polynomials split and by doing so we try to change the Galois groups. As an example we consider the family of  $a = -1$  twisted Edwards curves  $E_d$  with  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$ -torsion, these curves are exactly the ones with  $d = -e^4$  (see [5]). The technique might be used in any context.

3.4.1. *Looking for subfamilies.* For a generic  $d$ ,  $P_8^{\text{new}}$  splits into three irreducible factors: two of degree 4 and one of degree 16. If one takes  $d = -e^4$ , the polynomial of degree 16 splits into three factors: two of degree 4, called  $P_{8,0}$  and  $P_{8,1}$ , and one of degree 8, called  $P_{8,2}$ . By trying to force one of these three polynomials to split, we found four families, as shown in Table 3.

$d = -e^4$	“generic” $e$	$e = g^2$	$e = \frac{2g^2+2g+1}{2g+1}$	$e = \frac{g^2}{2}$	$e = \frac{g-\frac{1}{g}}{2}$
degree of factors of $P_{8,0}$	4	4	4	2, 2	2, 2
degree of factors of $P_{8,1}$	4	4	4	4	2, 2
degree of factors of $P_{8,2}$	8	4, 4	4, 4	8	8
average valuation of 2	$\frac{14}{3}$	$\frac{29}{6}$	$\frac{29}{6}$	$\frac{29}{6}$	$\frac{16}{3}$
for $p = 3 \pmod{4}$	4	4	4	4	5
for $p = 1 \pmod{4}$	$\frac{16}{3}$	$\frac{17}{3}$	$\frac{17}{3}$	$\frac{17}{3}$	$\frac{17}{3}$

TABLE 3. Subfamilies of twisted Edwards curves with torsion group isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$  and the degrees of the irreducible factors of  $P_{8,0}$ ,  $P_{8,1}$  and  $P_{8,2}$ .

In all these families the generic average valuation of 2 is increased by  $\frac{1}{6}$  ( $\frac{29}{6}$  instead of  $\frac{14}{3}$ ), except the family  $e = \frac{g-\frac{1}{2}}{2}$  for which it is increased by  $\frac{2}{3}$ , bringing it to the same valuation as for the family of twisted Edwards curves with  $a = 1$  and torsion isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/8\mathbb{Z}$ . Note that these four families cover all the curves presented in the first three columns of [5, Table 3.1], except the two curves with  $e = \frac{26}{7}$  and  $e = \frac{19}{8}$ , which have a generic Galois group for the 8-torsion.

3.4.2. *The family  $e = \frac{g-\frac{1}{2}}{2}$ .* In this section, we study in more detail the family  $e = \frac{g-\frac{1}{2}}{2}$ . Using Theorem 2.7 one can prove that the group order modulo all primes is divisible by 16. However, we give an alternative proof which is also of independent interest. We need the following theorem which computes the 8-torsion points that double to the 4-torsion points  $(\pm\sqrt[4]{-d^{-1}}, \pm\sqrt[4]{-d^{-1}})$ .

**Theorem 3.4.** *Let  $E_d$  be a twisted Edwards curve over  $\mathbb{Q}$  with  $d = -e^4$ ,  $e = \frac{g-\frac{1}{2}}{2}$  and  $g \in \mathbb{Q} \setminus \{-1, 0, 1\}$ . Let  $p > 3$  be a prime of good reduction. If  $t \in \{1, -1\}$  such that  $tg(g-1)(g+1)$  is a quadratic residue modulo  $p$  then the points  $(x, y) \in E_d(\mathbb{F}_p)$ , with  $w \in \{1, -1\}$ , and*

$$(5) \quad x = \pm g^w y, \quad y = \pm \sqrt{\frac{4tg^{2-w}}{(g-tw)^3(g+tw)}}$$

have order eight and double to  $(\pm e^{-1}, te^{-1})$ .

*Proof.* Note that all points  $(x, y)$  of order eight satisfy  $\infty \neq x \neq 0 \neq y \neq \infty$ . Following [6, Theorem 2.10] a point  $(x, y)$  doubles to  $((2xy : 1 + dx^2y^2), (x^2 + y^2 : 1 - dx^2y^2)) = ((2xy : -x^2 + y^2), (x^2 + y^2 : 2 - (-x^2 + y^2)))$ . Let  $s, t \in \{1, -1\}$  such that  $(x, y)$  doubles to  $(se^{-1}, te^{-1})$ , hence

$$\frac{2xy}{-x^2 + y^2} = \frac{s}{e} \quad \text{and} \quad \frac{x^2 + y^2}{2 - (-x^2 + y^2)} = \frac{t}{e}.$$

From the terms in the first equation we obtain  $\left(\frac{x}{y}\right)^2 + \frac{2exs}{y} + e^2 = 1 + e^2$ . Write  $e = \frac{g-\frac{1}{2}}{2}$  such that  $\left(\frac{x}{y} + se\right)^2 = \left(\frac{g+\frac{1}{2}}{2}\right)^2$ . Hence  $\frac{x}{y} \in \left\{\pm g, \pm \frac{1}{g}\right\}$  depending on the sign  $s$  and the sign after taking the square root. This gives  $x^2 = G^2 y^2$  with  $G^2 \in \{g^2, g^{-2}\}$ .

From the second equation we obtain  $(e-t)x^2 + (e+t)y^2 = 2t$  and substituting  $x^2$  results in  $((e-t)G^2 + (e+t))y^2 = 2t$ . This can be solved for  $y$  when  $2t((e-t)G^2 + (e+t))$  is a quadratic residue modulo  $p$ . This is equivalent to checking if any of

$$(6) \quad 2t((e-1)g^2 + (e+1)) = \frac{t(g-1)^3(g+1)}{g},$$

$$(7) \quad 2t((e-1) + (e+1)g^2) = \frac{t(g-1)(g+1)^3}{g}$$

is a quadratic residue modulo  $p$ . By assumption  $tg(g-1)(g+1)$  is a quadratic residue modulo  $p$ . Hence, both expression (6) and (7) are quadratic residues modulo  $p$ . Solving for  $y$  and keeping track of all the signs results in the formulae in (5).  $\square$

A direct consequence of this theorem is as follows.

**Corollary 3.5.** *Let  $E = E_d$  be a twisted Edwards curve over  $\mathbb{Q}$  with  $d = -\left(\frac{g-\frac{1}{2}}{2}\right)^4$ ,  $g \in \mathbb{Q} \setminus \{-1, 0, 1\}$  and  $p > 3$  a prime of good reduction. Then  $E(\mathbb{Q})$  has torsion group isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$  and the group order of  $E(\mathbb{F}_p)$  is divisible by 16.*

*Proof.* We consider two cases.

(1) If  $p \equiv 1 \pmod{4}$  then  $-1$  is a quadratic residue modulo  $p$ . Hence, the 4-torsion points  $(\pm i, 0)$  exist (see Figure 2) and  $16 \mid \#E(\mathbb{F}_p)$ .

(2) If  $p \equiv 3 \pmod{4}$  then  $-1$  is a quadratic non-residue modulo  $p$ . Then exactly one of  $\{g(g-1)(g+1), -g(g-1)(g+1)\}$  is a quadratic residue modulo  $p$ . Using Thm. 3.4 it follows that the curve  $E(\mathbb{F}_p)$  has eight 8-torsion points and hence  $16 \mid \#E(\mathbb{F}_p)$ .  $\square$

Corollary 3.5 explains the good behavior of the curve with  $d = -\left(\frac{77}{36}\right)^4$  and torsion group isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$  found in [5]. This parameter can be expressed as  $d = -\left(\frac{77}{36}\right)^4 = -\left(\frac{g-\frac{1}{2}}{2}\right)^4$  for  $g = \frac{9}{2}$  and, therefore, the group order is divisible by an additional factor two.

**Corollary 3.6.** *Let  $g \in \mathbb{Q} \setminus \{-1, 0, 1\}$ ,  $d = -\left(\frac{g-\frac{1}{2}}{2}\right)^4$  and  $p \equiv 1 \pmod{4}$  be a prime of good reduction. If  $g(g-1)(g+1)$  is a quadratic residue modulo  $p$  then the group order of  $E_d(\mathbb{F}_p)$  is divisible by 32.*

*Proof.* All 16 4-torsion points are in  $E_d(\mathbb{F}_p)$  (see Figure 2). By Thm. 3.4 we have at least one 8-torsion point. Hence,  $32 \mid \#E_d(\mathbb{F}_p)$ .  $\square$

We generated different values  $g \in \mathbb{Q}$  by setting  $g = \frac{i}{j}$  with  $1 \leq i < j \leq 200$  such that  $\gcd(i, j) = 1$ . This resulted in 12 231 possible values for  $g$  and Sage [23] found 614 non-torsion points. As expected, we observed that they behave similarly as the good curve found in [5].

**3.4.3. Parametrization.** In [5] a “generating curve” is specified which parametrizes  $d$  and the coordinates of the non-torsion points. Arithmetic on this curve can be used to generate an infinite family of twisted Edwards curves with torsion group isomorphic to  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$  and a non-torsion point. Using ideas from [10] we found a parametrization which does not involve a generating curve and hence no curve arithmetic.

**Theorem 3.7.** *Let  $t \in \mathbb{Q} \setminus \{0, \pm 1\}$  and  $d = -e^4$ ,  $e = \frac{3(t^2-1)}{8t}$ ,  $x_\infty = (4e^3 + 3e)^{-1}$  and  $y_\infty = \frac{9t^4 - 2t^2 + 9}{9t^4 - 9}$ . Then the twisted Edwards curve  $-x^2 + y^2 = 1 + dx^2y^2$  has torsion group  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$  and  $(x_\infty, y_\infty)$  is a non-torsion point.*

*Proof.* The twisted Edwards curve has torsion group  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$  because  $d = -e^4$  and  $e$  is not equal to 0 and  $\pm 1$ . The point  $(x_\infty, y_\infty)$  is on the curve and since  $x_\infty \notin \{0, \infty, e^{-1}, -e^{-1}\}$  this is a non-torsion point.  $\square$

This rational parametrization allowed us to impose additional conditions on the parameter  $e$ . For the four families, except  $e = g^2$  which is treated below, the parameter  $e$  is given by an elliptic curve of rank 0 over  $\mathbb{Q}$ .

**Corollary 3.8.** *Let  $P = (x, y)$  be a non-torsion point on the elliptic curve  $y^2 = x^3 - 36x$  having rank 1. Let  $t = \frac{x+6}{x-6}$ , using notations of Theorem 3.7, the curve  $E_{-e^4}$  belongs to the family  $e = g^2$  and has positive rank over  $\mathbb{Q}$ .*

**3.5. Better Suyama Curves by a Direct Change of the Galois Group.** In this section we will present two families that change the Galois group of the 4- and 8-torsion without modifying the factorization pattern of the 4- and 8-division polynomial.

3.5.1. *Suyama-11.* Kruppa observed in [14] that among the Suyama curves, the one corresponding to  $\sigma = 11$  finds exceptionally many primes. Barbulescu [3] extended it to an infinite family that we present in detail here.

Experiments show that the  $\sigma = 11$  curve differs from other Suyama curves only by its probabilities to have a given  $2^k$ -torsion when reduced modulo primes  $p \equiv 1 \pmod{4}$ . The reason is that the  $\sigma = 11$  curve satisfies Equation (3). Section 3.3 illustrates the changes in probabilities of the  $\sigma = 11$  curve when compared to curves which do not satisfy Equation (3) and shows that Equation (3) improves the average valuation of 2 from  $\frac{10}{3}$  to  $\frac{11}{3}$ .

Let us call *Suyama-11* the set of Suyama curves which satisfy Equation (3). When solving the system formed by Suyama's system plus Equation (3), we obtain an elliptic parametrization for  $\sigma$ . Given a point  $(u, v)$  on  $E_{\sigma_{11}} : v^2 = u^3 - u^2 - 120u + 432$ ,  $\sigma$  is obtained as  $\sigma = \frac{120}{u-24} + 5$ . The group  $E_{\sigma_{11}}(\mathbb{Q})$  is generated by the points  $P_\infty = (-6, 30)$ ,  $P_2 = (-12, 0)$  and  $Q_2 = (4, 0)$  of orders  $\infty$ , 2 and 2 respectively. We exclude  $0, \pm P_\infty, P_2, Q_2, P_2 + Q_2$ , and  $Q_2 \pm P_\infty$ , which are the points producing non-valid values of  $\sigma$ . The points  $\pm R, Q_2 \pm R$  lead to isomorphic curves. Note that the  $\sigma = 11$  curve corresponds to the point  $(44, 280) = P_\infty + P_2$ .

3.5.2. *Edwards  $\mathbb{Z}/6\mathbb{Z}$ : Suyama-11 in disguise.* In [5, Sec. 5] it is shown that the  $a = -1$  twisted Edwards curves with  $\mathbb{Z}/6\mathbb{Z}$ -torsion over  $\mathbb{Q}$  are precisely the curves  $E_d$  with  $d = \frac{-16u^3(u^2-u+1)}{(u-1)^6(u+1)^2}$  where  $u$  is a rational parameter.<sup>2</sup> In particular, according to [5, Sec. 5.3] one can translate any Suyama curve in Edwards language and then impose the condition that  $-a$  is a square to obtain curves of the  $a = -1$  type. Finally, [5, Sec. 5.5] points out that this family has exceptional torsion properties.

In order to understand the properties of this family, we translate it back to Montgomery language using Remark 3.1. Thus, we are interested in Suyama curves which satisfy equation  $A + 2 = -Bc^2$  (the Montgomery equivalent for  $-a$  being a square). This is the *Suyama-11* family, so its torsion properties were explained in Section 3.5.1. These two families have been discovered independently in [3] and [5].

3.5.3. *Suyama- $\frac{9}{4}$ .* In experiments by Zimmermann, new Suyama curves with exceptional torsion properties were discovered, such as  $\sigma = \frac{9}{4}$ . Further experiments show that their special properties are related to the  $2^k$ -torsion and concern exclusively primes  $p \equiv 1 \pmod{4}$ . Indeed, the  $\sigma = \frac{9}{4}$  curve satisfies Equation (4). Section 3.3 illustrates the changes in probabilities of the  $\sigma = \frac{9}{4}$  curve when compared to curves which do not satisfy Equation (4) and shows that Equation (4) improves the average valuation of 2 from  $\frac{10}{3}$  to  $\frac{11}{3}$ .

Let us call *Suyama- $\frac{9}{4}$*  the set of Suyama curves which satisfy Equation (4). When solving the system formed by Suyama's system plus Equation (4), we obtain an elliptic parametrization for  $\sigma$ . Given a point  $(u, v)$  on  $E_{\sigma_{9/4}} : v^2 = u^3 - 5u$ ,  $\sigma$  is obtained as  $\sigma = u$ . The group  $E_{\sigma_{9/4}}(\mathbb{Q})$  is generated by the points  $P_\infty = (-1, 2)$  and  $P_2 = (0, 0)$  of orders  $\infty$  and 2 respectively. We exclude the points  $0, \pm P_\infty, P_2$

<sup>2</sup>There is a typo in the proof of [5, Thm. 5.1]; the  $\frac{16u^3(u^2-u+1)}{(u-1)^6(u+1)^2}$  misses a minus sign.



Families	Curves	Average valuation of 2			Average valuation of 3		
		$n$	Th.	Exp.	$n$	Th.	Exp.
Suyama	$\sigma = 12$	2	$\frac{10}{3} \approx 3.333$	3.331	1	$\frac{27}{16} \approx 1.688$	1.689
Suyama-11	$\sigma = 11$	2	$\frac{11}{3} \approx 3.667$	3.669	1	$\frac{27}{16} \approx 1.688$	1.687
Suyama- $\frac{9}{4}$	$\sigma = \frac{9}{4}$	3	$\frac{11}{3} \approx 3.667$	3.664	1	$\frac{27}{16} \approx 1.688$	1.687
$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$	$E_{-11^4}$	3	$\frac{14}{3} \approx 4.667$	4.666	1*	$\frac{87}{128} \approx 0.680$	0.679
$e = \frac{g-\frac{1}{2}}{2}$	$E_{-\left(\frac{77}{36}\right)^4}$	3	$\frac{16}{3} \approx 5.333$	5.332	1*	$\frac{87}{128} \approx 0.680$	0.679
$e = g^2$	$E_{-9^4}$	3	$\frac{29}{6} \approx 4.833$	4.833	1*	$\frac{87}{128} \approx 0.680$	0.680
$e = \frac{g^2}{2}$	$E_{-\left(\frac{81}{8}\right)^4}$	3	$\frac{29}{6} \approx 4.833$	4.831	1*	$\frac{87}{128} \approx 0.680$	0.679
$e = \frac{2g^2+2g+1}{2g+1}$	$E_{-\left(\frac{5}{3}\right)^4}$	3	$\frac{29}{6} \approx 4.833$	4.833	1*	$\frac{87}{128} \approx 0.680$	0.679

TABLE 4. Experimental values (Exp.) are obtained with all primes below  $2^{25}$ . The case  $n = 1^*$  means that the Galois group is isomorphic to  $GL_2(\mathbb{Z}/\pi\mathbb{Z})$ .

and  $P_2 \pm P_\infty$  which produce non-valid values of  $\sigma$ . If two points in  $E_{\sigma_{94}}(\mathbb{Q})$  differ by  $P_2$  they correspond to isomorphic curves. We recognize the curve associated to  $\sigma = \frac{9}{4}$  when considering the point  $(\frac{9}{4}, -\frac{3}{8}) = [2]P_\infty$ .

**3.6. Comparison.** Table 4 gives a summary of all the families discussed in this article. The theoretical average valuations were computed with Theorem 2.19, Theorem 2.7 and Corollary 2.9 under some assumptions on Serre's exponent (see Example 2.21 for more information).

Note that, when we impose torsion points over  $\mathbb{Q}$ , the average valuation does not simply increase by 1, as can be seen in Table 4 for the average valuation of 3.

#### 4. CONCLUSION AND FURTHER WORK

We have used Galois theory in order to analyze the torsion properties of elliptic curves. We have determined the behavior of generic elliptic curves and explained the exceptional properties of some known curves (Edwards curves of torsion  $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$  and  $\mathbb{Z}/6\mathbb{Z}$ ). The new techniques suggested by the theoretical study have helped us to find infinite families of curves having exceptional torsion properties. We list some questions which were not addressed in this work:

- How does Serre's work relate to the independence of the  $m$ - and  $m'$ -torsion probabilities for coprime integers  $m$  and  $m'$ ?
- Is there a model predicting the success probability of ECM from the probabilities given in Theorem 2.19?
- Is it possible to effectively use the Resolvent Method [11] in order to compute equations which improve the torsion properties?

#### REFERENCES

- [1] A. O. L. Atkin and F. Morain. Finding suitable curves for the elliptic curve method of factorization. *Mathematics of Computation*, 60(201):399–405, 1993.
- [2] S. Bai, P. Gaudry, A. Kruppa, F. Morain, E. Thomé, and P. Zimmermann. Crible algébrique: distribution, optimisation (CADO-NFS). <http://cado-nfs.gforge.inria.fr/>.

- [3] R. Barbulescu. Familles de courbes adaptées à la factorisation des entiers. Research report version 2, <http://hal.inria.fr/inria-00419218/en/>, 2009.
- [4] D. J. Bernstein, P. Birkner, M. Joye, T. Lange, and C. Peters. Twisted Edwards curves. In S. Vaudenay, editor, *Africacrypt*, volume 5023 of *Lecture Notes in Computer Science*, pages 389–405. Springer, Heidelberg, 2008.
- [5] D. J. Bernstein, P. Birkner, and T. Lange. Starfish on strike. In M. Abdalla and P. S. L. M. Barreto, editors, *Latincrypt*, volume 6212 of *Lecture Notes in Computer Science*, pages 61–80. Springer, Heidelberg, 2010.
- [6] D. J. Bernstein, P. Birkner, T. Lange, and C. Peters. ECM using Edwards curves. Cryptology ePrint Archive, Report 2008/016, 2008. <http://eprint.iacr.org/>.
- [7] D. J. Bernstein and T. Lange. Faster addition and doubling on elliptic curves. In K. Kurosawa, editor, *Asiacrypt*, volume 4833 of *Lecture Notes in Computer Science*, pages 29–50. Springer, Heidelberg, 2007.
- [8] I. Blake, G. Seroussi, and N. Smart. *Elliptic curves in cryptography*, volume 265. Cambridge Univ Pr, 1999.
- [9] J. W. Bos, T. Kleinjung, A. K. Lenstra, and P. L. Montgomery. Efficient SIMD arithmetic modulo a Mersenne number. In *IEEE Symposium on Computer Arithmetic – ARITH-20*, pages 213–221. IEEE Computer Society, 2011.
- [10] E. Brier and C. Clavier. New families of ECM curves for Cunningham numbers. In G. Hanrot, F. Morain, and E. Thomé, editors, *Algorithmic Number Theory – ANTS-IX*, volume 6197 of *Lecture Notes in Computer Science*, pages 96–109. Springer, 2010.
- [11] H. Cohen. *A course in computational algebraic number theory*, volume 138. Springer-Verlag, 1993.
- [12] H. M. Edwards. A normal form for elliptic curves. *Bulletin of the American Mathematical Society*, 44:393–422, July 2007.
- [13] H. Hisil, K. K.-H. Wong, G. Carter, and E. Dawson. Twisted Edwards curves revisited. In J. Pieprzyk, editor, *Asiacrypt 2008*, volume 5350 of *Lecture Notes in Computer Science*, pages 326–343. Springer, Heidelberg, 2008.
- [14] A. Kruppa. *Speeding up Integer Multiplication and Factorization*. PhD thesis, Université Henri Poincaré - Nancy I, January 2010.
- [15] A. K. Lenstra and H. W. Lenstra, Jr. *The Development of the Number Field Sieve*, volume 1554 of *Lecture Notes in Mathematics*. Springer-Verlag, 1993.
- [16] H. W. Lenstra Jr. Factoring integers with elliptic curves. *Annals of Mathematics*, 126(3):649–673, 1987.
- [17] P. L. Montgomery. Speeding the Pollard and elliptic curve methods of factorization. *Mathematics of Computation*, 48(177):243–264, 1987.
- [18] J. Neukirch. *Class field theory*, volume 280. Springer-Verlag, 1986.
- [19] J. M. Pollard. The lattice sieve. pages 43–49 in [15].
- [20] J. Serre. Propriétés galoisiennes des points d’ordre fini des courbes elliptiques. *Inventiones mathematicae*, 15(4):259–331, 1971.
- [21] J. Serre. Quelques applications du théorème de Chebotarev. *Inst. Hautes Études Sci. Publ. Math.*, 54:323–401, 1981.
- [22] J. Silverman. *The arithmetic of elliptic curves*, volume 106. Springer-Verlag, 2009.
- [23] W. Stein et al. *Sage Mathematics Software (Version 4.7)*. The Sage Development Team, 2011. <http://www.sagemath.org>.
- [24] A. Sutherland. Computing the image of Galois. CNTA XII, <http://math.mit.edu/~drew/CNTA12.pdf>, 2012.
- [25] H. Suyama. Informal preliminary report (8), October 1985.
- [26] P. Zimmermann and B. Dodson. 20 years of ECM. In F. Hess, S. Pauli, and M. E. Pohst, editors, *Algorithmic Number Theory – ANTS-VII*, volume 4076 of *Lecture Notes in Computer Science*, pages 525–542. Springer, Heidelberg, 2006.
- [27] P. Zimmermann et al. GMP-ECM (elliptic curve method for integer factorization). <https://gforge.inria.fr/projects/ecm/>, 2010.
- [28] D. Zywina. On the surjectivity of mod  $\ell$  representations associated to elliptic curves. 2011.

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