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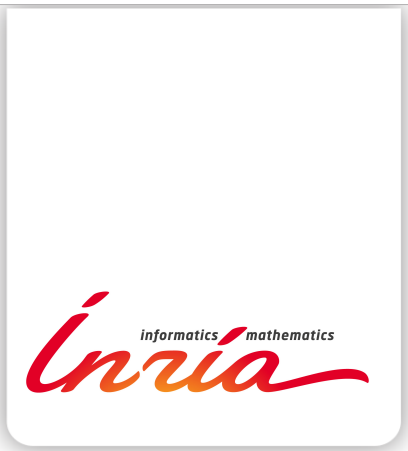

Global stability for SIR and $SIRS$ models with differential mortality

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Abstract: We consider SIR and $SIRS$ models with differential mortality. Global stability of equilibria is established by using Lyapunov's method.

Key-words: Nonlinear dynamical systems, global stability, Lyapunov methods.

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Stabilité globale pour les modèles SIR et $SIRS$ avec mortalités différentes

Résumé : On considère les modèles SIR et $SIRS$ avec des mortalités differentes. La stabilité globale des équilibres est etablie en utilisant les techniques de Lyapunov.

Mots-clés : Systèmes dynamiques non-linéaires, stabilité globale, méthodes de Lyapunov

1 Introduction

The SIR model is a classical model in mathematical epidemiology. Particularly Kermack and McKendrick [8] use a SIR model to prove the existence of threshold. The model of Kermack and McKendrick is without demography, i.e. without vital dynamics. The classic SIR models are very important as conceptual models (similar to predator-prey and competing species models in ecology). The SIR epidemic modeling yields the useful concept of the threshold.

When a vital dynamic is introduced the asymptotic behavior changes. When the death rates are equal in each compartments S,I and R, and equal to the birth rate, the global stability has been solved in [6, 5]. Actually, the total population is constant, hence the system reduces to a two dimensional system. Then using phase plane methods (Poincaré-Birkhoff) and Lyapunov functions the global stability is obtained.

Models with a variable total population size are often more difficult to analyze mathematically because the population size is an additional variable which is governed by a differential equation

The global stability using Lyapunov functions of SIR model with a total constant population is proved in [10]. However in this model the death rates of S and I are equal and the death rate of the removed compartment is adjusted relatively to the death rate of S and the constant birth rate. This adjustment is just done to have a constant total population. This is a little bit artificial. The model with constant population simplifies in two important ways :

- The mass action law $\dfrac{SI}{N}$ reduces to a bilinear law $\tilde{\beta} S\, I$
- The system is a two-dimensional system.

In this paper we propose a more realistic model, with constant population. We suppose, which is more or less observable, that the natality compensates for the mortality. Our model can deal with different death rates, and particularly with a over-mortality from the disease.

We denote $\mathcal{R}_0$ the basic reproduction number. It is defined as the expected number of new cases of infection caused by a typical infected in a population susceptible [3 , 16]. We prove in this paper the global stability of disease free equilibrium (DFE) if $\mathcal{R}_0 \leq 1$ and that there exists a unique endemic equilibrium (EE) if $\mathcal{R}_0 > 1$, which is globally asymptotically stable on the domain minus the stable manifold of the DFE.

The stability analysis of classical SIR model is well know since 1976 [5 , 7]. The reason was that study of stability for these models reduce to the study of 2-dimensionnal systems, hence phase methods can be used: Poincaré-Bendixon theorem. Periodic orbits are ruled out using Dulac criteria or a condition of Busenberg and Van Den Driessche [2].

In the recent litterature, the Lyapunov method is successfuly used to prove the global stability of endemic equilibrium. This method consists to find one function, called Lyapunov function and usually denoted by V, positive definite and its derivative along trajectories is negative definite. If the derivative $\dot{V}$ is only negative , the LaSalle's invariance principle extend the Lyapunov method in particular cases. This Lyapunov function is very difficult to exhibit. However, the class of Lyapunov function

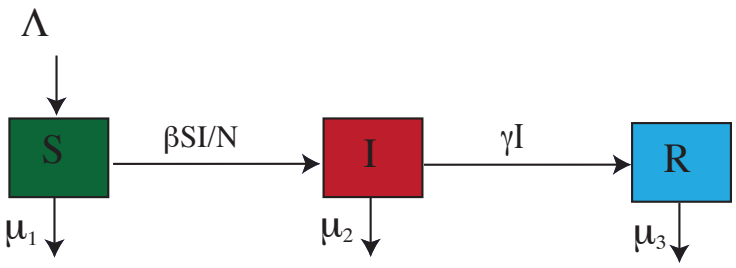
$$V = \sum_{i=1}^{n} a_i \left(x_i - \bar{x}_i \log x_i\right)$$

is used. This function has a long history of application to Lotka-Voltera models and was discovered by Voltera himself, although he did not use the vocabulary and the theory of Lyapunov functions. In 2002, Korobeinikov and Wake use this type of function to prove the global stability for SIR, $SIRS$ and SIS models [10] and in 2004, for $SEIR$ and $SEIS$ model [9] and give a simply proof of the result of Li and Muldowney [15].

We give a brief outline of the paper. In section 2, we formulate the model and study the global stability of the DFE if $\mathcal{R}_0 \leq 1$. In section 3, we study the global stability of endemic equilibrium if $\mathcal{R}_0 > 1$. In section 4, we deduce results of the next section to SIRS model. In finally we conclude in section 5.

2 Model formulation

We consider a population N divided into classes of susceptible, infectious and removed individuals, with numbers at time t denoted by $S(t)$, $I(t)$ and $R(t)$ respectively, that is $N = S(t)+I(t)+R(t)$. We assume that there is no vertical transmission, then all offsprings are susceptibles. We assume that the natality Λ compensates for the deaths. Then $\Lambda = \mu_1 S + \mu_2 I + \mu_3 R$. The parameter γ is the rate of recovery. Note that, in our model the disease confers a permanent immunity. The parameter β is the effective per capita contact rate of infective individuals. We modelize the contact by the classical law of mass action mass. We have the following flow graph:



The dynamic of this model is given by the following system:

$$\begin{cases} \dot{S} = \Lambda - \beta \frac{S\,I}{N} - \mu_1\, S \\ \dot{I} = \beta \frac{S\,I}{N} - \mu_2 I - \gamma\, I \\ \dot{R} = \gamma\, I - \mu_3\, R \end{cases} \tag{1}$$

Which reduces to

$$\begin{cases} \dot{S} = -\beta \frac{S\,I}{N} + \mu_2\, I + \mu_3\, R \\ \dot{I} = \beta \frac{S\,I}{N} - \mu_2\, I - \gamma\, I \\ \dot{R} = \gamma\, I - \mu_3\, R \end{cases} \tag{2}$$

The population size is constant, so that $S+I+R = N$, then we can omit the equation of removed population. We obtain the two-dimensional system:

$$\begin{cases} \dot{S} = -\beta \frac{SI}{N} + \mu_2 I + \mu_3\,(N - S - I) \\ \dot{I} = \beta \frac{SI}{N} - (\mu_2 + \gamma)\, I \end{cases}$$

For simplicity we can consider the prevalence, i.e. the proportions.

If we denote $\frac{S}{N}$, $\frac{I}{N}$, the susceptible and infectious fractions, again by S and I. Then the system (2) is reduced to

$$\begin{cases} \dot{S} = \mu_3 + (\mu_2 - \mu_3) I - \mu_3 S - \beta SI \\ \dot{I} = \beta SI - (\mu_2 + \gamma)\, I \end{cases} \tag{3}$$

We have $0 \leq S$, $0 \leq I$ and $S + I \leq 1$. The biological domain of this two-dimensional system is the standard simplex.

The set $\Omega = \{(S, I) : S \geq 0; I \geq 0; S + I \leq 1\}$ is a positively invariant compact set for (3). The system is well posed.

The basic reproduction ratio is given by

$$\mathcal{R}_0 = \frac{\beta}{\mu_2 + \gamma}.$$

2.1 Stability of DFE

System (3) has a disease free equilibrium state, which is given by $(S^*, 0) = (1, 0)$.

Theorem 2.1 *If $\mathcal{R}_0 \leq 1$ then the DFE is globally asymptotically stable on Ω.*

Proof:
We consider the Lyapunov-LaSalle function $V(S, I) = I$. We have:

$$\begin{aligned} \dot{V} &= \dot{I} \\ &= \beta SI - (\mu_2 + \gamma) I \\ &= I (\mathcal{R}_0 S - 1) (\mu_2 + \gamma) \\ &\leq 0 \end{aligned}$$

Furthermore $\dot{V} = 0$ if $I = 0$ or $S = S^*$ and $\mathcal{R}_0 = 1$. Hence the largest invariant set contained in the set $\mathcal{L} = \left\{(S, I) \in \Omega \ / \ \dot{V}(S, I) = 0\right\}$ is reduced to the DFE. Since we are in a compact positively invariant set, by the LaSalle's Invariance Principle [12 , 1] , the DFE is globaly asymtotically stable in Ω.

Remark 2.1 *Unlike Lyapunov's theorems, LaSalle's principle does not require the function $V(x)$ to be positive definite . If the largest invariant set M, contained in the set E of points where $\dot{V}$ vanishes, is reduced to the equilibrium point, i.e. if $M = \{x_0\}$, the LaSalle's principle allows to conlude that the equilibrium is attractive. But a drawback of Lasalle's principle, when significant, is that it proves only the attractivity of the equilibrium point. It is well known that in the nonlinear case attractivity does not imply stability. But when the function V is not positive definite, Lyapunov stability must be proven. This is why LaSalle's principle is often misquoted. Some additional condition enables, with LaSalle's principle, to ascertain asymptotic stability. To obtain stability from LaSalle's principle some additional work is needed. The most complete results, in the direction of Lasalle's principle to prove asymptotic stability, have been obtained by LaSalle himself (***LaSalle**:[13], in 1968, completed in 1976 [14] .)*

3 Global Stability of endemic equilibrium

An equilibrium for system (3), different from the DFE, is given by $\left(\bar{S}, \bar{I}\right)$, where

$$\bar{S} = \frac{\mu_2 + \gamma}{\beta} = \frac{1}{\mathcal{R}_0} \quad \text{and} \quad \bar{I} = \frac{\mu_3}{\mu_3 + \gamma}\left(1 - \frac{1}{\mathcal{R}_0}\right)$$

This equilibrium is in the simplex, i.e., $0 \leq \bar{S}$, $0 \leq \bar{I}$ and $\bar{S} + \bar{I} \leq 1$ iff $\mathcal{R}_0 > 1$.

Clearly $0 \leq \bar{I}$ is equivalent to $\mathcal{R}_0 \geq 1$. Now we can write

$$\bar{S} + \bar{I} = \frac{\frac{\gamma}{\mathcal{R}_0} + \mu_3}{\gamma + \mu_3}$$

When $\mathcal{R}_0 = 1$ this equilibrium coincides with the DFE. Then there is an unique equilibrium in the interior of the simplex iff $\mathcal{R}_0 > 1$.

Theorem 3.1 *If $\mathcal{R}_0 > 1$, the DFE is unstable and there exists a unique endemic equilibrium $(\bar{S}, \bar{I})$ and this endemic equilibrium is globally asymptotically stable on the domain $\Omega \backslash\ [0,1]\times\{0\}$. In other words on the simplex minus the stable manifold of the DFE*

Proof:

When $\mathcal{R}_0 > 1$ the instability of the DFE comes from [3].

Let Ω_1 the set defined by $\Omega_1 = \left\{(S,I)/\ S \geq \frac{\mu_2 - \mu_3}{\beta},\ I \geq 0,\ S + I \leq 1\right\}$. The set Ω_1 is a compact positively invariant. We Consider on $\overset{\circ}{\Omega}_1$ the Lyapunov function defined by

$$V(S,I) = \left(S - \bar{S}\right) - \frac{\mu_3 + \gamma}{\beta}\log\frac{-\mu_2 + \mu_3 + \beta S}{-\mu_2 + \mu_3 + \beta\bar{S}} + \left(I - \bar{I}\right) - \bar{I}\log\frac{I}{\bar{I}}$$

It is easy to verifiy that V is definite positive, that is $V(S,I) \geq 0$ and $V(\bar{S},\bar{I}) = 0$ if and only if

$(S,I)=(\bar{S},\bar{I})$. His derivative along trajectories of (3) is given by:

$$\begin{aligned}
\dot{V}(S,I) &= \dot{S}-(\mu_3+\gamma)\frac{\mu_3+(\mu_2-\mu_3)I-\mu_3 S-\beta SI}{-\mu_2+\mu_3+\beta S}+ \\
&\qquad \beta SI-(\mu_2+\gamma)I-\bar{I}(\beta S-(\mu_2+\gamma)) \\
&= \dot{S}-(\mu_3+\gamma)\frac{(\mu_3-\mu_3 S)}{-\mu_2+\mu_3+\beta S}+(\mu_3+\gamma)I+ \\
&\qquad \beta SI-(\mu_2+\gamma)I-\bar{I}(\beta S-(\mu_2+\gamma)) \\
&= \mu_3(1-S)-(\mu_3+\gamma)\frac{(\mu_3-\mu_3 S)}{-\mu_2+\mu_3+\beta S}-\bar{I}(\beta S-(\mu_2+\gamma)) \\
&= \mu_3(1-S)\left[1-\frac{\mu_3+\gamma}{-\mu_2+\mu_3+\beta S}\right]-\bar{I}(\beta S-(\mu_2+\gamma)) \\
&= \mu_3(1-S)\left(\frac{-\beta\bar{S}+\beta S}{-\mu_2+\mu_3+\beta S}\right)-\frac{\mu_3}{\mu_3+\gamma}\left(1-\bar{S}\right)\left(\beta S-\beta\bar{S}\right) \\
&= -\mu_3\beta\left(\bar{S}-S\right)\left[\frac{1-S}{-\mu_2+\mu_3+\beta S}-\frac{1-\bar{S}}{\mu_3+\gamma}\right] \\
&= -\mu_3\beta\left(\bar{S}-S\right)\left[\frac{1-S}{-\mu_2+\mu_3+\beta S}-\frac{1-\bar{S}}{-\mu_2+\mu_3+\beta\bar{S}}\right] \\
&= -\frac{\beta\mu_3}{\mu_3+\gamma}\left[\frac{-\mu_2+\beta+\mu_3}{-\mu_2+\beta+\beta S}\right]\left(S-\bar{S}\right)^2 \\
&\leq 0
\end{aligned}$$

Then we conclude $\dot{V}$ is semi-definite positive. Then the endemic equilibrium is stable by Lyapunov theorems. We prove the attractivity of endemic equilibrium using Lasalle's principle.
The set on which $\dot{V}=0$ is given by $E=\left\{(S,I)\in\overset{\circ}{\Omega}_1 \;/\; S=\bar{S}\right\}$. Then on this set, we have $\dot{S}=\mu_3+(\mu_2-\mu_3)I-\mu_3 S-\beta\bar{S}I=0$, then $I=\dfrac{\mu_3-\mu_3\bar{S}}{\beta S-\mu_2+\mu_3}=\bar{I}$. Furthermore the largest invariant set contained in the set $\left\{(S,I)\in\overset{\circ}{\Omega}_1 \;/\; \dot{V}(S,I)=0\right\}$ is reduced to the endemic equilibrium. Hence $(\bar{S},\bar{I})$ is attractive. Then EE is GAS on $\overset{\circ}{\Omega}_1$.
If $S\leq\dfrac{\mu_2-\mu_3}{\beta}$, we have:

$$\begin{aligned}
\dot{S} &= \mu_3+(\mu_2-\mu_3)I-\mu_3 S-\beta SI \\
&= \mu_3(1-S)+(\mu_2-\mu_3-\beta S)I \\
&> 0
\end{aligned}$$

Then $\dot{S}>0$. Furthermore all trajectories in $\overset{\circ}{\Omega}\setminus\overset{\circ}{\Omega}_1$ enter in $\overset{\circ}{\Omega}_1$. Then the set $\overset{\circ}{\Omega}_1$ is absorbant. Hence the EE is GAS on $\overset{\circ}{\Omega}$.

In the boundary $S=0$ et $S+I=1$, the vector field is strictly pointing inside Ω. Only the S-axis is invariant. The endemic equilibrium is GAS on $\Omega\setminus\{(S,I):I=0;0\leq S\leq 1\}$. This end the proof.

4 SIRS Model

In this section, we consider a SIRS model with different mortality. With the same notation that above, we have the following system:

$$\begin{cases} \dot{S} = \Lambda - \beta \frac{S\,I}{N} - \mu_1\, S + \nu R \\ \dot{I} = \beta \frac{S\,I}{N} - \mu_2 I - \gamma\, I \\ \dot{R} = \gamma\, I - (\mu_3 + \nu)\, R \end{cases} \tag{4}$$

Which reduces to

$$\begin{cases} \dot{S} = -\beta \frac{S\,I}{N} + \mu_2\, I + (\mu_3 + \nu)\, R \\ \dot{I} = \beta \frac{S\,I}{N} - (\mu_2 + \gamma)\, I \\ \dot{R} = \gamma\, I - (\mu_3 + \nu)\, R \end{cases} \tag{5}$$

The system (5) is exactly as system (2) where μ_3 is replaced by $\mu_3 + \nu$.

5 Conclusion

In this contribution, we have proved the global stability of SIR and $SIRS$ models with differential mortality by Lyapunov methods. Our results encompass and improve the results of [10].

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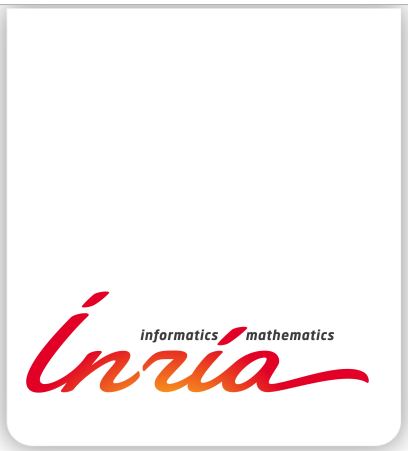

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