Set-Membership Estimation Improvement Applying HOSM Differentiators

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***Abstract*—This work is devoted to design of interval observers for a class of Linear-Parameter-Varying (LPV) systems. Applying High Order Sliding Mode (HOSM) techniques it is possible to decrease the initial level of uncertainty in the system, which leads to improvement of set-membership estimates generated by an interval observer. In addition, it is shown that HOSM techniques may relax the applicability conditions of the interval observer design methods. The efficiency of the proposed approach is demonstrated through computer simulations.**

I. Introduction

The problem of state estimation for nonlinear systems is very challenging and application important [4], [10], [18]. A complete palette of solutions exists for linear systems. In the nonlinear case, the most solutions are based on representation of the estimated system in a canonical form (frequently, partially linear), then particular approaches are available. In general case the LPV equivalent representation of nonlinear systems was found useful [15], [22], [25]. The basic idea is to replace the nonlinear complexity of the original system by an enlarged parametric variation in the LPV representation, which may simplify the observer design. There are several approaches to design observers for LPV systems [3], [11], [12], [17]. The present paper belongs to the framework of interval observers [3], [17]. That approach has been recently extended in [21] to nonlinear systems using LPV representations with known minorant and majorant matrices, and in [20] for observable nonlinear systems relaxing requirement on cooperativity (monotonicity) of the original system dynamics. The interval observers propagate the parameter uncertainty in the length of interval of the state estimation. The length of interval determines the estimation accuracy of the approach. This is why the uncertainty decreasing is very important for improvement of the interval (set-membership) estimation performance, which is the goal of the present work.

The HOSM techniques are very popular for design of observers for linear and nonlinear systems [1], [2], [5], [7], [23]. The sliding modes ensure a finite time of estimation error convergence to zero and complete insensitivity to a matched uncertainty [6], [16], [19]. Mainly these advances can be achieved under assumption that the systems is strongly observable or strongly detectable [2].

The objective of this work is to combine both approaches (the interval observers and the HOSM techniques) in order to improve accuracy of estimation achieved by the interval observers. Under a transformation of coordinates, an LPV system has a strongly observable subsystem. Applying HOSM differentiation approach it is possible to estimate the state and the state derivative for this subsystem, which can be further used for improved evaluation of the input and the parameter uncertainty in the rest part of the system. This combination improves the accuracy of the interval estimation. Additionally a relaxation of some applicability constraints usual for interval estimation can be obtained.

The paper is organized as follows. The system of interest, the basic facts from the theories of LPV systems, interval estimation and HOSM techniques are given in Section 2. The main result is described in Section 3. An example of computer simulation is presented in Section 4.

II. Preliminaries

Euclidean norm for a vector $x \in \mathbb{R}^n$ will be denoted as $|x|$, and for a measurable and locally essentially bounded input $u : \mathbb{R}_+ \to \mathbb{R}$ ($\mathbb{R}_+ = \{\tau \in \mathbb{R} : \tau \geq 0\}$) the symbol $||u||_{[t_0,t_1]}$ denotes its L_∞ norm:

$$||u||_{[t_0,t_1]} = ess\sup\{|u(t)|, t \in [t_0, t_1]\},$$

if $t_1 = +\infty$ then we will simply write $||u||$. Denote by $\mathcal{L}_\infty$ the set of all inputs u satisfying $||u|| < \infty$, and the sequence of integers $1, ..., k$ by $\overline{1,k}$.

In this work we consider the following LPV representation of a nonlinear system:

$$\dot{x} = A(\theta(t))x + B(\theta(t))u(t), \quad (1)$$
$$y = Cx,\ \psi(t) = y + v(t),$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, $y \in \mathbb{R}^p$, $v \in \mathbb{R}^p$ are the state, the input, the output and the measurement noise of the system (1), $\psi(t)$ is the signal available for on-line measurements; $\theta \in \Theta \subset \mathbb{R}^q$ is the scheduling parameter vector, the set Θ is known; the matrix functions $A : \Theta \to \mathbb{R}^{n\times n}$ and $B : \Theta \to \mathbb{R}^{n\times m}$ are given. The instant values of $u(t) \in \mathcal{L}_\infty$, $v(t) \in \mathcal{L}_\infty$ and $\theta(t) \in \mathcal{L}_\infty$ are not known. Almost all

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existent approaches assume that the vector θ is accessible for measurements, in the following this assumption is relaxed, and only the domain Θ is given.

Assumption 1. *$||x|| \leq X$, $||u|| \leq U$ and $||v|| \leq V$, the bounds $X > 0$, $U > 0$ and $V > 0$ are given.*

Boundedness of the state x and the inputs u, v is a standard assumption in the estimation theory. Under Assumption 1 the signal $\psi(t)$ is also bounded.

A. HOSM differentiation

Taking the s-th time differentiable output $y(t)$ of the system (1), its derivatives can be estimated by the HOSM differentiator [13], [14] based on the noisy measurements $\psi(t)$:

$$\begin{aligned} \dot{q}_0 = \nu_0,\ \nu_0 = -\lambda_0 |q_0 - \psi(t)|^{s/s+1}\mathrm{sign}[q_0 - \psi(t)] + q_1; \\ \dot{q}_i = \nu_i,\ i = \overline{1, s-1}, \\ \nu_i = -\lambda_i |q_i - \nu_{i-1}|^{s-i/s-i+1}\mathrm{sign}[q_i - \nu_{i-1}] + q_{i+1}; \\ \dot{q}_s = -\lambda_s \mathrm{sign}[q_s - \nu_{s-1}], \end{aligned} \tag{2}$$

where λ_k, $k = \overline{0, s}$ are positive parameters to be tuned.

Theorem 1. [14] *Let $y : \mathbb{R}_+ \to \mathbb{R}$ be s-th times continuously differentiable and $v(t) \in \mathcal{L}_\infty$ in (1), then there exist $0 \leq T < +\infty$ and some constants $\mu_k > 0$, $k = \overline{0, s}$ (dependent on λ_k, $k = \overline{0, s}$ only) such that in (2) for all $t \geq T$:*

$$|q_k(t) - y^{(k)}(t)| \leq \mu_k ||v||^{\frac{s-k+1}{s+1}},\ k = \overline{0, s}.$$

In particular, this result means that if $v(t) \equiv 0$ for all $t \geq 0$, then the differentiator (2) ensures the exact estimation of derivatives in a finite time. Application of HOSM differentiators for unknown input estimation and compensation in linear systems has been studied in [2], an extension to nonlinear systems is presented in [9].

B. Interval estimation

For two vectors $x_1, x_2 \in \mathbb{R}^n$ or matrices $A_1, A_2 \in \mathbb{R}^{n\times n}$, the relations $x_1 \leq x_2$ and $A_1 \leq A_2$ are understood elementwise. Given a matrix $A \in \mathbb{R}^{m\times n}$ or a vector $x \in \mathbb{R}^n$, define $A^+ = \max\{0, A\}$, $A^- = A^+ - A$ or $x^+ = \max\{0, x\}$, $x^- = x^+ - x$ respectively.

Lemma 1. *Let $x \in \mathbb{R}^n$ be a vector variable, $\underline{x} \leq x \leq \overline{x}$ for some $\underline{x}, \overline{x} \in \mathbb{R}^n$.*

1) If $A \in \mathbb{R}^{m\times n}$ is a constant matrix, then

$$A^+\underline{x} - A^-\overline{x} \leq Ax \leq A^+\overline{x} - A^-\underline{x}. \tag{3}$$

2) If $A \in \mathbb{R}^{m\times n}$ is a matrix variable, $\underline{A} \leq A \leq \overline{A}$ for some $\underline{A}, \overline{A} \in \mathbb{R}^{m\times n}$, then

$$\begin{aligned} \underline{A}^+\underline{x}^+ - \overline{A}^+\underline{x}^- - \underline{A}^-\overline{x}^+ + \overline{A}^-\overline{x}^- \leq Ax \leq \\ \overline{A}^+\overline{x}^+ - \underline{A}^+\overline{x}^- - \overline{A}^-\underline{x}^+ + \underline{A}^-\underline{x}^-. \end{aligned} \tag{4}$$

A matrix $A \in \mathbb{R}^{n\times n}$ is called Hurwitz if all its eigenvalues have negative real parts, it is called Metzler if all its elements outside the main diagonal are not negative. Any solution of the linear system

$$\dot{x} = Ax + \omega(t),\ \omega : \mathbb{R}_+ \to \mathbb{R}^n_+,$$

with $x \in \mathbb{R}^n$ and a Metzler matrix A, is elementwise nonnegative for all $t \geq 0$ provided that $x(0) \geq 0$ [24]. Such dynamical systems are called cooperative (monotone) [24].

III. Main result

For brevity of presentation the case $p = 1$ is considered only (the case of vector measurements can be treated similarly). We will need the following assumptions.

Assumption 2. *For all $\theta \in \Theta$, there is an invertible matrix $S(\theta) \in \mathbb{R}^{n\times n}$ such that the system (1) can be represented as follows:*

$$x = S(\theta)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix},\ y = c^T z_1,$$

$$\dim\{z_1\} = n_1,\ \dim\{z_2\} = n_2,\ n_1 + n_2 = n,$$

$$\dot{z}_1 = A_0 z_1 + b_0[a_{11}(\theta)^T z_1 + a_{12}(\theta)^T z_2 + b_1(\theta)^T u], \tag{5}$$

$$\dot{z}_2 = A_{21}(\theta) z_1 + A_{22}(\theta) z_2 + B_2(\theta) u, \tag{6}$$

where

$$c = [1\ 0 ... 0]^T,\ b_0 = [0 ... 0\ 1]^T,$$

$$A_0 = \begin{bmatrix} 0 & 1...0 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 0...0 & 1 \\ 0 & 0...0 & 0 \end{bmatrix}$$

is a canonical representation, the vector functions $a_{11}(\theta)$, $a_{12}(\theta)$, $b_1(\theta)$ and the matrix functions $A_{21}(\theta)$, $A_{22}(\theta)$, $B_2(\theta)$ have corresponding dimensions.

It is worth to stress that for $n_1 = 1$ this assumption is always true (at least the output coordinate can be chosen in the vector z_1).

Assumption 3. *Let there exist a vector function $f(\theta) \in \mathbb{R}^{n_2}$ such that*

$$\begin{aligned} [A_{22}(\theta) z_2 + B_2(\theta) u] - f(\theta)[a_{12}(\theta)^T z_2 \\ + b_1(\theta)^T u] = \Delta_1 z_2 + \Delta_2(\theta) u \end{aligned}$$

for some Hurwitz matrix $\Delta_1 \in \mathbb{R}^{n_2\times n_2}$ and $\Delta_2 : \Theta \to \mathbb{R}^{n_2\times m}$.

Assumption 4. *There exists a matrix $P \in \mathbb{R}^{n_2\times n_2}$ such that the matrix $D = P^{-1}\Delta_1 P$ is Hurwitz and Metzler (H&M).*

Assumption 2 states that there exists a transformation coordinates, which represents the system (1) as a pair of interconnected subsystems (5) and (6). The subsystem (5) is strongly observable since it has the canonical representation c, A_0, b_0 (the conditions of existence of such a transformation for linear time-invariant systems are analyzed in [2]). However, the system is not necessarily detectable (the dynamics of (1) could be non-minimum phase as in [23]) since there is no requirement on stability of the matrix function $A_{22}(\theta)$. This relaxation may be important for application of interval observer design method for estimation in uncertain non-minimum phase systems. Instead, Assumption 3 states that the matrix $\Delta_1 = A_{22}(\theta) - f(\theta)a_{12}(\theta)^{\mathrm{T}}$ is Hurwitz (the matrix $A_{22}(\theta)$ can be stabilized by an output feedback, or the pair of matrices $(A_{22}(\theta), a_{12}(\theta)^{\mathrm{T}})$ is observable for all $\theta \in \Theta$) and independent in θ. Under mild conditions of the main result in [20], in this case there is a matrix $P \in \mathbb{R}^{n_2 \times n_2}$ such that D is H&M, as it is stated in Assumption 4.

Under these assumptions it is proposed to use the differentiator (2) to estimate the state z_1 and its derivative $\dot{z}_1$, then from (5) we get an improved estimate on the signal $a_{12}(\theta)^{\mathrm{T}} z_2 + b_1(\theta)^{\mathrm{T}} u$, which can be applied for design of an interval observer for the system (6) in the new coordinates $r = P^{-1} z_2$. Let us consider these steps consequently.

Under Assumption 2 the output y of the system (5) has n_1 derivatives. Therefore according to Theorem 1 and Assumption 1, there exist parameters λ_k, $k = \overline{0, n_1}$ in (2) with $s = n_1$ and $T > 0$ such that for all $t \geq T$:

$$|q_k(t) - y^{(k)}(t)| \leq \mu_k V^{\frac{n_1 - k + 1}{n_1 + 1}},\ k = \overline{0, n_1}$$

for some constant μ_k, $k = \overline{0, n_1}$. Thus $z_1(t) = \hat{z}_1(t) + e_1(t)$ and $\dot{z}_{1,n_1}(t) = q_{n_1}(t) + e_2(t)$ for all $t \geq T$, where $\hat{z}_{1,i}(t) = q_{i-1}(t)$ and $|e_{1,i}(t)| \leq \mu_{i-1} V^{\frac{n_1 - i + 2}{n_1 + 1}}$ for $i = \overline{1, n_1}$, $|e_2(t)| \leq \mu_{n_1} V^{\frac{1}{n_1 + 1}}$. The variables $\hat{z}_1$ and q_{n_1} are available for a designer, the errors e_1 and e_2 are upper bounded by some functions of V. Substitution of these variables into the last equation of (5) gives:

$$q_{n_1} + e_2 = a_{11}(\theta)^{\mathrm{T}}[\hat{z}_1 + e_1] + a_{12}(\theta)^{\mathrm{T}} z_2 + b_1(\theta)^{\mathrm{T}} u,$$

or equivalently

$$a_{12}(\theta)^{\mathrm{T}} z_2 + b_1(\theta)^{\mathrm{T}} u = q_{n_1} + e_2 - a_{11}(\theta)^{\mathrm{T}}[\hat{z}_1 + e_1].$$

Substituting this equality in the differential equation (6) we obtain

$$\begin{aligned} \dot{z}_2 \ &= \ \Delta_1 z_2 + [A_{21}(\theta) - f(\theta)a_{11}(\theta)^{\mathrm{T}}](\hat{z}_1 + e_1) + \\ & \quad\ f(\theta)(q_{n_1} + e_2) + \Delta_2(\theta)u, \end{aligned} \tag{7}$$

which is a stable system according to Assumption 3.

Applying the transformation of coordinates $r = P^{-1} z_2$, the system (7) can be rewritten as follows

$$\begin{aligned} \dot{r} \ &= \ Dr + G_1(\theta)(\hat{z}_1 + e_1) + \\ & \quad\ G_2(\theta)(q_{n_1} + e_2) + G_3(\theta)u, \end{aligned} \tag{8}$$

where $G_1(\theta) = P^{-1}[A_{21}(\theta) - f(\theta)a_{11}(\theta)^{\mathrm{T}}]$, $G_2(\theta) = P^{-1} f(\theta)$ and $G_3(\theta) = P^{-1}\Delta_2(\theta)$. The dynamics of (8) is cooperative and stable, and all uncertain functions or variables in the right hand side of (8) belong to an interval for $\theta \in \Theta$:

$$\underline{G_j} \leq G_j(\theta) \leq \overline{G_j},\ j = \overline{1, 3};\ |u(t)| \leq U;$$
$$|e_{1,i}(t)| \leq \overline{e}_{1,i} = \mu_{i-1} V^{\frac{n_1 - i + 2}{n_1 + 1}},\ i = \overline{1, n_1};$$
$$|e_2(t)| \leq \overline{e}_2 = \mu_{n_1} V^{\frac{1}{n_1 + 1}}$$

for all $t \geq T$, where the matrices $\underline{G_j}, \overline{G_j}$, $j = \overline{1, 3}$ are known. Therefore the following interval observer can be synthesized for (8):

$$\begin{aligned} \dot{\overline{r}} \ &= \ D\overline{r} + (\overline{G_1}^{+} - \overline{G_1}^{-})\hat{z}_1^{+} + (\underline{G_1}^{-} - \underline{G_1}^{+})\hat{z}_1^{-} + \\ & \quad\ (\overline{G_1}^{+} + \underline{G_1}^{-})\overline{e}_1 + (\overline{G_2}^{+} - \overline{G_2}^{-})q_{n_1}^{+} + \\ & \quad\ (\underline{G_2}^{-} - \underline{G_2}^{+})q_{n_1}^{-} + (\overline{G_2}^{+} + \underline{G_2}^{-})\overline{e}_2 + \\ & \quad\ (\overline{G_3}^{+} + \underline{G_3}^{-})U, \end{aligned} \tag{9}$$

$$\begin{aligned} \dot{\underline{r}} \ &= \ D\underline{r} + (\underline{G_1}^{+} - \underline{G_1}^{-})\hat{z}_1^{+} + (\overline{G_1}^{-} - \overline{G_1}^{+})\hat{z}_1^{-} - \\ & \quad\ (\overline{G_1}^{+} + \underline{G_1}^{-})\overline{e}_1 + (\underline{G_2}^{+} - \underline{G_2}^{-})q_{n_1}^{+} + \\ & \quad\ (\overline{G_2}^{-} - \overline{G_2}^{+})q_{n_1}^{-} - (\overline{G_2}^{+} + \underline{G_2}^{-})\overline{e}_2 - \\ & \quad\ (\overline{G_3}^{+} + \underline{G_3}^{-})U, \end{aligned} \tag{10}$$

the properties (3), (4) have been used to calculate (9), (10). Introducing the interval estimation errors $\overline{\epsilon} = \overline{r} - r$, $\underline{\epsilon} = r - \underline{r}$, we obtain

$$\dot{\overline{\epsilon}} = D\overline{\epsilon} + \overline{\varepsilon},\ \dot{\underline{\epsilon}} = D\underline{\epsilon} + \underline{\varepsilon},$$

where $\overline{\varepsilon} = (\overline{G_1}^{+} - \overline{G_1}^{-})\hat{z}_1^{+} + (\underline{G_1}^{-} - \underline{G_1}^{+})\hat{z}_1^{-} + (\overline{G_1}^{+} + \underline{G_1}^{-})\overline{e}_1 + (\overline{G_2}^{+} - \overline{G_2}^{-})q_{n_1}^{+} + (\underline{G_2}^{-} - \underline{G_2}^{+})q_{n_1}^{-} + (\overline{G_2}^{+} + \underline{G_2}^{-})\overline{e}_2 + (\overline{G_3}^{+} + \underline{G_3}^{-})U - G_1(\theta)(\hat{z}_1 + e_1) - G_2(q_{n_1} + e_2) - G_3(\theta)u$, $\underline{\varepsilon} = G_1(\theta)(\hat{z}_1 + e_1) + G_2(q_{n_1} + e_2) + G_3(\theta)u - (\underline{G_1}^{+} - \underline{G_1}^{-})\hat{z}_1^{+} - (\overline{G_1}^{-} - \overline{G_1}^{+})\hat{z}_1^{-} + (\overline{G_1}^{+} + \underline{G_1}^{-})\overline{e}_1 + (\underline{G_2}^{+} - \underline{G_2}^{-})q_{n_1}^{+} + (\overline{G_2}^{-} - \overline{G_2}^{+})q_{n_1}^{-} - (\overline{G_2}^{+} + \underline{G_2}^{-})\overline{e}_2 + (\overline{G_3}^{+} + \underline{G_3}^{-})U$. It is an arithmetic exercise to verify that under assumptions 1 and 2 (and the result of Theorem 1) the residual terms $\overline{\varepsilon}$ and $\underline{\varepsilon}$ are elementwise positive and bounded. Then using the results of monotone system theory [24] we prove that for all $t \geq T$

$$\underline{r}(t) \leq r(t) \leq \overline{r}(t)$$

and the estimates $\underline{r}(t)$, $\overline{r}(t)$ are bounded, provided that

$$\underline{r}(T) \leq r(T) \leq \overline{r}(T). \tag{11}$$

The former relation for the initial conditions can be easily satisfied since $||x|| \leq X$ under Assumption 1. Using the property (3) we get for all $t \geq T$:

$$\underline{z}_2(t) \leq z_2(t) = Pr(t) \leq \overline{z}_2(t),$$
$$\underline{z}_2(t) = P^{+}\underline{r}(t) - P^{-}\overline{r}(t),\ \overline{z}_2(t) = P^{+}\overline{r}(t) - P^{-}\underline{r}(t);$$
$$\underline{z}_1(t) \leq z_1(t) \leq \overline{z}_1(t),$$
$$\underline{z}_1(t) = \hat{z}_1(t) - \overline{e}_1,\ \overline{z}_1(t) = \hat{z}_1(t) + \overline{e}_1.$$

Defining $\underline{z} = [\underline{z}_1^{\mathrm{T}}\ \underline{z}_2^{\mathrm{T}}]^{\mathrm{T}}$, $\overline{z} = [\overline{z}_1^{\mathrm{T}}\ \overline{z}_2^{\mathrm{T}}]^{\mathrm{T}}$and using (4) we finally formulate the interval estimates for the state x:

$$\begin{aligned} & \underline{S}^{+}\underline{z}^{+} - \overline{S}^{+}\underline{z}^{-} - \underline{S}^{-}\overline{z}^{+} + \overline{S}^{-}\overline{z}^{-} \leq x = S(\theta)z \leq \\ & \overline{S}^{+}\overline{z}^{+} - \underline{S}^{+}\overline{z}^{-} - \overline{S}^{-}\underline{z}^{+} + \underline{S}^{-}\underline{z}^{-}, \end{aligned} \tag{12}$$

which is satisfied for all $t \geq T$. Thus we have the following result.

Theorem 2. *Let assumptions 1, 2, 3, 4 hold for the system (1). Then there exist the set of parameters* λ_k, $k = \overline{0, n_1}$ *in (2) and a constant* $T > 0$ *such that for all* $t \geq T$ *the interval estimate (12) is true, provided that the condition (11) is satisfied for (9), (10).*

Remark 1. The assumptions 3 and 4 can be replaced with the following one: there exists a vector function $f(\theta) \in \mathbb{R}^{n_2}$ such that

$$[A_{22}(\theta)z_2 + B_2(\theta)u] - f(\theta)[a_{12}(\theta)^{\mathrm{T}} z_2 + b_1(\theta)^{\mathrm{T}} u] = \Delta_1(\theta)z_2 + \Delta_2(\theta)u$$

for some Hurwitz and Metzler matrix function $\Delta_1 : \Theta \to \mathbb{R}^{n_2 \times n_2}$ and some $\Delta_2 : \Theta \to \mathbb{R}^{n_2 \times m}$. Next, the result of Theorem 2 can be obtained using the same technique and an interval observer from the paper [21].

IV. Example

To illustrate improvement of accuracy achieved in interval estimation by application of HOSM techniques consider a non-minimum phase system (the conventional techniques for the interval observer design [3], [17], [20] can not be applied in this case):

$$\begin{aligned} \dot{x}_1 &= -a_{11}(\theta)x_1 + a_{12}(\theta)x_2 + b_1(\theta)u; \\ \dot{x}_2 &= a_{21}(\theta)x_1 + a_{22}(\theta)x_2 + b_2(\theta)u; \\ y &= x_1, \end{aligned} \tag{13}$$

where $x_1 \in \mathbb{R}$, $x_2 \in \mathbb{R}$ are the state variables, for all $\theta \in \Theta$

$$0.5 \leq a_{11}(\theta) \leq 1, \; -3 \leq a_{12}(\theta) \leq -1,$$
$$-0.5 \leq a_{21}(\theta) \leq 0.5, \; -0.5 \leq a_{22}(\theta) = 0.5a_{12}(\theta) + 1 \leq 0.5,$$
$$0.5 \leq b_1(\theta) \leq 1, \; 0.5 \leq b_2(\theta) \leq 1, \; U = 1, \; V = 0.1.$$

As we can see, the system (13) is already in the form (5), (6) with $x_1 = z_1$ and $x_2 = z_2$ (the matrix $S(\theta)$ equals to the identity, and Assumption 2 is satisfied). For simulation we use

$$a_{11}(\theta) = 0.75 + 0.25\sin(x_2 t), \; a_{12}(\theta) = -2 + \sin(x_1 t),$$
$$a_{21}(\theta) = 0.5\sin(t), \; b_1(\theta) = 0.5, \; b_2(\theta) = 0.75 + 0.25\cos(0.5t),$$
$$u(t) = U\sin(2t), \; v(t) = V\sin(10t),$$
$$\theta = [x_1 \; x_2 \; t]^{\mathrm{T}}.$$

For the system (13) with the chosen parameters and the given input u the state is bounded as follows $-3 \leq x_2 \leq 1$ (Assumption 1 holds). It is easy to verify that for $f = 0.5$ we have

$$[a_{22}(\theta)z_2 + b_2(\theta)u] - f[a_{12}(\theta)z_2 + b_1(\theta)u] = \Delta_1 z_2 + \Delta_2(\theta)u$$

for $\Delta_1 = -1$ and $0 \leq \Delta_2(\theta) \leq 1.5$ (Assumption 3 is satisfied). Since $\Delta_1 < 0$ Assumption 4 is true with the matrix P equals the identity.

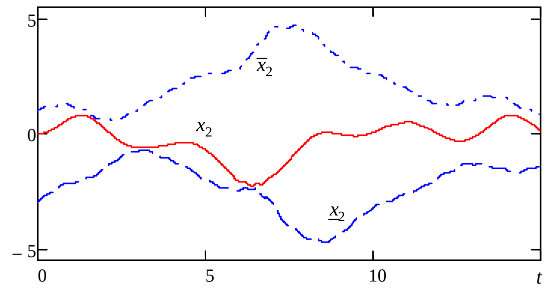

Figure 1. The results of simulation for non-minimum phase example

Therefore, according to Theorem 2 we may use the differentiator (2) to estimate x_1 and $\dot{x}_1$, that for $s = 2$ can be reduced to the conventional super-twisting differentiator [13]:

$$\begin{aligned} \dot{q}_0 &= -\lambda_0\sqrt{|q_0 - \psi(t)|}\mathrm{sign}[q_0 - \psi(t)] + q_1; \\ \dot{q}_1 &= -\lambda_1 \mathrm{sign}[q_0 - \psi(t)], \end{aligned} \tag{14}$$

where in our example $\lambda_0 = 20$, $\lambda_1 = 50$, $\hat{z}_1 = q_0$ and $\overline{e}_1 = |V|$, $\overline{e}_2 = 1.1\sqrt{|V|}$. In this case the finite time $T = 0.1$. Next, the interval observer (9), (10) generates the required set-membership estimates for the variable $x_2 = r$ ($-1 \leq G_1(\theta) = P^{-1}[a_{21}(\theta) - f a_{11}(\theta)] \leq 1$, $G_2 = 0.5$ and $G_3(\theta) = \Delta_2(\theta)$):

$$\dot{\overline{r}} = -\overline{r} + \hat{z}_1 + 2\overline{e}_1 + 0.5q_1 + 0.5\overline{e}_2 + 1.5U,$$
$$\dot{\underline{r}} = -\underline{r} - \hat{z}_1 - 2\overline{e}_1 + 0.5q_1 - 0.5\overline{e}_2 - 1.5U.$$

The results of this interval estimation are shown in Fig 1. It is worth to note that for the best knowledge of the authors, other existent approaches can not solve the problem of interval estimation for (13). In particular, application of a conventional interval observer design method [17], [21] is blocked by the non-minimum phase condition ($-0.5 \leq a_{22}(\theta) \leq 0.5$). A further application of the sliding-mode estimation approaches [6], [16], [19] is blocked by the uncertainty presented in θ.

V. Conclusion

The paper is devoted to application of the interval observers and the HOSM differentiation to LPV system estimation. The HOSM techniques allow us to improve the estimation accuracy of an interval observer designed for LPV systems, or enlarge the class of LPV systems having an interval observer. The efficiency is shown on a uncertain and non-minimum phase example by computer simulations.

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