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High order asymptotics for the electromagnetic scattering from thin periodic layers : the 3D Maxwell case

Bérangère Delourme

January 3, 2014

Abstract

This work deals with the scattering of electromagnetic waves by a thin periodic layer made of an array of regularly-spaced obstacles. The size of the obstacles and the spacing between two consecutive obstacles are of the same order δ, which is much smaller than the wavelength of the incident wave. We provide a complete description of the asymptotic behavior of the solution with respect to the small parameter δ: we use a method that mixes matched asymptotic expansions and homogenization techniques. We pay particular attention to the construction of the near field terms. Indeed, they satisfy electrostatic problems posed in an infinite 3D strip that require a careful analysis. Error estimates are carried out to justify the accuracy of our expansion.

1 Introduction

This work is dedicated to the study of an asymptotic model associated with electromagnetic waves scattering from a planar, thin, and periodic layer. This layer is made of an array of regularly spaced dielectric obstacles. In the situation we are interested in, the thickness of the layer and the distance between two consecutive obstacles are of the same order δ, which is much smaller than the wavelength of the incident wave. It is clear that direct numerical computations of such a problem become prohibitively expensive as the small parameter δ tends to 0. To overcome this kind of difficulty, approximate models (where the thin periodic layer is replaced by an approximate transmission condition) are derived.The numerical discretization of this approximate model is much less expensive than the exact one, since the mesh is no longer constrained by the small scale. One usual way to derive these approximate models is to construct (in a preliminary step) an asymptotic expansion of the solution of the exact problem with respect to the small parameter δ.

In this paper, we shall restrict ourselves to the construction of such an asymptotic expansion with respect to δ, the construction and analysis of an approximate model can be found in [1]. In our problem, the equispaced obstacles give rise to a boundary layer phenomenon: indeed, the solution oscillates more rapidly in the vicinity of the thin layer than far from it. Consequently, to build an asymptotic expansion, we distinguish different areas where the expansions are different. For that purpose, we shall employ a method that mixes the techniques of periodic-homogenization [2, 3] and the so-called matched asymptotic expansions. The latter method has been developed in [4] to treat singular perturbation problems which arise in fluid mechanics. A standard work on the matched asymptotic expansions applied to the Helmholtz equation can be found in [5, 6] and complex situations are studied in [7],[8], [9] and [10]. Note also that asymptotics associated with rough boundaries or thin periodic layers have been widely studied. For instance the two first terms of the expansion associated with the case of electromagnetic scattering problems from perfect conductors coated with periodic thin structures are derived in [11, 12] and [13] for planar geometry (Maxwell Equation). Their results have been extended by [14] and [15] for the Helmholtz equation in circular and smooth geometries. High order expansions have been derived in [16] and [17], [18] for the Laplace problem and in [19, 20] for the Helmholtz equation. A complete expansion for dielectric thin periodic layers for the Laplace equation can be found in [21]. The case of high order expansion to model highly conductive thin sheets is treated in [22]. Finally, a complete asymptotic asymptotic for the cases of 3D Maxwell strongly conducting obstacle and conductive sheets are carried out in [23] and [24], The goal of this work is to complement the work mentioned above by constructing and justifying an asymptotic expansion at any order for the thin periodic dielectric layer case.

The remainder of this article is organized as follows. In Section 2, we describe the scattering problem we are interested in. In particular, we prove a uniform (with respect to δ) stability result. Section 3 is dedicated to the formal construction of a matched asymptotic expansion of the solution. In Section 4 and 5, we set appropriate mathematical frameworks for the resolution of near and far field problems. Near field problems are electrostatic problems posed in the unbounded periodicity cell. These are solved using an augmented variational form with the help of a Friedrichs' Inequality. Then, existence and uniqueness of the terms of asymptotic expansion are proved in Section 6. Finally, Section 7 is dedicated to the convergence of the asymptotic expansion. In addition, some technical results are given in Appendix.

2 Setting of the problem

2.1 The scattering problem

In this paper, we are interested in the electromagnetic fields $\mathbf{E}^\delta$ and $\mathbf{H}^\delta$ solutions of the Maxwell's Equations

$$\begin{cases} \operatorname{curl} \mathbf{E}^\delta - i\omega\mu^\delta \mathbf{H}^\delta = 0 \text{ in } \Omega, \\ -\operatorname{curl} \mathbf{H}^\delta - i\omega\epsilon^\delta \mathbf{E}^\delta = -\dfrac{1}{i\omega} f \text{ in } \Omega. \end{cases} \tag{1}$$

where ω denotes the pulsation of time variations, μ^δ and ε^δ are the permeability and the permittivity of the medium and f is a given source term. The cubic domain Ω (see Figure 1) is defined by

$$\Omega := \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3, -\frac{L_1}{2} < x_1 < \frac{L_1}{2}, \quad -\frac{L_2}{2} < x_2 < \frac{L_2}{2}, \quad -\frac{L_3}{2} < x_3 < \frac{L_3}{2} \right\}, \tag{2}$$

where L_1 and L_2 and L_3 are the lengths of the edges of the cube (L_1, L_2 and L_3 are positive numbers). For $i = 1, 2, 3$, $\Sigma_i^\pm$ denoted the associated faces of the cube, that is to say

$$\Sigma_i^{\pm} := \left\{ (x_1, x_2, x_3) \in \partial\Omega, \pm x_i = \pm \frac{L_i}{2} \right\}; \tag{3}$$

Equations (1) are completed with periodic boundary conditions on the lateral boundaries $\Sigma_1^{\pm}$ and $\Sigma_2^{\pm}$

$$\begin{cases} \mathbf{H}^\delta \times e_{1|\Sigma_1^-} = \mathbf{H}^\delta \times e_{1|\Sigma_1^+}, & \mathbf{H}^\delta \times e_{2|\Sigma_2^-} = \mathbf{H}^\delta \times e_{2|\Sigma_2^-}, \\ \mathbf{E}^\delta \times e_{1|\Sigma_1^-} = \mathbf{E}^\delta \times e_{1|\Sigma_1^+}, & \mathbf{E}^\delta \times e_{2|\Sigma_2^-} = \mathbf{E}^\delta \times e_{2|\Sigma_2^-}, \end{cases} \tag{4}$$

together with an impedance condition on the lower and upper boundaries $\Sigma_3^{\pm}$,

$$\mathbf{H}^\delta \times n - \mathbf{E}_T^\delta = 0 \quad \text{on} \quad \Sigma_3^{\pm}. \tag{5}$$

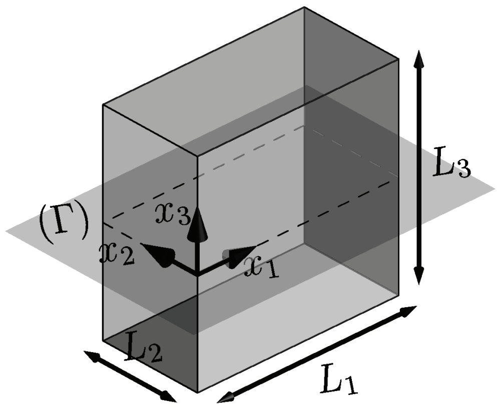

Figure 1: Domain Ω

In this work, the medium Ω is made of a thin periodic layer Ω_B^δ of thickness δ, namely

$$\Omega_B^\delta := \{(x_1, x_2, x_3) \in \Omega, |x_3| \leq \delta/2\},$$

embedded into an homogeneous medium $(\mu_\infty, \varepsilon_\infty)$. The parameter δ is a small geometrical parameter that can be arbitrarily close to 0. The thin layer consists of dielectric obstacles, regularly spaced in the directions x_1 and x_2 (see Figure 2), i.e. μ^δ and ε^δ are periodic with respect to the variables x_1 and x_2). The size of the obstacles and the spacing between two consecutive ones are proportional to the small parameter δ. More precisely, we assume that there exist two functions $\mu : \mathbb{R}^3 \to \mathbb{R}^+$ and $\varepsilon : \mathbb{R}^3 \to \mathbb{R}^+$ of the scaled (or fast) variables $(X_1, X_2, X_3) \in \mathbb{R}^3$ satisfying

$$\begin{cases} \mu(X_1 + 1, X_2, X_3) = \mu(X_1, X_2, X_3), \\ \mu(X_1, X_2 + \tau, X_3) = \mu(X_1, X_2, X_3), \\ \mu(X_1, X_2, X_3) = 1 \text{ if } |X_3| > \frac{1}{2}, \end{cases} \qquad \begin{cases} \epsilon(X_1 + 1, X_2, X_3) = \epsilon(X_1, X_2, X_3), \\ \epsilon(X_1, X_2 + \tau, X_3) = \epsilon(X_1, X_2, X_3), \\ \epsilon(X_1, X_2, X_3) = 1 \text{ if } |X_3| > \frac{1}{2}. \end{cases} \tag{6}$$

and such that, for any $(x_1, x_2, x_3) \in \Omega$,

$$\mu^\delta(x_1, x_2, x_3) = \mu(\frac{x_1}{\delta}, \frac{x_2}{\delta}, \frac{x_3}{\delta}), \qquad \epsilon^\delta(x_1, x_2, x_3) = \epsilon(\frac{x_1}{\delta}, \frac{x_2}{\delta}, \frac{x_3}{\delta}), \tag{7}$$

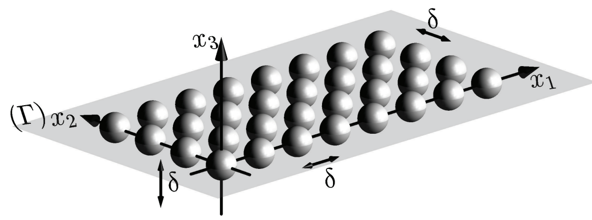

Figure 2: the periodic thin layer Ω_B^δ

As usual, we further assume that the permittivity and the permeability are bounded as follows:

$$0 < \epsilon^- < \epsilon < \epsilon^+ \quad \text{and} \quad 0 < \mu^- < \mu < \mu^+.$$

Here and in what follows, we consider the source term $f \in L^2(\Omega)^3$, and, for the sake of simplicity, we assume that its support does not intersect the thin layer Ω_B^δ. We shall denote by Γ the limit interface, that is the intersection of Ω and the plane of equation $x_3 = 0$:

$$\Gamma := \left\{ (x_1, x_1, x_3) \in]-\frac{L_1}{2}\frac{L_1}{2}[\times]-\frac{L_2}{2}\frac{L_2}{2}[\times\{0\} \right\}. \tag{8}$$

Remark 2.1 *Note that periodic boundary conditions* (4) *only make sense under the assumption that L_2 is a multiple of τL_1. Besides, periodic boundary conditions shall appreciably simplify the construction of the asymptotic expansion; indeed, unlike homogeneous Dirichlet or Neumann boundary conditions, they do not introduce boundary layers in the neighborhood of the lateral boundaries (which would make the asymptotic expansion much more involved).*

2.2 Mathematical investigation of the problem

To analyze the previous problem, we eliminate the magnetic field $\mathbf{H}^\delta$ ($\mathbf{H}^\delta = \operatorname{curl} \mathbf{E}^\delta / i\omega\mu^\delta$); we rewrite Maxwell's Equations (1) as a system of second order equations: the electric field $\mathbf{E}^\delta$ then satisfies

$$\operatorname{curl} \frac{1}{\mu^\delta} \operatorname{curl} \mathbf{E}^\delta - \omega^2 \epsilon^\delta \mathbf{E}^\delta = f \text{ in } \mathcal{D}'(\Omega), \tag{9}$$

together with periodic boundary conditions on the lateral boundaries

$$\frac{1}{\mu^\delta} \operatorname{curl} \mathbf{E}^\delta \times e_{i|\Sigma_i^+} = \frac{1}{\mu^\delta} \operatorname{curl} \mathbf{E}^\delta \times e_{i|\Sigma_i^-}, \qquad \mathbf{E}^\delta \times e_{i|\Sigma_i^+} = \mathbf{E}^\delta \times e_{i|\Sigma_i^-}, \quad i \in \{1, 2\}, \tag{10}$$

and an impedance condition on the lower and upper boundaries

$$\operatorname{curl} \mathbf{E}^\delta \times n - i\omega (\mathbf{E}^\delta)_T = 0 \quad \text{on} \quad \Sigma_3^\pm. \tag{11}$$

As usual (see for instance [25]), it is natural to find $\mathbf{E}^\delta$ in

$$V := \Big\{\varphi \in H(\mathrm{curl}\,,\Omega),\ (\varphi_T \times e_i)_{|\Sigma_i^+} = (\varphi_T \times e_i)_{|\Sigma_i^-} \text{ for } i \in \{1,2\}, \varphi_T \in L^2(\Sigma_3^\pm)\Big\}, \tag{12}$$

When equipped with the following (δ-dependent) dot product,

$$(\varphi,\psi) \mapsto \int_\Omega \big(\mathrm{curl}\,\varphi \cdot \overline{\mathrm{curl}\,\psi} + \epsilon^\delta \varphi \cdot \overline{\psi}\big)\ dx\ +\ \int_{\Sigma_3^\pm} \varphi_T \cdot \overline{\psi_T}\, ds, \tag{13}$$

V is an Hilbert space. We denote by $\|\cdot\|_{V_{\epsilon^\delta}}$ its associated norm,

$$\varphi \mapsto \|\varphi\|^2_{V_{\epsilon^\delta}} := \int_\Omega \big(|\mathrm{curl}\,\varphi|^2 + \epsilon^\delta \varphi \cdot \overline{\varphi}\big)\ dx\ +\ \int_{\Sigma_3^\pm} |\varphi_T|^2 ds. \tag{14}$$

Problem (9)-(10)-(11) is equivalent to the following variational problem: find $\mathbf{E}^\delta \in V$ such that

$$\forall \varphi \in V, \quad a^\delta(\mathbf{E}^\delta, \varphi) = \int_\Omega f \cdot \overline{\varphi}\, dx, \tag{15}$$

where

$$a^\delta(\psi,\varphi) = \int_\Omega (\frac{1}{\mu^\delta}\mathrm{curl}\,\psi \cdot \overline{\mathrm{curl}\,\varphi} - \omega^2 \epsilon^\delta \psi \cdot \overline{\varphi})\, dx - i\omega \int_{\Sigma_3^\pm} \psi_T \cdot \overline{\varphi_T}\, ds. \tag{16}$$

We can prove the following result:

Proposition 2.2 *Problem (15) is well-posed. Moreover, there exist $\delta_0 > 0$ and a positive constant C such that, for any $\delta < \delta_0$ and for any $\psi \in V$, the following continuity estimate holds:*

$$\|\psi\|_{V_{\epsilon^\delta}} \le C \sup_{\varphi \in V} \frac{a^\delta(\psi,\varphi)}{\|\varphi\|_{V_{\epsilon^\delta}}} \tag{17}$$

The proof of well-posedness for a fixed δ is well known (Theorems 4.7 and 4.12 in [25]) ; we focus on the proof of the uniform stability estimate (17). To do so, we start by writing an Helmholtz decomposition of V. Let

$$S := \Big\{p \in H^1(\Omega), \quad p \text{ is constant on } \Sigma_3^+,\ p = 0 \text{ on } \Sigma_3^-,\ p_{|\Sigma_i^+} - p_{|\Sigma_i^-} \in \mathbb{P}_0, \forall\, i \in \{1,2\}\ \Big\}. \tag{18}$$

∇S is a close subspace of V, thus the following Helmholtz decomposition holds

$$V = \nabla S \oplus V_0^{\epsilon^\delta}, \tag{19}$$

where $V_0^{\epsilon^\delta}$ is the orthogonal of ∇S with respect to the δ-dependent dot product (13):

$$\begin{aligned} V_0^{\epsilon^\delta} := \nabla S^\perp &= \Big\{u \in V,\ \int_\Omega \epsilon^\delta u \cdot \overline{\nabla p} = 0, \quad \forall p \in S\ \Big\}, \\ &= \Big\{u \in V,\ \mathrm{div}\,(\epsilon^\delta u) = 0 \text{ in } \Omega,\ \epsilon^\delta u \cdot e_{i|\Sigma_i^+} = \epsilon^\delta u \cdot e_{i|\Sigma_i^-}, i = 1,2\Big\}. \end{aligned} \tag{20}$$

The proof of stability estimate (17) is then based on the following 'compactness' lemma, whose proof is done below.

Lemma 2.3 *Let $(\delta_n)_{n\in\mathbb{N}}$ be a sequence going to 0 and $(u_n)_{n\in\mathbb{N}}$ a bounded sequence of $V_0^{\epsilon^{\delta_n}}$. Then $(u_n)_{n\in\mathbb{N}}$ has a subsequence that strongly converges in $L^2(\Omega)$.*

Proof of the Stability Estimate (17) As usual for this kind of estimate (see for instance [20], Theorem 2.1), the proof is by contradiction. Assume that there exist a sequence $(\delta_n)_{n\in\mathbb{N}}$ going to 0 and a sequence $(u_n)_{n\in\mathbb{N}}$ such that

$$\begin{cases} \|u_n\|_{V_{\epsilon_n}} = 1, \\ \lim\limits_{n\to+\infty} \sup\limits_{v\in V_{\epsilon_n}\setminus\{0\}} \dfrac{|a^{\delta_n}(u_n, v)|}{\|v\|_{V_{\epsilon_n}}} = 0, \end{cases} \tag{21}$$

where $\epsilon_n := \epsilon^{\delta_n}$. Applying Helmholtz decomposition (19), there exist two sequences $(w_n)_{n\in\mathbb{N}} \in V_0^{\epsilon_n}$ and $(p_n)_{n\in\mathbb{N}} \in S$ such that

$$u_n = w_n + \nabla p_n.$$

In view of the identity $\displaystyle\int_\Omega \epsilon_n \nabla p_n . \overline{\nabla p_n} = \int_\Omega \epsilon_n u_n \cdot \overline{\nabla p_n}$ it is clear that $\|p_n\|_{H^1(\Omega)}$ and consequently $\|w_n\|_{V_{\epsilon_n}}$ are bounded. Moreover,

$$a^\delta(u_n, \nabla p_n) = -\omega^2 \int_\Omega \epsilon_n(\nabla p_n) \cdot \overline{\nabla p_n}, \tag{22}$$

Taking the limit as n tends to $+\infty$, we have

$$\lim_{n\to+\infty} \|p_n\|_{H^1(\Omega)} = 0. \tag{23}$$

Besides, $(w_n)_{n\in\mathbb{N}}$ being bounded in $V_0^{\epsilon_n}$, Lemma 2.3 applies: w_n has a subsequence (still denoted by w_n) that converges strongly in $L^2(\Omega)$. We call its limit w. We are going to prove that $w = 0$. First, as $\operatorname{curl} w_n$ is bounded in $L^2(\Omega)$ and $(w_n)_T$ is bounded in $L^2(\Sigma_{x_3}^\pm)$

$$\begin{cases} \operatorname{curl} u_n = \operatorname{curl} w_n \rightharpoonup \operatorname{curl} w \text{ in } L^2(\Omega), \\ (u_n)_{T|\Sigma_{x_3}^\pm} = (w_n)_{T|\Sigma_{x_3}^\pm} \rightharpoonup (w_T)_{|\Sigma_{x_3}^\pm} \text{ in } L^2(\Sigma_{x_3}^\pm). \end{cases}$$

Moreover, since μ_n tends almost everywhere to 1, for any $v \in V$, $v_n := \frac{1}{\mu^{\delta_n}} \operatorname{curl} v$ converges strongly to $\operatorname{curl} v$ in $L^2(\Omega)$. Similarly $\epsilon_n v$ converges strongly to v in $L^2(\Omega)$. But, by assumption, $\frac{1}{\|v\|_{V_{\epsilon_n}}} a^{\delta_n}(u_n, v)$ tends to 0, so that

$$\forall v \in V, \quad \int_\Omega \operatorname{curl} w \cdot \operatorname{curl} v - \int_\Omega \omega^2 w \cdot \overline{v} - i\omega \int_{\Sigma_{x_3}^\pm} w_T \cdot \overline{v_T} = 0.$$

As well-known, the previous problem is well-posed; it follows that $w = 0$. As a direct consequence,

$$\lim_{n\to+\infty} \|u_n\|_{L^2(\Omega)} = 0.$$

In addition

$$\lim_{n\to+\infty} \omega \int_\Omega |(u_n)_T|^2 = \lim_{n\to+\infty} |\mathcal{I}m\ a^{\delta_n}(u_n, u_n)| = 0. \tag{24}$$

and similarly,

$$\lim_{n\to+\infty}\int_\Omega |\mathrm{curl}\, u_n|^2 \le C \lim_{n\in\mathbb{N}}\Big(\|u_n\|_{L^2(\Omega)} + |\mathcal{R}e\; a^{\delta_n}(u_n,u_n)|\Big)\,. \tag{25}$$

Combining equations (23), (24) and (25), we obtain

$$\lim_{n\to+\infty}\|u_n\|_{V_{\epsilon_n}} = 0,$$

which contradicts the initial assumption $\|u_n\|_{V_{\epsilon_n}} = 1$.

The stability proof is completed by the proof of the compactness Lemma 2.3.

Proof of Lemma 2.3 Note first that Lemma 2.3 holds if ϵ and μ are independent of δ. Our proof is an adaptation of the proof of Theorem 4.7 in [25] and relies on two key arguments:

- $V_0^1(\Omega)$ in compactly embedded in $L^2(\Omega)$ (see [25], theorem 3.47).
- The sequence $(\epsilon_n)_{n\in\mathbb{N}} := (\epsilon^{\delta_n})_{n\in\mathbb{N}}$ is uniformly bounded from below and from above and tends toward $\epsilon_0 = 1$ almost everywhere in Ω.

The Helmholtz decomposition (19) applied to the case $\epsilon^{\delta_n} = 1$ ensures the existence of $p_n \in S$ and $w_n \in V_0^1(\Omega)$ such that $u_n = w_n + \nabla p_n$. Moreover, since $\|\nabla p_n\|^2_{L^2(\Omega)} = \int_\Omega u_n\cdot\nabla p_n$, both $\|\nabla p_n\|_{L^2(\Omega)}$ and $\|w_n\|_{V_1(\Omega)}$ are bounded. Consequently, w_n has a subsequence (still denoted by (w_n)) that converges almost everywhere as well as strongly in $L^2(\Omega)$ to w.

The next step consists in proving that $\|\epsilon_n u_n - w\|_{L^2(\Omega)}$ goes to 0.

$$0 \le \int_\Omega (u_n - \frac{w}{\epsilon_n})\cdot\overline{(\epsilon_n u_n - w)} = \int_\Omega (u_n - \frac{w}{\epsilon_n})\cdot\overline{(\epsilon_n(w_n+\nabla p_n) - w)} = \int_\Omega (u_n - \frac{w}{\epsilon_n})\cdot\overline{(\epsilon_n w_n - w)}.$$

$u_n - \frac{w}{\epsilon_n}$ is bounded in $L^2(\Omega)$. Moreover, since ϵ_n tends to 1 almost everywhere, $\epsilon_n w_n$ tends to w almost everywhere. Applying the Lebesgue's theorem we obtain $\lim_{n\to+\infty}\|\epsilon_n w_n - w\|_{L^2(\Omega)} = 0$ and consequently

$$\lim_{n\to+\infty}\int_\Omega (u_n - \frac{w}{\epsilon_n})\cdot\overline{(\epsilon_n u_n - w)} = 0.$$

Finally since $\int_\Omega (u_n - \frac{w}{\epsilon_n})\cdot\overline{(\epsilon_n u_n - w)} \ge \frac{1}{\epsilon^+}\|\epsilon_n u_n - w\|^2_{L^2(\Omega)}$, we have

$$\lim_{n\to+\infty}\|\epsilon_n u_n - w\|_{L^2(\Omega)} = 0.$$

We use the triangular inequality to conclude:

$$\|u_n - w\|_{L^2(\Omega)} \le \underbrace{\|1/\epsilon_n\|_{L^\infty(\Omega)}}_{\le C}\Big(\underbrace{\|(\epsilon_n - 1)w\|_{L^2(\Omega)}}_{\to 0} + \underbrace{\|\epsilon_n u_n - w\|_{L^2(\Omega)}}_{\to 0}\Big).$$

2.3 General methodology and main results

This paper is devoted to the study of the asymptotical behavior of the electromagnetic fields $\mathbf{E}^\delta$ and $\mathbf{H}^\delta$ as δ tends to 0. In our case, due to the fast variations of ϵ^δ and μ^δ in the periodic layer, it does not seems possible to write a uniform expansion of the electromagnetic fields $\mathbf{E}^\delta$ and $\mathbf{H}^\delta$ in the whole domain: roughly speaking, $\mathbf{E}^\delta$ and $\mathbf{H}^\delta$ oscillate more rapidly in the neighborhood of the periodic layer than far from it ; this is a boundary layer phenomenon. It is nevertheless possible to write an asymptotic expansion using the method of matched asymptotics. Let us briefly explain how to apply this method in the present context; we follow five main steps:

Step 1: Far field Ansatz (Section 3.1.1): we start from an "ansatz" (a guess) of the asymptotic expansion of $\mathbf{E}_\delta$ and $\mathbf{H}_\delta$ far from the periodic thin layer: in the present case, we choose

$$\mathbf{E}^\delta = \sum_{n\in\mathbb{N}} \delta^n \mathbf{E}_n(x_1, x_2, x_3), \quad \mathbf{H}^\delta = \sum_{n\in\mathbb{N}} \delta^n \mathbf{H}_n(x_1, x_2, x_3) \quad |x_3| \gg \delta, \tag{26}$$

Inserting expansions (26) into Maxwell's equations (1), we formally derive equations satisfied by the far field terms $\mathbf{E}_n$ and $\mathbf{H}_n$. These equations are not well posed since transmission conditions are missing across the interface Γ.

Step 2: Near field Ansatz (Section 3.1.2): in order to obtain these missing transmission conditions, we study the expansion of the electromagnetic fields in the vicinity of the thin periodic layer. Because of the fast oscillations in this layer, it is not possible to have an expansion of the form (26). That is why we consider a different asymptotic expansion, inspired by the theory of periodic homogenization (see for instance Ref [26, 11, 16]):

$$\mathbf{E}^\delta = \sum_{n\in\mathbb{N}} \delta^n \mathcal{E}_n(\frac{x_1}{\delta}, \frac{x_2}{\delta}, \frac{x_3}{\delta}; x_1, x_2), \quad \mathbf{H}^\delta = \sum_{n\in\mathbb{N}} \delta^n \mathcal{H}_n(\frac{x_1}{\delta}, \frac{x_2}{\delta}, \frac{x_3}{\delta}; x_1, x_2) \quad |x_3| \sim \delta, \tag{27}$$

with periodic conditions with respect to the first two fast variables X_1 and X_2: for any integers n and m

$$\mathcal{E}_n(X_1 + m, X_2 + n\tau, X_3, x_1, x_2) = \mathcal{E}_n(X_1 + m, X_2 + n\tau, X_3, x_1, x_2) \tag{28}$$

$$\mathcal{H}_n(X_1 + m, X_2 + n\tau, X_3, x_1, x_2) = \mathcal{H}_n(X_1 + m, X_2 + n\tau, X_3, x_1, x_2) \tag{29}$$

Then, as for the far field equations, we obtain the near field equations by plugging Ansatz (28) into Maxwell's equations (1). Once again, these equations are not well-posed : we need to prescribe a particular behavior as X_3 tends to $\pm\infty$ to close these problems.

Step 3: Matching principle (Section 3.2): in order to obtain well posed problems for the far and near field terms, we have to specify the missing data, namely the behavior at infinity for the near fields and transmission conditions for the far fields. The matched asymptotic expansion method provides a procedure called "matching principle" in order to obtain "matching conditions" that express the fact that far field expansion and near field expansion coincide in some intermediate areas (also called matching areas). The matching conditions couple the behavior of the far field terms in the vicinity of Γ to the behavior of the near field terms as X_3 goes to $\pm\infty$.

We point out that those first three steps, described in Section 3, are formal.

Step 4: Well-posedness of the recurrent problems (Sections 4, 5 and 6): far field equations, near field equations completed by matching conditions give rise to a system of recurrent problems. We show that this system is well-posed in the sense that the terms of the asymptotic expansions are uniquely defined (Theorem 6.1). We emphasize that, in the present context, near field problems are not standard. Indeed, we have to solve electrostatic problems posed in an unbounded strip (with periodic conditions on the lateral boundaries of the strip). Section 4 is dedicated to the derivation of an appropriate variational framework associated with these problems.

Step 5: Error estimates (Section 7): as the first three steps rely on formal calculations, this last step consists in justifying a posteriori the definition of the asymptotic expansion by means of error estimates.With the help of a global approximation of the exact electromagnetic fields in the whole domain, we first obtain a global estimate: the global approximation coincides with the truncated (at order n) far field expansion far from the periodic ring and with the truncated near field in the vicinity of the periodic layer; it is obtained by means of a truncation function. We then deduce the following optimal estimate (Theorem 7.1) for the far fields terms:

Theorem 2.4 *Let* $0 < \gamma < \dfrac{L_3}{2}$ *and* $\Omega_\gamma := \{(x_1, x_2, z) \in \Omega, |z| > \gamma\}$. *For any* $n \in \mathbb{N}$, *there exist a constant* $C_n > 0$ *and a constant* $\delta_\gamma > 0$ *such that,*

$$\forall\, \delta < \delta_\gamma, \qquad \left\| \mathbf{E}^\delta - \sum_{k=0}^{n} \delta^k \mathbf{E}_k \right\|_{H(curl\,,\Omega_\gamma)} + \left\| \mathbf{H}^\delta - \sum_{k=0}^{n} \delta^k \mathbf{H}_k \right\|_{H(curl\,,\Omega_\gamma)} \leq C_n\ \delta^{n+1}.$$

3 Formal Asymptotic Expansion

As mentioned in Section 2.3 we start from the following two ansatzes:

- Far from the periodic thin layer, we assume that standard power series expansions hold:

$$\mathbf{E}^\delta = \sum_{n\in\mathbb{N}} \delta^n \mathbf{E}_n(x_1, x_2, x_3), \quad \mathbf{H}^\delta = \sum_{n\in\mathbb{N}} \delta^n \mathbf{H}_n(x_1, x_2, x_3) \quad |x_3| \gg \delta, \tag{30}$$

  where, the far field terms $\mathbf{E}_n$ and $\mathbf{H}_n$ are defined in the limit domain $\Omega^+ \cup \Omega^-$:

$$\Omega^+ := \left\{ (x_1, x_2, x_3) \in]-\frac{L_1}{2}, \frac{L_1}{2}[\times]-\frac{L_2}{2}, \frac{L_2}{2}[\times]0, \frac{L_3}{2}[\right\},$$

$$\Omega^- := \left\{ (x_1, x_2, x_3) \in]-\frac{L_1}{2}, \frac{L_1}{2}[\times]-\frac{L_2}{2}, \frac{L_2}{2}[\times]-\frac{L_3}{2}, 0[\right\}.$$

- In the vicinity of the thin periodic layer, we use a more complicated ansatz inspired by the periodic homogenization theory (see [3], [11], [26]):

$$\mathbf{E}^\delta = \sum_{n\in\mathbb{N}} \delta^n \mathcal{E}_n(\frac{x_1}{\delta}, \frac{x_2}{\delta}, \frac{x_3}{\delta}; x_1, x_2), \quad \mathbf{H}^\delta = \sum_{n\in\mathbb{N}} \delta^n \mathcal{H}_n(\frac{x_1}{\delta}, \frac{x_2}{\delta}, \frac{x_3}{\delta}; x_1, x_2) \quad |x_3| \sim \delta, \tag{31}$$

where $\mathcal{E}_n$ are $\mathcal{H}_n$ are complex valued functions defined in $\mathbb{R}^3\times]-L_1/2, L_1/2[\times]-L_2/2, L_2/2[$. Moreover, we impose $\mathcal{E}_n$ and $\mathcal{H}_n$ to be 1-periodic with respect to the first fast variable X_1 and τ-periodic with respect to second fast variable X_2 (cf. (28)). Consequently these functions will then be systematically identified to their restrictions to

$$B^* = B\times]-\frac{L_1}{2},\frac{L_1}{2}[\times]-\frac{L_2}{2},\frac{L_2}{2}[,\quad \text{where}\quad B := \left\{(X_1,X_2,X_3)\in]-\frac{1}{2},\frac{1}{2}[\times]-\frac{\tau}{2},\frac{\tau}{2}[\times\mathbb{R}\right\} \tag{32}$$

B is called the periodicity cell. It is unbounded in X_3.

- The expansions (30) and (31) are assumed to be valid in two overlapping areas

$$\Omega_{M,\delta^+} := \left\{(x_1,x_2,x_3)\in]-\frac{L_1}{2},\frac{L_1}{2}[\times]-\frac{L_2}{2},\frac{L_2}{2}[\times]\eta^-(\delta),\eta^+(\delta)[\right\},$$

$$\Omega_{M,\delta^-} := \left\{(x_1,x_2,x_3)\in]-\frac{L_1}{2},\frac{L_1}{2}[\times]-\frac{L_2}{2},\frac{L_2}{2}[\times]-\eta^+(\delta),-\eta^-(\delta)[\right\} \tag{33}$$

where the functions $\eta^\pm$ are such that $0<\eta^-<\eta^+$ and,

$$\lim_{\delta\to 0}\eta^\pm = 0,\quad \lim_{\delta\to 0}\frac{\eta^\pm}{\delta} = \pm\infty.$$

For instance, $\eta^-(\delta)=\sqrt{\delta}$ and $\eta^+(\delta)=2\sqrt{\delta}$ would be convenient. Note that, for the near field, overlapping areas correspond to X_3 going to $\pm\infty$. On the contrary, for the far field, the overlapping areas correspond to $x_3\approx 0$.

We emphasize that formal expansions (30) and (31) will be justified by the error analysis in section 7. Note also that these kind of two-scale expansions is well known (cf. [11, 27, 14, 15, 16]).

In the two following sections, we shall formally derive the equations satisfied by the far and near field terms.

3.1 Far field and near field equations

3.1.1 Far field equations

The derivation of these equations is immediate. There are directly obtained by substituting the electromagnetic fields by their expansions (30) in the Maxwell's Equations (1), (4), and (5), and formally separating the different powers of δ. The far field terms then satisfy the Maxwell equations

$$\begin{cases} -i\omega\mathbf{H}_n + \operatorname{curl}\mathbf{E}_n = 0 & \text{in } \Omega^\pm\,, \\ -i\omega\mathbf{E}_n - \operatorname{curl}\mathbf{H}_n = -\dfrac{\delta_0}{i\omega}F & \text{in } \Omega^\pm, \end{cases} \tag{34}$$

and an impedance condition on the lower and upper boundaries

$$\mathbf{H}_n\times n - (\mathbf{E}_n)_T = 0 \text{ sur } \Sigma^\pm_{x_3}. \tag{35}$$

Equations (34) and (35) do not entirely defined $\mathbf{E}_n$ and $\mathbf{H}_n$ since we have not prescribed any transmission condition on Γ yet. For instance, we need some information on the jumps of $[\mathbf{E}_n\times e_3]_\Gamma$ and $[\mathbf{H}_n\times e_3]_\Gamma$.

3.1.2 Near field expansion

Due to the two different scales, the derivation of these equations is more involved than the far field ones. To make the understanding easier, we need to introduce some additional notation. Let us first define the "surfacic" operators div_Γ, $\vec{\mathrm{curl}}_\Gamma$, curl_Γ. For any vector field

$$U(X_1, X_2, X_3; x_1, x_2) := \sum_{i=1}^{3} U^i(X_1, X_2, X_3; x_1, x_2) e_i,$$

we define

$$\mathrm{div}_\Gamma U := \partial_{x_1} U^1 + \partial_{x_2} U^2, \quad \mathrm{curl}_\Gamma U = \partial_{x_1} U^2 - \partial_{x_2} U^1.$$

and, for any function $p(X_1, X_2, X_3; x_1, x_2)$, we define

$$\vec{\mathrm{curl}}_\Gamma p := \partial_{x_2} p\, e_1 - \partial_{x_1} p\, e_2.$$

Note that these definitions are not usual since these operators apply to functions defined in $\mathbb{R}^3 \times \mathbb{R}^2$ although they usually apply to traces of functions.

In the same way, for any vector U, we define its "normal" and "tangential" part U_N et U_T by

$$U_T = (e_3 \times U) \times e_3, \quad U_N = U \cdot e_3. \tag{36}$$

Besides, we introduce some volumic operators (acting on fast variables)

$$\mathrm{Div}\, U = \partial_{X_1} U^1 + \partial_{X_1} U^2 + \partial_{X_3} U^3, \quad \mathrm{Curl}\, U = \left| \begin{array}{l} \partial_{X_2} U^3 - \partial_{X_3} U^2 \\ \partial_{X_3} U^1 - \partial_{X_1} U^3 \\ \partial_{X_1} U^2 - \partial_{X_2} U^1 \end{array} \right. \tag{37}$$

Finally, for any function $\mathcal{E}(X_1, X_2, X_3, x_1, x_2)$, we denote $(\mathcal{E})^\delta(x_1, x_2, x_3)$

$$(\mathcal{E})^\delta(x_1, x_2, x_3) := \mathcal{E}(\frac{x_1}{\delta}, \frac{x_2}{\delta}, \frac{x_3}{\delta}, x_1, x_2). \tag{38}$$

Note that

$$\mathrm{curl}\, (\mathcal{E})^\delta = \left(\frac{1}{\delta} \mathrm{Curl}\, \mathcal{E} + \mathcal{A}_0 \mathcal{E} \right)^\delta, \tag{39}$$

where,

$$\mathcal{A}_0 \mathcal{E} = \vec{\mathrm{curl}}_\Gamma (\mathcal{E}_N) + \mathrm{curl}_\Gamma (\mathcal{E}_T)\, e_3.$$

Introducing the near field expansions (31) in the Maxwell's equations, and formally separating the different powers of δ we get

$$\mathrm{Curl}\, \mathcal{E}_n = -\mathcal{A}_0 \mathcal{E}_{n-1} + i\omega\mu \mathcal{H}_{n-1}, \quad -\mathrm{Curl}\, \mathcal{H}_n = +\mathcal{A}_0 \mathcal{H}_{n-1} + i\omega\epsilon \mathcal{E}_{n-1}. \tag{40}$$

Since $\mathrm{Div}\, \mathrm{Curl}\, \mathcal{E}_n = \mathrm{Div}\, \mathrm{Curl}\, \mathcal{H}_n = 0$, the previous equations have no solution unless the following compatibility condition is satisfied:

$$\mathrm{Div}\ (-\mathcal{A}_0 \mathcal{E}_{n-1} + i\omega\mu \mathcal{H}_{n-1}) = 0 \ \text{ and } \ \mathrm{Div}\ (\mathcal{A}_0 \mathcal{H}_{n-1} + i\omega\epsilon \mathcal{E}_{n-1}) = 0.$$

In view of the formula $-\text{Div}\ (\mathcal{A}_0\mathcal{E}_{n-1}) = i\omega\,\mu\,\text{div}_\Gamma(\mathcal{H}_{n-2})_T$ (obtained by interchanging the derivations with respect to the fast and slow variables), the previous compatibility condition rewrites

$$\text{Div}\ (\epsilon\mathcal{E}_n) = -\epsilon\,\text{div}_\Gamma(\mathcal{E}_{n-1})_T \ \text{ and }\ \text{Div}\ (\mu\mathcal{H}_n) = -\mu\,\text{div}_\Gamma(\mathcal{H}_{n-1})_T.$$

Finally, we end up with the following near field equations

$$\begin{cases}\text{Curl}\,\mathcal{E}_n = -\mathcal{A}_0\mathcal{E}_{n-1} + i\omega\,\mu\,\mathcal{H}_{n-1}, \\ \text{Div}\,(\epsilon\mathcal{E}_n) = -\,\epsilon\,\text{div}_\Gamma\,(\mathcal{E}_{n-1})_T\,,\end{cases} \qquad \begin{cases}-\text{Curl}\,\mathcal{H}_n = +\mathcal{A}_0\mathcal{H}_{n-1} + i\omega\,\epsilon\,\mathcal{E}_{n-1}, \\ \text{Div}\,(\mu\mathcal{H}_n) = -\mu\,\text{div}_\Gamma\,(\mathcal{H}_{n-1})_T\,.\end{cases} \tag{41}$$

As is usual, Equations (41) are completed with periodicity conditions with respect to the first two fast variables X_1, X_2:

$$\begin{cases}\mathcal{E}_n(X_1+1,X_2,X_3,x_1,x_2) = \mathcal{E}_n(X_1,X_2+\tau,X_3,x_1,x_2) = \mathcal{E}_n(X_1,X_2,X_3,x_1,x_2), \\ \mathcal{H}_n(X_1+1,X_2,X_3,x_1,x_2) = \mathcal{H}_n(X_1,X_2+\tau,X_3,x_1,x_2) = \mathcal{H}_n(X_1,X_2,X_3,x_1,x_2).\end{cases} \tag{42}$$

As we shall see, near field equations (41, 42) are not well posed: they have a non-trivial null-space (Proposition 4.4). To define entirely the near field terms, we need to prescribe their behavior at as X_3 goes to $\pm\infty$.

3.2 Matching conditions

The missing information (near field behavior at infinity and transmission conditions for the far fields) will be provided by the matching conditions. The matching conditions express the fact that, far field and near field expansion "coincide" in the matching areas. We have seen that matching areas correspond to a neighborhood of Γ (x_3 close to 0) for the far field although they correspond to X_3 large for the near field. Before writing the matching conditions, we shall investigate, in turn, the behavior of far fields in the vicinity of Γ and the behavior of the near field for large X_3.

3.2.1 Behavior of the far fields in the matching areas

The behavior of far field terms, given in the following proposition, directly results from a Taylor expansion of the far field in the vicinity of Γ. Although technical, the proof simply follows from an identification process, plugging Taylor's expansions (43) into the homogeneous Maxwell's equations and collecting terms associated with the different powers of x_3 (see [28] and [29] for a detailed proof of this kind of result).

Proposition 3.1 *Let $E^\pm$ and $H^\pm$ be two smooth functions satisfying the homogeneous Maxwell's equations in a neighborhood $\mathcal{V}^\pm(\Gamma)$ of Γ:*

$$\begin{cases}\textit{curl}\,\mathbf{E} - i\omega\mathbf{H} = 0 \textit{ in } \mathcal{V}^\pm(\Gamma), \\ -\textit{curl}\,\mathbf{H} - i\omega\mathbf{E} = 0 \textit{ in } \mathcal{V}^\pm(\Gamma).\end{cases}$$

Then, their Taylor's expansion is given by

$$\mathbf{E}^\pm(x_1,x_2,x_3) = \sum_{k\in\mathbb{N}} x_3^k\ (\mathbf{E}^k)^\pm(x_1,x_2), \quad \mathbf{H}^\pm(x_1,x_2,x_3) = \sum_{k\in\mathbb{N}} x_3^k\ (\mathbf{H}^k)^\pm(x_1,x_2), \tag{43}$$

where,

$$(\mathbf{E}^0)^{\pm}_T(x_1,x_2) = \mathbf{E}^{\pm}_T(x_1,x_2,0) \textit{ undetermined},$$
$$(\mathbf{H}^0)^{\pm}_T(x_1,x_2) = \mathbf{H}^{\pm}_T(x_1,x_2,0) \textit{ undetermined},$$

$$(\mathbf{E}^0)^{\pm}_N(x_1,x_2) = \mathbf{E}^{\pm}_N(x_1,x_2,0) = \frac{-1}{i\omega} curl_\Gamma((\mathbf{H}^0)^{\pm}_T)(x_1,x_2)),$$
$$(\mathbf{H}^0)^{\pm}_N(x_1,x_2) = \mathbf{H}^{\pm}_N(x_1,x_2,0) = \frac{1}{i\omega} curl_\Gamma((\mathbf{E}^0)^{\pm}_T)(x_1,x_2)),$$

and, for any $k \geq 1$,

$$(\mathbf{E}^k)^{\pm}_T(x_1,x_2) = \frac{1}{k!}\frac{\partial^k \mathbf{E}_T(x_1,x_2,0)}{\partial x_3^k} = \frac{1}{k}\left(\nabla_\Gamma(\mathbf{E}^{k-1})^{\pm}_N + i\omega(\mathbf{H}^{k-1})^{\pm}_T \times e_3\right)(x_1,x_2),$$
$$(\mathbf{H}^k)^{\pm}_T(x_1,x_2) = \frac{1}{k!}\frac{\partial^k \mathbf{H}_T(x_1,x_2,0)}{\partial x_3^k} = \frac{1}{k}\left(\nabla_\Gamma(\mathbf{H}^{k-1})^{\pm}_N - i\omega(\mathbf{E}^{k-1})^{\pm}_T \times e_3\right)(x_1,x_2), \tag{44}$$

$$(\mathbf{E}^k)^{\pm}_N(x_1,x_2) = \frac{1}{k!}\frac{\partial^k \mathbf{E}_N(x_1,x_2,0)}{\partial x_3^k} = -\frac{1}{k} div_\Gamma(\mathbf{E}^{k-1})^{\pm}_T(x_1,x_2),$$
$$(\mathbf{H}^k)^{\pm}_N(x_1,x_2) = \frac{1}{k!}\frac{\partial^k \mathbf{H}_N(x_1,x_2,0)}{\partial x_3^k} = \frac{1}{k} div_\Gamma(\mathbf{H}^{k-1})^{\pm}_T(x_1,x_2). \tag{45}$$

3.2.2 Behavior of the near fields in the matching areas

The behavior of near field terms is obtained using a Fourier decomposition in the areas where ϵ and μ are constant. As usually, we shall assume that the near field terms do not increase exponentially in X_3. Consequently we will define $\mathcal{E}_n(\cdot,\cdot,\cdot;x_1,x_2)$ and $\mathcal{H}_n(\cdot,\cdot,\cdot;x_1,x_2)$ respectively in the spaces $V^+_\epsilon(B)$ and $V^+_\mu(B)$:

$$V^+_\epsilon(B) := \Big\{\mathcal{E} \in L^2_{loc}(\mathbb{R}^3), \mathcal{E} \text{ 1-periodic in } X_1 \text{ and } \tau\text{-periodic in } X_2 \text{ such that}$$
$$\text{curl}\,\mathcal{E} \in L^2_{loc}(\mathbb{R}^3), \text{div}\,(\epsilon\mathcal{E}) \in L^2_{loc}(\mathbb{R}^3) \text{ and } \int_B (|\mathcal{E}|^2 + |\text{curl}\,\mathcal{E}|^2 + |\text{div}\,\epsilon\mathcal{E}|^2)e^{-|X_3|/2} < +\infty\Big\}, \tag{46}$$

$$V^+_\mu(B) := \Big\{\mathcal{H} \in L^2_{loc}(\mathbb{R}^3), \mathcal{E} \text{ 1-periodic } X_1 \text{ and } \tau\text{-periodic } X_2 \text{ such that}$$
$$\text{curl}\,\mathcal{H} \in L^2_{loc}(\mathbb{R}^3), \text{div}\,(\mu\mathcal{H}) \in L^2_{loc}(\mathbb{R}^3) \text{ and } \int_B (|\mathcal{H}|^2 + |\text{curl}\,\mathcal{H}|^2 + |\text{div}\,\mu\mathcal{H}|^2)e^{-|X_3|/2} < +\infty\Big\}. \tag{47}$$

In this part, it is convenient to introduce some notation (Fourier coefficients).

Definition 3.2 *Let $u(X_1, X_2, X_3)$ be a function in $L^2_{loc}(\mathbb{R}^3)$ 1-periodic in X_1 and τ-periodic X_2. For any $(p,q) \in \mathbb{Z}^2$, we denote by $\{u\}_{p,q}$ the Fourier coefficient of u associated with the Fourier mode $e^{2i\pi(pX_1+\frac{q}{\tau}X_2)}$:*

$$u(X_1, X_2, X_3) = \sum_{(p,q)\in\mathbb{Z}^2} \{u\}_{p,q}(X_3)\, e^{2i\pi(pX_1+\frac{q}{\tau}X_2)}.$$

with,

$$\{u\}_{p,q}(X_3) = \frac{1}{\tau}\int_{-1/2}^{1/2}\int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} u(X_1, X_2, X_3) e^{-2i\pi(pX_1+\frac{q}{\tau}X_2)}\, dX_1 dX_2.$$

Definition 3.3 (property $\mathcal{P}^\infty$) *Let $\mathcal{U}$ a function of $L^2_{loc}(B)$. We say that $\mathcal{U}$ satisfies property $\mathcal{P}^\infty$ if there exist two subsequences of polynomials $(p_{l,k}(X_3)^\pm)_{(l,k)\in\mathbb{Z}^2}$, such that*

$$\mathcal{U} = p^\pm_{0,0}(X_3) + \sum_{(l,k)\in\mathbb{Z}^2\setminus(0,0)} p^\pm_{l,k}(X_3) e^{2i\pi(lX_1+k\frac{X_2}{\tau})} e^{-2\pi\sqrt{l^2{X_1}^2+\frac{k^2}{\tau^2}{X_2}^2}|X_3|} \quad \text{if } \pm X_3 > \frac{1}{2}, \tag{48}$$

We then define $\ell^\pm_T(\mathcal{U})$ and $\ell^\pm_N(\mathcal{U})$, two linear forms by

$$\ell^\pm_T(\mathcal{U}) := e_3 \times (p^\pm(0) \times e_3) \qquad \ell^\pm_N(\mathcal{U}) := e_3 \cdot p^\pm(0). \tag{49}$$

as well as the tangential and normal jump and mean values of $\mathcal{U}$:

$$[\ell_T(\mathcal{U})] := \ell^+_T(\mathcal{U}) - \ell^-_T(\mathcal{U}), \quad \langle \ell_T(\mathcal{U})\rangle = \frac{1}{2}\left(\ell^+_T(\mathcal{U}) + \ell^-_T(\mathcal{U})\right),$$

$$[\ell_N(\mathcal{U})] := \ell^+_N(\mathcal{U}) - \ell^-_N(\mathcal{U}), \quad \langle \ell_N(\mathcal{U})\rangle = \frac{1}{2}\left(\ell^+_N(\mathcal{U}) + \ell^-_N(\mathcal{U})\right). \tag{50}$$

Remark 3.4 *If $\mathcal{U}(\cdot,\cdot,\cdot;x_1,x_2)$ satisfies property $\mathcal{P}^\infty$ for any $(x_1,x_2) \in]-L_1/2, L_1/2[\times]-L_2/2, L_2/2[$, we define $\ell^\pm_T(\mathcal{U})(x_1,x_2)$ and $\ell^\pm_N(\mathcal{U})(x_1,x_2)$ by*

$$\ell^\pm_T(\mathcal{U})(x_1,x_2) := \ell^\pm_T(\mathcal{U}(\cdot,\cdot,\cdot;x_1,x_2)), \quad \ell^\pm_T(U)(x_1,x_2) := \ell^\pm_T(\mathcal{U}(\cdot,\cdot,\cdot;x_1,x_2)).$$

and,

$$[\,\ell_T(\mathcal{U})\,](x_1,x_2) = [\,\ell_T(\mathcal{U}(\cdot,\cdot,\cdot;x_1,x_2))\,], \quad [\,\ell_N(\mathcal{U})\,](x_1,x_2) = [\,\ell_N(\mathcal{U}(\cdot,\cdot,\cdot;x_1,x_2))\,].$$

Proceeding to a decomposition of the near fields in term of Fourier series and using near field equations (41), we obtain the following result:

Proposition 3.5 *Let $(\mathcal{E}_n)_{n\in\mathbb{N}}$ and $(\mathcal{H}_n)_{n\in\mathbb{N}}$ two sequences of functions, respectively in $V^+_\epsilon(B)$ and $V^+_\mu(B)$, that satisfy near field equations (41). Then, for any $n \in \mathbb{N}$, $\mathcal{E}_n$ and $\mathcal{H}_n$ satisfy property $\mathcal{P}^\infty$: more precisely, there exists some functions $C_{n,k}(x_1,x_2)$ $(\in \mathbb{C}^3)$ and $D_{n,k}(x_1,x_2)$, and four sequences of polynomials in X_3, $(p^\pm_{n,l,k}(X_3;x_1,x_2))_{(l,k)\in Z^2\setminus(0,0)}$, $(q^\pm_{n,l,k}(X_3;x_1,x_2))_{(l,k)\in Z^2\setminus(0,0)}$ such that, if $\pm Z > \frac{1}{2}$,*

$$\mathcal{E}_n = \sum_{k=0}^n C^\pm_{n,k}(x_1,x_2){X_3}^k + \sum_{(l,k)\in\mathbb{Z}^2\setminus(0,0)} p^\pm_{n,l,k}(\nu;x_1,x_2) e^{2i\pi(lX_1+k\frac{X_2}{\tau})} e^{-2\pi\sqrt{l^2{X_1}^2+\frac{k^2}{\tau^2}{X_2}^2}|X_3|},$$

$$\mathcal{H}_n = \sum_{k=0}^n D^\pm_{n,k}(x_1,x_2)X_3^k + \sum_{(l,k)\in\mathbb{Z}^2\setminus(0,0)} q^\pm_{n,l,k}(\nu;x_1,x_2) e^{2i\pi(lX_1+k\frac{X_2}{\tau})} e^{-2\pi\sqrt{l^2{X_1}^2+\frac{k^2}{\tau^2}{X_2}^2}|Z|}. \tag{51}$$

Moreover, for any $k \geq 1$,

$$(C_{n,k})_T^{\pm} = \frac{1}{k}\left(\nabla_\Gamma (C_{n-1,k-1})_N^{\pm} + i\omega (D_{n-1,k-1})_T^{\pm} \times e_3\right),$$
$$(C_{n,k})_N^{\pm} = \frac{-1}{k} div_\Gamma (C_{n-1,k-1})_T^{\pm}, \tag{52}$$

and

$$(D_{n,k})_T^{\pm} = \frac{1}{k}\left(\nabla_\Gamma (D_{n-1,k-1})_N^{\pm} - i\omega (C_{n-1,k-1})_T^{\pm} \times e_3\right),$$
$$(D_{n,k})_N^{\pm} = \frac{-1}{k} div_\Gamma (D_{n-1,k-1})_T. \tag{53}$$

At this point, it is interesting to note the similarities between the formulas (52, 53) (Polynomial coefficients of the near fields) and Formulas (44, 45) (coefficients of the Taylor's expansions of the far fields).

3.2.3 Matching conditions

The derivation of the matching conditions is formal but will be justified a posteriori by the error estimate. In the matchings areas, both expansions (51) and (43) hold ; Consequently, substituting (51) and (43) into the far and near field expansions (31) and (30), neglecting the exponentially decaying terms, and formally identifying the terms of the form $x_3^k \delta^n$, $n, k \in \mathbb{N}$, we get

$$\begin{cases} C_{n,k}^{\pm} = \dfrac{1}{k!}\dfrac{\partial^k \mathbf{E}_{n-k}}{\partial x_3^k}(x_1, x_2, 0^{\pm}), \\ D_{n,k}^{\pm} = \dfrac{1}{k!}\dfrac{\partial^k \mathbf{H}_{n-k}}{\partial x_3^k}(x_1, x_2, 0^{\pm}), \end{cases} \tag{54}$$

which can also be written in a condensed way:

$$\begin{cases} [(\mathbf{E}_n)_T]_\Gamma = [\ell_T^{\pm}(\mathcal{E}_n)], & [(\mathbf{H}_n)_T]_\Gamma = [\ell_T(\mathcal{H}_n)], \\ \langle \ell_T^{\pm}(\mathcal{E}_n)\rangle = \langle (\mathbf{E}_n)_T\rangle_\Gamma, & \langle \ell_T^{\pm}(\mathcal{H}_n)\rangle = \langle (\mathbf{H}_n)_T\rangle_\Gamma, \\ \langle \ell_N^{\pm}(\mathcal{E}_n)\rangle = \langle (\mathbf{E}_n)_N\rangle_\Gamma, & \langle \ell_N^{\pm}(\mathcal{H}_n)\rangle = \langle (\mathbf{H}_n)_T\rangle_\Gamma. \end{cases} \tag{55}$$

The procedure to obtain matching conditions is well-known, see [5], [30], [31] for more detailed explanations.

Using matching conditions (55), we can write $[e_3 \times \mathbf{E}_n]_\Gamma$ and $[e_3 \times \mathbf{H}_n]_\Gamma$ in a more explicit way: Let $n \in \mathbb{N}$ and assume for a while that $\mathbf{E}_k$, $\mathbf{H}_k$, $\mathcal{E}_k$ and $\mathcal{H}_k$ are known for any $k \leq n$. We consider the truncated periodicity cell B_{h_0} and its upper and lower boundary $\Gamma_{h_0}^{\pm}$:

$$B_{h_0} := \{(X_1, X_2, X_3) \in B_0 \text{ such that } -h_0 < X_3 < h_0\}, \tag{56}$$
$$\Gamma_{h_0}^{\pm} := \{(X_1, X_2, X_3) \in B_0 \text{ such that } X_3 = \pm h_0\}.$$

Integrating the rotational part of near field equation (39) over B_{h_0}, we get

$$\begin{aligned}
\frac{1}{\tau}\int_{B_{h_0}} \operatorname{curl}\mathcal{E}_n &= \frac{1}{\tau}\left\{\int_{\Gamma^+_{h_0}}(e_3\times\mathcal{E}_n) - \int_{\Gamma^-_{h_0}}(e_3\times\mathcal{E}_n)\right\},\\
&= [e_3\times(C^0_n)]_\Gamma + \sum_{k=1}^{n}(e_3\times C^+_{n,k})(h_0)^k - \sum_{k=1}^{n}(e_3\times C^-_{n,k})(-h_0)^k,\\
&= [e_3\times\mathbf{E}_n]_\Gamma + \sum_{k=1}^{n}(e_1\times C^+_{n,k})(h_0)^k - \sum_{k=1}^{n}(e_3\times C^-_{n,k})(-h_0)^k.
\end{aligned}$$

Therefore,

$$\begin{aligned}
[e_3\times\mathbf{E}_n]_\Gamma &= \frac{1}{\tau}\int_{B_{h_0}}\operatorname{curl}\mathcal{E}_n + \sum_{k=1}^{n}(e_3\times C^-_{n,k})^1(-h_0)^k - \sum_{k=1^n}(e_3\times C^+_{n,k})^1(h_0)^k,\\
&= \frac{1}{\tau}\left\{\int_{B_{h_0}} -\mathcal{A}_0\left(\mathcal{E}_{n-1}\right) + i\omega\mu\mathcal{H}_{n-1}\right\} + \sum_{k=1}^{n}(e_3\times C^-_{n,k})(-h_0)^k - \sum_{k=1^n}(e_3\times C^+_{n,k})(h_0)^k.
\end{aligned}$$

As the right is known, we obtain an explicit expression of $[e_3\times\mathbf{E}_n]_\Gamma$ that only depends on the lower order terms $\mathbf{E}_k$, $\mathbf{H}_k$, $\mathcal{E}_k$ et $\mathcal{H}_k$, $k<n$. In the same, way, we can have an explicit expression of $[e_3\times\mathbf{H}_n]_\Gamma$. Finally, we get

$$[e_3\times\mathbf{E}_n]_\Gamma = g_n \quad\text{and}\quad [e_3\times\mathbf{H}_n]_\Gamma = h_n, \tag{57}$$

where

$$g_n = \frac{1}{\tau}\left(\int_{B_{h_0}}\big(-\mathcal{A}_0(\mathcal{E}_{n-1}) + i\omega\mu\mathcal{H}_{n-1}\big)_T\right) - \left(\sum_{k=1}^{n}(e_3\times C^+_{n,k})(h_0)^k - \sum_{k=1}^{n}(e_3\times C^-_{n,k})(-h_0)^k\right),$$

$$h_n = \frac{1}{\tau}\left(\int_{B_{h_0}}\big(-\mathcal{A}_0(\mathcal{H}_{n-1}) - i\omega\epsilon\mathcal{E}_{n-1}\big)_T\right) - \left(\sum_{k=1}^{n}(e_3\times D^+_{n,k})(h_0)^k - \sum_{k=1}^{n}(e_3\times D^-_{n,k})(-h_0)^k\right).$$

4 Variational framework for the near fields problems

Near field problems (41) do not fit any particular standard framework. Indeed, they are posed in an unbounded strip and their right-hand side does not remain bounded at infinity. Consequently, we stop our asymptotic procedure for a while and we dedicate the present section to the settlement of an appropriate functional framework to solve near fields problems (41).

4.1 Model problem for the near fields

Here and in what follows we say that a function $\mathcal{U}$ is periodic if $\mathcal{U}$ is periodic of period 1 with respect to X_1 and periodic of period τ with respect to X_2. Near field problems are electrostatic kind problems, namely, find a periodic function $\mathcal{U}$ such that

$$\begin{cases}\text{Curl } \mathcal{U} = f \text{ in } \mathcal{D}'(\mathbb{R}^3),\\ \text{Div } (a\mathcal{U}) = g \text{ in } \mathcal{D}'(\mathbb{R}^3),\end{cases}$$

where

- $f \in (L^2_{per}(\mathbb{R}^3)^3)_{loc}$ and $g \in (L^2_{per}(\mathbb{R}^3))_{loc}$ where

$$L^2_{per}(\mathbb{R}^3) := \big\{ f \in \mathcal{D}'(\mathbb{R}^3), f \text{ periodic and } \int_B f^2 < +\infty \big\}.$$

- $\operatorname{Div} f = 0$.
- a belongs to $L^\infty_{per}(\mathbb{R}^3) := \big\{ a \in L^\infty(\mathbb{R}^3), a \text{ is periodic} \big\}$.
- $a = 1$ if $|X_3| > \frac{1}{2}$.

Remark 4.1 *In the context of near field problems (41), the function a is equal to ϵ or to μ and f and g may have a polynomial growth in X_3.*

In a first step we will investigate the case $f \in (L^2_{per}(\mathbb{R}^3)^3)$ and $g \in (L^2_{per}(\mathbb{R}^3))$, which can be solved with the help of a variational form. We first restrict ourselves to functions that only depend on the fast variables X_1, X_2 et X_3, so that we shall abusively use div instead of Div as well as curl instead of Curl. Note that, the electrostatic problems have been widely studied in bounded domains, see for instance, [32], [33], [34] [35]. The objective of this section is to adapt the results mentioned above to the case of an unbounded periodic strip.

In view of the geometry of B (which is infinite in the X_3 direction), it seems natural to find $\mathcal{U}$ in the weighted space $X_a(\mathbb{R}^3)$

$$X_a(\mathbb{R}^3) := \left\{ \mathcal{U} \in \mathcal{D}'(\mathbb{R}^3)^3, \mathcal{U} \text{ periodic}, \ \operatorname{curl} \mathcal{U} \in L^2(B)^3, \ \operatorname{div}(a\mathcal{U}) \in L^2(B), \ \frac{\mathcal{U}}{\sqrt{1+(X_3)^2}} \in L^2(B)^3 \right\}, \tag{58}$$

We introduce the dot-product

$$(\mathcal{U}_1, \mathcal{U}_2) \to \int_B \left(\frac{1}{1+(X_3)^2} \mathcal{U}_1 \cdot \overline{\mathcal{U}_2} + \operatorname{curl} \mathcal{U}_1 \cdot \overline{\operatorname{curl} \mathcal{U}_2} + \operatorname{div}(a\mathcal{U}_1) \, \overline{\operatorname{div}}(a\mathcal{U}_2) \right) dx, \tag{59}$$

as well as the associate norm

$$\|\mathcal{U}\|^2_{X_a} = \left\| \frac{\mathcal{U}}{\sqrt{1+(X_3)^2}} \right\|^2_{L^2(B)} + \|\operatorname{curl} \mathcal{U}\|^2_{L^2(B)} + \|\operatorname{div}(a\mathcal{U})\|^2_{L^2(B)} . \tag{60}$$

Let us also introduce $X_a(B)$

$$X_a(B) := \Big\{ \mathcal{U} \in \mathcal{D}'(B)^3, \quad \mathcal{U} \times e_{i|\Gamma_i^-} = \mathcal{U} \times e_{i|\Gamma_i^+}, \ a\mathcal{U} \cdot e_{i|\Gamma_i^-} = a\mathcal{U} \cdot e_{i|\Gamma_i^+}, \ i = 1, 2,$$
$$\operatorname{curl} \mathcal{U} \in L^2(B)^3, \ \operatorname{div}(a\mathcal{U}) \in L^2(B), \ \frac{\mathcal{U}}{\sqrt{1+(X_3)^2}} \in L^2(B)^3 \Big\},$$

where $\Gamma_i^\pm$ $(i = 1, 2)$ are the lateral boundaries of B of outward unit $\pm e_i$. $X_a(B)$, equipped with the norm (60) is a Hilbert space.

Remark 4.2

- *If $\mathcal{U}$ is in $X_a(\mathbb{R}^3)$, then $\mathcal{U}_{|B}$ is in $X_a(B)$. Conversely, if $\mathcal{U}$ is in $X_a(B)$ then $\tilde{\mathcal{U}}$, the periodic extension of $\mathcal{U}$ to $\mathbb{R}^3$ is in $X_a(\mathbb{R}^3)$.*
- *Let*

$$W_1(\mathbb{R}^3) = \left\{ p \in \mathcal{D}'(\mathbb{R}^3), p \text{ periodic } \nabla p \in L^2(B)^3,\ \frac{p}{\sqrt{1+(X_3)^2}} \in L^2(B) \right\},$$

$$W_1(B) = \left\{ p \in \mathcal{D}'(B),\ p_{\Gamma^-_{x_1}} = p_{\Gamma^+_{x_1}},\ p_{\Gamma^-_{x_2}} = p_{\Gamma^+_{x_2}},\ \nabla p \in L^2(B)^3,\ \frac{p}{\sqrt{1+(X_3)^2}} \in L^2(B) \right\}.$$

Then the following inequality holds (see [11]):

$$X_1(\mathbb{R}^3) = W_1(\mathbb{R}^3)^3 \text{ and } X_1(B) = W_1(B)^3. \tag{61}$$

Finally we will solve the following problem: find $\mathcal{U} \in X_a(\mathbb{R}^3)$ such that

$$\mathcal{P}: \begin{cases} \text{curl } \mathcal{U} = f \text{ dans } \mathcal{D}'(\mathbb{R}^3), \\ \text{div } (a\mathcal{U}) = g \text{ dans } \mathcal{D}'(\mathbb{R}^3). \end{cases} \tag{62}$$

Of course, this problem is equivalent to the following one, posed in B: find $\mathcal{U} \in X_a(B)$ such that

$$\mathcal{P}_B: \begin{cases} \text{curl } \mathcal{U} = f \text{ dans } \mathcal{D}'(B), \\ \text{div } (a\mathcal{U}) = g \text{ dans } \mathcal{D}'(B). \end{cases} \tag{63}$$

The remainder of this section is organized as follows. First, we characterize the Kernel

$$\mathcal{N}_a(\mathcal{P}) := \left\{ \mathcal{U} \in X_a(\mathbb{R}^3) \text{ such that curl} \mathcal{U} = 0 \text{ and div} \mathcal{U} = 0 \right\} \tag{64}$$

of the problem $\mathcal{P}$ (Section 4.2). Then, we are able to prove that $(\mathcal{P})$ is well posed in the subspace $X_a^0 := \mathcal{N}_a(\mathcal{P})^\perp$ (Section 4.4). The well-posedness result is mainly based on a Friedrichs'inequality presented in Section 4.3.

Before starting this analysis, we will remind about a well-known result that will be subsequently use:

Proposition 4.3 *Let $f \in L^2_{per}(\mathbb{R}^3)$ such $\sqrt{1+(X_3)^2} f \in L^2(B)$, $\int_B f = 0$ and let $g \in L^2_{per}(\mathbb{R}^3)^3$. There exists a unique function $p \in W_1(\mathbb{R}^3)|\mathbb{C}$ such that*

$$\textit{div}\,(a\nabla p) = f + \textit{div}\, g \textit{ in } \mathcal{D}'(\mathbb{R}^3).$$

4.2 Characterization of $\mathcal{N}_a(\mathcal{P})$

Proposition 4.4

$$\mathcal{N}_a(\mathcal{P}) := \text{span} \left\{ \nabla p_1^a, \nabla p_2^a, \nabla p_3^a \right\} \tag{65}$$

where $p_i^a := \tilde{p}_i^a + X_i$ *and* $\tilde{p}_i^a$ *is the unique function of* $W_1(\mathbb{R}^3)$ *that satisfies*

$$\begin{cases} \tilde{p}_i^a \in W_1(\mathbb{R}^3)|\mathbb{R}, \\ div\ (a\nabla\tilde{p}_i^a) = -\dfrac{\partial a}{\partial X_i} \ in\ \mathcal{D}'(\mathbb{R}^3), \\ \tilde{p}_i^a = \pm C_i + g_i^{\pm}(X_1, X_2, X_3) \ if\ \pm X_3 > \frac{1}{2}. \end{cases}$$

where $g_i^{\pm}$ *decays exponentially as* X_3 *goes to* $\pm\infty$*.*

Remark 4.5

- *If* $a = 1$*,* $\mathcal{N}_1(\mathcal{P}) = \mathbb{C}^3$*.*
- ∇p_1^a*,* ∇p_2^a *et* ∇p_3^a *satisfy the property* $\mathcal{P}^\infty$*, which means that they can be factorized in a constant and an exponentially decreasing function. Moreover,* $\forall\ i \in \{1,2,3\}$*,*
$$[\ell_T(\nabla p_i^a)] = 0, \quad [\ell_N(\nabla p_i^a)] = 0 \tag{66}$$
as well as
$$\left\{ \begin{array}{lll} \langle\ell_T(\nabla p_1^a)\rangle\cdot e_1 = 1, & \langle\ell_T(\nabla p_1^a)\rangle\cdot e_2 = 0, & \langle\ell_N(\nabla p_1^a)\rangle = 0, \\ \langle\ell_T(\nabla p_2^a)\rangle\cdot e_1 = 0, & \langle\ell_T(\nabla p_2^a)\rangle\cdot e_2 = 1, & \langle\ell_N(\nabla p_2^a)\rangle = 0, \\ \langle\ell_T(\nabla p_3^a)\rangle\cdot e_1 = 0, & \langle\ell_T(\nabla p_3^a)\rangle\cdot e_2 = 0, & \langle\ell_N(\nabla p_3^a)\rangle = 1. \end{array} \right. \tag{67}$$
- *If the periodic thin layer is a homogenous thin layer, that is*
$$a(X_1, X_2, X_3) := \left\{ \begin{array}{ll} a_0 & si\ |X_3| \le 1/2, \\ 1 & otherwise. \end{array} \right.$$
the functions p_1^a*,* p_2^a *and* p_3^a *are explicitly known:*
$$p_1^a = X_1, \quad p_2^a = X_2, \quad p_3^a = \left\{ \begin{array}{ll} \dfrac{1}{a_0} X_3 & if\ |X_3| \le 1/2, \\ X_3 + \dfrac{1}{2a_0}(1 - a_0) & otherwise. \end{array} \right. \tag{68}$$

Proof First, it is clear that any function belonging to (65) also belongs to $\mathcal{N}_a(\mathcal{P})$. Thus, it remains to show the converse inclusion. Proposition 3.5 (applied for $n = 0$) gives

$$\mathcal{U}(X_1, X_2, X_3) = \mathbf{U}^{\pm} + \sum_{(p,q)\in(X_3)^2\setminus(0,0)} \mathbf{U}^{\pm}_{m,n} e^{-2\pi\sqrt{m^2 + \frac{n^2}{\tau^2}}|X_3|}\ e^{2i\pi(mX_1 + \frac{n}{\tau}X_2)} \text{ if } \pm X_3 > \frac{1}{2}. \tag{69}$$

where $\mathbf{U}^{\pm}$ and $\mathbf{U}^{\pm}_{p,q}$ are some complex valued constant vectors. Note that $\mathbf{U}^+$ and $\mathbf{U}^-$ are not independent: integrating equations $\mathrm{curl}\,\mathcal{U} = 0$ and $\mathrm{div}\,(a\mathcal{U}) = 0$ over the truncated periodicity cell $B_{h_0} := \{(X_1, X_2, X_3) \in B$ such that $|X_3| \le h_0\}$, it is easily seen that $\mathbf{U}^+ - \mathbf{U}^- = 0$, so that $\mathbf{U}^{\pm} = \mathbf{U}$. Moreover, the following equalities hold: for all $(m,n) \in X_3^2 \setminus (0,0)$

$$\sqrt{m^2 + \frac{n^2}{\tau^2}}\ (\mathbf{U}_{m,n}\cdot e_1) + im\ (\mathbf{U}_{m,n}\cdot e_3) = 0, \qquad m\ (\mathbf{U}_{m,n}\cdot e_2) - \frac{n}{\tau}(\mathbf{U}_{m,n}\cdot e_1) = 0. \tag{70}$$

Besides, since curl$\mathcal{U} = 0$, there exists a function $p \in H^1_{loc}(\mathbb{R}^3)$ (defined up to a constant), such that $\mathcal{U} = \nabla p$. Let us characterize p. In view of formula (69), the behavior of p for $|X_3| > \frac{1}{2}$ is given by

$$p(X_1, X_2, X_3) = C^{\pm} + U^1 X_1 + U^2 X_2 + U^2 X_3 + \sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} p^{\pm}_{m,n} e^{-2\pi\sqrt{m^2+\frac{n^2}{\tau^2}}|X_3|} e^{2i\pi(mX_1+\frac{n}{\tau}X_2)} \tag{71}$$

where (see formula 70)

$$p^{\pm}_{m,n} = \begin{cases} \dfrac{\mathbf{U}_{m,n}\cdot e_1}{2\,i\,\pi\,m} & \text{if } m \neq 0, \\ \dfrac{\mathbf{U}_{m,n}\cdot e_2}{2\,i\,\pi\,n} & \text{if } m = 0. \end{cases} \tag{72}$$

For the sake of uniqueness, we impose $C^+ = -C^-$ (this arbitrary choice has no importance since we are interested in the gradient of p). Let us introduce $\tilde{p} := p - U^1X_1 - U^2X_2 - U^3X_3$. In view of Lemma A.1, it is clear that $\tilde{p}$ is periodic: indeed, A.1 ensures the existence of two complex valued constants C_1 and C_2 such that $p - C_1X_1 - C_2X_2$ is periodic. But, $p - U^1X_1 - U^2X_2 - U^3X_3$ is periodic for $|X_3| > 1/2$ (see 71), so that $C_1 = U^1$ and $C_2 = U^2$. Consequently, $\tilde{p}$ is $W_1(B)$ and satisfies

$$\text{div}\,(a\nabla\tilde{p}) = -U^1\frac{\partial a}{\partial X_1} - U^2\frac{\partial a}{\partial X_2} - U^3\frac{\partial a}{\partial X_3} \text{ in } \mathcal{D}'(\mathbb{R}^3).$$

Then, Proposition 4.3 ensures the well-posedness of the previous problem. So $\tilde{p} = U^1\nabla\tilde{p}^a_1 + U^2\nabla\tilde{p}^a_2 + U^3\nabla\tilde{p}^a_3$ where $\tilde{p}^a_i, i = 1,2,3$ are defined in Proposition 4.4. Finally

$$p = U^1(\tilde{p}^a_1 + X_1) + U^2(\tilde{p}^a_2 + X_2) + U^3(\tilde{p}^a_3 + X_3).$$

4.3 A new Friedrichs' Inequality

It is now clear that $\mathcal{P}$ is not well posed in $X_a(\mathbb{R}^3)$ since it has non-trivial solutions. However, it would be rational to prove well-posedness in the subspace $X^0_a(\mathbb{R}^3)$ of $X_a(\mathbb{R}^3)$ orthogonal to $\mathcal{N}_a(\mathcal{P})$ (with respect to the dot product (59)) defined by

$$X^0_a(\mathbb{R}^3) := \left\{\mathcal{U} \in X_a(\mathbb{R}^3) \text{ such that} \int_B \frac{\mathcal{U}}{1+(X_3)^2}\cdot\nabla p^a_i = 0 \quad \forall i \in \{1,2,3\}\right\}. \tag{73}$$

We also consider

$$X^0_a(B) := \left\{\mathcal{U} \in X_a(B) \text{ such that} \int_B \frac{\mathcal{U}}{1+(X_3)^2}\cdot\nabla p^a_i = 0 \quad \forall i \in \{1,2,3\}\right\}. \tag{74}$$

To this end, we prove the following Friedrichs' inequality:

Proposition 4.6 *There exists a constant $C > 0$ such that, for any $\mathcal{U} \in X^0_a(B)$*

$$\left\|\frac{\mathcal{U}}{\sqrt{1+(X_3)^2}}\right\|_{L^2(B)^3} \leq C\left(\|div\,(a\mathcal{U})\|_{L^2(B)} + \|curl\mathcal{U}\|_{L^2(B)}\right) \tag{75}$$

Proof This kind of result is well-known for bounded domains ([36],[37], [38], [35]).We remind the following proposition, which is the key ingredient of our proof:

Proposition 4.7 *Let $\Omega \subset \mathbb{R}^3$ be a bounded simply connected domain and ϵ a positive, definite, piecewise continuous matrix in Ω. Let u be a function of $H(curl\,,\Omega)$ such that $div\,(\epsilon u)$ also belongs to $L^2(\Omega)$ and that satisfies $n \times u = 0$ on $\partial\Omega$. Then, there exists a positive constant C (which depends only of Ω and ϵ) such*

$$\|u\|_{L^2(\Omega)} \le C \left(\|curl\, u\|_{L^2(\Omega)} + \|div\,(\epsilon u)\|_{L^2(\Omega)} \right).$$

We prove Proposition 4.6 by a contradiction argument. Let us assume that there is a sequence $(\mathcal{U}_n)_{n\in\mathbb{N}}$ such that

$$\text{(a)}: \left\| \frac{\mathcal{U}_n}{\sqrt{1+(X_3)^2}} \right\|_{L^2(B)} = 1,$$

$$\text{(b)}: \lim_{n\to+\infty} \|\text{div}\ (a\mathcal{U}_n)\|_{L^2(B)} = 0 \text{ and } \lim_{n\to+\infty} \|\text{curl}\,\mathcal{U}_n\|_{L^2(B)} = 0\,.$$

Step 1: we prove that $\mathcal{U}_n/\sqrt{1+(X_3)^2}$ weakly tends to 0: since $\mathcal{U}_n/\sqrt{1+(X_3)^2}$ is bounded $L^2(B)^3$, then, up to a subsequence, $\mathcal{U}_n/\sqrt{1+(X_3)^2}$, weakly converges to $\mathcal{V}$ in $L^2(B)$. Let us denote $\mathcal{U} := \sqrt{1+(X_3)^2}\mathcal{V}$. $\mathcal{U}$ is in $X_a^0(B)$. Besides, the assumption (b) yields curl$\,\mathcal{U} = 0$ and div$\,(a\mathcal{U}) = 0$. Consequently $\mathcal{U} \in X_a^0(B) \cap \mathcal{N}_a(\mathcal{P})$ so that $\mathcal{U} = 0$.

Step 2: localization process: let us now consider two smooth truncation functions χ_1 and χ_2 (Fig.3) that take values in $[0,1]$ and that satisfy: for any $(X_1, X_2, X_3) \in B$,

$$\chi_1(X_1,X_2,X_3) = \begin{cases} 1 & \text{if } 3 \le |X_3| \le 4, \\ 0 & \text{if } |X_3| \le 2 \text{ ou } |X_3| \ge 5. \end{cases} \qquad \chi_2(X_1,X_2,X_3) = \begin{cases} 1 & \text{if } |X_3| \le 3, \\ 0 & \text{if } |X_3| \ge 4, \end{cases} \tag{76}$$

Note that the supports of $\nabla\chi_1$ and $\nabla\chi_2$ are both included in the area where a is constant. Let us introduce the domains B_1, B_2 B_1^∇, B_2^∇ and B_4:

$$B_1 := supp(\chi_1), \quad B_2 := supp(\chi_2), \quad B_1^\nabla := supp(\nabla\chi_1), \quad B_2^\nabla := supp(\nabla\chi_2)$$
$$B_{ext} := \{(X_1,X_2,X_3) \in B \text{ tels que } |X_3| \ge 3\}$$

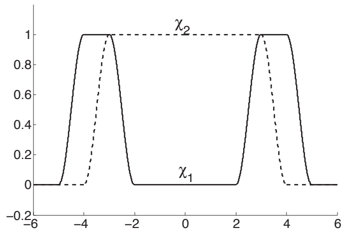

Figure 3: Truncation functions

Let $\mathcal{W}_n = (1-\chi_2)\mathcal{U}_n$ and $\mathcal{Z}_n = \chi_2 \mathcal{U}_n$. The objective of the next three steps is to prove that

$$\lim_{n\to+\infty} \left\| \frac{\mathcal{W}_n}{\sqrt{1+(X_3)^2}} \right\|_{L^2(B)} = \lim_{n\to+\infty} \left\| \frac{\mathcal{Z}_n}{\sqrt{1+(X_3)^2}} \right\|_{L^2(B)} = 0, \tag{77}$$

which contradicts the original assumption since

$$\lim_{n\to+\infty} \left\| \frac{\mathcal{U}_n}{\sqrt{1+(X_3)^2}} \right\|_{L^2(B)} \leq \lim_{n\to+\infty} \left\| \frac{\mathcal{W}_n}{\sqrt{1+(X_3)^2}} \right\|_{L^2(B)} + \lim_{n\to+\infty} \left\| \frac{\mathcal{Z}_n}{\sqrt{1+(X_3)^2}} \right\|_{L^2(B)} .$$

Step 3: estimate of $\left\| \mathcal{W}_n / \sqrt{1+(X_3)^2} \right\|_{L^2(B)}$ by means of the Hardy inequality: let us consider

$$\begin{aligned} W_0^1(B_{ext}) := \Big\{ & w \in H^1_{loc}(B_{ext}) \text{ periodic such that } w(X_1, X_2, \pm 3) = 0, \\ & \int_{B_{ext}} \frac{|w|^2}{1+(X_3)^2} + |\nabla w|^2 < +\infty \Big\} \end{aligned} \tag{78}$$

equipped with the norm $\|w\|_{W_1^0(B_{ext})} = \|\nabla w\|_{W_1^0(B_{ext})}$. In view of the Hardy inequality (see [39]) , the semi-norm of the gradient is a norm. Moreover, by integration by parts (cf. lemma 5.4.2 in [40]) and using the density of $\mathcal{C}^\infty_{per}(B_{ext})$ into $W_0^1(B_{ext})$ (cf. [41]), we can prove that any function $w \in W_1^0(B_{ext})$ satisfies:

$$\|\nabla w\|^2_{L^2(B_{ext})} = \|\mathrm{curl}\, w\|^2_{L_2(B_{ext})} + \|\mathrm{div}\, w\|^2_{L^2(B_{ext})} .$$

Since $a = 1$ in B_{ext}, it is clear that $\mathcal{W}_n \in W_1^0(B_{ext})$. Consequently,

$$\|\nabla \mathcal{W}_n\|^2_{L^2(B_{ext})} = \|\mathrm{curl}\, \mathcal{W}_n\|^2_{L_2(B_{ext})} + \|\mathrm{div}\, \mathcal{W}_n\|^2_{L^2(B_{ext})} .$$

Note that,

$$\mathrm{div}\, \mathcal{W}_n = (1-\chi_2)\mathrm{div}\, \mathcal{U}_n + \nabla\chi_2 \cdot \mathcal{U}_n,$$

that $\nabla\chi_2 \cdot \mathcal{U}_n$ is compactly supported in B_2^∇ and $(1-\chi_2)\mathrm{div}\, \mathcal{U}_n = (1-\chi_2)\mathrm{div}\,(a\mathcal{U}_n)$. So,

$$\|\mathrm{div}\, \mathcal{W}_n\|^2_{L^2(B_{ext})} \leq \|\mathrm{div}\,(a\mathcal{U}_n)\|^2_{L^2(B)} + \|\chi_2'\|^2_{L^\infty(\mathbb{R})} \|\mathcal{U}_n\|^2_{L^2(B_2^\nabla)}$$

Similarly,

$$\|\mathrm{curl}\, \mathcal{W}_n\|^2_{L^2(B_{ext})} \leq \|\mathrm{curl}\, \mathcal{U}\|^2_{L^2(B)} + \|\chi_2'\|^2_{L^\infty(\mathbb{R})} \|\mathcal{U}_n\|^2_{L^2(B_2^\nabla)}$$

so that

$$\|\nabla \mathcal{W}_n\|^2_{L^2(B)} \leq \|\mathrm{curl}\, \mathcal{U}_n\|^2_{L^2(B)} + \|\mathrm{div}\ (a\mathcal{U}_n)\|^2_{L^2(B)} + 2\,\|\chi_2'\|^2_{L^\infty(\mathbb{R})} \|\mathcal{U}_n\|^2_{L^2(B_2^\nabla)} .$$

It follows from the Hardy's inequality that

$$\begin{aligned} \left\| \frac{\mathcal{W}_n}{\sqrt{1+(X_3)^2}} \right\|^2_{L^2(B)} &= \left\| \frac{\mathcal{W}_n}{\sqrt{1+(X_3)^2}} \right\|^2_{L^2(B_{ext})} \\ &\leq C \left(\|\mathrm{curl}\, \mathcal{U}_n\|^2_{L^2(B)} + \|\mathrm{div}\ (a\mathcal{U}_n)\|^2_{L^2(B)} + 2\,\|\chi_2'\|^2_{L^\infty(\mathbb{R})} \|\mathcal{U}_n\|^2_{L^2(B_2^\nabla)} \right) . \end{aligned} \tag{79}$$

Step 4: estimate of $\left\| \mathcal{Z}_n / \sqrt{1+(X_3)^2} \right\|_{L^2(B)}$ with the help of Proposition 4.7: $\mathcal{Z}_n = \chi_2 \mathcal{U}_n$ is compactly supported in B_2 and satisfies $\mathcal{Z}_n \times e_3 = 0$ one the upper and lower boundaries of B_2, Proposition 4.7 (easily generalized to periodic functions) gives

$$\begin{aligned} \|\mathcal{Z}_n\|^2_{L^2(B)} &= \|\mathcal{Z}_n\|^2_{L^2(B_2)} \leq C \left(\|\text{curl}\, \mathcal{Z}_n\|^2_{L_2(B_2)} + \|\text{div}\ (a\mathcal{Z}_n)\|^2_{L^2(B_3)} \right), \\ &\leq C \left(\|\text{curl}\, \mathcal{U}_n\|^2_{L_2(B)} + \|\text{div}\ (a\mathcal{U}_n)\|^2_{L^2(B)} + \|\chi_2'\|^2_{L^\infty(\mathbb{R})} \|\mathcal{U}_n\|^2_{L^2(B_2^\nabla)} \right) \quad (80) \end{aligned}$$

Step 5: estimate of $\|\mathcal{U}_n\|^2_{L^2(B_2^\nabla)}$ and conclusion: to end the proof and to obtain (77), it remains to prove that $\|\mathcal{U}_n\|^2_{L^2(B_2^\nabla)}$ goes to 0. Indeed, if $\|\mathcal{U}_n\|^2_{L^2(B_2^\nabla)}$ tends to 0, then the right hand sides of inequalities (80) and (80) tends also to 0 (by assumption (b) $\|\text{div}\ (a\mathcal{U}_n)\|_{L^2(B)}$ and $\|\text{curl}\,\mathcal{U}_n\|_{L_2(B)}$ go to 0). To do so, we consider $\mathcal{V}_n = \chi_1 \mathcal{U}_n$. Since $\mathcal{U}_n$ is bounded in the weighted $L^2(B)$ norm, then the L^2 norm of $\mathcal{U}_n$ is bounded is any bounded domain, in particular in B_1. So,

$$\|\mathcal{V}_n\|_{L^2(B_1)} + \|\text{div}\, \mathcal{V}_n\|_{L^2(B_1)} + \|\text{curl}\, \mathcal{V}_n\|_{L^2(B_1)} \leq C, \quad \text{and} \quad \mathcal{V}_n \times n = 0 \text{ on } \partial\Gamma_1$$

where Γ_1 denotes the subspace of ∂B_1 with outward normal equal to $\pm e_1$. Consequently, $\mathcal{V}_n$ is bounded in $H^1(B_1)$ (It suffises to apply Proposition 4.7, since B_1 is made of two convexe polyhedrons). So $\mathcal{V}_n$ converges strongly to $\mathcal{W}$ in $L^2(B_1)$. Since, besides, $\mathcal{U}_n$ weakly tends to 0 in any bounded subspace of B, then $\mathcal{V}_n$ tends to 0 in $L^2(B_1)$. Since $B_2^\nabla \subset B_1$ We get

$$\lim_{n\to+\infty} \|\mathcal{U}_n\|_{L^2(B_2^\nabla)} = 0, \tag{81}$$

which ends the proof.

From Friedrichs' inequality (75), we immediately deduce a first well-posedness result:

Proposition 4.8 *Let f, g and h be three functions such that $f \in L^2(B)$, $g \in L^2(B)$, and $\sqrt{1+(X_3)^2}h \in L^2(B)$. Then, the following is well posed: find $\mathcal{U} \in X_a^0$ such that, $\forall\, \mathcal{V} \in X_a^0$,*

$$\int_B \mathit{curl}\,\mathcal{U} \cdot \overline{\mathit{curl}\,\mathcal{V}} + \int_B \mathit{div}\,(a\mathcal{U}) \cdot \overline{\mathit{div}\,(a\mathcal{V})} = \int_B f \cdot \overline{\mathit{curl}\,\mathcal{V}} + \int_B g \cdot \overline{\mathit{div}\,(a\mathcal{V})} + \int_B h \cdot \overline{\mathcal{V}}$$

4.4 Well-posedness result

Proposition 4.9 *Let f and g be two functions such that $f \in L^2_{per}(\mathbb{R}^3)$, $\mathit{div}\, f = 0$ and $g \in L^2_{per}(\mathbb{R}^3)$.Then, the following problem is well posed: find $\mathcal{U} \in X_a^0(B)$ such that*

$$\begin{cases} \mathit{div}\, a\mathcal{U} = g \text{ in } \mathcal{D}'(B), \\ \mathit{curl}\,\mathcal{U} = f \text{ in } \mathcal{D}'(B). \end{cases} \tag{82}$$

Proof The proof is an adaptation of the proof of theorem 5 in [34] to the unbounded domain B.

1. Variational form associated with problem (82): Let $\mathcal{U}$ be a solution of problem (82) and let $\varphi \in X_a^0(B)$ be a test function. Then, it is easily seen that

   $$\forall \varphi \in X_a^0(B) \quad \int_B \text{curl}\,\mathcal{U} \cdot \text{curl}\,\varphi + \int_B \text{div}\,(a\mathcal{U}) \cdot \text{div}\,(a\varphi) = \int_B f \cdot \text{curl}\,\varphi + \int_B g\text{div}\,(a\varphi) \quad (83)$$

   By proposition 4.8, (83) has a unique solution.

2. From the variational form to the PDE. This is the most complicated step.

    - Divergence Equation: let $\mathcal{U}$ be the unique solution of the variational problem (83). Note first that variational form (83) remains valid for test functions φ belonging $X_a(B)$: indeed, for $i = 1, 2$ or 3, div $a\nabla p_i^a =$ curl $\nabla p_i^a = 0$.

      Let us consider $h \in \mathcal{D}(B)$ such $\int_B h = 0$. Then there exists $p \in W_1(B)$ such that div $(a\nabla p) = h$. In equation (83), we can take $\varphi = \nabla p$. So,

      $$\int_B (\text{div}\,(a\mathcal{U}) - g)h = 0 \quad \forall h \in \mathcal{D}(B) \text{ such that } \int_B h = 0. \tag{84}$$

      But, in addition, we can proof the following lemma (Lemma A.2 in Appendix):

      Lemma A.2 *Let $h \in \mathcal{D}(B)$. Then, there exists a sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(B)$ such that*

      – $\displaystyle\int_B h_n = 0,$

      – $\displaystyle\lim_{n\to+\infty} \|h_n - h\|_{L^2(B)} = 0.$

      It follows that, since div $(a\mathcal{U}) - g \in L^2(B)$, equality (84) still holds for any function h in $\mathcal{D}(B)$, which exactly means that

      $$\text{div}\,(a\mathcal{U}) = g \quad \text{dans } \mathcal{D}'(B).$$

    - Rotational Equation: Let us first give a lemma, whose proof is given in Appendix (Lemma A.5) (This lemma is an adaptation of theorem 3.39 in [25]):

      Lemma A.5 *Soit $f \in L^2_{per}(\mathbb{R}^3)^3$ such that div $f = 0$. Then, there exists a function $w_a \in X_a(\mathbb{R}^3)$ such that*

      $$\begin{cases} \text{div}\,(aw_a) = 0 \text{ in } \mathcal{D}'(\mathbb{R}^3), \\ \text{curl}\,(w_a) = f \text{ in } \mathcal{D}'(\mathbb{R}^3)^3. \end{cases}$$

      Since div $f = 0$, the previous lemma ensures that there exists a function $\mathcal{W} \in X_a(B)$ such that

      $$\text{curl}\,\mathcal{W} = f \quad \text{et} \quad \text{div}\,(a\mathcal{W}) = 0.$$

      So, $\mathcal{W} - \mathcal{U} \in X_a(B)$ and

      $$\int_B \text{curl}\,\mathcal{W} \cdot \overline{\text{curl}\,\varphi} = \int_B f \cdot \overline{\text{curl}\,\varphi} \;\; \forall \varphi \in X_a,$$

      $$\int_B \text{curl}\,\mathcal{U} \cdot \overline{\text{curl}\,\varphi} = \int_B f \cdot \overline{\text{curl}\,\varphi} \;\; \forall \varphi \in X_a.$$

      Subtracting the two previous equalities and taking $\varphi = \mathcal{W} - \mathcal{U}$, we get

      $$\int_B |\text{curl}\,(\mathcal{W} - \mathcal{U})|^2 = 0.$$

      Consequently, we recover the rotational equation curl $\mathcal{U} =$ curl $\mathcal{W} = f$ in $\mathcal{D}'(B)$.

We deduce immediately the following property:

Proposition 4.10 *Let us consider f and g two functions such that $f \in L^2_{per}(\mathbb{R}^3)$, $div\, f = 0$, $g \in L^2_{per}(\mathbb{R}^3)$, and f and g satisfy property $\mathcal{P}^\infty$(see 48), and two constants $a_T = (a^1, a^2) \in \mathbb{C}^2$, $a_N \in \mathbb{C}$. Then, the following problem has a unique solution: find $\mathcal{U} \in X_a(B)$ such that*

$$\begin{cases} div\, a\mathcal{U} = g \text{ in } \mathcal{D}'(B), \\ curl\mathcal{U} = f \text{ in } \mathcal{D}'(B), \\ \langle \ell_T(\mathcal{U})\rangle = a_T, \\ \langle \ell_N(\mathcal{U})\rangle = a_N, \end{cases} \tag{85}$$

Moreover,

$$\|\mathcal{U}\|_{X_a} \leq C\left(|a_N| + |a_T| + \|g\|_{L^2(B)} + \|f\|_{L^2(B)}\right) \tag{86}$$

4.5 Summary

To summarize, near field problems (41) defined the near field terms up to the specification of three constants associated with the three functions of the kernel $\mathcal{N}_a(\mathcal{P})$. If we further specify quantities $\langle \ell_T(\mathcal{E}_n)\rangle$, $\langle \ell_N(\mathcal{E})\rangle$, $\langle \ell_T(\mathcal{H}_n)\rangle$, $\langle \ell_N(\mathcal{H})\rangle$, we completely determine the solution. These additional quantities are precisely given by the matching conditions (55).
Note also that jump values $[\ell_T(\mathcal{E}_n)]$ and $[\ell_T(\mathcal{E}_n)]$ (in opposition to the mean values) cannot be forced since the functions of the kernel $\mathcal{N}_a(\mathcal{P})$ does not have a jump (cf. (67)). In fact, jump values result from the resolution of near field problems and will be used as source terms in the far field problems.

5 Variational framework for the far field problem

The framework to solve far field problems is much simpler. Eliminating the magnetic field $\mathbf{H}_n$ in equations (34), reminding about the transmission conditions (57), we can see that far field problems are of the following form: find $\mathbf{E}$ satisfying

$$\begin{cases} \text{curl}\,\text{curl}\,\mathbf{E} - \omega^2\mathbf{E} = f \text{ in } \Omega^\pm, \\ [(\mathbf{E})_T]_\Gamma = g_1 \text{ on } \Gamma, \\ [\text{curl}\,\mathbf{E}\times n]_\Gamma = g_2 \text{ sur } \Gamma, \\ \text{curl}\,\mathbf{E}\times n = i\omega(\mathbf{E})_T \text{ on } \Sigma_3^\pm. \end{cases} \tag{87}$$

together with periodic boundary conditions

$$\mathbf{E}\times e_{i|\Sigma_i^-} = \mathbf{E}\times e_{i|\Sigma_i^+}, \quad \text{curl}\,\mathbf{E}\times e_{i|\Sigma_i^-} = \text{curl}\,\mathbf{E}\times e_{i|\Sigma_i^+}, \quad i = 1,2 \tag{88}$$

Here n denotes the outward unit normal on $\Sigma_3^\pm$, and $n := e_3$ on the interface Γ. It is natural to find $\mathbf{E}$ in the space $H(\Omega)$:

$$H(\Omega) := \Big\{\mathbf{E} \in H(\text{curl}\,,\Omega^+)\cap H(\text{curl}\,,\Omega^-) \text{ such that } (\mathbf{E})_T \in L^2_t(\Sigma^\pm_{x_3}), [(\mathbf{E})_T] \in H^{1/2}(\Gamma), \\ \mathbf{E}\times e_{i|\Sigma_i^-} = \mathbf{E}\times e_{i|\Sigma_i^+}, \quad i = 1,2\Big\}. \tag{89}$$

equipped with the norm $\|\mathbf{E}\|^2_{H(\Omega)} := \|\mathbf{E}\|^2_{L^2(\Omega)} + \|\text{curl}\,\mathbf{E}\|^2_{L^2(\Omega)} + \|\mathbf{E}_T\|^2_{L^2(\Sigma_3^\pm)} + \|[\mathbf{E}_T]\|^2_{H^{1/2}(\Gamma)}$. We have the following well-known well-posedness result (see for instance [29]).

Proposition 5.1 *If $g_1 \in H^{1/2}(\Gamma)$, $g_2 \in H^{-1/2}(div_\Gamma, \Gamma)$ and $f \in L^2(\Omega^\pm)$ is compactly supported in $\Omega^\pm$, then there exists a unique solution $\mathbf{E} \in H(\Omega)$ that satisfies* (87) *and* (88)*. Moreover, there exists a positive constant C such that*

$$\|\mathbf{E}\|_{H(\Omega)} \leq C\big(\|g_1\|_{H^{1/2}(\Gamma)} + \|g_2\|_{H^{1/2}(div_\Gamma,\Gamma)} + \|f\|_{L^2(\Omega^\pm)}\big)$$

6 Existence and uniqueness of the asymptotic expansion

We are now in a position to define all the terms of the asymptotic expansion. The construction is done by induction starting from the explicit construction of the zeroth order terms.
As we have already said, far field terms $\mathbf{E}_n$ and $\mathbf{H}_n$ are in $H(\Omega)$ (see definition (89)). As for the near fields terms $\mathcal{E}_n$ and $\mathcal{H}_n$, they are respectively in the spaces $\mathcal{C}^\infty(\Gamma, V_\epsilon^+(B))$ and $\mathcal{C}^\infty(\Gamma, V_\mu^+(B))$ (see definitions (46)-(47)).

Reformulation of the recurrent problems

For any $n \in \mathbb{N}$, we want to solve the system of equations made of far field equations (34), near field equations (41) and matching conditions (55). We shall reformulate this system taking into account the analysis carried out in the previous two sections (sections 4 and 5): in the end, we consider the following system: $\forall n \in \mathbb{N}$, find $\mathbf{E}_n \in H(\Omega)$, $\mathbf{H}_n \in H(\Omega)$, $\mathcal{E}_n \in \mathcal{C}^\infty(\Gamma, V_\epsilon^+(B))$, $\mathcal{H}_n \in \mathcal{C}^\infty(\Gamma, V_\mu^+(B))$ such that

$$\begin{cases} \text{curl}\,\text{curl}\,\mathbf{E}_n - \omega^2\mathbf{E}_n = \delta_0^n F, \\ [(\mathbf{E}_n)_T]_\Gamma = g_n \times e_3, \\ [\text{curl}\,\mathbf{E}_n \times e_3]_\Gamma = i\omega h_n, \\ \text{curl}\,\mathbf{E}_n \times e_{i|\Sigma_i^+} = \text{curl}\,\mathbf{E}_n \times e_{i|\Sigma_i^-}, \quad i = 1, 2, \\ \text{curl}\,\mathbf{E}_n \times n - i\omega(\mathbf{E}_n^T) = 0 \text{ on } \Sigma_3^\pm, \end{cases}$$

$$\mathbf{H}_n := \frac{1}{i\omega}\text{curl}\,\mathbf{E}_n,$$

$$\begin{cases} \text{curl}\,\mathcal{E}_n = -\mathcal{A}_0\mathcal{E}_{n-1} + i\omega\,\mu\,\mathcal{H}_{n-1}, \\ \text{div}\,(\epsilon\mathcal{E}_n) = -\,\epsilon\,\text{div}_\Gamma\,(\mathcal{E}_{n-1})_T\,. \end{cases} \tag{90}$$

$$\begin{cases} -\text{curl}\,\mathcal{H}_n = +\mathcal{A}_0\mathcal{H}_{n-1} + i\omega\,\epsilon\,\mathcal{E}_{n-1}, \\ \text{div}\,(\mu\mathcal{H}_n) = -\mu\,\text{div}_\Gamma\,(\mathcal{H}_{n-1})_T\,. \end{cases}$$

$$\begin{cases} \langle \ell_T^{\pm}(\mathcal{E}_n) \rangle = \langle (\mathbf{E}_n)_T \rangle_\Gamma \\ \langle \ell_T^{\pm}(\mathcal{H}_n) \rangle = \langle (\mathbf{H}_n)_T \rangle_\Gamma \\ \langle \ell_N^{\pm}(\mathcal{E}_n) \rangle = -\frac{1}{i\omega} \mathrm{curl}_\Gamma \langle (\mathbf{H}_n)_T \rangle_\Gamma \\ \langle \ell_N^{\pm}(\mathcal{H}_n) \rangle = \frac{1}{i\omega} \mathrm{curl}_\Gamma \langle (\mathbf{E}_n)_T \rangle_\Gamma \end{cases}$$

where,

$$g_n = \frac{1}{\tau} \left(\int_{B_{h_0}} \big(-\mathcal{A}_0(\mathcal{E}_{n-1}) + i\omega\mu\mathcal{H}_{n-1} \big)_T \right) - \left(\sum_{k=1}^{n} (e_3 \times C_{n,k}^{+}) h_0^k - \sum_{k=1}^{n} (e_3 \times C_{n,k}^{-})(-h_0)^k \right),$$

$$h_n = \frac{1}{\tau} \left(\int_{B_{h_0}} \big(-\mathcal{A}_0(\mathcal{H}_{n-1}) - i\omega\epsilon\mathcal{H}_{n-1} \big)_T \right) - \left(\sum_{k=1}^{n} (e_3 \times D_{n,k}^{+}) h_0^k - \sum_{k=1}^{n} (e_3 \times D_{n,k}^{-})(-h_0)^k \right). \tag{91}$$

6.1 Construction of $\mathbf{E}_0$, $\mathbf{H}_0$, $\mathcal{E}_0$ et $\mathcal{H}_0$

6.1.1 Construction of $\mathbf{E}_0$ et $\mathbf{H}_0$

From (90), $\mathbf{E}_0$ satisfies homogeneous jump conditions, namely $[n \times \mathbf{E}_0]_\Gamma = [n \times \mathrm{curl}\, \mathbf{E}_0]_\Gamma = 0$ sur Γ. It follows that $\mathbf{E}_0$ is the unique solution of the following problem: find $\mathbf{E}_0 \in V$ (V is defined by (12)) such that

$$\begin{cases} \mathrm{curl}\,\mathrm{curl}\, \mathbf{E}_0 - \omega^2 \mathbf{E}_0 = F \text{ in } \Omega \\ \mathrm{curl}\, \mathbf{E}_0 \times n - i\omega \mathbf{E}_0 = 0 \text{ on } \Sigma_3^{\pm}. \end{cases} \tag{92}$$

We deduce immediately $\mathbf{H}_0$,

$$\mathbf{H}_0 := \frac{1}{i\omega} \mathrm{curl}\, \mathbf{E}_0. \tag{93}$$

6.1.2 Construction of $\mathcal{E}_0$ et $\mathcal{H}_0$

$\mathcal{E}_0$ satisfies the following electrostatic problem: find $\mathcal{E}_0 \in V_\epsilon^+(B)$

$$\begin{cases} \mathrm{curl}\, \mathcal{E}_0 = 0 \text{ in } B, \\ \mathrm{div}\, (\epsilon \mathcal{E}_0) = 0 \text{ in } B, \end{cases}$$

and the matching conditions $\langle \ell_T(\mathcal{E}_0) \rangle = \langle (\mathbf{E}_0)_T \rangle_\Gamma$ and $\langle \ell_N^{\pm}(\mathcal{E}_0) \rangle = -\frac{1}{i\omega} \mathrm{curl}_\Gamma \langle (\mathbf{H}_0)_T \rangle_\Gamma$. Then, Proposition 4.4 directly yields

$$\mathcal{E}_0 \;=\; \langle \mathbf{E}_0^1 \rangle_\Gamma \nabla p_1^\epsilon \;+\; \langle \mathbf{E}_0^2 \rangle_\Gamma \nabla p_2^\epsilon \;-\; \frac{1}{i\omega} \mathrm{curl}_\Gamma \langle (\mathbf{H}_0)_T \rangle_\Gamma \nabla p_3^\epsilon. \tag{94}$$

Note that the fast and slow variables are separated. As for $\mathcal{H}_0 \in V_\mu^+(B)$ it satisfies

$$\begin{cases} \mathrm{curl}\, \mathcal{H}_0 = 0 \text{ in } B, \\ \mathrm{div}\, (\mu \mathcal{H}_0) = 0 \text{ in } B. \end{cases}$$

and the matching conditions $\ell_T(\mathcal{H}_0) = \langle \mathbf{H}_{0T} \rangle$, and $\langle \ell_N^{\pm}(\mathcal{E}_0) \rangle = \frac{1}{i\omega} \mathrm{curl}_\Gamma \langle (\mathbf{E}_0)_T \rangle$. So, applying again Proposition 4.4, we get

$$\mathcal{H}_0 \;=\; \langle \mathbf{H}_0^1 \rangle_\Gamma \nabla p_1^\mu \;+\; \langle \mathbf{H}_0^2 \rangle_\Gamma \nabla p_2^\mu \;+\; \frac{1}{i\omega} \mathrm{curl}_\Gamma \langle (\mathbf{E}_0)_T \rangle_\Gamma \nabla p_3^\mu. \tag{95}$$

6.2 A general result of existence and uniqueness of the asymptotic expansion

Theorem 6.1 *For any $n \in \mathbb{N}$, Problem (90) is well-posed.*

Proof The uniqueness of (90) is obvious. It remains to prove the existence by induction. The initialization has been done in the previous paragraph. Let us assume that, for any $k < n$, system (90) is well-posed.

In a first step, we build $\mathbf{E}_n$: by assumption, the right hand sides g_n and h_n, completely defined by (91) are in $\mathcal{C}^\infty(\Gamma)$. Consequently, $g_n \times e_3$ is in $H^{1/2}(\Gamma)$ and h_n is in $H^{-1/2}(\Gamma)$. Then $\mathbf{E}_n$ is the unique solution of the following problem: find $\mathbf{E}_n \in H(\Omega)$ such that

$$\begin{cases} \operatorname{curl}\operatorname{curl}\mathbf{E}_n - \omega^2\mathbf{E}_n = \delta_0^n F, \\ [(\mathbf{E}_n)_T]_\Gamma = g_n \times e_3, \\ [\operatorname{curl}\mathbf{E}_n \times e_3]_\Gamma = i\omega h_n, \\ \operatorname{curl}\mathbf{E}_n \times e_{i|\Sigma_i^+} = \operatorname{curl}\mathbf{E}_n \times e_{i|\Sigma_i^-}, \quad i = 1, 2, \\ \operatorname{curl}\mathbf{E}_n \times n - i\omega(E_n^T) = 0 \text{ on} \Sigma_{x_3}^\pm, \end{cases}$$

We point out that $\mathbf{E}_n$ is smooth in the vicinity of Γ. Then $\mathbf{H}_n$ is defined $\mathbf{H}_n := \frac{1}{i\omega}\operatorname{curl}\mathbf{E}_n$. We still have to build the near field terms $\mathcal{E}_n$ and $\mathcal{H}_n$. We remind that $\mathcal{E}_n$ satisfies property $\mathcal{P}^\infty$: for large X_3, $\mathcal{E}_n$ is given by

$$\mathcal{E}_n = \sum_{k=0}^{n} C_{n,k}^\pm(x_1, x_2)X_3^k + g^\pm,$$

where the functions $C_{n,k}^\pm$ are known for $k \geq 1$ and $g^\pm$ are exponentially decreasing functions. Let χ be a truncation function in $\mathcal{C}^\infty(\mathbb{R})$ that satisfies

$$\chi(X_3) = \begin{cases} 1 \text{ if } X_3 > 2, \\ 0 \text{ if } X_3 < 1. \end{cases}$$

We consider $\mathcal{P}_n$:

$$\mathcal{P}_n = \chi(X_3)\sum_{k=1}^{n} C_{n,k}^+(x_1, x_2)X_3^k + \chi(-X_3)\sum_{k=1}^{n} C_{n,k}^-(x_1, x_2)X_3^k.$$

It is clear that $\ell_T^\pm(\mathcal{P}_n) = 0$ and $\ell_N^\pm(\mathcal{P}_n) = 0$. Besides, using formulas (52) and (53), it is easily seen that $(-\mathcal{A}_0(\mathcal{E}_{n-1}) + i\omega\mu\mathcal{H}_{n-1} - \operatorname{curl}\mathcal{P}_n)$ and $(-\epsilon\operatorname{div}_\Gamma(\mathcal{E}_{n-1}) - \epsilon_\infty\operatorname{div}\mathcal{P}_n)$ are exponentially decreasing for large X_3. Applying proposition 4.10, we can define $\mathcal{V}_n$, as the unique solution in $\mathcal{C}^\infty(\Gamma, X_a(B))$ of the following problem:

$$\begin{cases} \operatorname{curl}\mathcal{V}_n = -\mathcal{A}_0(\mathcal{E}_{n-1}) + i\omega\mu\mathcal{H}_{n-1} - \operatorname{curl}\mathcal{P}_n, \\ \operatorname{div}(\epsilon\mathcal{V}_n) = -\epsilon\operatorname{div}_\Gamma(\mathcal{E}_{n-1}) - \epsilon_\infty\operatorname{div}\mathcal{P}_n, \\ \langle \ell_T(\mathcal{V}_n)\rangle = 0, \\ \langle \ell_N(\mathcal{V}_n)\rangle = 0. \end{cases}$$

Then we seek $\mathcal{E}_n$ of the form

$$\mathcal{E}_n = \mathcal{V}_n + \mathcal{P}_n + a\nabla p_1^\epsilon + b\nabla p_2^\epsilon + c\nabla p_3^\epsilon.$$

The matching conditions $\langle \ell_N(\mathcal{E}_n)\rangle = \langle (\mathbf{E}_n)_N\rangle_\Gamma$ and $\langle \ell_T(\mathcal{E}_n)_T\rangle = \langle (\mathbf{E}_n)_T\rangle_\Gamma$ give $a = \langle \mathbf{E}_n \cdot e_1\rangle_\Gamma$, $b = \langle \mathbf{E}_n \cdot e_2\rangle_\Gamma$ and $c = \langle \mathbf{E}_n \cdot e_3\rangle_\Gamma$, which ends the construction of $\mathcal{E}_n$. $\mathcal{H}_n$ is build in the same way.

7 Error estimates: asymptotic expansion justification

To end our investigation, it remains to prove that the truncated expansion tends toward the exact solution $\mathbf{E}^\delta$. Our main result gives an optimal error estimate on the far field error, namely when the error between the exact solution and the far field truncated series $\sum_{k=0}^{n} \delta^k \mathbf{E}_k$.

Theorem 7.1 *Let $0 < \gamma < \frac{L_3}{2}$ and $\Omega_\gamma := \{(x_1, x_2, z) \in \Omega, |z| > \gamma\}$. For any $n \in \mathbb{N}$, there exist a constant $C_n > 0$ and a constant $\delta_\gamma > 0$ such that,*

$$\forall\, \delta < \delta_\gamma, \qquad \left\| \mathbf{E}^\delta - \sum_{k=0}^{n} \delta^k \mathbf{E}_k \right\|_{H(curl\,,\Omega_\gamma)} \leq C_n\ \delta^{n+1}.$$

We only sketch the proof (A detailed proof can be found in [29], similar results can be found in [6]).This estimation is obtained in three main steps:

- In a first step, for any $n \in \mathbb{N}$, we build a global approximation of the exact solution that coincides with the first n terms of the far field expansion far from the thin layer

$$\mathbf{E}^n_{e,\delta} := \sum_{k=0}^{n} \delta^k \mathbf{E}_k, \tag{96}$$

  and with the first n terms of the near field expansion in the vicinity of the periodic thin layer:

$$\mathcal{E}^n_{i,\delta} := \sum_{k=0}^{n} \delta^k \mathcal{E}_k. \tag{97}$$

  This approximation is built with the help of a truncation function χ that satisfies

$$\chi(s) = \begin{cases} 1 \text{ if } |s| \leq 1, \\ 0 \text{ if } |s| \geq 2, \end{cases} \tag{98}$$

  and a positive distance function $\eta(\delta)$ such that

$$\lim_{\delta \to 0} \eta = 0 \quad \text{et} \quad \lim_{\delta \to 0} \frac{\eta}{\delta} = +\infty. \tag{99}$$

  Then, considering $\chi_\eta(z) := \chi(\frac{z}{\eta})$, we define our global approximation by,

$$\mathbf{E}^n_{\eta,\delta} := (1 - \chi_\eta)\mathbf{E}^n_{e,\delta} + \chi_\eta \left(\mathcal{E}^n_{i,\delta}\right)^\delta, \tag{100}$$

  where notation $(\,.\,)^\delta$ is defined by (38). η can be seen as a parameter that we shall set later.

- In a second step, we bounded $|a^\delta(\mathbf{E}^\delta - \mathbf{E}^n_{\eta,\delta}, \varphi)|$. Then Stability estimate (17) (Proposition 2.2), gives an estimation of the error $\left\|\mathbf{E}^\delta - \mathbf{E}^n_{\eta,\delta}\right\|_{H(\text{curl}\,,\Omega)}$: indeed,

$$\left\|\mathbf{E}^\delta - \mathbf{E}^n_{\eta,\delta}\right\|_{H(\text{curl}\,,\Omega)} \leq \sup_{\varphi\in V\setminus\{0\}} \frac{|a^\delta(\mathbf{E}^\delta - \mathbf{E}^n_{\eta,\delta}, \varphi)|}{\|\varphi\|_{V_{\epsilon^\delta}}}.$$

  Besides,

$$a^\delta(\mathbf{E}^\delta - \mathbf{E}^n_{\eta,\delta}, \varphi) := \mathcal{D}^r_{\eta,\delta,n} + \mathcal{D}^c_{\eta,\delta,n},$$

  where $\mathcal{D}^r_{\eta,\delta,n}$ represents the matching error,

$$\mathcal{D}^r_{\eta,\delta,n} := \int_\Omega \frac{1}{\mu^\delta} \left(\nabla\chi_\eta \times (\mathbf{E}^n_{e,\delta} - (\mathcal{E}^n_{i,\delta})^\delta)\right) \cdot \overline{\text{curl}\,\varphi} - \int_\Omega \frac{1}{\mu^\delta} \text{curl}\,\left(\mathbf{E}^n_{e,\delta} - (\mathcal{E}^n_{i,\delta})^\delta\right) \cdot \overline{\nabla\chi_\eta \times \varphi}, \tag{101}$$

  and $\mathcal{D}^c_{\eta,\delta,n}$ represents the near field error,

$$\mathcal{D}^c_{\eta,\delta,n} = a^\delta((\mathcal{E}^n_{i,\delta})^\delta, \chi_\eta\varphi). \tag{102}$$

  The near field error, also called consistency error, measures how the near field expansion fails to satisfy the Maxwell equations. Bounding successively these two terms, we obtain a global error: there exist two constants $C_n > 0$ and $\tau_n > 0$ such that

$$\left\|\mathbf{E}^\delta - \mathbf{E}^n_{\eta,\delta}\right\|_{H(\text{curl}\,,\Omega)} \leq C\left(\eta^{n-1/2} + \frac{1}{\delta}e^{-\tau_n\frac{\eta}{\delta}}\right)\|\varphi.\|_{V_{\epsilon^\delta}}$$

- Finally, using a localization process (in order to only take to account the far field error), we obtain Proposition 7.1. In this step, we set the parameter η.

A Technical results

Lemma A.1 *Let p be a function of $H^1_{loc}(B)$ such that ∇p is a 1-periodic function in X_1 and a τ-periodic function in X_2. Then, there exist two constants C_1 and C_2 such that $\tilde{p}(X_1, X_2\, X_3) = p - C_1X_1 - C_2X_2$ is a a 1-periodic function in X_1 and a τ-periodic function in X_2.*

Proof Let us consider the translation operators $\mathcal{T}_1$ and $\mathcal{T}_2$

$$\mathcal{T}_2 : \begin{cases} L^2_{loc}(\mathbb{R}^3) \to L^2_{loc}(\mathbb{R}^3), \\ (T_1u)(X_1, X_2, X_3) = u(X_1 + 1, X_2, X_3), \end{cases}$$

$$\mathcal{T}_1 : \begin{cases} L^2_{loc}(\mathbb{R}^3) \to L^2_{loc}(\mathbb{R}^3), \\ (T_2u)(X_1, X_2, X_3) = u(X_1, X_2 + \tau, X_3), \end{cases}$$

∇p being periodic, it is clear that $\mathcal{T}_1\nabla p = \nabla p$ and $\mathcal{T}_2\nabla p = \nabla p$. Since the operators $\mathcal{T}_1$ and $\mathcal{T}_2$ commute with the operator ∇, we obtain

$$\nabla(\mathcal{T}_1 p - p) = \nabla(\mathcal{T}_2 p - p) = 0,$$

which means that there exist two complex-valued constants C_1 and C_2 such that

$$\mathcal{T}_1 p = p + C_1 \quad \mathcal{T}_2 p = p + C_2 \tau.$$

Let $\tilde{p} = p - C_1 X_1 - C_2 X_2$. Then

$$\mathcal{T}_1 \tilde{p} = p + C_1 - C_1(X_1 + 1) - C_2 X_2 = \tilde{p},$$

and similarly, $\mathcal{T}_2 \tilde{p} = \tilde{p}$.

Lemma A.2 *Let $h \in \mathcal{D}(B)$ (h is a smooth function compactly supported in B). Then, there exists a sequence $(h_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(B)$ such that*

- $\displaystyle\int_B h_n = 0,$
- *h_n is compactly supported (its support varies with n),*
- $\displaystyle\lim_{n\to+\infty} \|h_n - h\|_{L^2(B)} = 0.$

Proof As h is smooth and vanishes for large X_3, h is in $L^1(B)$. Let $\alpha = \displaystyle\int_B h\, dX_1$. We consider a cut-off function $\chi(x)$ (χ: $\mathbb{R} \to \mathbb{R}$) satisfying

- χ is compactly supported in $]-\frac{1}{2}, \frac{1}{2}[$,
- $0 \leq \chi \leq 2$,
- $\int_{\mathbb{R}} \chi(x)\, dx = 1$.

We consider the sequence of smooth functions $(h_n)_{n\in\mathbb{N}}$:

$$h_n = h - \frac{\alpha}{\tau n}\, \chi(X_1)\, \chi\left(\frac{X_2}{\tau}\right)\, \chi\left(\frac{X_3}{n}\right).$$

It is clear that h_n is compactly supported. Moreover, $\displaystyle\int_B h_n = 0$, and

$$\begin{aligned} \|h - h_n\|^2_{L^2(B)} &= \frac{\alpha^2}{(\tau n)^2} \int_B \left(\chi(X_1)\, \chi\left(\frac{X_2}{\tau}\right) \chi\left(\frac{X_3}{n}\right)\right)^2 dX_1\, dX_2\, dX_3, \\ &\leq \frac{\alpha^2}{(\tau n)^2}\, \tau n \left\{\int_{-\frac{1}{2}}^{\frac{1}{2}} \chi^2\,(x)\, dx\right\}^3 \leq \frac{16\, \alpha^2}{\tau n}. \end{aligned}$$

so that $\displaystyle\lim_{n\to+\infty} \|h_n - h\|_{L^2(B)} = 0.$

Proposition A.3 *Let $f \in L^2_{per}(\mathbb{R}^3)^3$ et $g \in L^2_{per}(\mathbb{R}^3)$. There exists a unique function $w \in X_1(\mathbb{R}^3)|\mathbb{R}^3$ such that*

$$\begin{cases} curl\, curl\, w = curl\, f \quad in\ \mathcal{D}'(\mathbb{R}^3)^3, \\ div\, w = g \quad in\ \mathcal{D}'(\mathbb{R}^3). \end{cases} \tag{103}$$

Proof Let us first consider $X_{1,c}(\mathbb{R}^3)$:

$$X_{1,c}(\mathbb{R}^3) := \Big\{\varphi \in \mathcal{D}'(\mathbb{R}^3), \text{ such that }, \frac{\varphi}{\sqrt{1+(X_3)^2}} \in L^2(\mathbb{R}^3), \operatorname{curl}\varphi \in L^2(\mathbb{R}^3), \operatorname{div}\varphi \in L^2(\mathbb{R}^3),$$
$$\exists\, K = [a,b]\times[c,d]\times\mathbb{R} \text{ such that } \operatorname{supp}\varphi \subset K\}. \tag{104}$$

Assume that w satisfies (103). Then,

$$\forall \varphi \in X_{1,c}(\mathbb{R}^3), \quad \int_{\mathbb{R}^3} \operatorname{curl} w \cdot \operatorname{curl}\varphi + \operatorname{div} w \operatorname{div}\varphi = \int_{\mathbb{R}^3} g \operatorname{div}\varphi + f\cdot\operatorname{curl}\varphi. \tag{105}$$

Let us introduce the surjective operator $\mathcal{S}$:

$$\begin{cases} X_{1,c}(\mathbb{R}^3) \to X_1(B), \\ \varphi \mapsto \mathcal{S}(\varphi) = \displaystyle\sum_{j\in\mathbb{Z}}\sum_{i\in\mathbb{Z}} \varphi(X_1+i, X_2+\tau j, X_3). \end{cases} \tag{106}$$

Note that the summations on i et j are finite since $\varphi \in X_{1,c}(\mathbb{R}^3)$. Equation (105) can also be rewritten with the help of the operator $\mathcal{S}$:

$$\forall \varphi \in X_{1,c}(\mathbb{R}^3), \quad \int_B \operatorname{curl} w \cdot \operatorname{curl}\mathcal{S}(\varphi) + \operatorname{div} w \operatorname{div}\mathcal{S}(\varphi) = \int_B g \operatorname{div}\mathcal{S}(\varphi) + f\cdot\operatorname{curl}\mathcal{S}(\varphi). \tag{107}$$

Since $\mathcal{S}$ is surjective,

$$\forall \varphi \in X_1(B), \quad \int_B \operatorname{curl} w \cdot \operatorname{curl}\varphi + \operatorname{div} w \operatorname{div}\varphi = \int_B g \operatorname{div}\varphi + f\cdot\operatorname{curl}\varphi, \tag{108}$$

and so,

$$\forall \varphi \in X_1(B)|\mathbb{R}^3, \quad \int_B \operatorname{curl} w \cdot \operatorname{curl}\varphi + \operatorname{div} w \operatorname{div}\varphi = \int_B g \operatorname{div}\varphi + f\cdot\operatorname{curl}\varphi. \tag{109}$$

The previous problem has a unique solution $w_1 \in X_1(B)|\mathbb{R}^3$ (see Proposition 4.8). Conversely, we have to prove that if w_1 is the unique solution of (109), it satisfies (103). First, let us remark that (109) is still valid for any test function φ in $X_1(B)$ (instead of $X_1(B)|\mathbb{R}^3$) so that

$$\forall \varphi \in X_1(B), \quad \int_B \operatorname{curl} w_1 \cdot \operatorname{curl}\varphi + \operatorname{div} w_1 \operatorname{div}\varphi = \int_B g \operatorname{div}\varphi + f\cdot\operatorname{curl}\varphi. \tag{110}$$

Taking $\varphi = \nabla p$, we get

$$\int_B (\operatorname{div}(w_1) - g)h = 0 \quad \forall\, h \in \mathcal{D}(B) \text{ such that } \int_B h = 0.$$

From Lemma A.2 and since $\operatorname{div}(w_1) - g \in L^2(B)$, the previous equality holds in $\mathcal{D}(B)$, which means that

$$\operatorname{div}(w_1) = g \quad \text{in } \mathcal{D}'(B).$$

Consequently, since $g \in L^2_{per}(\mathbb{R}^3)$, the previous inequality holds in $\mathcal{D}'(\mathbb{R}^3)$. It remains to prove that $\operatorname{curl}\operatorname{curl} w_1 = \operatorname{curl} f$ in $\mathcal{D}'(\mathbb{R}^3)^3$. Let $\varphi \in \mathcal{D}(\mathbb{R}^3)^3$. Then, since $\mathcal{S}(\varphi) \in X_1(B)$ we have

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^3)^3, \qquad \int_B \operatorname{curl} w_1 \cdot \operatorname{curl}\mathcal{S}(\varphi) = \int_B f\cdot\operatorname{curl}\mathcal{S}(\varphi).$$

The previous equality exactly means that

$$\forall \varphi \in \mathcal{D}(\mathbb{R}^3)^3, \qquad \int_{\mathbb{R}^3} \mathrm{curl}\, w_1 \cdot \mathrm{curl}\, \varphi = \int_{\mathbb{R}^3} f \cdot \mathrm{curl}\, \varphi.$$

So $\mathrm{curl}\,\mathrm{curl}\, w_1 = \mathrm{curl}\, f \in \mathcal{D}'(R^3)^3$.

We can then deduce the following proposition (it corresponds to an adaptation of Theorem 3.38 in [25] for the unbounded periodicity cell B, see also Theorem 3.4 in [42]):

Proposition A.4 *Let $f \in L^2_{per}(\mathbb{R}^3)^3$ such that $div\, f = 0$. Then, there exists $w \in X_1(\mathbb{R}^3)$ such that*

$$\begin{cases} \mathit{div}\,(w) = 0 \ \mathit{in}\ \mathcal{D}'(\mathbb{R}^3), \\ \mathit{curl}\,(w) = f \ \mathit{in}\ \mathcal{D}'(\mathbb{R}^3)^3. \end{cases}$$

Proof In view of Proposition A.3, there exists $w \in X_1(\mathbb{R}^3)$ such that

$$\begin{cases} \mathrm{curl}\,\mathrm{curl}\, w = \mathrm{curl}\, f \ \text{in}\ \mathcal{D}'(\mathbb{R}^3)^3, \\ \mathrm{div}\, w = 0 \ \text{in}\ \mathcal{D}'(\mathbb{R}^3). \end{cases}$$

We prove that $\mathrm{curl}\, w = f$. Let $d = f - \mathrm{curl}\, w$. It is clear that d belongs to $L^2_{per}(\mathbb{R}^3)$. In addition,

$$\begin{cases} \mathrm{curl}\, d = 0 \ \text{in}\ \mathcal{D}'(\mathbb{R}^3)^3, \\ \mathrm{div}\, d = 0 \ \text{in}\ \mathcal{D}'(\mathbb{R}^3). \end{cases}$$

So, using Proposition 4.4, d is a constant in $\mathbb{R}^3$. But, since $d \in L^2_{per}(\mathbb{R}^3)$, $d = 0$, which completes the proof.

Lemma A.5 *Let $f \in L^2_{per}(\mathbb{R}^3)^3$ such that $div\, f = 0$. Then, there exists $w_a \in X_a(\mathbb{R}^3)$ such that*

$$\begin{cases} \mathit{div}\,(a w_a) = 0 \ \mathit{in}\ \mathcal{D}'(\mathbb{R}^3), \\ \mathit{curl}\,(w_a) = f \ \mathit{in}\ \mathcal{D}'(\mathbb{R}^3)^3. \end{cases} \tag{111}$$

Proposition A.4 ensures the existence of $w \in X_1(\mathbb{R}^3)$ such that $\mathrm{curl}\, w = f$ and $\mathrm{div}\, w = 0$. Integrating $\mathrm{div}\, w = 0$ over a bounded domain, we remark that

$$\int_{-1/2}^{1/2} \int_{-\tau/2}^{\tau/2} w(X_1, X_2, h_0) \cdot e_3 - w(X_1, X_2, -h_0) \cdot e_3 dX_1 dX_2 = 0 \quad \forall\ h_0 > 0 \tag{112}$$

However $\mathrm{div}\,(aw) \neq 0$. It is nevertheless rational to search w_a of the form

$$w_a = w + \nabla p, \tag{113}$$

where p satisfies $\mathrm{div}\,(a\nabla p) = -\mathrm{div}\,(aw)$. In fact, we define $p \in W_1(\mathbb{R}^3)|\mathbb{R}$ as the unique solution of the problem

$$\mathrm{div}\,(a\nabla p) = -\mathrm{div}\,(aw) \ \text{in}\ \mathcal{D}'(\mathbb{R}^3). \tag{114}$$

We point out that Problem (114) is well posed. Indeed, let χ be a smooth truncation function such that

$$\chi(z) = \begin{cases} 1 \text{ si } |z| > 2, \\ 0 \text{ si } |z| < 1. \end{cases} \tag{115}$$

Then,

$$\operatorname{div}(aw) = \operatorname{div}\underbrace{(a(1-\chi)w)}_{g_0} + \underbrace{\nabla\chi \cdot w}_{f_0},$$

where

- $f_0 \in L^2_{per}(\mathbb{R}^3)$ is compactly supported. In view of equality (112)), it satisfies

$$\int_B f_0 = \int_{-1/2}^{1/2} \int_{-\tau/2}^{\tau/2} w(X_1, X_2, h_0) \cdot e_3 - w(X_1, X_2, -h_0) \cdot e_3 = 0.$$

- $g_0 \in L^2_{per}(\mathbb{R}^3)$ is compactly supported.

Consequently Proposition 4.3 applies and Problem (114) is well posed and we have obtained a function $w_a \in X_a(\mathbb{R}^3)$ satisfying (111).

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