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Mireille Bossy* Joaquin Fontbona† Pierre-Emmanuel Jabin‡

Jean-François Jabir§

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**Abstract**

We consider an incompressible kinetic Fokker Planck equation in the flat torus. This equation is a simplified version of the Lagrangian stochastic models for turbulent flows introduced by S.B. Pope in the context of computational fluid mechanics. The main difficulties in its treatment arise from the pressure type force in the equation that couples the Fokker Planck equation with a Poisson equation. We prove short time existence of analytic solutions in the one-dimensional case, where we are able to explicit the pressure force and use techniques and functional norms recently introduced in the study of a related singular model .

**Keywords** Fluid particle model; Incompressibility; Analytic solution; Singular kinetic equation.

## Contents



*INRIA Sophia Antipolis France, EPI TOSCA;Mireille.Bossy@inria.fr

†DIM–CMM, UMI 2807 UChile-CNRS, Universidad de Chile, Casilla 170-3, Correo 3, Santiago, Chile ; Partially supported by Fondecyt Grant 1110923 and Basal-Conicyt-Chile; fontbona@dim.uchile.cl.

‡Department of Mathematics, University of Maryland, College Park; pjabin@cscamm.umd.edu

§IMT, Université Paul Sabatier, Toulouse, France; Partially supported by Fondecyt Grant 3100132. Conicyt-Chile; jfjabir@math-univ.toulouse.fr.

# 1 Introduction

Let $(Y_0, U_0)$ be a random variable in $\mathbb{R}^d \times \mathbb{R}^d$ and $\beta \geq 0$, $\sigma \geq 0$, and $\alpha \in \{0,1\}$ be given constants. We consider the following stochastic differential equation of kinetic type in $\mathbb{T}^d \times \mathbb{R}^d$, where $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ is the flat $d$-dimensional torus:

$$X_t = \left[ Y_0 + \int_0^t U_s\, ds \right], \tag{1.1a}$$

$$U_t = U_0 - \int_0^t \nabla_x P(s, X_s) ds - \beta \int_0^t (U_s - \alpha \mathbb{E}(U_s | X_s)) ds + \sigma W_t, \tag{1.1b}$$

$$\mathbb{P}(X_t \in dx) = dx, \text{ for all } t \in [0,T]. \tag{1.1c}$$

Here, $W$ is a standard $d$ dimensional Brownian motion, $[x] \in \mathbb{T}^d$ denotes the class of $x \in \mathbb{R}^d$, $dx$ is the uniform measure on $\mathbb{T}^d$ and $P : [0,T] \times \mathbb{T}^d \to \mathbb{R}$ is an unknown (deterministic) scalar field.

The stochastic differential equation (1.1) constitutes a laboratory example of the class of Lagrangian stochastic models for incompressible turbulent flows, mainly introduced by S.B. Pope in the eighties in order to provide a fluid-particle description of turbulent flows, and to develop probabilistic numerical methods for their simulation. We refer the reader to [13] for a general presentation of this turbulent model approach in the framework of computational fluid dynamics, and to [2], [7] for a survey on mathematical problems related to the Lagrangian stochastic models.

In physical terms, the drift term $\nabla_x P(t,x)$ is interpreted as the gradient of a pressure field, which is intended to accomplish the homogeneous mass distribution constraint specified by Equation (1.1c). When $\alpha = 0$, the process $U_t$, representing the velocity of a fluid particle, reverses towards the origin like an Ornstein-Uhlenbeck process with a potential given by the standard kinetic energy $\mathbb{E}|U_t|^2$. When $\alpha = 1$, reversion towards the origin in (1.1b) is replaced by a reversion toward the *averaged velocity* or *bulk-velocity*, $\mathbb{E}(U_t|X_t = x)$, which can be associated to the local-in-space potential $\mathbb{E}(|U_t - \mathbb{E}(U_t|X_t)|^2 | X_t = x)$ interpreted as the *turbulent* kinetic energy. In the two cases, the additional potential $\nabla_x P$ forces the particle position $X_t$ to have a macroscopically uniform spacial distribution in the torus. In the Lagrangian modeling of turbulent flow, the constraint (1.1c) is formulated heuristically (see e.g. [13]) by rather imposing a divergence free condition to the flow, which in the case of system (1.1) writes as a divergence free condition on the bulk velocity field:

$$\nabla_x \cdot \mathbb{E}(U_t | X_t = x) = 0. \tag{1.2}$$

By a classical projection argument on the divergence free space, it is then assumed that the field $P$ should verify the elliptic PDE

$$\triangle_x P = - \sum_{i,j=1}^d \partial_{x_i x_j} \mathbb{E}\left( U_t^{(i)} U_t^{(j)} | X_t = x \right).$$

The equation above is obtained by taking the divergence of the formal equation of the bulk velocity derived form the Fokker Planck equation associated to $(X, U)$, (we refer to [14] for the precise formulation and related numerical issues).

In spite of its relevance for the simulation of complex fluid dynamics (see e.g. [14], [11] and the references therein), a rigorous mathematical formulation of the Lagrangian stochastic models, and in particular of the uniform mass distribution constraint, has not yet been given. In [7], first well-posedness results on a simpler kinetic model were obtained, which featured nonlinearity of conditional type. From a probabilistic point of view, the conditional expectation was treated as a McKean-Vlasov equation. This enabled the authors to also construct a mean field stochastic particle approximation of the nonlinear model. Combined with an heuristic numerical procedure to deal with the constraint (1.1c) and the pressure term, that particle scheme gave raise to a stochastic numerical downscaling method studied and implemented in [2]. Extension to some of those results to a special case of boundary value problem were obtained in [5] and [6]. As regards the uniform mass distribution constraint, in spite of the formal resemblance with the Navier-Stokes equations, there is no rigorous mathematical evidence that such constraint can be satisfied, for instance in the case of the linear Langevin process, by adding some force term of the form $\nabla_x P(t,x)$ (a trivial exception is the situation $\nabla_x P \equiv 0$ of the stationary Langevin process, considered as a benchmark for the stochastic downscaling method in [2]).

The aim of this paper is to address for the first time the well-posedness of a relatively simple instance of Lagrangian stochastic model yet satisfying in a non trivial way the uniform mass distribution constraint (1.1c). In the case of system (1.1), the main underlying problem is of course the question of well-posedness of the nonlinear Fokker-Planck equation that the time-marginal laws $f(t,u,x)$ of a solution $(X_t,U_t)$ to the system of equations (1.1) should satisfy, namely

$$\partial_t f(t,x,u) - \mathcal{A}^\star f(t,x,u) - \nabla_u f(t,x,u)\cdot\left(\nabla_x P(t,x) - \beta\alpha\int_{\mathbb{R}^d} v f(t,x,v)\,dv\right) = 0,\ \text{on } (0,T]\times\mathbb{T}^d\times\mathbb{R}^d, \tag{1.3a}$$

$$f(0,x,u) = f_0(x,u),\ \text{on } \mathbb{T}^d\times\mathbb{R}^d, \tag{1.3b}$$

$$\int_{\mathbb{R}^d} f(t,x,u)\,du = 1,\ \text{on } [0,T]\times\mathbb{T}^d, \tag{1.3c}$$

where $P(t,x)$ is unknown and $\mathcal{A}^\star$ denotes the formal adjoint of the linear kinetic Ornstein-Uhlenbeck operator $\mathcal{A}\phi(x,u) = \frac{\sigma^2}{2}\triangle_u\phi(x,u) - \beta u\cdot\nabla_u\phi(x,u) + u\cdot\nabla_x\phi(x,u)$. Note that under condition (1.3c) $\int_{\mathbb{R}^d} vf(t,x,v)dv$ is the conditional expectation $\mathbb{E}(U_t|X_t = x)$. The previous equation exhibits several conceptual and technical difficulties, and to our knowledge there is so far no direct strategy for its study nor mathematical results in the fields of stochastic processes or kinetic PDE.

A first step in our study will be to establish an alternative formulation of the previous equation. Indeed, we will see below that under natural assumptions on the initial data, any pair $(f,P)$ that is a classical solution to (1.3) is also a solution to the system

$$\begin{cases} \partial_t f(t,x,u) + u\cdot\nabla_x f(t,x,u) = \dfrac{\sigma^2}{2}\triangle_u f(t,x,u) + \beta\, d\, f(t,x,u) + \beta u\cdot\nabla_u f(t,x,u) \\ \qquad + \nabla_u f(t,x,u)\cdot\left(\nabla_x P(t,x) - \beta\alpha\displaystyle\int_{\mathbb{R}^d} v f(t,x,v)\,dv\right) = 0,\ \text{on } (0,T]\times\mathbb{T}^d\times\mathbb{R}^d, \\ f(0,x,u) = f_0(x,u),\ \text{on } \mathbb{T}^d\times\mathbb{R}^d, \\ \triangle_x P(t,x) = -\displaystyle\sum_{i,j=1}^d \partial_{x_i x_j}\int_{\mathbb{R}^d} v_i v_j f(t,x,v)dv,\ \text{on } [0,T]\times\mathbb{T}^d, \end{cases} \tag{1.4}$$

where, plainly, condition (1.3c) has been replaced by the above Poisson equation. The two systems however seem not to be equivalent in general. The formulation (1.4) allows us to see the original problem as an instance of Vlasov-Fokker-Planck equation, which nevertheless is highly singular. More precisely, the gradient of the pressure field can be expressed as the convolution of the derivative of the periodic Poisson kernel with the function $-\sum_{i,j=1}^d \partial_{x_i x_j}\int_{\mathbb{R}^d} v_i v_j f(t,x,v)dv$. The existence of smooth (analytical) solutions to kinetic equations with singular potential has been addressed in several situations, in particular for the nonlinear Vlasov-Poisson equation see e.g. Benachour [1], Mouhot and Villani [12]; for gyro-kinetic models see e.g [4, 3] and [9].

In the case $d = 1$, we can specify $P(t,x)$ on $[0,T]\times\mathbb{R}$ by $P(t,x) = -\int_{\mathbb{R}} u^2 f(t,x,u)du$. Hence, in the present paper, we restrict ourself to the simpler situation of the one-dimensional equation

$$\begin{cases} \partial_t f(t,x,u) + u\cdot\partial_x f(t,x,u) = \dfrac{\sigma^2}{2}\partial_u^2 f(t,x,u) + \beta f(t,x,u) + \beta u\partial_u f(t,x,u) \\ \qquad + \partial_u f(t,x,u)\left(\partial_x P(t,x) - \beta\alpha\displaystyle\int_{\mathbb{R}} v f(t,x,v)dv\right) = 0,\ \text{on } (0,T]\times\mathbb{T}\times\mathbb{R}, \\ f(t,x,u) = f_0(x,u),\ \text{on } \mathbb{T}\times\mathbb{R}, \\ P(t,x) = -\displaystyle\int_{\mathbb{R}} u^2 f(t,x,u)du,\ \text{on } [0,T]\times\mathbb{T}. \end{cases} \tag{1.5}$$

Till recently, a closed formulation (in terms of regularity) of such Vlasov type nonlinear system was not available. To tackle the system (1.5), we will follow recent ideas introduced by one of the authors in [10], in order to obtain a local existence result of analytical solutions.

Our main results can be summarized in a simplified way in the following statement:

**Theorem 1.1.** *Let $\bar\lambda > 0$ and $s \geq 2$ be an even integer. There exists a constant $\kappa_0 = \kappa_0(\bar\lambda, s)$ and a positive function $r \mapsto \kappa_1(r,\bar\lambda,s)$ such that if $f_0 : \mathbb{T}\times\mathbb{R}\to\mathbb{R}$ of class $\mathcal{C}^\infty$ and $T > 0$ satisfy:*

- $\int_{\mathbb{R}} f_0(x,u)du = 1$ *and* $\partial_x \int_{\mathbb{R}^d} u f_0(x,u)du = 0$ *for all* $x \in \mathbb{T}$,
- $\|(1+|u|^2)^{\frac{s}{2}} \partial_x^l \partial_u^k f_0\|_\infty \leq \frac{C_0 (k+m)!(l+n)!}{\bar{\lambda}^{k+l}}$ *for some* $n, m \in \mathbb{N}$, *all pair of indices* $k, l \in \mathbb{N}$ *and a constant* $C_0 < \kappa_0(\bar{\lambda}, s)$, *and*
- $T < \kappa_1(C_0, \bar{\lambda}, s)$,

*then, a solution* $f$ *of class* $\mathcal{C}^{1,\infty}$ *to equation* (1.5) *in* $[0,T] \times \mathbb{T} \times \mathbb{R}$ *exists.*

The remainder of the paper is organized as follows:
In Subsection 1.1 we briefly establish the validity of system (1.4) for any solution to equation (1.3) in arbitrary space dimension, and state additional conditions required in order that, reciprocally, a solution to the former also solves the latter. From Section 2 on, we restrain ourselves to the one-dimensional case. We recall therein the analytical norms introduced in [10] and we state useful properties of them. Following their strategy, in the case $\beta = 0$ we then introduce an equivalent formulation of equation (1.5), in order to deal with the integrability problems posed by the first and second order velocity moments involved in the equation. We then show that solutions to (1.4) in these particular spaces of analytical functions actually do satisfy the conditions required to be solutions of (1.3). Using the fixed point argument of [10], we will then prove a local existence result in these analytical spaces, which indeed is a slightly more general version of Theorem 1.1 restricted to the case $\beta = 0$. In Section 3, we then briefly extend the previous result to the case $\beta \neq 0$. Finally, some technical results are proved in Appendix.

## 1.1 The Lagrangian stochastic model coupled with a Poisson equation

We start by establishing connections between conditions related to the homogeneous mass distribution constraint, which are valid in arbitrary dimension:

**Lemma 1.2.** *Assume that* $f$ *is a classical solution to equations* (1.3a) *and* (1.3b) *for some function* $P : [0,T] \times \mathbb{T}^d \to \mathbb{R}^d$ *of class* $\mathcal{C}^{0,2}$. *Moreover, assume that*

$$\rho(t,x) := \int_{\mathbb{R}^d} f(t,x,u)du, \quad V(t,x) := \int_{\mathbb{R}^d} u f(t,x,u)du$$

*are functions of class* $\mathcal{C}^{1,1}$ *in* $[0,T] \times \mathbb{T}^d$, *that* $\int_{\mathbb{R}^d} |u|^2 |D^m f(t,x,u)| du < +\infty$ *for each multiindex* $|m| \leq 2$ *and each* $(t,x) \in [0,T] \times \mathbb{T}^d$ *(where* $D$ *is the derivative operator), and further that for all* $t \in [0,T]$ *the function*

$$x \mapsto \int_{\mathbb{R}^d} v_i v_j f(t,x,v)dv$$

*is of class* $\mathcal{C}^2$. *Then, the following system of equations is satisfied for* $(t,x) \in (0,T] \times \mathbb{T}^d$:

$$\begin{cases} \partial_t \rho(t,x) + \nabla_x \cdot V(t,x) = 0, \\ \partial_t (\nabla_x \cdot V(t,x)) + \beta \nabla_x \cdot V(t,x) + \nabla_x \cdot \big(\rho(t,x) \left(\nabla_x P(t,x) - \beta\alpha V(t,x)\right)\big) + \displaystyle\sum_{i,j=1}^d \partial_{x_i x_j} \int_{\mathbb{R}^d} v_i v_j f(t,x,v)dv = 0 \end{cases}$$

*We deduce:*

*a)* $\rho(t,x) = \rho(0,x)$ *for every* $(t,x) \in [0,T] \times \mathbb{T}^d$ *if and only if* $\nabla_x \cdot V(t,x) = 0$ *for every* $(t,x) \in [0,T] \times \mathbb{T}^d$.

*b)* $\nabla_x \cdot V(t,x) = e^{-\beta t} \nabla_x \cdot V(0,x)$ *for every* $(t,x) \in [0,T] \times \mathbb{T}^d$ *if and only if* $P$ *satisfies the equation of elliptic type:*

$$\nabla_x \cdot \big(\rho(t,x) \left(\nabla_x P(t,x) - \beta\alpha V(t,x)\right)\big) = -\sum_{i,j=1}^d \partial_{x_i x_j} \int_{\mathbb{R}^d} v_i v_j f(t,x,v)dv\,, \quad (t,x) \in (0,T] \times \mathbb{T}^d.$$

c) *If in addition to* (1.3a) *and* (1.3b)*, condition* (1.3c) *is verified, then* $P(t,x)$ *is a solution to the Poisson equation*

$$\triangle_x P(t,x) = -\sum_{i,j=1}^{d} \partial_{x_i x_j} \int_{\mathbb{R}^d} v_i v_j f(t,x,v) dv, \quad (t,x) \in (0,T] \times \mathbb{T}^d.$$

d) *Set* $\bar{\rho}(t,x) := \rho(t,x) - 1$*. If in addition to* (1.3a) *and* (1.3b) *we assume that the Poisson equation in part c) holds, we have:*

*when* $\alpha = 1$, $\partial_t(\nabla_x \cdot V(t,x)) + \nabla_x \cdot \big(\bar{\rho}(t,x)\,(\nabla_x P(t,x) - \beta V(t,x))\,\big) = 0$*;*

*when* $\alpha = 0$, $\partial_t(\nabla_x \cdot V(t,x)) + \nabla_x \cdot \big(\bar{\rho}(t,x)\nabla_x P(t,x)\big) + \beta \nabla_x \cdot V(t,x) = 0$.

*Proof.* The first equation is obtained by integrating equation (1.3a) with respect to $u \in \mathbb{R}^d$, and using the assumptions in order to integrate by parts and get rid of integrals of divergence type terms. To get the second equation, we first take the derivative with respect to the variable $x_i$ in equation (1.3a), then multiply it by $u_i$ and sum over $i = 1, \dots, d$, before integrating and proceeding as before. Statements a),b),c) and d) are then easily deduced. □

**Remark 1.3.** a) *According to Lemma 1.2 part c), finding a solution to equation* (1.3) *requires in particular to find a solution to the highly singular Vlasov-Fokker-Planck equation* (1.4).

b) *If conditions* (1.3a) *and* (1.3b) *hold, and the Poisson equation in Lemma 1.2 part c) is satisfied, the equation obtained in Lemma 1.2 part d) together with the continuity equation*

$$\partial_t \bar{\rho}(t,x) + \nabla_x \cdot V(t,x) = 0$$

*furnish a system of two equations (where* $P$ *is fixed) that the pair* $(\bar{\rho}, V(t,x))$ *must satisfy. Thus, in that situation a strategy to prove that* (1.3c) *also holds is to prove that this system starting from* $(0,0)$ *has the unique solution* $\bar{\rho}(t,x) = \nabla_x \cdot V(t,x) \equiv 0$*. We will be able to do this in the functional setting that we will consider, deducing thus a solution to* (1.3) *from a solution to* (1.4).

# 2 Local analytic well-posedness in the vanishing kinetic potential case ($\beta = 0$)

In this section, we construct an analytical solution to the nonlinear Vlasov-Fokker-Planck equation associated with the incompressible Lagrangian stochastic model up to some small time horizon $T$, in the case $\beta = 0$. Using the weighted analytical functional space introduced in [10] and a fixed point argument developed therein, we shall give in Theorem 2.5 below a local-in-time well-posedness result for the nonlinear Vlasov-Fokker-Planck equation:

$$\begin{cases} \partial_t f(t,x,u) + u\partial_x f(t,x,u) - \partial_x P(t,x)\partial_u f(t,x,u) - \dfrac{\sigma^2}{2}\partial_u^2 f(t,x,u) = 0, \text{ on } (0,T] \times \mathbb{R}^2, \\ f(0,x,u) = f_0(x,u), \text{ on } \mathbb{R}^2, \end{cases} \tag{VFP}$$

where

$$P(t,x) = -\int_{\mathbb{R}} u^2 f(t,x,u) du, \ x \in \mathbb{R}.$$

Notice that periodicity is not yet imposed. Then, we will show in Corollary 2.6 that if at $t = 0$ the following conditions are satisfied:

($\mathrm{H}_{\mathrm{unif}(t)}$) at time $t \in [0,T]$:

$f(t,x,u)$ is $1-$periodic in $x$ for all $u \in \mathbb{R}$,

$\int_{\mathbb{R}} f(t,x,u) du = 1$, for all $x \in \mathbb{T}$ (*Uniform mass repartition in* $\mathbb{T}$),

$\partial_x \int_{\mathbb{R}} u f(t,x,u) du = 0$, for all $x \in \mathbb{T}$ (*Mean incompressibility in* $\mathbb{T}$),

the obtained local solution $f(t,x,u)$ of (VFP) *a fortiori* satisfies the same properties for all $t \in [0,T]$. To establish the latter result, the choice of analytical functional spaces and the use of the analytic norms in [10] will also be fundamental.

## 2.1 The nonlienar Vlasov-Fokker-Planck equation in analytic spaces

We start by defining the functional spaces where an equivalent version of equation (VFP) will be studied.

A function $\mathbb{R}^2 \ni (x,u) \mapsto \psi(x,u) \in \mathbb{R}$ having bounded derivatives of all order is said to be analytic if there exists $C > 0$ and $\lambda > 0$ such that for all $k,l \in \mathbb{N}$ $\|\partial_x^k \partial_u^l \psi\|_\infty \leq C\lambda^{k+l} k!l!$, for $\|\cdot\|_\infty$ the uniform norm on $\mathbb{R}^2$ and using the convention $0! = 1$. For such functions, we introduce the analytic norm:

$$\|\psi\|_\lambda := \sum_{k,l\in\mathbb{N}} \frac{\lambda^{k+l}}{k!l!} \left\|\partial_x^k \partial_u^l \psi\right\|_\infty .$$

We further introduce the $\lambda$-derivatives of these norms for each $a \in \mathbb{N}$,

$$\|\psi\|_{\lambda,a} := \frac{d^a}{d\lambda^a}\|\psi\|_\lambda = \sum_{k+l\geq a} \frac{(k+l)!}{(k+l-a)!} \frac{\lambda^{k+l-a}}{k!l!} \left\|\partial_x^k \partial_u^l \psi\right\|_\infty .$$

Notice that $\|\psi\|_{\lambda,0} = \|\psi\|_\lambda$. We then define

$$\|\psi\|_{\mathcal{H},\lambda} := \sum_{a\in\mathbb{N}} \frac{1}{(a!)^2}\|\psi\|_{\lambda,a}, \qquad \|\psi\|_{\widetilde{\mathcal{H}},\lambda} := \sum_{a\geq 1} \frac{a^2}{(a!)^2}\|\psi\|_{\lambda,a}.$$

Last, we define the functional spaces associated with these norms:

$$\mathcal{H}(\lambda) := \left\{\psi \in \mathcal{C}^\infty(\mathbb{R}^2) \text{ such that } \|\psi\|_{\mathcal{H},\lambda} < +\infty\right\}, \tag{2.2a}$$

$$\widetilde{\mathcal{H}}(\lambda) := \left\{\psi \in \mathcal{C}^\infty(\mathbb{R}^2) \text{ such that } \|\psi\|_{\widetilde{\mathcal{H}},\lambda} < +\infty\right\}. \tag{2.2b}$$

The next two lemmas give some insight about the meaning of these norms, and will be useful later on:

**Lemma 2.1.** *Let $v : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^\infty$ be such that $\|\partial_x^l \partial_u^k v\|_\infty \leq \frac{C(k+m)!(l+n)!}{\bar{\lambda}^{k+l}}$ for some $C, \bar{\lambda} > 0$, some $m,n,j \in \mathbb{N}$ and all $k,l \geq j$.*

*a) If the previous holds for $j = 0$, (in particular if $\|v\|_{\bar{\lambda},0} < +\infty$), then $v \in \mathcal{H}(\lambda)$ for all $\lambda \in [0,\bar{\lambda})$.*

*b) If the previous holds for $j = 1$, (in particular if $\|v\|_{\bar{\lambda},1} < +\infty$), then $v \in \widetilde{\mathcal{H}}(\lambda)$ for all $\lambda \in [0,\bar{\lambda})$.*

*Proof.* For $a \geq j$ and $\lambda \in [0,\bar{\lambda})$ we obtain from the assumption that

$$\begin{aligned}\frac{d^a}{d\lambda^a}\|v\|_{\lambda,0} \leq& \frac{C}{\bar{\lambda}^a} \sum_{k+l\geq a} \frac{(k+l)!(k+m)!(l+n)!}{k!l!(k+l-a)!} \left(\lambda/\bar{\lambda}\right)^{(k+l-a)} \\ =& \frac{C}{\bar{\lambda}^a} \sum_{k+l\geq a+m+n, k\geq m, l\geq n} \frac{(k+l-(m+n))!k!l!}{(k-m)!(l-n)!(k+l-(a+m+n))!} \left(\lambda/\bar{\lambda}\right)^{(k+l-(a+m+n))}\end{aligned}$$

changing indexes $k+m$ to $k$ and $l+n$ to $l$. Since $\frac{(k+l-(m+n))!k!l!}{(k-m)!(l-n)!(k+l)!} \leq 1$ for $k \geq m, l \geq n$, we deduce that

$$\frac{d^a}{d\lambda^a}\|v\|_{\lambda,0} \leq \frac{C}{\bar{\lambda}^a} \sum_{k+l\geq a+m+n} \frac{(k+l)!}{(k+l-(a+m+n))!} \left(\lambda/\bar{\lambda}\right)^{(k+l-(a+m+n))} = \frac{C}{\bar{\lambda}^a}\left[\frac{d^{a+m+n}}{dr^{a+m+n}}\left(\sum_{k,l\in\mathbb{N}} r^{k+l}\right)\right]\Bigg|_{r=\lambda/\bar{\lambda}}.$$

Observing that for $r \in [0,1)$, $\sum_{k,l\in\mathbb{N}} r^{k+l} = (\sum_{j\in\mathbb{N}} r^j)^2 = \frac{1}{(1-r)^2}$ and $|\frac{d^a}{dr^a}\frac{1}{(1-r)^2}| \leq (1+a)!$, we conclude that

$$\|v\|_{\mathcal{H},\lambda} = \sum_{a=0}^\infty \frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\|v\|_{\lambda,0} \leq C\sum_{a=0}^\infty \frac{1}{\bar{\lambda}^a}\frac{(a+1)\cdots(a+m+n+1)}{a!} < +\infty \text{ and}$$

$$\|v\|_{\widetilde{\mathcal{H}},\lambda} = \sum_{a=1}^\infty \frac{a^2}{(a!)^2}\frac{d^a}{d\lambda^a}\|v\|_{\lambda,0} \leq C\sum_{a=0}^\infty \frac{1}{\bar{\lambda}^a}\frac{(a+1)\cdots(a+m+n+2)}{a!} < +\infty.$$

□

**Lemma 2.2.** *Let $\psi$ be an analytic function defined on $\mathbb{R}^2$. Then:*

*(i) For each $a \in \mathbb{N}$ one has $\|\psi\|_{\lambda,a+1} = \|\partial_x \psi\|_{\lambda,a} + \|\partial_u \psi\|_{\lambda,a}$. We deduce that*

$$\|\psi\|_{\widetilde{\mathcal{H}},\lambda} = \|\partial_x \psi\|_{\mathcal{H},\lambda} + \|\partial_u \psi\|_{\mathcal{H},\lambda}.$$

*(ii) Moreover,*

$$\frac{d}{d\lambda}\|\psi\|_{\mathcal{H},\lambda} = \|\psi\|_{\widetilde{\mathcal{H}},\lambda}.$$

*(iii) Last, for any pair $\psi_1, \psi_2$ of analytic functions defined on $\mathbb{R}^2$*

$$\|\psi_1\psi_2\|_\lambda \le \|\psi_1\|_\lambda \|\psi_2\|_\lambda.$$

*Proof.* (i). The first identity follows from

$$\begin{aligned}
\|\psi\|_{\lambda,a+1} = \frac{d^{a+1}}{d\lambda^{a+1}}\|\psi\|_{\lambda,0} &= \sum_{m+l\ge a+1} \frac{(m+l)\cdots(m+l-a-1)\lambda^{m+l-a-1}}{m!l!}\|\partial_x^m \partial_u^l \psi\|_\infty \\
&= \sum_{m+l\ge a+1, l\ge 1} \frac{(m+l-1)\cdots(m-a+l-1)\lambda^{m-a+l-1}}{m!(l-1)!}\|\partial_x^m \partial_u^l \psi\|_\infty \\
&\quad + \sum_{m+l\ge a+1, m\ge 1} \frac{(m-1+l)\cdots(l-a+m-1)\lambda^{l-a+m-1}}{(m-1)!l!}\|\partial_x^m \partial_u^l \psi\|_\infty,
\end{aligned}$$

by respectively changing the indexes $l$ to $l+1$ and $m$ to $m+1$ in the first and second sums in the last expression. Multiplying by $\frac{a^2}{(a!)^2}$ both sides of the previously established identity and summing the resulting expressions over $a \ge 1$ yields the identity for $\|\psi\|_{\widetilde{\mathcal{H}},\lambda}$. (ii) readily follows from

$$\frac{d}{d\lambda}\|\psi\|_{\mathcal{H},\lambda} = \frac{d}{d\lambda}\sum_{a\in\mathbb{N}} \frac{1}{(a!)^2}\|\psi\|_{\lambda,0} = \sum_{a\in\mathbb{N}} \frac{1}{(a!)^2}\|\psi\|_{\lambda,a+1} = \sum_{a\ge 1} \frac{1}{((a-1)!)^2}\|\psi\|_{\lambda,a} = \sum_{a\ge 1} \frac{a^2}{(a!)^2}\|\psi\|_{\lambda,a} = \|\psi\|_{\widetilde{\mathcal{H}},\lambda}.$$

Finally, since $\|\partial_x^k \partial_u^l (\psi_1\psi_2)\|_\infty \le \sum_{r=0}^k \sum_{n=0}^l C_k^r C_l^n \|\partial_x^r \partial_u^n \psi_1\|_\infty \|\partial_x^{k-r}\partial_u^{l-n}\psi_2\|_\infty$, we have

$$\begin{aligned}
\|\psi_1\psi_2\|_{\lambda,0} = \sum_{k,l\in\mathbb{N}} \frac{\lambda^{k+l}}{k!l!}\|\partial_x^k \partial_u^l (\psi_1\psi_2)\|_\infty &\le \sum_{r,n\in\mathbb{N}} \|\partial_x^r \partial_u^n \psi_1\|_\infty \sum_{k\ge r}\sum_{l\ge n} \frac{C_k^r C_l^n \lambda^{k+l}}{k!l!}\|\partial_x^{k-r}\partial_u^{l-n}\psi_2\|_\infty \\
&\le \sum_{r,n\in\mathbb{N}} \frac{\lambda^{r+n}}{r!n!}\|\partial_x^r \partial_u^n \psi_1\|_\infty \sum_{k\ge r}\sum_{l\ge n} \frac{\lambda^{(k-r)+(l-n)}}{(k-r)!(l-n)!}\|\partial_x^{k-r}\partial_u^{l-n}\psi_2\|_\infty
\end{aligned}$$

which provides (iii) by changing the indexes $k$ to $k+r$ and $l$ to $l+n$ in the inner sums. □

We now observe that finiteness of the analytical norm of a solution $f$ to (VFP) is not enough to provide a control of the function $(t,x) \mapsto \int_{\mathbb{R}} u^2 f(t,x,u)du$. This is the reason why we introduce a weight function intended to truncate the velocity state space in a suitable sense. More precisely, assume that $f : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ is a $\mathcal{C}^{1,\infty}$ solution of equation (VFP) with bounded derivatives of all order, and set

$$g(t,x,u) := \omega(u) f(t,x,u), \tag{2.3}$$

where $\omega : \mathbb{R} \to (0,+\infty)$ is a weight function such that $\int_{\mathbb{R}} \frac{u^2}{\omega(u)}du < +\infty$. Then, the regularity of velocity moments of $f$ is easily controlled in terms of the regularity of $g$:

$$\begin{aligned}
\sup_{(t,x)\in[0,T]\times\mathbb{R}} \left|\partial_x^k \int_{\mathbb{R}} |u|^2 f(t,x,u)\,du\right| &= \sup_{(t,x)\in[0,T]\times\mathbb{R}} \left|\int_{\mathbb{R}} \frac{|u|^2}{\omega(u)}\partial_x^k g(t,x,u)\,du\right| \\
&\le \sup_{(t,x,u)\in[0,T]\times\mathbb{R}^2} \left|\partial_x^k g(t,x,u)\right| \int_{\mathbb{R}} \frac{u^2}{\omega(u)}du.
\end{aligned}$$

Moreover, since $\partial_u f = \partial_u g - g(\partial_u \ln\omega)$ and $\omega\partial_u^2 f = \partial_u^2 g - 2(\partial_u \ln(\omega))(\partial_u g) + g(\frac{2|\partial_u\omega|^2}{\omega^2} - \frac{\partial_u^2\omega}{\omega})$, the function $g$ defined in (2.3) is seen to satisfy the equation

$$\begin{cases} \partial_t g(t,x,u) + u\partial_x g(t,x,u) - [\partial_x P(t,x) - \partial_u(\ln\omega(u))]\,\partial_u g(t,x,u) - \dfrac{\sigma^2}{2}\partial_u^2 g(t,x,u) \\ \quad = \partial_x P(t,x)\partial_u(\ln\omega(u))g(t,x,u) - g(t,x,u)h(u), \text{ on } 0,T] \times \mathbb{R}^2, \\ P(t,x) = -\displaystyle\int_{\mathbb{R}} \frac{u^2}{\omega(u)} g(t,x,u)du, \qquad h(u) := \frac{\partial_u^2\omega(u)}{2\omega(u)} - |\partial_u \ln(\omega(u))|^2 \\ g(0,x,u) = g_0(x,u), \text{ on } \mathbb{R}^2; \end{cases} \tag{VFP$\omega$}$$

reciprocally, given a solution $g$ to (VFP$\omega$), the function $f$ defined by (2.3) is a solution to (VFP).

In all the sequel, we shall assume that $\omega : \mathbb{R} \to (0,+\infty)$ is a function of class $\mathcal{C}^\infty$ such that

(H$_\omega$)
$$\lim_{|u|\to+\infty} \frac{\omega(u)}{u} = +\infty \text{ and } \int_{\mathbb{R}} \frac{u^2}{\omega(u)}du = 1.$$
$$\limsup_{|u|\to\infty} \Big|\frac{\omega'(u)}{\omega(u)}\Big| < \infty, \ \limsup_{|u|\to\infty} \Big|\frac{\omega''(u)}{\omega'(u)}\Big| < \infty.$$

Moreover, $u \mapsto \ln(\omega(u)) \in \widetilde{\mathcal{H}}(\lambda_0)$ and $u \mapsto h(u) \in \mathcal{H}(\lambda_0)$ for some $\lambda_0 > 0$.

For instance, we have

**Lemma 2.3.** *For any $s > 3$, (H$_\omega$)holds for the weight function $\omega(u) := c(s)(1+u^2)^{\frac{s}{2}}$, where $c(s)$ is the normalizing constant such that $\int_{\mathbb{R}} \frac{u^2}{\omega(u)}du = 1$. In particular, (H$_\omega$)holds for every $\lambda_0 \in (0, \frac{1}{4})$.*

The proof that $\omega$ as in Lemma 2.3 satisfies (H$_\omega$)relies on Lemma 2.1, and is given in Appendix A.1. Condition (2.5) on $f_0$, introduced next, provides tractable bounds on the analytic norms of $g_0$ that will be required in the study of equation (VFP$\omega$):

**Remark 2.4.** *For the choice $\omega(u) = c(1+u^2)^{\frac{s}{2}}$ with $s$ an even positive integer, by interchanging the sums over $a$ and $j$ and using the fact that $\partial_u^n\omega = 0$ for all $n > s$, one has*

$$\begin{aligned} \|g_0\|_{\mathcal{H},\lambda} &= \sum_{a\in\mathbb{N}} \frac{1}{(a!)^2} \sum_{j=0}^{a} \frac{a!}{j!(a-j)!} \sum_{n\geq a-j} \frac{\lambda^{n-(a-j)}}{n-(a-j)!} \sum_{k+l\geq j} \frac{\lambda^{k+l-j}(k+l)!}{k!l!(k+l-j)!} \|\partial_u^n\omega\partial_u^l\partial_x^k f_0\|_\infty \\ &\leq \kappa(s) \sum_{j\in\mathbb{N}} \frac{1}{(j!)^2} \sum_{k+l\geq j} \frac{\lambda^{k+l-j}(k+l)!}{k!l!(k+l-j)!} \|(1+u^2)^{\frac{s}{2}}\partial_u^l\partial_x^k f_0\|_\infty \left( \sum_{a\geq j} \frac{j!}{a!(a-j)!} \sum_{n=a-j}^{s} \frac{\lambda^{n-(a-j)}}{(n-(a-j))} \right) \end{aligned}$$

*where $\kappa(s) > 0$ is a bound for the absolute values of the coefficients of the polynomials $\omega, \partial_u\omega, \ldots, \partial_u^s\omega$, and the term in parentheses is bounded by $\exp(1+\lambda)$. Arguing as in the proof of Lemma 2.1, we see that*

$$\exists\, C_0 > 0,\ m,n \in \mathbb{N} \text{ such that: } \|(1+u^2)^{\frac{s}{2}}\partial_x^l\partial_u^k f_0\|_\infty \leq \frac{C_0(k+m)!(l+n)!}{\bar{\lambda}^{k+l}} \quad \forall k,l \geq 0, \tag{2.5}$$

*is a sufficient condition for $g_0$ to belong to $\mathcal{H}(\lambda)$ for all $\lambda \in [0,\bar{\lambda})$ and to deduce the control of the norm: $\|g_0\|_{\mathcal{H},\lambda} \leq C_0\kappa(s)\exp(1+\lambda)\sum_{a\in\mathbb{N}} \frac{1}{\bar{\lambda}^a}\frac{(a+1)\cdots(a+m+n+1)}{a!}$. Similarly, we obtain*

$$\|g_0\|_{\widetilde{\mathcal{H}},\lambda} \leq \kappa(s) \sum_{j\in\mathbb{N}} \frac{j^2}{(j!)^2} \sum_{k+l\geq j} \frac{\lambda^{k+l-j}(k+l)!}{k!l!(k+l-j)!} \|(1+u^2)^{\frac{s}{2}}\partial_u^l\partial_x^k f_0\|_\infty \left( \sum_{a\geq j} \frac{a^2 j!}{j^2 a!(a-j)!} \sum_{n=a-j}^{s} \frac{\lambda^{n-(a-j)}}{(n-(a-j))} \right),$$

*where the first sum in parentheses can be seen to be bounded by $\frac{1}{j}\sum_{a\in\mathbb{N}} \frac{a+j}{a!} \leq 2e$, and we see that condition (2.5) implies also that $g_0 \in \widetilde{\mathcal{H}}(\lambda)$ for all $\lambda \in [0,\bar{\lambda})$, with $\|g_0\|_{\widetilde{\mathcal{H}},\lambda} \leq 2C_0\kappa(s)\exp(1+\lambda)\sum_{a\in\mathbb{N}} \frac{1}{\bar{\lambda}^a}\frac{(a+1)\cdots(a+m+n+2)}{a!}$.*

## 2.2 Main results

Given $K, T$ and $\lambda_0$ strictly positive real numbers such that $\lambda_0 > T(1+K)$, and the function

$$\lambda(t) := \lambda_0 - (1+K)t,$$

we now define the spaces

$$\mathcal{H}_{\lambda_0,K,T} := \left\{ \psi \in \mathcal{C}^{1,\infty}([0,T]\times\mathbb{R}^2) \text{ such that } \sup_{t\in[0,T]} \|\psi(t)\|_{\mathcal{H},\lambda(t)} < +\infty \right\},$$
$$\widetilde{\mathcal{H}}_{\lambda_0,K,T} := \left\{ \psi \in \mathcal{C}^{1,\infty}([0,T]\times\mathbb{R}^2) \text{ such that } \int_0^T \|\psi(t)\|_{\widetilde{\mathcal{H}},\lambda(t)} dt < +\infty \right\}$$

and their bounded subsets defined for a positive constant $M$:

$$\mathcal{B}^M_{\lambda_0,K,T} := \left\{ \psi \in \mathcal{H}_{\lambda_0,K,T} \text{ such that } \sup_{t\in[0,T]} \|\psi(t)\|_{\mathcal{H},\lambda(t)} \le M \right\},$$
$$\widetilde{\mathcal{B}}^M_{\lambda_0,K,T} := \left\{ \psi \in \widetilde{\mathcal{H}}_{\lambda_0,K,T} \text{ such that } \int_0^T \|\psi(t)\|_{\widetilde{\mathcal{H}},\lambda(t)} dt \le M \right\}.$$

We are ready to state the main result of this section:

**Theorem 2.5.** *Let $\lambda_0, M, T$ be positive constants and $\omega : \mathbb{R} \to (0,+\infty)$ be a function of class $\mathcal{C}^\infty$ satisfying* ($\mathrm{H}_\omega$). *Introduce the finite constants $\gamma_0 := \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda_0}$ and $\gamma_1 := \|h\|_{\mathcal{H},\lambda_0}$ and assume that*

- *a) $T < \frac{\lambda_0}{2+\lambda_0+4\gamma_0}$,*
- *b) $M \le \frac{1}{16}(K - \lambda_0 - 4\gamma_0 - 1)$ for some $K$ in the nonempty interval $(1+\lambda_0+4\gamma_0, \frac{\lambda_0}{T} - 1)$ and*
- *c) $M(1+\gamma_0)\exp\{(M\gamma_0+\gamma_1)T\} < 1$.*

*Assume moreover that $f_0 : \mathbb{R}^2 \to \mathbb{R}$ is a function of class $\mathcal{C}^\infty$ and that $g_0(x,u) := \omega(u)f_0(x,u)$ satisfies*

- *d) $\max\{\|g_0\|_{\mathcal{H},\lambda_0}, T\|g_0\|_{\widetilde{\mathcal{H}},\lambda_0}\} \le M$ and*
- *e) $\|g_0\|_{\mathcal{H},\lambda_0} \exp(T(\gamma_1 + 16\gamma_0)) \le M\exp(-(16+\gamma_0)M)$.*

*Then, equation* (VFP$\omega$) *has a unique smooth solution $g \in \mathcal{B}^M_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^M_{\lambda_0,K,T}$. In particular, under the previous assumptions, a solution $f \in \mathcal{C}^{1,\infty}$ to* (VFP) *with initial condition $f_0$ exists.*

**Corollary 2.6.** *Let $f$ be the solution to* (VFP) *given in Theorem 2.5 and assume that* ($\mathrm{H}_{\mathrm{unif}(0)}$) *holds. Then, $f(t,x,u)$ satisfies* ($\mathrm{H}_{\mathrm{unif}(t)}$)*, for all $t$ in $[0,T]$. In particular, assume* ($\mathrm{H}_{\mathrm{unif}(0)}$) *in addition to the assumptions of Theorem 2.5, then a solution to* (1.3) *with $\beta = 0$ exists.*

**Remark 2.7.** *Using Lemma 2.3, one can check that if $f_0 \in \mathcal{C}^\infty$ and $\lambda_1 > 0$ are such that $g_0 \in \mathcal{H}(\lambda_1) \cap \widetilde{\mathcal{H}}(\lambda_1)$ and $\|g_0\|_{\mathcal{H},\lambda_1} < \frac{1}{2(1+\|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda_2})}$ for some $\lambda_2 \in (0,\frac{1}{4})$, where $\omega(u) = c(1+u^2)^{\frac{s}{2}}$ for some $s > 3$ and the suitable normalizing constant $c > 0$, then the conclusion of Theorem 2.5 holds for $\lambda_0 := \min\{\lambda_1,\lambda_2\}$, $M = 2\|g_0\|_{\mathcal{H},\lambda_0}$ and $K := 32\|g_0\|_{\mathcal{H},\lambda_0} + \lambda_0 + 4\gamma_0 + 1$, as soon as*

$$T < \min\left\{ \frac{\lambda_0}{2+\lambda_0+4\gamma_0}, \frac{\lambda_0}{32\|g_0\|_{\mathcal{H},\lambda_0}+\lambda_0+4\gamma_0+1}, \frac{2\|g_0\|_{\mathcal{H},\lambda_0}}{\|g_0\|_{\widetilde{\mathcal{H}},\lambda_0}}, -\frac{\ln(2(1+\gamma_0))\|g_0\|_{\mathcal{H},\lambda_0})}{2\gamma_0\|g_0\|_{\mathcal{H},\lambda_0}+\gamma_1}, \frac{\ln 2 - 2(16+\gamma_0)\|g_0\|_{\mathcal{H},\lambda_0}}{\gamma_1+16\gamma_0} \right\},$$

*all five terms in the bracket being strictly positive in this case. Moreover, if* (2.5) *holds, thanks to Remark 2.4 explicit conditions depending only on the positive constants $C_0$ and $\bar{\lambda}$ can be given in order that conditions d) and e) of Theorem 2.5 are satisfied. In that case it is then possible to exhibit, for each real number $\bar{\lambda} > 0$ and even integer $s \ge 4$, a constant $\kappa_0 = \kappa_0(\bar{\lambda}, s)$ and a function $C_0 \mapsto \kappa_1(C_0,\bar{\lambda},s)$ such that the conclusion of Theorem 2.5 holds for every $f_0$ and $T > 0$ such that* (2.5) *holds with $C_0 < \kappa_0(\bar{\lambda},s)$ and that $T < \kappa_1(C_0,\bar{\lambda},s)$.*

The steps of the proof of Theorem 2.5 are the following: first we will establish in Section 2.3 the existence of an analytic solution to a suitable linear version of (VFP) in a small time interval, along with useful estimates. Then, under additional constraints we construct in Section 2.4 a solution to the nonlinear equation (VFP) by means of a fixed point argument.

Before proceeding, let us prove Corollary 2.6:

*Proof of Corollary 2.6.* Periodicity of the solution is an easy consequence of the fixed point method employed in the proof of Theorem 2.5 (see remark 2.16 in Section 2.4).

Now, thanks to the assumptions on $\omega$ and the fact that $f(t,x,u)\omega(u)$ belongs to $\mathcal{H}(\lambda(t))$ for each $t\in[0,T]$, the assumptions of Lemma 1.2 are satisfied (in particular the integrals $\int_{\mathbb{R}} u\partial_u^2 f(t,x,u)du = \int_{\mathbb{R}} \partial_u f(t,x,u)du = \int_{\mathbb{R}} \partial_u^2 f(t,x,u)du$ exist and vanish; moreover, we have $\int_{\mathbb{R}} u\partial_u f(t,x,u)du = -\int_{\mathbb{R}} f(t,x,u)du$). Therefore, the following system of equations is verified: for all $(t,x)\in(0,T]\times\mathbb{R}$,

$$\begin{cases} \partial_t\overline{\rho}(t,x) = -\partial_x V(t,x), \\ \partial_t(\partial_x V(t,x)) = -\partial_x(\overline{\rho}(t,x)\partial_x P(t,x)), \end{cases}$$

where $\overline{\rho}(t,x) := \rho(t,x)-1 = \int_{\mathbb{R}} f(t,x,u)du-1$, $V(t,x) := \int_{\mathbb{R}} uf(t,x,u)\,du$ and $P(t,x) = -\int_{\mathbb{R}} u^2 f(t,x,u)\,du$. From the latter and from Lemma 2.2-(iii), we obtain, for each $\lambda\in[0,\lambda_0)$,

$$\begin{cases} \partial_t\|\overline{\rho}(t)\|_\lambda \le \|\partial_x V(t)\|_\lambda, \\ \partial_t\|\partial_x V(t)\|_\lambda \le \|\partial_x P(t)\|_\lambda\|\partial_x\overline{\rho}(t)\|_\lambda + \|\partial_x^2 P(t)\|_\lambda\|\overline{\rho}(t)\|_\lambda. \end{cases} \tag{2.6}$$

Since $\|\partial_x\overline{\rho}(t)\|_\lambda = \frac{d}{d\lambda}\|\overline{\rho}(t)\|_\lambda$ by Lemma 2.2-(ii), (2.6) rewrites as

$$\begin{cases} \partial_t A(t,\lambda) \le B(t,\lambda), \\ \partial_t B(t,\lambda) \le \|\partial_x P(t)\|_\lambda\partial_\lambda A(t,\lambda) + \|\partial_x^2 P(t))\|_\lambda A(t,\lambda), \end{cases}$$

for $A(t,\lambda) := \|\overline{\rho}(t)\|_\lambda$ and $B(t,\lambda) := \|\partial_x V(t)\|_\lambda$. Since $t\mapsto\lambda(t)$ is decreasing and, by Theorem 2.5, $P\in\mathcal{B}^M_{\lambda_0,K,T}\cap\widetilde{\mathcal{B}}^M_{\lambda_0,K,T}$, we have

$$\|\partial_x P(t)\|_{\lambda(T)} \le \|\partial_x P(t)\|_{\lambda(t)} \le \max_{t\in[0,T]}\|P(t)\|_{\mathcal{H},\lambda(t)} \le M,$$
$$\|\partial_x^2 P(t)\|_{\lambda(T)} \le \|\partial_x^2 P(t)\|_{\lambda(t)} \le 4\max_{t\in[0,T]}\|P(t)\|_{\mathcal{H},\lambda(t)} \le 4M.$$

We deduce that for all $\lambda\in[0,\lambda(T)]$, $t\in[0,T]$,

$$\begin{cases} \partial_t A(t,\lambda) \le B(t,\lambda), \\ \partial_t B(t,\lambda) \le M\partial_\lambda A(t,\lambda) + 4MA(t,\lambda). \end{cases} \tag{2.7}$$

Let us now set $\mathcal{Y}(t,\lambda) := A(t,\lambda)+bB(t,\lambda)$ where $b$ is a positive constant that we will specify later. From (2.7) we obtain

$$\partial_t\mathcal{Y}(t,\lambda) \le B(t,\lambda) + 4bMA(t,\lambda) + bM\partial_\lambda A(t,\lambda) \le (1\vee 4bM)\,\mathcal{Y}(t,\lambda) + bM\partial_\lambda\mathcal{Y}(t,\lambda).$$

That is, with $b_2 := bM > 0$ and $b_1 := (1\vee 4bM) > 0$, it holds that

$$\partial_t\mathcal{Y}(t,\lambda) \le b_1\mathcal{Y}(t,\lambda) + b_2\partial_\lambda\mathcal{Y}(t,\lambda),\ \ \forall t\in[0,T],\ \forall\lambda\in[0,\lambda(T)).$$

We now observe that the function $t\mapsto\mathcal{Y}(t,\gamma(t))$ with $\gamma(t) := \lambda(T)-b_2t$ is constant for all $t\in[0,\frac{\lambda(T)}{b_2})$. Indeed, we have

$$\partial_t\mathcal{Y}(t,\gamma(t)) = (\partial_t\mathcal{Y})(t,\gamma(t)) - b_2\partial_\lambda\mathcal{Y}(t,\gamma(t)) \le b_1\mathcal{Y}(t,\gamma(t)),$$

and Gronwall's lemma, together with assumption ($\mathrm{H}_{\mathrm{unif}(0)}$) implying that $\mathcal{Y}(0,\lambda)=0$ for all non negative $\lambda$, yield $\mathcal{Y}(t,\gamma(t))=0$ for all $t\in[0,\frac{\lambda(T)}{b_2})$. This shows that $\overline{\rho}(t,x) = |\partial_x V(t,x)| = 0$ for all $t\in[0,\frac{\lambda(T)}{b_2})$. Choosing $b=\lambda(T)/(MT)$, we conclude the result, using also the uniform bounds available up to time $t=T$. ☐

## 2.3 The linearized equation

Consider the linear equation

$$\begin{cases} \partial_t g(t,x,u) + u\partial_x g(t,x,u) - (\partial_x Q(t,x) - \partial_u(\ln\omega(u)))\,\partial_u g(t,x,u) - \dfrac{\sigma^2}{2}\partial_u^2 g(t,x,u) \\ = \partial_x Q(t,x)\partial_u(\ln\omega(u))g(t,x,u) + g(t,x,u)h(u), \text{ on } (0,T)\times\mathbb{R}^2, \\ g(0,x,u) = g_0(x,u) := \omega(u)f_0(x,u), \text{ on } \mathbb{R}^2, \end{cases} \tag{FP$\omega$}$$

where $Q : [0,T]\times\mathbb{R} \to \mathbb{R}$ is a given function, with uniformly in $t \in [0,T]$ bounded derivatives of all order in $x \in \mathbb{R}$. Equation (FP$\omega$) is easily seen to be equivalent, through the relation (2.3), to the linear version of (VFP):

$$\begin{cases} \partial_t f(t,x,u) + u\partial_x f(t,x,u) - \partial_x Q(t,x)\partial_u f(t,x,u) - \dfrac{\sigma^2}{2}\partial_u^2 f(t,x,u) = 0, \text{ on } (0,T)\times\mathbb{R}^2 \\ f(0,x,u) = f_0(x,u), \text{ on } \times\mathbb{R}^2. \end{cases} \tag{FP}$$

Existence and uniqueness of a $\mathcal{C}^\infty$-solution to the two previous equations is recalled in Theorem A.1 in Appendix A.2. We next prove that the solution $g$ to (FP$\omega$) is indeed analytic whenever the inputs $g_0$ and $Q$ have small enough analytic norms and the time horizon $T > 0$ is small enough:

**Theorem 2.8.** *Assume that for some $\lambda_0 > 0$ (H$_\omega$)holds, and that $g_0 : \mathbb{R}^2 \to \mathbb{R}$ is a function of class $\mathcal{C}^\infty$ such that $\|g_0\|_{\mathcal{H},\lambda_0} < +\infty$. For $\gamma_0$ and $\gamma_1$ as in Theorem 2.5, let $T > 0$ and $M_1$>0 be a time horizon and a constant satisfying*

*a) $T < \frac{\lambda_0}{2+\lambda_0+4\gamma_0}$ and*

*b) $M_1 \le \frac{1}{16}(K - \lambda_0 - 4\gamma_0 - 1)$ for some $K$ in the nonempty set $(1+\lambda_0+4\gamma_0, \frac{\lambda_0}{T} - 1)$.*

*Then, for any $M_2 > 0$ and $Q \in \mathcal{B}^{M_1}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{M_2}_{\lambda_0,K,T}$, equation (FP$\omega$) has a solution $g$ of class $\mathcal{C}^{1,\infty}$ such that*

$$g \in \mathcal{B}^{\hat{M}}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{\hat{M}}_{\lambda_0,K,T}$$

*where $\hat{M} = \|g_0\|_{\mathcal{H},\lambda_0}\exp\{T(\gamma_1 + 16\gamma_0) + (16+\gamma_0)M_2\}$.*

In the proof we need to deal with truncated versions of the analytic norms previously introduced. For an arbitrary function $\psi$ of class $\mathcal{C}^\infty$ and a fixed $A \in \mathbb{N}$, set

$$\mathbb{A} := \{0,\cdots,A\}, \qquad \|\psi\|_{\lambda;A} := \sum_{k,l\in\mathbb{A}} \frac{\lambda^{k+l}}{k!l!}\left\|\partial_x^k\partial_u^l\psi\right\|_\infty,$$

$$\|\psi\|_{\lambda,a;A} := \frac{d^a}{d\lambda^a}\|\psi\|_{\lambda;A} = \sum_{k,l\in\mathbb{A};k+l\ge a} \frac{(k+l)!}{(k+l-a)!}\frac{\lambda^{k+l-a}}{k!l!}\left\|\partial_x^k\partial_u^l\psi\right\|_\infty,$$

$$\|\psi\|_{\mathcal{H},\lambda;A} := \sum_{a\in\mathbb{A}} \frac{1}{(a!)^2}\|\psi\|_{\lambda,a;A}, \qquad \|\psi\|_{\widetilde{\mathcal{H}},\lambda;A} := \sum_{a\in\mathbb{A}} \frac{a^2}{(a!)^2}\|\psi\|_{\lambda,a;A}.$$

Using a maximum principle for kinetic Fokker-Planck equation, stated in Appendix A.2, we start the proof by establishing estimates for the time evolution of the norms $\|g(t)\|_{\mathcal{H},\lambda(t);A}$ and $\|g(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A}$ along a solution $g$ of the linear equation (FP$\omega$), in terms of $\|Q(t)\|_{\mathcal{H},\lambda(t)}$, $\|\partial_u\ln(\omega)\|_{\mathcal{H},\lambda(t)}$, $\|h\|_{\widetilde{\mathcal{H}},\lambda(t)}$ and $\|Q(t)\|_{\widetilde{\mathcal{H}},\lambda(t)}$.

### 2.3.1 Regularity estimates

Let $g$ be a smooth solution to (FP$\omega$). Observe that

$$\partial_x^k\partial_u^l(u\partial_x g(t,x,u)) = \sum_{n=0}^{l} C_l^n(\partial_u^n u)(\partial_x^{k+1}\partial_u^{l-} g(t,x,u)) = u\partial_x^{k+1}\partial_u^l g(t,x,u) + l\partial_x^{k+1}\partial_u^{l-1} g(t,x,u),$$

$$\begin{aligned}\partial_x^k\partial_u^l(\partial_x Q(t,x)\partial_u g(t,x,u)) &= \sum_{m=0}^{k} C_k^m(\partial_x^{m+1}Q(t,x))(\partial_x^{k-m}\partial_u^{l+1} g(t,x,u))\\ &= \partial_x Q(t,x)\partial_x^k\partial_u^{l+1}g(t,x,u) + \mathbb{1}_{\{k\geq 1\}}\sum_{m=0}^{k-1} C_k^m\partial_x^{k-m+1}Q(t,x)\partial_x^m\partial_u^{l+1}g(t,x,u),\end{aligned}$$

$$\begin{aligned}\partial_x^k\partial_u^l\left(\partial_u\ln(\omega(u))\partial_u g(t,x,u)\right) &= \sum_{n=0}^{l} C_l^n\partial_u^{l-n+1}\ln(\omega(u))\partial_u^{n+1}\partial_x^k g(t,x,u)\\ &= \partial_u\ln\omega(u)\partial_u^{l+1}\partial_x^k g(t,x,u) + \mathbb{1}_{\{l\geq 1\}}\sum_{n=0}^{l-1} C_l^n\,\partial_u^{l-n+1}\ln\omega(u)\,\partial_u^{n+1}\partial_x^k g(t,x,u),\end{aligned}$$

$$\partial_x^k\partial_u^l\left(\partial_x Q(t,x)\partial_u\ln(\omega(u))g(t,x,u)\right) = \sum_{n=0}^{l}\sum_{m=0}^{k} C_l^nC_k^m\partial_x^{k-m+1}Q(t,x)\,\partial_u^{l-n+1}\ln\omega(u)\,\partial_x^m\partial_u^n g(t,x,u),$$

$$\partial_x^k\partial_u^l(g(t,x,u)h(u)) = \sum_{n=0}^{l} C_l^n\partial_x^k\partial_u^n g(t,x,u)\,\partial_u^{l-n}h(u).$$

By applying the differential operator $\partial_x^k\partial_u^l$ to (FP$\omega$), we deduce,that

$$\begin{aligned}&\partial_t(\partial_x^k\partial_u^l g(t,x,u)) + u\partial_x(\partial_x^k\partial_u^l g(t,x,u)) - (\partial_x Q(t,x) - \partial_u\ln\omega(u))\,\partial_u(\partial_x^k\partial_u^l g(t,x,u)) - \frac{\sigma^2}{2}\partial_u^2(\partial_x^k\partial_u^l g(t,x,u))\\ &= -l\partial_x^{k+1}\partial_u^{l-1}g(t,x,u)\mathbb{1}_{\{l\geq 1\}} + \mathbb{1}_{\{k\geq 1\}}\sum_{m=0}^{k-1} C_k^m\partial_x^{k-m}\partial_x Q(t,x)\,\partial_x^m\partial_u^{l+1}g(t,x,u)\\ &\quad - \mathbb{1}_{\{l\geq 1\}}\sum_{n=0}^{l-1} C_l^n\partial_u^{l-n+1}\ln(\omega(u))\partial_u^{n+1}\partial_x^k g(t,x,u) + \sum_{n=0}^{l} C_l^n\partial_x^k\partial_u^n g(t,x,u)\,\partial_u^{l-n}h(u)\\ &\quad + \sum_{n=0}^{l}\sum_{m=0}^{k} C_l^nC_k^m\partial_x^{k-m+1}Q(t,x)\partial_u^{l-n+1}\ln(\omega(u))\partial_x^m\partial_u^n g(t,x,u).\end{aligned}$$

The function $\partial_x^k\partial_u^l g$ is thus a classical solution to a linear Fokker-Planck equation. Applying the maximum principle stated in Theorem A.1 in the appendix section A.2, it follows that

$$\begin{aligned}\frac{d}{dt}\|\partial_x^k\partial_u^l g(t)\|_\infty &\leq \mathbb{1}_{\{l\geq 1\}}l\|\partial_x^{k+1}\partial_u^{l-1}g(t)\|_\infty + \mathbb{1}_{\{k\geq 1\}}\sum_{m=0}^{k-1} C_k^m\|\partial_x^m\partial_u^{l+1}g(t)\|_\infty\|\partial_x^{k-m+1}Q(t)\|_\infty\\ &\quad + \mathbb{1}_{\{l\geq 1\}}\sum_{n=0}^{l-1} C_l^n\|\partial_u^{n+1}\partial_x^k g(t)\|_\infty\|\partial_u^{l-n+1}\ln(\omega)\|_\infty + \sum_{n=0}^{l} C_l^n\|\partial_x^k\partial_u^n g(t)\|_\infty\|\partial_u^{l-n}h\|_\infty\\ &\quad + \sum_{n=0}^{l}\sum_{m=0}^{k} C_l^nC_k^m\|\partial_x^m\partial_u^n g(t)\|_\infty\|\partial_x^{k-m+1}Q(t)\|_\infty\|\partial_u^{l-n+1}\ln(\omega)\|_\infty.\end{aligned} \tag{2.10}$$

We now obtain estimates for the function $t\mapsto\|g(t)\|_{\lambda,a;A}$ for fixed $\lambda>0$ and $A\in\mathbb{N}$.

**Lemma 2.9.** *For each $A \in \mathbb{N}$, $a \in \mathbb{A} = \{0, ..., A\}$ and $\lambda > 0$, a smooth solution $g$ to (FP$\omega$) satisfies:*

$$\frac{d}{dt}\|g(t)\|_{\lambda,a;A} \le \lambda\|g(t)\|_{\lambda,a+1;A} + a\|g(t)\|_{\lambda,a;A} + \frac{d^a}{d\lambda^a}\Big(\|g(t)\|_{\lambda,1;A}\big\{\|Q(t)\|_{\lambda,1;A} + \|\ln(\omega)\|_{\lambda,1;A}\big\}\Big)$$
$$+ \frac{d^a}{d\lambda^a}\Big(\|g(t)\|_{\lambda,0;A}\big\{\|h\|_{\lambda,0} + \|Q(t)\|_{\lambda,1;A}\|\ln(\omega)\|_{\lambda,1;A}\big\}\Big).$$

*Proof.* Multiplying both sides of the inequality (2.10) by $\frac{d^a}{d\lambda^a}\frac{\lambda^{k+l}}{k!l!} = \frac{(k+l)!\lambda^{k+l-a}}{(k+l-a)!k!l!}\mathbb{1}_{\{k+l\ge a\}}$ and summing over $k, l \in \mathbb{A}$ with $k + l \ge a$, we get

$$\frac{d}{dt}\|g(t)\|_{\lambda,a;A} = \sum_{k,l\in\mathbb{A}:k+l\ge a}\frac{d^a}{d\lambda^a}\frac{\lambda^{k+l}}{k!l!}\frac{d}{dt}\|\partial_x^k\partial_u^l g(t)\|_\infty$$
$$\le \sum_{k,l\in\mathbb{A}:k+l\ge a,l\ge 1}\frac{d^a}{d\lambda^a}\frac{l\lambda^{k+l}}{k!l!}\|\partial_x^{k+1}\partial_u^{l-1}g(t)\|_\infty + \sum_{k,l\in\mathbb{A}:k+l\ge a,k\ge 1}\frac{d^a}{d\lambda^a}\frac{\lambda^{k+l}}{k!l!}\sum_{m=0}^{k-1}C_k^m\|\partial_x^m\partial_u^{l+1}g(t)\|_\infty\|\partial_x^{k-m+1}Q(t)\|_\infty$$
$$+ \sum_{k,l\in\mathbb{A}:k+l\ge a,l\ge 1}\frac{d^a}{d\lambda^a}\frac{\lambda^{k+l}}{k!l!}\sum_{n=0}^{l-1}C_l^n\|\partial_u^{n+1}\partial_x^k g(t)\|_\infty\|\partial_u^{l-n+1}\ln(\omega)\|_\infty$$
$$+ \sum_{k,l\in\mathbb{A}:k+l\ge a}\frac{d^a}{d\lambda^a}\frac{\lambda^{k+l}}{k!l!}\sum_{n=0}^{l}C_l^n\|\partial_x^k\partial_u^n g(t)\|_\infty\|\partial_u^{l-n}h\|_\infty$$
$$+ \sum_{k,l\in\mathbb{A}:k+l\ge a}\frac{d^a}{d\lambda^a}\frac{\lambda^{k+l}}{k!l!}\sum_{n=0}^{l}\sum_{m=0}^{k}C_l^nC_k^m\|\partial_x^m\partial_u^n g(t)\|_\infty\|\partial_x^{k-m+1}Q(t)\|_\infty\|\partial_u^{l-n+1}\ln(\omega)\|_\infty. \tag{2.11}$$

To bound from above the first term on the r.h.s. of (2.11) we observe that

$$\sum_{k,l\in\mathbb{A};k+l\ge a,l\ge 1}\frac{d^a}{d\lambda^a}\frac{l\lambda^{k+l}}{k!l!}\|\partial_x^{k+1}\partial_u^{l-1}g(t)\|_\infty = \frac{d^a}{d\lambda^a}\sum_{k,l\in\mathbb{A};l\ge 1}\frac{l\lambda^{k+l}}{k!l!}\|\partial_x^{k+1}\partial_u^{l-1}g(t)\|_\infty$$

with

$$\sum_{k,l\in\mathbb{A}:l\ge 1}\frac{l\lambda^{k+l}}{k!l!}\|\partial_x^{k+1}\partial_u^{l-1}g(t)\|_\infty = \sum_{k,l\in\mathbb{A}:l\ge 1}\frac{\lambda^{k+l}}{k!(l-1)!}\|\partial_x^{k+1}\partial_u^{l-1}g(t)\|_\infty$$
$$= \sum_{k,l\in\mathbb{A}}\frac{\lambda^{k+l+1}}{k!l!}\|\partial_x^{k+1}\partial_u^{l}g(t)\|_\infty = \lambda\|\partial_x g(t)\|_{\lambda,0;A}.$$

Since

$$\frac{d^a}{d\lambda^a}\left(\lambda\|\partial_x g(t)\|_{\lambda,0;A}\right) = \sum_{r=0}^{a}C_a^r\left(\frac{d^r}{d\lambda^r}\lambda\right)\left(\frac{d^{a-r}}{d\lambda^{a-r}}\|\partial_x g(t)\|_{\lambda,0;A}\right) = C_a^a\lambda\|\partial_x g(t)\|_{\lambda,a;A} + C_a^{a-1}\|\partial_x g(t)\|_{\lambda,a-1;A}$$
$$= \lambda\|\partial_x g(t)\|_{\lambda,a;A} + a\|\partial_x g(t)\|_{\lambda,a-1;A},$$

it follows that

$$\sum_{k,l\in\mathbb{A}:k+l\ge a,l\ge 1}\frac{d^a}{d\lambda^a}\frac{l\lambda^{k+l}}{k!l!}\|\partial_x^{k+1}\partial_u^{l-1}g(t)\|_\infty = \lambda\|g(t)\|_{\lambda,a+1;A} + a\|g(t)\|_{\lambda,a;A}.$$

The remaining terms on the r.h.s of (2.11) rewrite as

$$
\begin{aligned}
&\frac{d^a}{d\lambda^a}\left(\sum_{k,l\in\mathbb{A}:k\geq 1}\frac{\lambda^{k+l}}{k!l!}\sum_{m=0}^{k-1}C_k^m\|\partial_x^{k-m+1}Q(t)\|_\infty\|\partial_x^m\partial_u^{l+1}g(t)\|_\infty\right)\\
&+\frac{d^a}{d\lambda^a}\left(\sum_{k,l\in\mathbb{A}}\frac{\lambda^{k+l}}{k!l!}\sum_{n=0}^{l-1}C_l^n\|\partial_u^{l-n+1}\ln(\omega)\|_\infty\|\partial_u^{n+1}\partial_x^k g(t)\|_\infty\right)+\frac{d^a}{d\lambda^a}\left(\sum_{k,l\in\mathbb{A}}\frac{\lambda^{k+l}}{k!l!}\sum_{n=0}^{l}C_l^n\|\partial_x^k\partial_u^n g(t)\|_\infty\|\partial_u^{l-n}h\|_\infty\right)\\
&+\frac{d^a}{d\lambda^a}\left(\sum_{k,l\in\mathbb{A}:l\geq 1}\frac{\lambda^{k+l}}{k!l!}\sum_{n=0}^{l}\sum_{m=0}^{k}C_l^nC_k^m\|\partial_x^{k-m+1}Q(t)\|_\infty\|\partial_u^{l-n+1}\ln(\omega)\|_\infty\|\partial_x^m\partial_u^n g(t)\|_\infty\right)\\
&:= I_1+I_2+I_3+I_4.
\end{aligned}
\tag{2.12}
$$

For $I_1$, we notice that

$$
\begin{aligned}
\sum_{k,l\in\mathbb{A}:k\geq 1}\frac{\lambda^{k+l}}{k!l!}\sum_{m=0}^{k-1}C_k^m\|\partial_x^{k-m+1}Q(t)\|_\infty\|\partial_x^m\partial_u^{l+1}g(t)\|_\infty &= \sum_{m,l\in\mathbb{A}}\|\partial_x^m\partial_u^{l+1}g(t)\|_\infty\left(\sum_{k=m+1}^{A}\frac{C_k^m\lambda^{k+l}}{k!l!}\|\partial_x^{k-m+1}Q(t)\|_\infty\right)\\
&= \sum_{m,l\in\mathbb{A}}\frac{\lambda^{m+l}}{m!l!}\|\partial_x^m\partial_u^{l+1}g(t)\|_\infty\left(\sum_{k=m+1}^{A}\frac{\lambda^{k-m}}{(k-m)!}\|\partial_x^{k-m+1}Q(t)\|_\infty\right)\\
&= \sum_{m,l\in\mathbb{A}}\frac{\lambda^{m+l}}{m!l!}\|\partial_x^m\partial_u^{l+1}g(t)\|_\infty\left(\sum_{k=1}^{A-m}\frac{\lambda^{k}}{k!}\|\partial_x^{k+1}Q(t)\|_\infty\right)\\
&= \sum_{m,l\in\mathbb{A}}\frac{\lambda^{m+l}}{m!l!}\|\partial_x^m\partial_u^{l+1}g(t)\|_\infty\|Q(t)\|_{\lambda,1;A-m}.
\end{aligned}
$$

Taking the $a$-th derivative with respect to $\lambda$, and noting that $\sum_{m,l\in\mathbb{A}}\frac{\lambda^{m+l}}{m!l!}\|\partial_x^m\partial_u^{l+1}g(t)\|_\infty=\|\partial_u g(t)\|_{\lambda,0;A}=\|g(t)\|_{\lambda,1;A}-\|\partial_x g(t)\|_{\lambda,0;A}$ (by similar computations as proof of Lemma 2.2-(i)), we deduce that

$$
I_1\leq\frac{d^a}{d\lambda^a}\Big(\|Q(t)\|_{\lambda,1;A}\|g(t)\|_{\lambda,1;A}\Big)
$$

using also the fact that $\frac{d^b}{d\lambda^b}\|\partial_x g(t)\|_{\lambda,0;A}\geq 0$ and $\|Q(t)\|_{\lambda,a-b+1;A-m}\leq\|Q(t)\|_{\lambda,a-b+1;A}$ for all $b\in\{0,\ldots,a\}$. In the same way, we obtain the estimate

$$
I_2\leq\frac{d^a}{d\lambda^a}\Big(\|\ln(\omega)\|_{\lambda,1;A}\|g(t)\|_{\lambda,1;A}\Big).
$$

For $I_3$, one can directly check that

$$
\begin{aligned}
\sum_{k,l\in\mathbb{A}}\frac{\lambda^{k+l}}{k!l!}\sum_{n=0}^{l}C_l^n\|\partial_x^k\partial_u^n g(t)\|_\infty\|\partial_u^{l-n}h\|_\infty &= \sum_{k,n\in\mathbb{A}}\|\partial_x^k\partial_u^n g(t)\|_\infty\sum_{l=n}^{A}\frac{C_l^n\lambda^{k+l}}{k!l!}\|\partial_u^{l-n}h\|_\infty\\
&= \sum_{k,n\in\mathbb{A}}\|\partial_x^k\partial_u^n g(t)\|_\infty\sum_{l=0}^{A}\frac{C_{l+n}^n\lambda^{k+l+n}}{k!(l+n)!}\|\partial_u^{l}h\|_\infty\\
&= \sum_{k,n\in\mathbb{A}}\frac{\lambda^{k+n}}{k!n!}\|\partial_x^k\partial_u^n g(t)\|_\infty\sum_{l=0}^{A}\frac{\lambda^{l}}{l!}\|\partial_u^{l-n}h\|_\infty\\
&= \|h\|_{\lambda,0;A}\|g(t)\|_{\lambda,0;A}.
\end{aligned}
$$

As for $I_4$,

$$\sum_{k,l\in\mathbb{A}} \frac{\lambda^{k+l}}{k!l!} \sum_{n=0}^{l}\sum_{m=0}^{k} C_l^n C_k^m \|\partial_x^{k-m+1} Q(t)\|_\infty \|\partial_u^{l-n+1}\ln(\omega)\|_\infty \|\partial_x^m\partial_u^n g(t)\|_\infty$$
$$= \sum_{m,n\in\mathbb{A}} \frac{\lambda^{m+n}}{m!n!}\|\partial_x^m\partial_u^n g(t)\|_\infty \left(\sum_{k=m}^{A}\frac{\lambda^{k-m}}{(k-m)!}\|\partial_x^{k-m+1}Q(t)\|_\infty\right)\left(\sum_{l=n}^{A}\frac{\lambda^{l-n}}{(l-n)!}\|\partial_u^{l-n+1}\ln(\omega)\|_\infty\right)$$
$$= \|g(t)\|_{\lambda,0;\mathbb{A}}\|Q(t)\|_{\lambda,1;A}\|\ln(\omega)\|_{\lambda,1;A}.$$

Hence, (2.12) is bounded from above by

$$\frac{d^a}{d\lambda^a}\Big(\|g(t)\|_{\lambda,0;A}\left(\|h\|_{\lambda,0;A}+\|Q(t)\|_{\lambda,1;A}\|\ln(\omega)\|_{\lambda,1;A}\right)\Big)+\frac{d^a}{d\lambda^a}\Big(\|g(t)\|_{\lambda,1;A}\left(\|Q(t)\|_{\lambda,1;A}+\|\ln(\omega)\|_{\lambda,1;A}\right)\Big).$$

Coming back to (2.11), the above estimates prove Lemma 2.9. □

### 2.3.2 Evolution and control of the time-inhomogeneous analytic norms

Next Lemmas 2.10 and 2.11 are preliminaries for the bounds of the time derivative of $\|g(t)\|_{\mathcal{H},\lambda(t);A}$ in Proposition 2.12 below. Their proof is given in Appendix A.3.

**Lemma 2.10.** *Let $f, v, w$ be functions of class $\mathcal{C}^\infty$ with bounded derivatives at all order. Then, for all $\lambda > 0$ and $A \in \mathbb{N}$,*

$$\sum_{a\in\mathbb{A}}\frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|f\|_{\lambda,0;A}\|v\|_{\lambda,1;A}\|w\|_{\lambda,1;A}\right)\le\|f\|_{\mathcal{H},\lambda;A}\|v\|_{\widetilde{\mathcal{H}},\lambda;A}\|w\|_{\widetilde{\mathcal{H}},\lambda;A}. \tag{2.13}$$

*Suppose moreover that for some $\bar\lambda > 0$, one has $f \in \mathcal{H}(\bar\lambda)$ and $v, w \in \widetilde{\mathcal{H}}(\bar\lambda)$. Then, for all $\lambda \in [0, \bar\lambda)$,*

$$\sum_{a\in\mathbb{N}}\frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|f\|_{\lambda,0}\|v\|_{\lambda,1}\|w\|_{\lambda,1}\right)\le\|f\|_{\mathcal{H},\lambda}\|v\|_{\widetilde{\mathcal{H}},\lambda}\|w\|_{\widetilde{\mathcal{H}},\lambda}.$$

**Lemma 2.11.** *Let $f, w$ be functions of class $\mathcal{C}^\infty$ with bounded derivatives at all order.*

*(i) For all $\lambda > 0$ and $A \in \mathbb{N}$, one has*

$$\sum_{a\in\mathbb{A}}\frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|f\|_{\lambda,1;A}\|v\|_{\lambda,1;A}\right)\le 16(\|f\|_{\mathcal{H},\lambda;A}\|v\|_{\widetilde{\mathcal{H}},\lambda;A}+\|f\|_{\widetilde{\mathcal{H}},\lambda;A}\|v\|_{\mathcal{H},\lambda;A}). \tag{2.14}$$

*Moreover if for some $\bar\lambda > 0$ we have $f, v \in \mathcal{H}(\bar\lambda)\cap\widetilde{\mathcal{H}}(\bar\lambda)$ then, for all $\lambda \in [0, \bar\lambda)$*

$$\sum_{a\in\mathbb{N}}\frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|f\|_{\lambda,1}\|v\|_{\lambda,1}\right)\le 16(\|f\|_{\mathcal{H},\lambda}\|v\|_{\widetilde{\mathcal{H}},\lambda}+\|f\|_{\widetilde{\mathcal{H}},\lambda}\|v\|_{\mathcal{H},\lambda}).$$

*(ii) For all $\lambda > 0$ and $A \in \mathbb{N}$, one has*

$$\sum_{a\in\mathbb{A}}\frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|f\|_{\lambda,1;A}\|v\|_{\lambda,1;A}\right)\le 4\|v\|_{\widetilde{\mathcal{H}},\lambda;A}(4\|f\|_{\mathcal{H},\lambda;A}+\|f\|_{\widetilde{\mathcal{H}},\lambda;A}). \tag{2.15}$$

*Moreover for some $\bar\lambda > 0$, $f \in \mathcal{H}(\bar\lambda)\cap\widetilde{\mathcal{H}}(\bar\lambda)$ and $v \in \widetilde{\mathcal{H}}(\bar\lambda)$, for all $\lambda \in [0, \bar\lambda)$,*

$$\sum_{a\in\mathbb{N}}\frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|f\|_{\lambda,1}\|v\|_{\lambda,1}\right)\le 4\|v\|_{\widetilde{\mathcal{H}},\lambda}(4\|f\|_{\mathcal{H},\lambda}+\|f\|_{\widetilde{\mathcal{H}},\lambda}).$$

**Proposition 2.12.** *For each $A \in \mathbb{N}$, the $\mathcal{C}^{1,\infty}$ function $g$ solution to (FP$\omega$) satisfies*

$$\frac{d}{dt}\|g(t)\|_{\mathcal{H},\lambda(t);A}\le\left(\lambda(t)+1+\lambda'(t)+4\gamma_0+16\|Q(t)\|_{\mathcal{H},\lambda(t)}\right)\|g(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A}$$
$$+\Big(\gamma_1+16\gamma_0+(\gamma_0+16)\,\|Q(t)\|_{\widetilde{\mathcal{H}},\lambda(t)}\Big)\|g(t)\|_{\mathcal{H},\lambda(t);A},$$

*where $\gamma_0 := \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda_0}$ and $\gamma_1 := \|h\|_{\mathcal{H},\lambda_0}$.*

*Proof.* Differentiating in time the norm $\|g(t)\|_{\mathcal{H},\lambda(t);A}$, we get

$$\begin{aligned}\frac{d}{dt}\|g(t)\|_{\mathcal{H},\lambda(t);A} &= \sum_{a\in\mathbb{A}} \frac{1}{(a!)^2}\left(\lambda'(t)\frac{d^{a+1}}{d\lambda^{a+1}}\|g(t)\|_{\lambda,0;A}\right) + \sum_{a=0}^{A}\frac{1}{(a!)^2}\left(\frac{d}{dt}\|g(t)\|_A\right)\Big|_{\lambda=\lambda(t)} \\ &= \lambda'(t)\sum_{a=0}^{A}\frac{1}{(a!)^2}\|g(t)\|_{\lambda(t),a+1;A} + \sum_{a=0}^{A}\frac{1}{(a!)^2}\left(\frac{d}{dt}\|g(t)\|_{\lambda,a;A}\right)\Big|_{\lambda=\lambda(t)},\end{aligned}$$

Dividing both sides of the inequality in Lemma 2.9 by $(a!)^2$ and summing the resulting expression over $a\in\mathbb{A}$, it follows that

$$\begin{aligned}\frac{d}{dt}\|g(t)\|_{\mathcal{H},\lambda(t);A} \le& \sum_{a\in\mathbb{A}}\frac{\lambda'(t)+\lambda(t)}{(a!)^2}\|g(t)\|_{\lambda(t),a+1;A} + \sum_{a=0}^{A}\frac{a}{(a!)^2}\|g(t)\|_{\lambda(t),a;A} \\ &+ \sum_{a=0}^{A}\frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\Big(\|g(t)\|_{\lambda,0;A}\left(\|h\|_{\lambda,0;A} + \|Q(t)\|_{\lambda,1}\|\ln(\omega)\|_{\lambda,1;A}\right)\Big)\Big|_{\lambda=\lambda(t)} \\ &+ \sum_{a=0}^{A}\frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\Big(\|g(t)\|_{\lambda,1;A}\left(\|Q(t)\|_{\lambda,1;A} + \|\ln(\omega)\|_{\lambda,1;A}\right)\Big)\Big|_{\lambda=\lambda(t)}.\end{aligned} \tag{2.16}$$

For the first term in (2.16), we have

$$\begin{aligned}\sum_{a=0}^{A}\frac{\lambda'(t)+\lambda(t)}{(a!)^2}\|g(t)\|_{\lambda(t),a+1;A} &= (\lambda'(t)+\lambda(t))\sum_{a=0}^{A}\frac{(a+1)^2}{((a+1)!)^2}\|g(t)\|_{\lambda(t),a+1;A} \\ &= (\lambda'(t)+\lambda(t))\sum_{a=0}^{A}\frac{(a)^2}{((a)!)^2}\|g(t)\|_{\lambda(t),a;A},\end{aligned}$$

and, for the second term

$$\sum_{a=0}^{A}\frac{a}{(a!)^2}\|g(t)\|_{\lambda(t),a;A} \le \sum_{a\in\mathbb{A}}\frac{(a)^2}{(a!)^2}\|g(t)\|_{\lambda(t),a;A},$$

so that

$$\sum_{a=0}^{A}\frac{\lambda'(t)+\lambda(t)}{(a!)^2}\|g(t)\|_{\lambda(t),a+1;A} + \sum_{a=0}^{A}\frac{a}{(a!)^2}\|g(t)\|_{\lambda(t),a;A} \le (1+\lambda'(t)+\lambda(t))\,\|g(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A}. \tag{2.17}$$

For the third term, observe that on one hand

$$\begin{aligned}\sum_{a=0}^{A}\frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|g(t)\|_{\lambda,0;A}\|h\|_{\lambda,0;A}\right) &= \sum_{a=0}^{A}\frac{1}{(a!)^2}\sum_{r=0}^{a}C_a^r\left(\frac{d^r}{d\lambda^r}\|g(t)\|_{\lambda,0;A}\right)\left(\frac{d^{a-r}}{d\lambda^{a-r}}\|h\|_{\lambda,0;A}\right) \\ &= \sum_{a=0}^{A}\frac{1}{(a!)^2}\sum_{r=0}^{a}C_a^r\|g(t)\|_{\lambda,r;A}\|h\|_{\lambda,a-r;A} \\ &= \sum_{r=0}^{A}\|g(t)\|_{\lambda,r;A}\sum_{a=r}^{A}\|h\|_{\lambda,a-r;A}\frac{C_a^r}{(a!)^2} \\ &= \sum_{r=0}^{A}\|g(t)\|_{\lambda,r;A}\sum_{a=0}^{A}\|h\|_{\lambda,a;A}\frac{C_{a+r}^r}{((a+r)!)^2} \\ &= \sum_{r=0}^{A}\frac{\|g(t)\|_{\lambda,r;A}}{(r!)^2}\sum_{a=0}^{A}\frac{\|h\|_{\lambda,a-r;A}}{(a!)^2}\frac{a!r!}{(a+r))!},\end{aligned}$$

and since $\frac{a!r!}{(a+r)!} \leq 1$, for all $a, r \in \mathbb{N}$, we get that

$$\sum_{a=0}^{A} \frac{1}{(a!)^2} \frac{d^a}{d\lambda^a} \left( \|g(t)\|_{\lambda,0;A} \|h\|_{\lambda,0;A} \right) \leq \|g(t)\|_{\mathcal{H},\lambda;A} \|h\|_{\mathcal{H},\lambda;A}. \tag{2.18}$$

On the other hand, inequality 2.13 provides a bound for the remaining summand in the third term of (2.16):

$$\sum_{a\in\mathbb{A}} \frac{1}{(a!)^2} \frac{d^a}{d\lambda^a} \Big( \|g(t)\|_{\lambda,0;A} \|Q(t)\|_{\lambda,1;A} \|\ln(\omega)\|_{\lambda,1;A} \Big) \leq \|g(t)\|_{\mathcal{H},\lambda(t);A} \|Q(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A} \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda(t);A}. \tag{2.19}$$

For the fourth term in (2.16) we use (2.14) and (2.15) in order to get the estimate

$$\begin{aligned} &\sum_{a=0}^{A} \frac{1}{(a!)^2} \frac{d^a}{d\lambda^a} \left( \|g(t)\|_{\lambda,1;A} \left( \|Q(t)\|_{\lambda,1;A} + \|\ln(\omega)\|_{\lambda,1;A} \right) \right) \\ &\leq 16 \|g(t)\|_{\mathcal{H},\lambda(t);A} \Big( \|Q(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A} + \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda(t);A} \Big) \\ &\quad + 4 \|g(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A} \Big( 4 \|Q(t)\|_{\mathcal{H},\lambda(t);A} + \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda(t);A} \Big). \end{aligned} \tag{2.20}$$

Inserting (2.17), (2.18), (2.19) and (2.20) in (2.16), we conclude that

$$\begin{aligned} \frac{d}{dt} \|g(t)\|_{\mathcal{H},\lambda(t);A} &\leq \left( \lambda'(t) + \lambda(t) + 1 \right) \|g(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A} + \|g(t)\|_{\mathcal{H},\lambda(t);A} \|h\|_{\mathcal{H},\lambda(t);A} \\ &\quad + \|g(t)\|_{\mathcal{H},\lambda(t);A} \|Q(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A} \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda(t);A} + 16 \|g(t)\|_{\mathcal{H},\lambda(t);A} \Big( \|Q(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A} + \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda(t);A} \Big) \\ &\quad + \|g(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A} \Big( 16 \|Q(t)\|_{\mathcal{H},\lambda(t);A} + 4 \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda(t);A} \Big). \end{aligned}$$

We end the proof by using the obvious upper bounds for the truncated norms. □

### 2.3.3 Proof of Theorem 2.8

Applying Gronwall's lemma to the inequality in Proposition 2.12, we obtain that, for all $t \in [0, T]$ and $A \in \mathbb{N}$,

$$\begin{aligned} &\|g(t)\|_{\mathcal{H},\lambda(t);A} \leq \|g_0\|_{\mathcal{H},\lambda_0} \exp\left\{ \int_0^t \Big( \gamma_1 + 16\gamma_0 + (16+\gamma_0) \|Q(\theta)\|_{\widetilde{\mathcal{H}},\lambda(\theta)} \Big)\, ds \right\} \\ &+ \int_0^t \left( \lambda(\theta) + 1 + \lambda'(\theta) + 4\gamma_0 + 16 \|Q(\theta)\|_{\mathcal{H},\lambda(\theta)} \right) \|g(\theta)\|_{\widetilde{\mathcal{H}},\lambda(\theta);A} \exp\left\{ \int_\theta^t \Big( \gamma_1 + 16\gamma_0 + (16+\gamma_0) \|Q(\theta')\|_{\widetilde{\mathcal{H}},\lambda(\theta')} \Big)\, d\theta' \right\} d\theta \\ &\leq \|g_0\|_{\mathcal{H},\lambda_0} \exp\left\{ T(\gamma_1 + 16\gamma_0) + (16+\gamma_0) M_2 \right\} \\ &\quad + \exp\left\{ T(\gamma_1 + 16\gamma_0) + (16+\gamma_0) M_2 \right\} \left( \lambda_0 - K + 4\gamma_0 + 16 M_1 \right) \int_0^t \|g(\theta)\|_{\widetilde{\mathcal{H}},\lambda(\theta);A} d\theta. \end{aligned} \tag{2.21}$$

where in the second inequality we use the facts that $Q \in \mathcal{B}^{M_1}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{M_2}_{\lambda_0,K,T}$ and that

$$\lambda(t) + 1 + \lambda'(t) + 4\gamma_0 + 16 \|Q(t)\|_{\mathcal{H},\lambda(t)} \leq \lambda_0 - K + 4\gamma_0 + 16 \|Q(t)\|_{\mathcal{H},\lambda(t)} \leq \lambda_0 - K + 4\gamma_0 + 16 M_1$$

for all $t \in [0, T]$. From the assumptions we can choose $K > 0$ such that $K < \frac{\lambda_0}{T} - 1$ and

$$K - \lambda_0 - 4\gamma_0 - 16 M_1 \geq 1.$$

Then we deduce with (2.21) and the latter inequality that

$$\begin{aligned} \|g(t)\|_{\mathcal{H},\lambda(t);A} + \int_0^t \|g(\theta)\|_{\widetilde{\mathcal{H}},\lambda(\theta);A} d\theta &\leq \|g(t)\|_{\mathcal{H},\lambda(t);A} + \exp\left\{ T(\gamma_1 + 16\gamma_0) + (16+\gamma_0) M_2 \right\} \int_0^t \|g(\theta)\|_{\widetilde{\mathcal{H}},\lambda(\theta);A} d\theta \\ &\leq \|g_0\|_{\mathcal{H},\lambda_0} \exp\left\{ T(\gamma_1 + 16\gamma_0) + (16+\gamma_0) M_2 \right\}. \end{aligned}$$

After letting $A \to \infty$ we conclude that

$$\max_{t\in[0,T]} \|g(t)\|_{\mathcal{H},\lambda(t)} \le \|g_0\|_{\mathcal{H},\lambda_0} \exp\{T(\gamma_1+16\gamma_0)+(16+\gamma_0)M_2\}\,,$$
$$\int_0^T \|g(s)\|_{\widetilde{\mathcal{H}},\lambda(t)}\,dt \le \|g_0\|_{\mathcal{H},\lambda_0} \exp\{T(\gamma_1+16\gamma_0)+(16+\gamma_0)M_2\}\,. \tag{2.22}$$

## 2.4 Proof of Theorem 2.5 : solving the Vlasov-Fokker-Planck equation (VFP)

Relying upon Theorem 2.8, we construct now, by means of a Banach's fixed point method, a solution to the nonlinear Vlasov-Fokker-Planck equation (VFP).

**Remark 2.13.** *Since we are assuming in* ($\mathrm{H}_\omega$)*that* $\int_{\mathbb{R}} \frac{|u|^2}{\omega(u)}du = 1$*, for all* $\lambda \ge 0$ *and* $a \in \mathbb{N}$ *it holds that*

$$\left\| \int_{\mathbb{R}} \frac{|u|^2}{\omega(u)} \varphi(t,\cdot,u)du \right\|_{\lambda,a} \le \|\varphi(t,\cdot,\cdot)\|_{\lambda,a}$$

*for any function* $\varphi : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ *of class* $\mathcal{C}^{1,\infty}$ *and every* $t \in [0,T]$*. Therefore, if we denote by* $\Phi$ *the mapping associating to a function* $\varphi$ *the solution* $\Phi(\varphi)$ *of the linear equation* (FP$\omega$) *with potential* $\partial_x Q(t,x)$ *given by*

$$Q(t,x) := -\int_{\mathbb{R}} \frac{|u|^2}{\omega(u)} \varphi(t,x,u)\,du\,,$$

*the inclusion*

$$\Phi\left(\mathcal{B}^{M_1}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{M_2}_{\lambda_0,K,T}\right) \subseteq \mathcal{B}^{\hat{M}}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{\hat{M}}_{\lambda_0,K,T}$$

*holds under the conditions on the constants* $T, \lambda_0, K, M_1, M_2$ *and* $\hat{M}$ *established in Theorem 2.8.*

**Corollary 2.14.** *If in addition to the assumptions of Theorem 2.8, the constants* $M := M_1$ *and* $T > 0$ *satisfy the constraint*

$$\|g_0\|_{\mathcal{H},\lambda_0} \exp(T(\gamma_1+16\gamma_0)) \le M\exp(-(16+\gamma_0)M),$$

*then* $\Phi\left(\mathcal{B}^{M}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T}\right) \subseteq \mathcal{B}^{M}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T}$.

*Proof.* Taking $M_2 = M = M_1$ in Theorem 2.8 we get that $\Phi\left(\mathcal{B}^{M}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T}\right) \subseteq \mathcal{B}^{\hat{M}}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{\hat{M}}_{\lambda_0,K,T}$ for $\hat{M} = \|g_0\|_{\mathcal{H},\lambda_0} \exp\{T(\gamma_1+16\gamma_0)+(16+\gamma_0)M\}$. The additional constraint ensures that $\hat{M} \le M$. ☐

**Theorem 2.15.** *Under the assumptions of Corollary 2.14 and, moreover, that*

$$M(1+\gamma_0)\exp\{(M\gamma_0+\gamma_1)T\} < 1 \tag{2.23}$$

*the mapping*

$$\Phi : \mathcal{B}^{M}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T} \to \mathcal{B}^{M}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T}$$

*is well defined and is a contraction for the norm*

$$\max\left\{ \max_{t\in[0,T]} \|\psi(t)\|_{\lambda(t),0}\,, \int_0^T \|\psi(t)\|_{\lambda(t),1}\,dt \right\}.$$

*If in addition to all the previous assumptions, we have*

$$\max\{\|g_0\|_{\mathcal{H},\lambda_0}, T\|g_0\|_{\widetilde{\mathcal{H}},\lambda_0}\} \le M,$$

*then* $g_0 \in \mathcal{B}^{M}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T}$ *and a solution to the nonlinear Vlasov-Fokker-Planck equation (VFP$\omega$) exists in* $\mathcal{B}^{M}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T}$.

*Proof.* Given $f_i \in \mathcal{B}^M_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^M_{\lambda_0,K,T}$, $i=1,2$, we set $P_i(t,x) := \int_{\mathbb{R}} \frac{|u|^2}{\omega(u)} f_i(t,x,u)\,du$ for $i=1,2$. The difference $\Phi(f_1)-\Phi(f_2)$ satisfies

$$\partial_t \left(\Phi(f_1)-\Phi(f_2)\right) + \left(u\partial_x \left(\Phi(f_1)-\Phi(f_2)\right)\right) - \left[\left(\partial_x P_1 - \partial_u \ln(\omega)\right)\partial_u \left(\Phi(f_1)-\Phi(f_2)\right)\right] - \frac{1}{2}\partial_u^2 \left(\Phi(f_1)-\Phi(f_2)\right)$$
$$= \partial_u \Phi(f_2)\left(\partial_x P_1 - \partial_x P_2\right) + \Phi(f_2)\partial_u \ln(\omega)\left(\partial_x P_1 - \partial_x P_2\right) + \left(\partial_u \ln(\omega)\partial_x P_1 + h\right)\left(\Phi(f_1)-\Phi(f_2)\right).$$

Writing $\bar{\Phi} := \Phi(f_1)-\Phi(f_2)$ and $\bar{P} := P_1 - P_2$, we get

$$\begin{aligned}
&\partial_t \bar{\Phi} + \left(u\partial_x \bar{\Phi}\right) - \left(\left(\partial_x P_1 - \partial_u \ln(\omega)\right)\partial_u \hat{\Phi}\right) - \frac{1}{2}\partial_u^2 \bar{\Phi} \\
&= \left(\Phi(f_2)\partial_u \ln(\omega) + \partial_u \Phi(f_2)\right)\partial_x \bar{P} + \left(\partial_x P_1 \partial_u \ln(\omega) + h\right)\bar{\Phi}.
\end{aligned}$$

Then, by similar computations as in the proof of Theorem 2.8, we successively obtain:
- by applying the operator $\partial_x^k \partial_u^l$,

$$\begin{aligned}
&\partial_t(\partial_x^k\partial_u^l\bar{\Phi}) + u\partial_x\left(\partial_x^k\partial_u^l\bar{\Phi}\right) - \left(\partial_x P_1 - \partial_u \ln(\omega)\right)\partial_u\left(\partial_x^k\partial_u^l\bar{\Phi}\right) - \frac{1}{2}\partial_u^2(\partial_x^k\partial_u^l\bar{\Phi}) \\
&= -l\partial_x^{k+1}\partial_u^{l-1}\bar{\Phi} + \mathbb{1}_{\{k\geq 1\}}\sum_{m=0}^{k-1} C_k^m \left(\partial_x^{k-m+1}P_1\right)\partial_x^m\partial_u^{l+1}\bar{\Phi} - \mathbb{1}_{\{l\geq 1\}}\sum_{n=0}^{l-1} C_l^n\left(\partial_u^{l-n+1}\ln(\omega)\right)\partial_x^k\partial_u^{n+1}\bar{\Phi} \\
&\quad + \sum_{n=0}^{l}\sum_{m=0}^{k} C_k^m C_l^n \partial_x^{k-m}\partial_u^n \Phi(f_2)\partial_u^{l-n+1}\ln\omega(u)\partial_x^{m+1}\bar{P} + \sum_{m=0}^{k} C_k^m\left(\partial_x^{k-m}\partial_u^{l+1}\Phi(f_2)\right)\left(\partial_x^{m+1}\bar{P}\right) \\
&\quad + \sum_{n=0}^{l}\sum_{m=0}^{k} C_k^m C_l^n\left(\partial_x^{k-m+1}P_1\right)\left(\partial_u^{l-n+1}\ln(\omega)\right)\left(\partial_x^m\partial_u^n\bar{\Phi}\right) + \sum_{n=0}^{l} C_l^n\left(\partial_x^k\partial_u^n\Phi\right)\partial_u^{l-n}h;
\end{aligned}$$

- by a maximum principle, and the fact that for all $m \in \mathbb{N}$: $\|\partial^m P_i\|_\infty \leq \|\partial^m f_i\|_\infty$, $i=1,2$, and $\|\partial^m \bar{P}\|_\infty \leq \|\partial^m \bar{f}\|_\infty$ for $\bar{f} := f_1 - f_2$, we get

$$\begin{aligned}
&\frac{d}{dt}\|\partial_x^k\partial_u^l\bar{\Phi}(t)\|_\infty \\
&\leq l\|\partial_x^{k+1}\partial_u^{l-1}\bar{\Phi}(t)\|_\infty + \mathbb{1}_{\{k\geq 1\}}\sum_{m=0}^{k-1} C_k^m\|\partial_x^{k-m+1}f_1(t)\|_\infty\|\partial_x^m\partial_u^{l+1}\bar{\Phi}(t)\|_\infty \\
&\quad + \mathbb{1}_{\{l\geq 1\}}\sum_{n=0}^{l-1} C_l^n\|\partial_u^{l-n+1}\ln(\omega)\|_\infty\|\partial_x^k\partial_u^{n+1}\bar{\Phi}(t)\|_\infty \\
&\quad + \sum_{n=0}^{l}\sum_{m=0}^{k} C_k^m C_l^n\|\partial_u^{l-n+1}\ln(\omega)\|_\infty\|\partial_x^{k-m}\partial_u^n\Phi(f_2)(t)\|_\infty\|\partial_x^{m+1}\bar{f}(t)\|_\infty + \sum_{m=0}^{k} C_k^m\|\partial_x^{k-m}\partial_u^{l+1}\Phi(f_2)(t)\|_\infty\|\partial_x^{m+1}\bar{f}(t)\|_\infty \\
&\quad + \sum_{n=0}^{l}\sum_{m=0}^{k} C_k^m C_l^n\|\partial_u^{l-n+1}\ln(\omega)\|_\infty\|\partial_x^{k-m+1}f_1(t)\|_\infty\|\partial_x^m\partial_u^n\bar{\Phi}(t)\|_\infty + \sum_{n=0}^{l} C_l^n\|\partial_x^k\partial_u^n\bar{\Phi}(t)\|_\infty\|\partial_u^{l-n}h\|_\infty.
\end{aligned}$$

- Replicating the computations in the proof of Lemma 2.9 for $a=0$, $A=+\infty$, we then obtain

$$\begin{aligned}
\frac{d}{dt}\|\bar{\Phi}(t)\|_{\lambda,0} &\leq \lambda\|\bar{\Phi}(t)\|_{\lambda,1} + \|\bar{\Phi}(t)\|_{\lambda,1}\left(\|f_1(t)\|_{\lambda,1} + \|\ln(\omega)\|_{\lambda,1}\right) \\
&\quad + \|\bar{\Phi}(t)\|_{\lambda,0}\left(\|f_1(t)\|_{\lambda,1}\|\ln(\omega)\|_{\lambda,1} + \|h\|_{\lambda,0}\right) \\
&\quad + \|\bar{f}(t)\|_{\lambda,1}\left(\|\Phi(f_2)(t)\|_{\lambda,1} + \|\Phi(f_2)(t)\|_{\lambda,0}\|\ln(\omega)\|_{\lambda,1}\right).
\end{aligned} \tag{2.24}$$

Hence,

$$\begin{aligned}
\frac{d}{dt}\|\bar{\Phi}(t)\|_{\lambda(t),0} &\leq \left(\lambda'(t) + \lambda(t) + \|f_1(t)\|_{\lambda,1} + \|\ln(\omega)\|_{\lambda(t),1}\right)\|\bar{\Phi}(t)\|_{\lambda(t),1} \\
&\quad + \|\bar{\Phi}(t)\|_{\lambda(t),0}\left(\|f_1(t)\|_{\lambda(t),1}\|\ln(\omega)\|_{\lambda(t),1} + \|h\|_{\lambda(t),0}\right) \\
&\quad + \|\bar{f}(t)\|_{\lambda(t),1}\left(\|\Phi(f_2)(t)\|_{\lambda(t),1} + \|\Phi(f_2)(t)\|_{\lambda(t),0}\|\ln(\omega)\|_{\lambda(t),1}\right).
\end{aligned}$$

Since, by our assumptions,

$$\max_{t\in[0,T]} \|f_1(t)\|_{\lambda(t),1} \left( \leq \max_{t\in[0,T]} \|f_1(t)\|_{\mathcal{H},\lambda(t)} \right) \leq M, \quad \text{and} \quad \max_{t\in[0,T]} \|\Phi(f_2(t))\|_{\lambda(t),1} \leq M,$$

we deduce that for all $t \in [0,T]$,

$$\frac{d}{dt}\|\bar{\Phi}(t)\|_{\lambda(t),0} \leq (\lambda_0 - K + M + \gamma_0)\, \|\bar{\Phi}(t)\|_{\lambda(t),1} + M(1+\gamma_0)\|\bar{\Phi}(t)\|_{\lambda(t),0}\,(M\gamma_0 + \gamma_1) + \|\bar{f}(t)\|_{\lambda(t),1}$$

thanks also to the upper-bounds $\|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda(t)} \leq \gamma_0 = \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda_0}$, $\|h\|_{\mathcal{H},\lambda(\theta),} \leq \gamma_1 = \|h\|_{\mathcal{H},\lambda_0}$. It follows then by Gronwall's inequality that

$$\|\bar{\Phi}(t)\|_{\lambda(t),0} \leq \exp\left\{T\,(M\gamma_0 + \gamma_1)\right\} \int_0^t \left(\|\bar{\Phi}(\theta)\|_{\lambda(\theta),1}(\lambda_0 - K + M + \gamma_0) + \|\bar{f}(\theta)\|_{\lambda(\theta),1} M(1+\gamma_0)\right) d\theta.$$

Observe that the current assumptions of Theorem 2.8 ensure that we can choose $K \in \left(0, \frac{\lambda_0}{T} - 1\right)$ such that

$$K - \lambda_0 - M - \gamma_0 > 1.$$

We thus get from the previous that for each $t \in [0,T]$,

$$\begin{aligned}\|\bar{\Phi}(t)\|_{\lambda(t),0} + \int_0^t \|\bar{\Phi}(\theta)\|_{\lambda(\theta),1}\, d\theta &\leq \|\bar{\Phi}(t)\|_{\lambda(t),0} + \exp\left\{T\,(M\gamma_0 + \gamma_1)\right\} \int_0^t \|\bar{\Phi}(\theta)\|_{\lambda(\theta),1}\, d\theta \\ &\leq M(1+\gamma_0)\exp\left\{(M\gamma_0+\gamma_1)\,T\right\} \int_0^T \|\bar{f}(t)\|_{\lambda(t),1} dt.\end{aligned}$$

In particular,

$$\max_{t\in[0,T]} \|\Phi(f_1)(t) - \Phi(f_2)(t)\|_{\lambda(t),0} \leq M(1+\gamma_0)\exp\left\{(M\gamma_0+\gamma_1)\,T\right\} \int_0^T \|f_1(t) - f_2(t)\|_{\lambda(t),1} dt.$$
$$\int_0^T \|\Phi(f_1)(\theta) - \Phi(f_2)(\theta)\|_{\lambda(\theta),1}\, d\theta \leq M(1+\gamma_0)\exp\left\{(M\gamma_0+\gamma_1)\,T\right\} \int_0^T \|f_1(t) - f_2(t)\|_{\lambda(t),1} dt.$$

The contractivity property is thus granted by (2.23). □

*Proof of Theorem 2.5.* Under the assumptions on $\lambda_0, M$ and $T$, Theorem 2.15 holds and, moreover, the assumptions on $f_0$ imply that $g_0 \in \mathcal{B}^M_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^M_{\lambda_0,K,T}$. Therefore, by Banach's fixed point theorem the sequence $\Phi^n(g_0)$ converges to a function $g \in \mathcal{B}^M_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^M_{\lambda_0,K,T}$ which is a solution of (VFP$\omega$). □

**Remark 2.16.** *If $g_0(x,u)$ is 1-periodic in $x$, uniqueness of classical solutions to the linear equation (FP$\omega$) implies that $\Phi^n(g_0)$ too is 1-periodic in $x$ for each $n \in \mathbb{N}$. Consequently, so is the limit $g$.*

# 3 The kinetic potential case

In this section we extend the previous results to the situation $\beta \neq 0$ and $\alpha = 0$ (corresponding to the standard kinetic energy potential) or $\alpha = 1$ (corresponding to the turbulent kinetic energy). We consider the nonlinear Vlasov-Fokker-Planck equation with additional kinetic potential

$$\begin{cases} \partial_t g(t,x,u) + u\partial_x g(t,x,u) - \left[\partial_x P(t,x) + \beta(u - \alpha V(t,x)) - \partial_u \ln(\omega(u))\right] \partial_u g(t,x,u) \\ -\dfrac{\sigma^2}{2}\partial_u^2 g(t,x,u) = \left[\partial_x P(t,x) - \alpha\beta V(t,x)\partial_u \ln(\omega(u))\right] g(t,x,u) - g(t,x,u)\hat{h}(u), \\ \qquad \text{on } (0,T]\times\mathbb{R}^2, \\ P(t,x) = -\displaystyle\int_{\mathbb{R}} \frac{u^2}{\omega(u)} g(t,x,u)\, du, \quad V(t,x) = \int_{\mathbb{R}} \frac{u}{\omega(u)} g(t,x,u) du \\ g(0,x,u) = g_0(x,u) \text{ on } \mathbb{R}^2, \end{cases} \tag{VFP$\omega$K}$$

where
$$\hat{h}(u) := \frac{\partial_u^2 \omega(u)}{2\omega(u)} - |\partial_u \ln(\omega(u))|^2 - \beta - \beta u \partial_u (\ln \omega(u)) \,.$$
Through the relation $g(t,x,u) = \omega(u) f(t,x,u)$, equation (VFP$\omega$K) is seen to be equivalent to

$$\begin{cases} \partial_t f(t,x,u) + u\partial_x f(t,x,u) - (\partial_x P(t,x) + \beta(u - \alpha V(t,x)))\, \partial_u f(t,x,u) - \beta f(t,x,u) \\ -\dfrac{\sigma^2}{2}\partial_u^2 f(t,x,u) = 0, \text{ on } (0,T]\times\mathbb{R}^2, \\ P(t,x) = -\displaystyle\int_{\mathbb{R}} u^2 f(t,x,u)du, \quad V(t,x) = \int_{\mathbb{R}} u f(t,x,u)du \\ f(0,x,u) = f_0(x,u) \text{ on } \mathbb{R}^2. \end{cases} \tag{VFPK}$$

We now prove

**Theorem 3.1.** *Let $\lambda_0, M, T > 0$ be positive constants and $\omega : \mathbb{R} \to (0,+\infty)$ a function of class $\mathcal{C}^\infty$ satisfying* (H$_\omega$) *and moreover that $u\partial_u(\ln\omega(u)) \in \mathcal{H}(\lambda_0)$. Define the finite constants*
$$\gamma_0 := \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda_0}, \quad \hat{\gamma}_1 := \|\hat{h}\|_{\mathcal{H},\lambda_0} \quad \text{and } C_\omega := \int_{\mathbb{R}} \frac{|u|}{\omega(u)} du,$$
*and assume that*

*a)* $T < \frac{(1+\beta)\lambda_0}{1+4\gamma_0+(1+\beta)(1+\lambda_0)}$,

*b)* $M \le \frac{(1+\beta)(K-\lambda_0)-4\gamma_0-1}{16+\alpha\beta}$ *for some* $K \in (\frac{1+4\gamma_0}{1+\beta} + \lambda_0, \frac{\lambda_0}{T} - 1)\,(\neq \emptyset)$ *and*

*c)* $M(1+\gamma_0)(1+TC_\omega\alpha\beta)\exp\{(M(1+C_\omega\alpha\beta)\gamma_0 + \hat{\gamma}_1)T\} < 1$.

*Assume moreover that $f_0 : \mathbb{R}^2 \to \mathbb{R}$ is a function of class $\mathcal{C}^\infty$ and that $g_0(x,u) := \omega(u) f_0(x,u)$ satisfies*

*d)* $\max\{\|g_0\|_{\mathcal{H},\lambda_0}, T\|g_0\|_{\widetilde{\mathcal{H}},\lambda_0}\} \le M$ *and*

*e)* $\|g_0\|_{\mathcal{H},\lambda_0} \exp\{T(\hat{\gamma}_1 + 16\gamma_0 + \alpha\beta\gamma_0 C_\omega M) + (16+\gamma_0)M\} < M$.

*Then, equation (VFP$\omega$K) has a unique smooth solution $g \in \mathcal{B}^M_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^M_{\lambda_0,K,T}$. In particular, under the previous assumptions, a solution $f \in \mathcal{C}^{1,\infty}$ to (VFPK) with initial condition $f_0$ exists.*

It is checked in Appendix A.1 that the function $u \mapsto \hat{h}(u)$ belongs to $\mathcal{H}(\lambda_0)$ for every $\lambda_0 \in (0, \frac{1}{4})$ when $\omega(u) := c(1+u^2)^{\frac{s}{2}}$.

**Corollary 3.2.** *Let $f$ be the solution to (VFPK) given above and assume that* (H$_{\text{unif}(0)}$) *holds. Then, $f(t,x,u)$ satisfies* (H$_{\text{unif}(t)}$) *for all $t$ in $[0,T]$. In particular, under the assumptions of Theorem 3.1 and* (H$_{\text{unif}(0)}$) *a solution to (1.3) for $\beta \ge 0$ exists.*

**Remark 3.3.** *As discussed in Remark 2.7, given $f_0 \in \mathcal{C}^\infty$ and $\lambda_1 > 0$ such that $g_0 \in \mathcal{H}(\lambda_1) \cap \widetilde{\mathcal{H}}(\lambda_1)$, it is again possible to exhibit small constants $\kappa_0' > 0$ and $\kappa_1' > 0$, the former depending on $\omega$ and the latter on $\lambda_1$, $\beta \ge 0$, the function $\omega$, and the norms $\|g_0\|_{\mathcal{H},\lambda_0}$ and $\|g_0\|_{\widetilde{\mathcal{H}},\lambda_0}$, such that the conclusion of Theorem 3.1 holds if $\|g_0\|_{\mathcal{H},\lambda_0} < \kappa_0'$ and $T < \kappa_1'$. Using Remark 2.4, smallness conditions depending only on $f_0$ and $T > 0$ can also be exhibited.*

Most of the computations required in the proofs are the same as in the previous section, so we only provide some details about the additional terms that the case $\beta > 0$ requires to deal with.

*Proof of Theorem 3.1.* Consider the linear equation obtained by respectively replacing in (VFP$\omega$K) the functions $P$ and $V$ by fixed given functions $Q, H : [0,T]\times\mathbb{R} \to \mathbb{R}$:

$$\begin{cases} \partial_t g(t,x,u) + u\partial_x g(t,x,u) - [\partial_x Q(t,x) + \beta(u - \alpha H(t,x)) - \partial_u \ln(\omega(u))]\, \partial_u g(t,x,u) \\ -\dfrac{\sigma^2}{2}\partial_u^2 g(t,x,u) = [\partial_x Q(t,x) - \alpha\beta H(t,x)\partial_u \ln(\omega(u))]\, g(t,x,u) - g(t,x,u)\hat{h}(u), \\ \qquad \text{on } (0,T]\times\mathbb{R}^2, \\ g(0,x,u) = g_0(x,u) \text{ on } \mathbb{R}^2, \end{cases} \tag{FP$\omega$K}$$

First we notice that

$$\partial_x^k \partial_u^l (u \partial_u g(t,x,u)) = \sum_{n=0}^{l} C_l^n (\partial_u^n u)(\partial_u \partial_x^k \partial_u^{l-n} g(t,x,u)) = u \partial_x^k \partial_u^{l+1} g(t,x,u) + l \partial_x^k \partial_u^l g(t,x,u).$$

Therefore, application of the differential operator $\partial_x^k \partial_u^l$ to the linear equation (FP$\omega$K) yields the identity

$$\begin{aligned}
&\partial_t \partial_x^k \partial_u^l g(t,x,u) + u \partial_x (\partial_x^k \partial_u^l g(t,x,u)) \\
&\quad - (\partial_x Q(t,x) - \partial_u \ln \omega(u) + \beta(u - \alpha V(t,x)))\, \partial_u (\partial_x^k \partial_u^l g(t,x,u)) - \frac{\sigma^2}{2} \partial_u^2 (\partial_x^k \partial_u^l g(t,x,u)) \\
&= \beta l \partial_x^k \partial_u^l g(t,x,u) - l \partial_x^{k+1} \partial_u^{l-1} g(t,x,u) \\
&\quad + \mathbb{1}_{\{k\geq 1\}} \sum_{m=0}^{k-1} C_k^m \partial_x^{k-m} (\partial_x Q(t,x) - \alpha\beta H(t,x)) \partial_u (\partial_x^m \partial_u^l g(t,x,u)) - \mathbb{1}_{\{l\geq 1\}} \sum_{n=0}^{l-1} C_l^n \left(\partial_u^{l-n+1} \ln(\omega(u)) \partial_u^{n+1} \partial_x^k g(t,x,u)\right) \\
&\quad + \sum_{n=0}^{l} \sum_{m=0}^{k} C_l^n C_k^m \left(\partial_x^{k-m} \partial_x Q(t,x) - \alpha\beta H(t,x)\right) \partial_u^{l-n+1} \ln \omega(u) \partial_x^m \partial_u^n g(t,x,u) + \sum_{n=0}^{l} C_l^n \partial_x^k \partial_u^n g(t,x,u) \partial_u^{l-n} \hat{h}(u).
\end{aligned}$$

By the maximum principle we deduce that for all $A \in \mathbb{N}$ and $a \in \{0, \ldots, A\}$, a smooth solution $g$ to equation (FP$\omega$K) must satisfy

$$\begin{aligned}
\frac{d}{dt} \|g(t)\|_{\lambda,a;A} &\leq \lambda(1+\beta) \|g(t)\|_{\lambda,a+1;A} + a(1+\beta)\|g(t)\|_{\lambda,a;A} \\
&\quad + \frac{d^a}{d\lambda^a} \Big( \|g(t)\|_{\lambda,1;A} \left( \|Q(t)\|_{\lambda,1;A} + \alpha\beta \|H(t)\|_{\lambda,0;A} + \|\ln(\omega)\|_{\lambda,1;A} \right) \Big) \\
&\quad + \frac{d^a}{d\lambda^a} \Big( \|g(t)\|_{\lambda,0;A} \left[ \|\hat{h}\|_{\lambda,0;A} + (\|Q(t)\|_{\lambda,1;A} + \alpha\beta \|H(t)\|_{\lambda,0;A}) \|\ln(\omega)\|_{\lambda,1;A} \right] \Big).
\end{aligned}$$

By similar computation as in the proof of Lemma 2.10, it is also possible to establish

**Lemma 3.4.** *(i) Suppose that for some $\bar{\lambda} > 0$ we have $f \in \mathcal{H}(\bar{\lambda})$ and $v \in \widetilde{\mathcal{H}}(\bar{\lambda})$. Then, for all $\lambda \in [0, \bar{\lambda})$ one has*

$$\sum_{a \in \mathbb{N}} \frac{1}{(a!)^2} \frac{d^a}{d\lambda^a} \left( \|f\|_{\lambda,0} \|v\|_{\lambda,1} \right) \leq \|f\|_{\mathcal{H},\lambda} \|v\|_{\widetilde{\mathcal{H}},\lambda}.$$

*(ii) Suppose that for some $\bar{\lambda} > 0$, $f, w \in \mathcal{H}(\bar{\lambda})$ and $v \in \widetilde{\mathcal{H}}(\bar{\lambda})$. Then, for all $\lambda \in [0, \bar{\lambda})$ one has*

$$\sum_{a \in \mathbb{N}} \frac{1}{(a!)^2} \frac{d^a}{d\lambda^a} \left( \|f\|_{\lambda,0} \|w\|_{\lambda,0} \|v\|_{\lambda,1} \right) \leq \|f\|_{\mathcal{H},\lambda} \|w\|_{\mathcal{H},\lambda} \|v\|_{\widetilde{\mathcal{H}},\lambda}.$$

Truncated version of these estimates, combined with the already obtained ones yield:

**Proposition 3.5.** *For each $A \in \mathbb{N}$, the $\mathcal{C}^{1,\infty}$ function $g$ solution to (FP$\omega$K) satisfies the estimate*

$$\begin{aligned}
\frac{d}{dt} \|g(t)\|_{\mathcal{H},\lambda(t);A} &\leq \left( (1+\beta)[\lambda(t) + 1 + \lambda'(t)] + 4\gamma_0 + 16 \|Q(t)\|_{\mathcal{H},\lambda(t)} + \alpha\beta \|H(t)\|_{\mathcal{H},\lambda(t)} \right) \|g(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A} \\
&\quad + \left( \gamma_1 + 16\gamma_0 + (\gamma_0 + 16) \|Q(t)\|_{\widetilde{\mathcal{H}},\lambda(t)} + \alpha\beta\gamma_0 \|H(t)\|_{\mathcal{H},\lambda(t)} \right) \|g(t)\|_{\mathcal{H},\lambda(t);A},
\end{aligned}$$

*where $\gamma_0 := \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda_0}$ and $\hat{\gamma}_1 := \|\hat{h}\|_{\mathcal{H},\lambda_0}$.*

Applying Gronwall's lemma and using the fact that

$$\lambda(t) + 1 + \lambda'(t) + 4\gamma_0 + 16 \|P(t)\|_{\mathcal{H},\lambda(t)} + \alpha\beta \|V(t)\|_{\mathcal{H},\lambda(t)} \leq (1+\beta)[\lambda_0 - K] + 4\gamma_0 + (16 + \alpha\beta) M_1$$

we then obtain that, for all $t \in [0,T]$ and $A \in \mathbb{N}$,

$$\|g(t)\|_{\mathcal{H},\lambda(t);A} \le \|g_0\|_{\mathcal{H},\lambda_0} \exp\{T(\hat\gamma_1 + 16\gamma_0 + \alpha\beta\gamma_0 M_1) + (16+\gamma_0)M_2\}$$
$$+ \exp\{T(\hat\gamma_1 + 16\gamma_0 + \alpha\beta\gamma_0 M_1) + (16+\gamma_0)M_2\}\left((1+\beta)[\lambda_0 - K] + 4\gamma_0 + (16+\alpha\beta)M_1\right)\int_0^t \|g(s)\|_{\widetilde{\mathcal{H}},\lambda(s);A} ds. \tag{3.4}$$

From assumptions a) and b) of Theorem 3.1 we can choose $K>0$ such that $K < \frac{\lambda_0}{T} - 1$ and

$$(1+\beta)(K-\lambda_0) - 4\gamma_0 - (16+\alpha\beta)M_1 \ge 1$$

in which case we obtain

$$\|g(t)\|_{\mathcal{H},\lambda(t);A} + \int_0^t \|g(s)\|_{\widetilde{\mathcal{H}},\lambda(s);A} ds \le \|g_0\|_{\mathcal{H},\lambda_0} \exp\{T(\hat\gamma_1 + 16\gamma_0 + \alpha\beta\gamma_0 M_1) + (16+\gamma_0)M_2\},$$

and then

$$\|g(t)\|_{\mathcal{H},\lambda(t)} + \int_0^t \|g(s)\|_{\widetilde{\mathcal{H}},\lambda(s)} ds \le \|g_0\|_{\mathcal{H},\lambda_0} \exp\{T(\hat\gamma_1 + 16\gamma_0 + \alpha\beta\gamma_0 M_1) + (16+\gamma_0)M_2\}.$$

Therefore, since for any function $\varphi : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^{1,\infty}$ and every $t\in[0,T]$ we have

$$\left\|\int_{\mathbb{R}} \frac{u^2}{\omega(u)}\varphi(t,\cdot,u)du\right\|_{\lambda,a} \le \|\varphi(t,\cdot,\cdot)\|_{\lambda,a} \quad\text{and}\quad \left\|\int_{\mathbb{R}} \frac{|u|}{\omega(u)}\varphi(t,\cdot,u)du\right\|_{\lambda,a} \le C_\omega \|\varphi(t,\cdot,\cdot)\|_{\lambda,a}$$

for all $\lambda \ge 0$ and $a\in\mathbb{N}$, the mapping $\Phi$ associating with a function $\varphi$ the solution $\Phi(\varphi)$ of equation (FP$\omega$K) with the data

$$Q(t,x) := -\int_{\mathbb{R}} \frac{u^2}{\omega(u)}\varphi(t,x,u)du \quad\text{and}\quad H(t,x) := \int_{\mathbb{R}} \frac{u}{\omega(u)}\varphi(t,x,u)du$$

satisfies the inclusion

$$\Phi\left(\mathcal{B}^{M_1}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{M_2}_{\lambda_0,K,T}\right) \subseteq \mathcal{B}^{\hat M}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{\hat M}_{\lambda_0,K,T}$$

if $M_1, T, \lambda_0 > 0$ are as previously, $M_2>0$ is arbitrary and

$$\hat M = \|g_0\|_{\mathcal{H},\lambda_0} \exp\{T(\hat\gamma_1 + 16\gamma_0 + \alpha\beta\gamma_0 C_\omega M_1) + (16+\gamma_0)M_2\}.$$

In particular, one has $\Phi\left(\mathcal{B}^{M}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T}\right) \subseteq \mathcal{B}^{M}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T}$ if in addition to conditions a) and b) of Theorem 3.1, the constants $M>0$ and $T>0$ satisfy condition d). Now, writing $\bar\Phi := \Phi(f_1) - \Phi(f_2)$, $\bar P := P_1 - P_2$ and $\bar V := V_1 - V_2$ where

$$P_i(t,x) := -\int_{\mathbb{R}} \frac{u^2}{\omega(u)} f_i(t,\cdot,u)\,du \text{ and } V_i(t,x) := \int_{\mathbb{R}} \frac{u}{\omega(u)} f_i(t,\cdot,u)du \quad i=1,2,$$

we have

$$\partial_t(\partial_x^k\partial_u^l\bar\Phi) + u\partial_x\left(\partial_x^k\partial_u^l\bar\Phi\right) - \left(\partial_x Q_1 - \partial_u \ln\omega(u) + \beta(u - \alpha V_1(t,x))\right)\partial_u\left(\partial_x^k\partial_u^l\bar\Phi\right) - \frac{\sigma^2}{2}\partial_u^2(\partial_x^k\partial_u^l\bar\Phi)$$
$$= -l\partial_x^{k+1}\partial_u^{l-1}\bar\Phi + \mathbb{1}_{\{k\ge1\}}\sum_{m=0}^{k-1} C_k^m \partial_x^{k-m}\left(\partial_x Q_1(t,x) - \alpha\beta V_1(t,x)\right)\partial_x^m\partial_u^{l+1}\bar\Phi - \mathbb{1}_{\{l\ge1\}}\sum_{n=0}^{l-1} C_l^n\left(\partial_u^{l-n+1}\ln(\omega)\right)\partial_x^k\partial_u^{n+1}\bar\Phi$$
$$+\sum_{n=0}^{l}\sum_{m=0}^{k} C_k^m C_l^n\left(\partial_x^{k-m}\partial_u^n\Phi(f_2)\right)\left(\partial_u^{l-n+1}\ln(\omega)\right)\partial_x^m\left(\partial_x\bar P - \alpha\beta\bar V(t,x)\right)$$
$$+\sum_{m=0}^{k} C_k^m\left(\partial_x^{k-m}\partial_u^{l+1}\Phi(f_2)\right)\partial_x^m\left(\partial_x\bar P - \alpha\beta\bar V(t,x)\right))$$
$$+\sum_{n=0}^{l}\sum_{m=0}^{k} C_k^m C_l^n \partial_x^{k-m}\left(\partial_x P_1 - \alpha\beta V_1(t,x)\right)\partial_u^{l-n+1}\ln\omega(u)\partial_x^m\partial_u^n\bar\Phi + \sum_{n=0}^{l} C_l^n\left(\partial_x^k\partial_u^n\bar\Phi\right)\partial_u^{l-n}\hat h(u);$$

From this and the maximum principle we deduce that $\frac{d}{dt}\|\partial_x^k\partial_u^l\bar{\Phi}(t)\|_\infty$ is bounded above by

$$
\begin{aligned}
& l\|\partial_x^{k+1}\partial_u^{l-1}\bar{\Phi}(t)\|_\infty + \mathbb{1}_{\{k\geq 1\}}\sum_{m=0}^{k-1} C_k^m(\|\partial_x^{k-m+1}f_1(t)\|_\infty + C_\omega\alpha\beta\|\partial_x^{k-m}f_1(t)\|_\infty)\|\partial_x^m\partial_u^{l+1}\bar{\Phi}(t)\|_\infty \\
& \quad + \mathbb{1}_{\{l\geq 1\}}\sum_{n=0}^{l-1} C_l^n\|\partial_u^{l-n+1}\ln(\omega)\|_\infty\|\partial_x^k\partial_u^{n+1}\bar{\Phi}(t)\|_\infty \\
& \quad + \sum_{n=0}^{l}\sum_{m=0}^{k} C_k^m C_l^n\|\partial_u^{l-n+1}\ln(\omega)\|_\infty\|\partial_x^{k-m}\partial_u^n\Phi(f_2)(t)\|_\infty(\|\partial_x^{m+1}\bar{f}(t)\|_\infty + C_\omega\alpha\beta\|\partial_x^m\bar{f}(t)\|_\infty) \\
& \quad + \sum_{m=0}^{k} C_k^m\|\partial_x^{k-m}\partial_u^{l+1}\Phi(f_2)(t)\|_\infty(\|\partial_x^{m+1}\bar{f}(t)\|_\infty + C_\omega\alpha\beta\|\partial_x^m\bar{f}(t)\|_\infty) \\
& \quad + \sum_{n=0}^{l}\sum_{m=0}^{k} C_k^m C_l^n\|\partial_u^{l-n+1}\ln(\omega)\|_\infty(\|\partial_x^{k-m+1}f_1(t)\|_\infty + C_\omega\alpha\beta\|\partial_x^{k-m}f_1(t)\|_\infty)\|\partial_x^m\partial_u^n\bar{\Phi}(t)\|_\infty \\
& \quad + \sum_{n=0}^{l} C_l^n\|\partial_x^k\partial_u^n\bar{\Phi}(t)\|_\infty\|\partial_u^{l-n}\hat{h}(u)\|_\infty.
\end{aligned}
$$

This yields

$$
\begin{aligned}
\frac{d}{dt}\|\bar{\Phi}(t)\|_{\lambda(t),0} \leq & \left(\lambda(t) + \lambda'(t) + \|f_1(t)\|_{\lambda(t),1} + C_\omega\alpha\beta\|f_1(t)\|_{\lambda(t),0} + \|\ln(\omega)\|_{\lambda(t),1}\right)\|\bar{\Phi}(t)\|_{\lambda(t),1} \\
& + \|\bar{\Phi}(t)\|_{\lambda(t),0}\left[\left(\|f_1(t)\|_{\lambda(t),1} + C_\omega\alpha\beta\|f_1(t)\|_{\lambda(t),0}\right)\|\ln(\omega)\|_{\lambda(t),1} + \|\hat{h}\|_{\lambda(t),0}\right] \\
& + \left(\|\bar{f}(t)\|_{\lambda(t),1} + C_\omega\alpha\beta\|\bar{f}(t)\|_{\lambda(t),0}\right)\left(\|\Phi(f_2)(t)\|_{\lambda(t),1} + \|\Phi(f_2)(t)\|_{\lambda(t),0}\|\ln(\omega)\|_{\lambda(t),1}\right)
\end{aligned} \tag{3.5}
$$

and therefore, for all $t \in [0, T]$,

$$
\begin{aligned}
\frac{d}{dt}\|\bar{\Phi}(t)\|_{\lambda(t),0} \leq & \left((1+\beta)(\lambda_0 - K) + (1 + C_\omega\alpha\beta)M + \gamma_0\right)\|\bar{\Phi}(t)\|_{\lambda(t),1} \\
& + \|\bar{\Phi}(t)\|_{\lambda(t),0}\left(M(1 + C_\omega\alpha\beta)\gamma_0 + \hat{\gamma}_1\right) \\
& + \left(\|\bar{f}(t)\|_{\lambda(t),1} + C_\omega\alpha\beta\|\bar{f}(t)\|_{\lambda(t),0}\right)M(1+\gamma_0).
\end{aligned} \tag{3.6}
$$

Since our assumptions allow us to choose $K \in (0, \frac{\lambda_0}{T} - 1)$ such that

$$(1+\beta)(K - \lambda_0) - (1 + C_\omega\alpha\beta)M - \gamma_0 > 1,$$

we get from the previous after applying Gronwall's lemma that

$$
\begin{aligned}
\|\bar{\Phi}(t)\|_{\lambda(t),0} + \int_0^t \|\bar{\Phi}(s)\|_{\lambda(t),1}\,ds & \leq \|\bar{\Phi}(t)\|_{\lambda(t),0} + \exp\left\{T\left(M(1 + C_\omega\alpha\beta)\gamma_0 + \hat{\gamma}_1\right)\right\}\int_0^t \|\bar{\Phi}(s)\|_{\lambda(s),1}\,ds \\
& \leq M(1+\gamma_0)\exp\left\{\left(M(1 + C_\omega\alpha\beta)\gamma_0 + \hat{\gamma}_1\right)T\right\}\int_0^T \|\bar{f}(t)\|_{\lambda(t),1} + C_\omega\alpha\beta\|\bar{f}(t)\|_{\lambda(t),0}dt.
\end{aligned}
$$

This implies that $\Phi : \mathcal{B}^M_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^M_{\lambda_0,K,T} \to \mathcal{B}^M_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^M_{\lambda_0,K,T}$ is a contraction for the norm

$$\max\left\{\max_{t\in[0,T]}\|\psi(t)\|_{\lambda(t),0}\,,\int_0^T \|\psi(t)\|_{\lambda(t),1}\,dt\right\}$$

under condition c) of Theorem 3.1, and condition d) allows us to conclude the existence of a solution starting from $g_0$. □

*Proof of Corollary 3.2.* By Lemma 1.2 we now obtain in the case $\alpha = 1$ that, for all $(t,x) \in (0,T] \times \mathbb{R}$,

$$
\begin{cases}
\partial_t\bar{\rho}(t,x) = -\partial_x V(t,x), \\
\partial_t(\partial_x V(t,x)) = -\partial_x(\bar{\rho}(t,x)\partial_x P(t,x)) + \beta\partial_x\left(V(t,x)\bar{\rho}(t,x)\right),
\end{cases}
$$

where $\overline{\rho}(t,x) := \rho(t,x) - 1 = \int_{\mathbb{R}} f(t,x,u)du - 1$, $V(t,x) := \int_{\mathbb{R}} uf(t,x,u)\,du$ and $\partial_x P(t,x) = -\partial_x \int_{\mathbb{R}} u^2 f(t,x,u)\,du$. Hence, for all $\lambda > 0$,

$$\begin{cases} \partial_t\|\overline{\rho}(t)\|_\lambda \leq \|\partial_x V(t)\|_\lambda, \\ \partial_t\|\partial_x V(t)\|_\lambda \leq \|\partial_x P(t)\|_\lambda\|\partial_x\overline{\rho}(t)\|_\lambda + \|\partial_x^2 P(t)\|_\lambda\|\overline{\rho}(t)\|_\lambda + \beta\left(\|\partial_x V(t)\|_\lambda\|\overline{\rho}(t)\|_\lambda + \|V(t)\|_\lambda\|\partial_x\overline{\rho}(t)\|_\lambda\right). \end{cases} \tag{3.7}$$

With $A(t,\lambda) := \|\overline{\rho}(t)\|_\lambda$ and $B(t,\lambda) := \|\partial_x V(t)\|_\lambda$ we have

$$\begin{cases} \partial_t A(t,\lambda) \leq B(t,\lambda), \\ \partial_t B(t,\lambda) \leq (\|\partial_x P(t)\|_\lambda + \beta\|V(t)\|_\lambda)\partial_\lambda A(t,\lambda) + (\|\partial_x^2 P(t)\|_\lambda + \beta B(t,\lambda))A(t,\lambda). \end{cases}$$

From these inequalities, since the terms in parentheses are bounded, the conclusion is obtained by similar arguments as in the case $\beta = 0$. If now $\alpha = 0$, we obtain the equations, for all $(t,x) \in (0,T] \times \mathbb{R}$,

$$\begin{cases} \partial_t\overline{\rho}(t,x) = -\partial_x V(t,x), \\ \partial_t(\partial_x V(t,x)) = -\partial_x(\overline{\rho}(t,x)\partial_x P(t,x)) + \beta\partial_x V(t,x) \end{cases}$$

whit the same notation as before. This yields

$$\begin{cases} \partial_t A(t,\lambda) \leq B(t,\lambda), \\ \partial_t B(t,\lambda) \leq \|\partial_x P(t)\|_\lambda\partial_\lambda A(t,\lambda) + \|\partial_x^2 P(t)\|_\lambda A(t,\lambda) + \beta B(t,\lambda). \end{cases}$$

Since the remainder of the proof in the case $\beta = 0$ relies on the inequality satisfied by the sum $\mathcal{Y}(t,\lambda) := A(t,\lambda) + bB(t,\lambda)$, by suitably modifying the constants therein one can conclude in a similar way. □

# A Appendix

## A.1 A weight function of analytic type

In this section, we show that for the weight function $\omega(u) = c(1+u^2)^{\frac{s}{2}}$ where $c, s > 0$ are fixed constants, the functions $u \mapsto \partial_u \ln(\omega(u)) = \frac{su}{(1+u^2)}$, $u \mapsto h(u) = \frac{\partial_u^2\omega(u)}{2\omega(u)} - |\partial_u \ln(\omega(u))|^2 = \frac{s-(s+s^2)u^2}{2(1+u^2)^2}$ and $u \mapsto \hat{h}(u) = h(u) - \beta(1 + u\partial_u(\ln\omega(u)))$ satisfy ($\mathrm{H}_\omega$)for $\lambda_0 \in [0, 1/4)$. Observe also that $\int_{\mathbb{R}} \frac{u^2}{\omega(u)}du < +\infty$ as soon as $s > 3$.

Let us first consider $\partial_u \ln(\omega)$. We are going to identify $\partial_u^l \ln(\omega)$ for $l \geq 1$ with a function of the form $\frac{q_l(u)}{(1+|u|^2)^l}$ where $q_l$ is a polynomial function of order $l$ satisfying $q_1(u) = su$ and, for all $l \geq 1$,

$$\frac{q_{l+1}(u)}{(1+|u|^2)^{l+1}} = \partial_u\left(\frac{q_l(u)}{(1+|u|^2)^l}\right) = \frac{(1+|u|^2)\partial_u q_l(u) - 2luq_l(u)}{(1+|u|^2)^{l+1}},$$

or, equivalently, $q_{l+1}(u) = (1+|u|^2)\partial_u q_l(u) - 2luq_l(u)$. We can now determine the coefficients $\{a_n^{(l)}\}_{0\leq n\leq l}$ such that $q_l(u) = \sum_{n=0}^l a_n^{(l)}u^n$ observing that, for $l \geq 1$,

$$\begin{aligned} (1+|u|^2)\partial_u q_l(u) - 2luq_l(u) &= (1+|u|^2)\sum_{n=1}^{l} na_n^{(l)}u^{n-1} - 2lu\sum_{n=0}^{l} a_n^{(l)}u^n \\ &= \sum_{n=1}^{l} na_n^{(l)}u^{n-1} + \sum_{n=1}^{l} na_n^{(l)}u^{n+1} - 2l\sum_{n=0}^{l} a_n^{(l)}u^{n+1} \\ &= \sum_{n=0}^{l-1}(n+1)a_{n+1}^{(l)}u^n + \sum_{n=2}^{l+1}(n-1)a_{n-1}^{(l)}u^n - 2l\sum_{n=1}^{l+1} a_{n-1}^{(l)}u^n. \end{aligned}$$

Therefore, we have $a_0^{(1)} = 0$, $a_1^{(1)} = s$, $a_0^{(2)} = s, a_1^{(2)} = 0, a_2^{(2)} = -s$ and, for $l \geq 2$,

$$\begin{aligned} &a_0^{(l+1)} = a_1^{(l)}, \quad a_1^{(l+1)} = 2a_2^{(l)} - 2la_0^{(l)}, \\ &a_n^{(l+1)} = (n+1)a_{n+1}^{(l)} + (n-1)a_{n-1}^{(l)} - 2la_{n-1}^{(l)}, \text{ if } 2 \leq n \leq l-1, \\ &\text{and } a_l^{(l+1)} = -(l+1)a_{l-1}^{(l)}, \quad a_{(l+1)}^{l+1} = -la_l^{(l)}. \end{aligned} \tag{A.1}$$

Setting $a^{(l)} := \max_{n\in\{0,\dots,l\}} a_n^{(l)}$, we deduce the rough estimates: $a^{(l+1)} \leq 4(l+1)a^{(l)}$ for $l \geq 2$, and then: $a^{(l)} \leq \frac{s}{4}4^l l!$ for $l \geq 1$. Thus, for $l \geq 1$

$$|\partial_u^l(\partial_u(\ln(\omega))| \leq \frac{\sum_{n=0}^{l+1} a_n^{(l+1)}|u|^n}{(1+u^2)^{l+1}} \leq \frac{s}{4}\sum_{n=0}^{l+1}\frac{4^{l+1}(l+1)!|u|^n}{(1+u^2)^{l+1}} \leq s4^l(l+2)!. \tag{A.2}$$

Consequently, by Lemma 2.1 we have $\partial_u(\ln(\omega)) \in \mathcal{H}(\lambda)$ for all $\lambda \in [0,1/4)$, and from Lemma 2.2-(i) we conclude that $\ln(\omega) \in \widetilde{\mathcal{H}}(\lambda)$ for all $\lambda \in [0,1/4)$.

As for the function $h$, it is similarly checked in this case that for all $l \geq 0$

$$\partial_u^l h(u) = \frac{r_{l+2}(u)}{2(1+|u|^2)^{l+2}},$$

where $r_l$ is a polynomial function of order $l$, defined for $l \geq 2$. The coefficients $\{b_n^{(l)}\}_{0\leq n\leq l}$ such that $r_l(u) = \sum_{n=0}^{l} b_n^{(l)} u^n$ satisfy $b_0^{(2)} = s/2$, $b_1^{(2)} = 0$, $b_2^{(2)} = -(s+s^2)/2$ and moreover, for all $l \geq 2$, the recurrence relations (A.1) with $a_n^{(l)}$ replaced by $b_n^{(l)}$. It follows in a similar way as before that

$$|\partial_u^l h(u)| \leq \frac{s+s^2}{4}4^l(l+3)!$$

and we conclude as well by Lemma 2.1 that $h \in \mathcal{H}(\lambda)$ for all $\lambda \in [0,1/4)$.

Finally, we have $u\partial_u(\ln\omega(u)) = s - \frac{s}{1+u^2}$ so that we only need to check that the function $\frac{s}{1+u^2}$ belongs to the space $\mathcal{H}(\lambda)$ for $\lambda \in [0,1/4)$. Plainly, for each $l \geq 0$, $\partial_u^l\left(\frac{s}{1+u^2}\right) = \frac{j_l(u)}{(1+|u|^2)^{l+1}}$ for some polynomial $j_l$ of order $l$. This yields a recurrence relation for the coefficients that only differs from (A.1) in that the factors $-2l$ are replaced by $-2(l+1)$. The conclusion thus follows as previously.

## A.2 A maximum principle for kinetic Fokker-Planck equations

We next give for completeness a brief proof of the version of the maximum principle that has been used throughout.

**Theorem A.1.** *Let $d \geq 1$ and $\sigma \geq 0$. Consider bounded functions $\varrho_0 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $F, c : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and a function $\phi : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ that grows linearly in $(x,u)$ uniformly in $t \in [0,T]$. Assume moreover that these functions are of class $\mathcal{C}^{0,\infty}$ and have bounded derivatives of all order. Then, there exists a unique solution $\varrho$ of class $\mathcal{C}^{1,\infty}$ to the linear Fokker-Planck equation*

$$\begin{cases} \partial_t\varrho(t,x,u) + u\cdot\nabla_x\varrho(t,x,u) - \phi(t,x,u)\cdot\nabla_u\varrho(t,x,u) - \frac{\sigma^2}{2}\triangle_u\varrho(t,x,u) + c(t,x,u)\varrho(t,x,u) = F(t,x,u) \text{ on } Q_T, \\ \varrho(0,x,u) = \varrho_0(x,u), \text{ on } \mathbb{R}^{2d}, \end{cases}$$

*which is bounded in $Q_T := [0,T] \times \mathbb{R}^{2d}$. Moreover, the function $t \mapsto \|\varrho(t)\|_\infty$ is absolutely continuous, and for almost every $t \in [0,T]$ one has*

$$\frac{d}{dt}\|\varrho(t)\|_\infty \leq \|c(t)\|_\infty\|\varrho(t)\|_\infty + \|F(t)\|_\infty. \tag{A.3}$$

*Proof.* In the case $\sigma > 0$, existence of a bounded solution of class $\mathcal{C}^{1,\infty}$ can be obtained by probabilistic methods, considering the unique pathwise solution $(X_t^{s,x,u}, U_t^{s,x,u})_{s\leq t\leq T}$ to the stochastic differential equation in $\mathbb{R}^d \times \mathbb{R}^d$:

$$X_t^{T;s,x,u} = x - \int_s^t U_r^{T;s,x,u}dr$$
$$U_t^{T;s,x,u} = u - \int_s^t \phi(T-r, X_r^{T;s,x,u}, U_r^{T;s,x,u})dr + \sigma(W_t - W_s)$$

where $W$ is a standard $d$-dimensional Brownian motion defined in some filtered probability space. Following Friedmann [8] one shows that

$$\begin{aligned}(T,x,u)\mapsto&\mathbb{E}\left[\rho_0(X_T^{T;0,x,u},U_T^{T;0;x,u})\exp\left\{\int_0^T c(T-\theta,X_\theta^{T;0,x,u},U_\theta^{T;0,x,u})\,d\theta\right\}\right]\\&+\mathbb{E}\left[\int_0^T F(T-\theta,X_\theta^{T;0,x,u},U_\theta^{T;0,x,u})\exp\left\{\int_0^\theta c(T-s,X_s^{T;0,x,u},U_s^{T;0,x,u})ds\right\}d\theta\right]\end{aligned}$$

is a solution to the Fokker-Planck equation (regularity of this function is then a direct consequence of the regularity of $\rho_0,\phi,c,F$ and of the flow $(t,x,u)\mapsto(X_t^{T;0,x,u},U_t^{T;0,x,u}))$.

In order to check that (A.3) holds, we set $\bar{\varrho}(t,x,u):=\varrho(T-t,x,u)$ and apply Itô's formula to get

$$\bar{\varrho}(t,X_t^{T;s,x,u},U_t^{T;s,x,u})=\bar{\varrho}(s,x,u)+\int_s^t(F-c\rho)(T-r,X_r^{T;s,x,u},U_r^{T;s,x,u})dr+\sigma\int_s^t\nabla\bar{\varrho}(r,X_r^{T;s,x,u},U_r^{T;s,x,u})dW_r.$$

Since $\nabla\bar{\varrho}$ is locally bounded there exists a sequence $(\tau_n)$ of stopping times with $\tau_n\nearrow+\infty$ such that $\int_s^{t\wedge\tau_n}\nabla\bar{\varrho}(r,X_r^{s,x,u},U_r^{s,x,u})dW_r$ is a martingale. Taking expectations and letting $n\to\infty$ using the fact that $\varrho$ is continuous and bounded, we deduce that

$$\mathbb{E}\varrho(T-t,X_t^{T;s,x,u},U_t^{T;s,x,u})=\varrho(T-s,x,u)+\int_s^t\mathbb{E}\left[(F-c\varrho)(T-r,X_r^{T;s,x,u},U_r^{T;s,x,u}))\right]dr.$$

Taking $\theta=T-t\le\alpha=T-s$, we get

$$\mathbb{E}\varrho(\theta,X_{T-\theta}^{T;T-\alpha,x,u},U_{T-\theta}^{T;T-\alpha,x,u})=\varrho(\alpha,x,u)+\int_\theta^\alpha\mathbb{E}\left[(c\varrho-F)(r,X_{T-r}^{T;T-\alpha,x,u},U_{T-r}^{T;T-\alpha,x,u})\right]dr.$$

Writing this identity for $\theta=0$ and $\alpha=t$, then for $\theta=0$ and $\alpha=t+h$, and taking the difference between the two obtained identities, we deduce that

$$\Big|\|\varrho(t+h)\|_\infty-\|\varrho(t)\|_\infty\Big|\le\|\varrho(t+h)-\varrho(t)\|_\infty\le\left|\int_t^{t+h}\left(\|c(\theta)\|_\infty\|\varrho(\theta)\|_\infty+\|F(\theta)\|_\infty\right)d\theta\right|$$

for all $0\le t\le T$ and $h\in\mathbb{R}$ such that $t+h\in[0,T]$. This inequality implies that $t\mapsto\|\varrho(t)\|_\infty$ is absolutely continuous and that (A.3) holds.

In the case $\sigma=0$ the same arguments go through by considering the solution of the corresponding ordinary differential equation.

□

### A.3 Proof of Lemmas 2.10 and 2.11

Here, we provide the proofs of Lemmas 2.10 and 2.11 following arguments of [10]. Their truncated versions used in the proof of Proposition 2.12 are obtained in a similar way, namely replacing in the next proofs the norms $\|\cdot\|_{\lambda,a}$ by their truncated versions $\|\cdot\|_{\lambda,a,A}$ for each $A\in\mathbb{N}$, and the sums over $\mathbb{N}$ by sums over the set $\{0,\dots,A\}$.

*Proof of Lemma 2.10.* By definition, we have

$$\sum_{a\in\mathbb{N}}\frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|f\|_{\lambda,0}\|v\|_{\lambda,1}\|w\|_{\lambda,1}\right)=\sum_{a\in\mathbb{N}}\frac{1}{(a!)^2}\sum_{r=0}^a C_a^r\frac{d^r}{d\lambda^r}\|f\|_{\lambda,0}\frac{d^{a-r}}{d\lambda^{a-r}}\left(\|v\|_{\lambda,1}\|w\|_{\lambda,1}\right).$$

Then we see that

$$\begin{aligned}
&\sum_{a\in\mathbb{N}} \frac{1}{(a!)^2} \sum_{r=0}^{a} C_a^r \frac{d^r}{d\lambda^r}\|f\|_{\lambda,0} \frac{d^{a-r}}{d\lambda^{a-r}} \left(\|v\|_{\lambda,1}\|w\|_{\lambda,1}\right)\\
&= \sum_{r\in\mathbb{N}} \|f\|_{\lambda,r} \sum_{a=r}^{+\infty} \frac{C_a^r}{(a!)^2} \frac{d^{a-r}}{d\lambda^{a-r}} \left(\|v\|_{\lambda,1}\|w\|_{\lambda,1}\right) \quad \left(\text{since } \frac{d^p}{d\lambda^p}\|\psi\|_{\lambda,0} = \|\psi\|_{\lambda,p} \text{ by definition}\right)\\
&= \sum_{r\in\mathbb{N}} \|f\|_{\lambda,r} \sum_{a=r}^{+\infty} \frac{C_a^r}{(a!)^2} \sum_{q=0}^{a-r} C_{a-r}^q \left(\|v\|_{\lambda,q+1}\|w\|_{\lambda,a-q-r+1}\right)\\
&= \sum_{r\in\mathbb{N}} \|f\|_{\lambda,r} \sum_{a=0}^{+\infty} \sum_{q=0}^{a} \frac{C_{a+r}^r C_a^q}{((a+r)!)^2} \left(\|v\|_{\lambda,q+1}\|w\|_{\lambda,a-q+1}\right) \quad (\text{by a change of variables})\\
&= \sum_{r\in\mathbb{N}} \|f\|_{\lambda,r} \sum_{q=0}^{+\infty} \left( \|v\|_{\lambda,q+1} \sum_{a=q}^{+\infty} \|w\|_{\lambda,a-q+1} \frac{C_{a+r}^r C_a^q}{((a+r)!)^2} \right)\\
&= \sum_{r\in\mathbb{N}} \|f\|_{\lambda,r} \sum_{q=0}^{+\infty} \left( \|v\|_{\lambda,q+1} \sum_{a=0}^{+\infty} \|w\|_{\lambda,a+1} \frac{C_{a+r+q}^r C_{a+q}^q}{((a+r+q)!)^2} \right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{a\in\mathbb{N}} \frac{1}{(a!)^2} \frac{d^a}{d\lambda^a} \left(\|f\|_{\lambda,0}\|v\|_{\lambda,1}\|w\|_{\lambda,1}\right)\\
&= \sum_{r\in\mathbb{N}} \frac{\|f\|_{\lambda,r}}{(r!)^2} \sum_{q=0}^{+\infty} \frac{(q+1)^2}{((q+1)!)^2}\|v\|_{\lambda,q+1} \sum_{a=0}^{+\infty} \frac{(a+1)^2}{((a+1)!)^2}\|w\|_{\lambda,a+1} \frac{(r!)^2((q+1)!)^2((a+1)!)^2 C_{a+r+q}^r C_{a+q}^q}{(a+1)^2(q+1)^2((a+r+q)!)^2},
\end{aligned}$$

where

$$\frac{(r!)^2((q+1)!)^2((a+1)!)^2 C_{a+r+q}^r C_{a+q}^q}{(a+1)^2(q+1)^2((a+r+q)!)^2} = \frac{a!q!r!}{(a+q+r)!}.$$

The claim follows since $\frac{a!q!r!}{(a+q+r)!} \leq 1,\ \forall\, a, q, r \in \mathbb{N}$. □

*Proof of Lemma 2.11.* One has

$$\begin{aligned}
\sum_{a\in\mathbb{N}} \frac{1}{(a!)^2} \frac{d^a}{d\lambda^a} \left(\|f\|_{\lambda,1}\|v\|_{\lambda,1}\right) &= \sum_{a\in\mathbb{N}} \frac{1}{(a!)^2} \left( \sum_{r=0}^{a} C_a^r \frac{d^r}{d\lambda^r}\|f\|_{\lambda,1} \frac{d^{a-r}}{d\lambda^{a-r}}\|v\|_{\lambda,1} \right)\\
&= \sum_{a\in\mathbb{N}} \frac{1}{(a!)^2} \left( \sum_{r=0}^{a} C_a^r \|f\|_{\lambda,r+1}\|v\|_{\lambda,a-r+1} \right) \quad \left(\text{since } \frac{d^p}{d\lambda^p}\|\psi\|_{\lambda,0} = \|\psi\|_{\lambda,p} \text{ by definition}\right)\\
&= \sum_{r\in\mathbb{N}} \|f\|_{\lambda,r+1} \left( \sum_{a=r}^{+\infty} \frac{C_a^r}{(a!)^2}\|v\|_{\lambda,a-r+1} \right)\\
&= \sum_{r\in\mathbb{N}} \|f\|_{\lambda,r+1} \left( \sum_{a=0}^{+\infty} \frac{C_{a+r}^r}{((a+r)!)^2}\|v\|_{\lambda,a+1} \right)\\
&= \sum_{r\in\mathbb{N}} \frac{\|f\|_{\lambda,r+1}}{((r+1)!)^2} \sum_{a\in\mathbb{N}} \frac{\|v\|_{\lambda,a+1}}{((a+1)!)^2} \left( \frac{C_{a+r}^r((a+1)!)^2((r+1)!)^2}{((a+r)!)^2} \right),
\end{aligned}$$

where

$$\frac{C_{a+r}^r((a+1)!)^2((r+1)!)^2}{((a+r)!)^2} = \frac{(a+1)(r+1)(a+1)!(r+1)!}{(a+r)!}.$$

As in [10], we observe that when $a \geq 2$ and $r \geq 2$ one has

$$\frac{(r+1)(a+1)(a+1)!(r+1)!}{(a+r)!} \leq 24.$$

Indeed, for $r \geq 2$ and $a \geq 3$,

$$\begin{aligned}
\frac{(r+1)(a+1)(a+1)!(r+1)!}{(a+r)!} &= \frac{1\times\cdots\times(r+1)\times(r+1)}{1\times\cdots\times r\times(r+1)\times(r+2)}\times 1\times 2\times 3\times 4\times\frac{5\times\cdots\times a\times(a+1)\times(a+1)}{(r+3)\cdots\times(a+r-1)\times(a+r)}\\
&\leq 4!\times\frac{5\times\cdots\times(a+1)\times(a+1)}{(r+3)\times\cdots\times(a+r-1)\times(a+r)}\\
&\leq 24,
\end{aligned}$$

where $a \geq 3$ was used in the first expansion and $r \geq 2$ in the second inequality. If $r \geq 2$ and $a = 2$ then

$$\frac{(r+1)(a+1)(a+1)!(r+1)!}{(a+r)!} = 18\times\frac{(r+1)(r+1)!}{(r+2)!} \leq 18.$$

If $a \leq 1$ or $r \leq 1$ we have to separate the corresponding terms in the estimation. Then, we get

$$\begin{aligned}
&\sum_{a\in\mathbb{N}}\frac{1}{(a!)^2}\sum_{r=0}^{a}C_a^r\|f\|_{\lambda,r+1}\|v\|_{\lambda,a-r+1} = \sum_{r\in\mathbb{N}}\frac{\|f\|_{\lambda,r+1}}{((r+1)!)^2}\sum_{a=0}^{+\infty}\frac{\|v\|_{\lambda,a+1}}{((a+1)!)^2}\left(\frac{(r+1)(a+1)(a+1)!(r+1)!}{(a+r)!}\right)\\
&= \|v\|_{\lambda,1}\sum_{r\in\mathbb{N}}\frac{\|f\|_{\lambda,r+1}}{((r+1)!)^2}(r+1)^2 + 4\|v\|_{\lambda,2}\sum_{r\in\mathbb{N}}\frac{\|f\|_{\lambda,r+1}}{((r+1)!)^2}(r+1)\\
&\quad + \|f\|_{\lambda,1}\sum_{a\geq 2}\frac{\|v\|_{\lambda,a+1}}{((a+1)!)^2}(a+1)^2 + 4\|f\|_{\lambda,2}\sum_{a\geq 2}\frac{\|v\|_{\lambda,a+1}}{((a+1)!)^2}(a+1)\\
&\quad + 24\sum_{r\geq 2}\frac{\|f\|_{\lambda,r+1}}{((r+1)!)^2}\sum_{a\geq 2}\frac{\|v\|_{\lambda,a+1}}{((a+1)!)^2}\\
&\leq \left(\|v\|_{\lambda,1}+4\|v\|_{\lambda,2}\right)\|f\|_{\widetilde{\mathcal{H}},\lambda} + \left(\|f\|_{\lambda,1}+4\|f\|_{\lambda,2}\right)\|v\|_{\widetilde{\mathcal{H}},\lambda} + 24\sum_{r\geq 2}\frac{\|f\|_{\lambda,r+1}}{((r+1)!)^2}\sum_{a\geq 2}\frac{\|v\|_{\lambda,a+1}}{((a+1)!)^2}
\end{aligned}$$

Since $\sum_{a\geq 3}\frac{\|v\|_{\lambda,a}}{(a!)^2} \leq \|v\|_{\widetilde{\mathcal{H}},\lambda}\wedge\|v\|_{\mathcal{H},\lambda}$ and $\sum_{r\geq 3}\frac{\|f\|_{\lambda,r}}{(r!)^2} \leq \|f\|_{\widetilde{\mathcal{H}},\lambda}\wedge\|f\|_{\mathcal{H},\lambda}$, the latter expression can be bounded above by

$$\begin{aligned}
&\left(\|v\|_{\lambda,1}+4\|v\|_{\lambda,2}+12\sum_{a\geq 3}\frac{\|v\|_{\lambda,a}}{(a!)^2}\right)\|f\|_{\widetilde{\mathcal{H}},\lambda} + \left(\|f\|_{\lambda,1}+4\|f\|_{\lambda,2}+12\sum_{r\geq 3}\frac{\|f\|_{\lambda,r}}{(r!)^2}\right)\|v\|_{\widetilde{\mathcal{H}},\lambda}\\
&\qquad\leq 16(\|f\|_{\widetilde{\mathcal{H}},\lambda}\|v\|_{\mathcal{H},\lambda}+\|f\|_{\mathcal{H},\lambda}\|v\|_{\widetilde{\mathcal{H}},\lambda})
\end{aligned}$$

and with the bound $\|w\|_{\lambda,1}+4\|w\|_{\lambda,2}+12\sum_{a\geq 3}\frac{\|w\|_{\lambda,a}}{(a!)^2} \leq 16\|w\|_{\mathcal{H},\lambda}$ for $w = f, v$ we establish (i). Using the bound $\|v\|_{\lambda,1}+4\|v\|_{\lambda,2}+12\sum_{a\geq 3}\frac{\|v\|_{\lambda,a}}{(a!)^2} \leq 4\|v\|_{\widetilde{\mathcal{H}},\lambda}$ we alternatively obtain (ii). □

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