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Abstract

We consider an incompressible kinetic Fokker Planck equation in the flat torus, which is a simplified version of the Lagrangian stochastic models for turbulent flows introduced by S.B. Pope in the context of computational fluid mechanics. The main difficulties in its treatment arise from the pressure type force in the equation that couples the Fokker Planck equation with a Poisson equation involving second order moments. In this paper we prove short time existence of analytic solutions in the one-dimensional case, where the pressure force is explicit, and for which we are able to use techniques and functional norms that have been recently introduced in the study of a related singular model.

Keywords Fluid particle model; Incompressibility; Singular kinetic equation; Analytic solution.

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Contents

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1 Introduction

Let $\mathbb{T}^d := \mathbb{R}^d/\mathbb{Z}^d$ denote the flat d-dimensional torus and $\beta \geq 0$, $\sigma \geq 0$ and $\alpha \in \{0,1\}$ be fixed constants. We consider the following partial differential equation in $\mathbb{T}^d \times \mathbb{R}^d$, with unknown (scalar) functions $f(t,x,u)$ and $P(t.x)$:

$$\partial_t f(t,x,u) + u\cdot\nabla_x f(t,x,u) = \nabla_u f(t,x,u)\cdot\left(\nabla_x P(t,x) + \beta\left(u - \alpha\int_{\mathbb{R}^d} v f(t,x,v)\,dv\right)\right)$$
$$+ \frac{\sigma^2}{2}\triangle_u f(t,x,u) + \beta\, d\, f(t,x,u) \qquad \text{on } (0,T]\times\mathbb{T}^d\times\mathbb{R}^d, \tag{1.1a}$$
$$f(0,x,u) = f_0(x,u) \text{ on } \mathbb{T}^d\times\mathbb{R}^d \text{ and} \tag{1.1b}$$
$$\int_{\mathbb{R}^d} f(t,x,u)\,du = 1 \text{ on } [0,T]\times\mathbb{T}^d. \tag{1.1c}$$

This "constrained" equation of kinetic type can be understood as a the Fokker-Planck equation associated with the following stochastic differential equation in $\mathbb{T}^d \times \mathbb{R}^d$,

$$X_t = \left[X_0 + \int_0^t U_s\,ds\right], \quad U_t = U_0 + \sigma W_t - \int_0^t \nabla_x P(s,X_s)ds - \beta\int_0^t (U_s - \alpha\mathbb{E}(U_s|X_s))ds \tag{1.2a}$$
$$law(X_0,U_0) = f_0(x,u)dx\,du, \tag{1.2b}$$
$$\mathbb{P}(X_t \in dx) = dx, \text{ for all } t\in[0,T], \tag{1.2c}$$

($x \mapsto [x]$ denoting the projection on the torus) which constitutes a laboratory example of the class of Lagrangian stochastic models for incompressible turbulent flows, introduced mainly by S.B. Pope in the eighties in order to provide a fluid-particle description of turbulent flows and develop probabilistic numerical methods for their simulation. We refer the reader to [16] for a general presentation of this turbulent model approach in the framework of computational fluid dynamics, and to [2], [8] for a survey on mathematical problems raised by the Lagrangian stochastic models. In physical terms, when $\alpha = 0$, the process U_t representing the velocity of a fluid particle reverts towards the origin like an Ornstein-Uhlenbeck process with a potential given by the standard kinetic energy $\mathbb{E}|U_t|^2$. When $\alpha = 1$, reversion towards the origin in (1.2a) is replaced by reversion towards the *averaged velocity* or *bulk-velocity*, $\mathbb{E}(U_t|X_t = x)$, which can be associated to the local-in-space potential $\mathbb{E}(|U_t - \mathbb{E}(U_t|X_t)|^2|X_t = x)$, interpreted as the *turbulent* kinetic energy (notice that under condition (1.1c) or (1.2c) $\int_{\mathbb{R}^d} vf(t,x,v)dv$ is the conditional expectation $\mathbb{E}(U_t|X_t = x)$). In both cases, the additional drift term $\nabla_x P(t,x)$ is interpreted as the gradient of a pressure field intended to accomplish the homogeneous mass distribution constraint specified by equations (1.1c) or (1.2c), in other words to force the particle position X_t to have a macroscopically uniform spacial distribution.

In spite of its relevance for the simulation of complex fluid dynamics (see e.g. [17], [13] and the references therein), a rigorous mathematical formulation of the Lagrangian stochastic models, and in particular of the uniform mass distribution constraint, has not yet been given. Indeed, equations (1.1) and (1.2) exhibit several conceptual and technical difficulties, and to our knowledge there are so far no direct strategy for its study or mathematical results about it, neither in the field of stochastic processes nor in that of kinetic PDE. In [8], first well-posedness results on a simpler kinetic model were obtained, which featured nonlinearity of conditional type. From a probabilistic point of view, the conditional expectation was treated as a McKean-Vlasov equation. This enabled the authors to also construct a mean field stochastic particle approximation of the nonlinear model. Combined with an heuristic numerical procedure to deal with the constraint (1.2c) and the pressure term, that particle scheme gave raise to a stochastic numerical downscaling method studied and implemented in [2]. Extensions of some of those results to a relevant instance of boundary value problem were obtained in [6] and [7]. However, in spite of the formal resemblance of the uniform mass distribution with the case of the incompressible Navier-Stokes equations, there is so far no rigorous mathematical evidence that (1.2c) can be satisfied by adding a force term of the form $\nabla_x P(t,x)$ in the linear Langevin process (a trivial exception is the situation $\nabla_x P \equiv 0$ of the stationary Langevin process, considered as a benchmark for the stochastic downscaling method in [2]).

The aim of this paper is to address for the first time the well-posedness of a relatively simple instance of Lagrangian stochastic models, yet satisfying in a non trivial way the uniform mass distribution constraint.

A first step in our study will be to establish an alternative formulation of the previous equation. In the Lagrangian modeling of turbulent flow, the constraint (1.2c) is indeed formulated heuristically (see e.g. [16]) by rather imposing some divergence free property on the flow, which in the case of system (1.2) would correspond to a divergence free condition on the bulk velocity field:

$$\nabla_x \cdot \mathbb{E}(U_t|X_t = x) = 0.$$

By taking the divergence of a formal equation for the bulk velocity derived from the Fokker-Planck equation, and a classical projection argument on the space of divergence free fields, it is then assumed that the field P verifies an elliptic PDE, which in our notation is written as

$$\triangle_x P = -\sum_{i,j=1}^{d} \partial_{x_i x_j} \mathbb{E}\left(U_t^{(i)} U_t^{(j)} | X_t = x\right) \tag{1.3}$$

(see [17] for a precise formulation and related numerical issues). Consistently with this heuristic point of view, we will rigorously show below that, under natural assumptions on the initial data, any smooth pair (f, P) that is a classical solution to (1.1) must also be a solution to the system

$$\begin{cases} \partial_t f(t,x,u) + u \cdot \nabla_x f(t,x,u) = \dfrac{\sigma^2}{2} \triangle_u f(t,x,u) + \beta\, d\, f(t,x,u) + \beta u \cdot \nabla_u f(t,x,u) \\ \qquad + \nabla_u f(t,x,u) \cdot \left(\nabla_x P(t,x) - \beta\alpha \displaystyle\int_{\mathbb{R}^d} v f(t,x,v)\, dv \right) = 0, \text{ on } (0,T] \times \mathbb{T}^d \times \mathbb{R}^d, \\ f(0,x,u) = f_0(x,u), \text{ on } \mathbb{T}^d \times \mathbb{R}^d, \\ \triangle_x P(t,x) = -\displaystyle\sum_{i,j=1}^{d} \partial_{x_i x_j} \int_{\mathbb{R}^d} v_i v_j f(t,x,v) dv, \text{ on } [0,T] \times \mathbb{T}^d, \end{cases} \tag{1.4}$$

where, plainly, condition (1.1c) has been replaced by the above Poisson equation. The two systems however seem not to be equivalent in general. From the PDE point of view, the interest of formulation (1.4) is that it allows us to see the original problem as an instance of Vlasov-Fokker-Planck type equation, albeit highly singular: the gradient of the pressure field turns out to be the convolution of the derivative of the periodic Poisson kernel with the function $-\sum_{i,j=1}^{d} \partial_{x_i x_j} \int_{\mathbb{R}^d} v_i v_j f(t,x,v) dv$. Existence of smooth solutions to nonlinear kinetic equations with singular potential has been addressed in several situations, mainly recently. In particular for the nonlinear Vlasov-Poisson equation see e.g. [1], [14]; for gyro-kinetic models see e.g [4, 3] and [11].

In the case $d = 1$, we can specify $P(t,x)$ on $[0,T] \times \mathbb{R}$ by $P(t,x) = -\int_{\mathbb{R}} u^2 f(t,x,u) du$. Hence, in the present paper, we restrict ourself to the simpler situation of the one-dimensional equation

$$\begin{cases} \partial_t f(t,x,u) + u \cdot \partial_x f(t,x,u) = \dfrac{\sigma^2}{2} \partial_u^2 f(t,x,u) + \beta f(t,x,u) + \beta u \partial_u f(t,x,u) \\ \qquad + \partial_u f(t,x,u) \left(\partial_x P(t,x) - \beta\alpha \displaystyle\int_{\mathbb{R}} v f(t,x,v) dv \right) = 0, \text{ on } (0,T] \times \mathbb{T} \times \mathbb{R}, \\ f(t,x,u) = f_0(x,u), \text{ on } \mathbb{T} \times \mathbb{R}, \\ P(t,x) = -\displaystyle\int_{\mathbb{R}} u^2 f(t,x,u) du, \text{ on } [0,T] \times \mathbb{T}. \end{cases} \tag{1.5}$$

Till recently, a closed formulation (in terms of regularity) of such Vlasov type nonlinear system was not available. To tackle the system (1.5), we will follow new ideas introduced in [12], in order to obtain a local existence result of analytical solutions. Our main results are valid irrespective of wether $\sigma > 0$ or $\sigma = 0$, and are indeed valid for any $\beta \in \mathbb{R}$. We summarize them in the following simplified statement:

Theorem 1.1. *Let $\bar\lambda > 0$ and $s \geq 4$ be an even integer. There exists a constant $\kappa_0 = \kappa_0(\bar\lambda, s)$ and a positive function $r \mapsto \kappa_1(r, \bar\lambda, s)$ such that if $f_0 : \mathbb{T} \times \mathbb{R} \to \mathbb{R}$ of class $\mathcal{C}^\infty$ and $T > 0$ satisfy:*

- *$\int_{\mathbb{R}} f_0(x,u) du = 1$ and $\partial_x \int_{\mathbb{R}^d} u f_0(x,u) du = 0$ for all $x \in \mathbb{T}$,*
- *$\|(1+u^2)^{\frac{s}{2}} \partial_x^l \partial_u^k f_0\|_\infty \leq \frac{C_0 (k+m)!(l+n)!}{\bar\lambda^{k+l}}$ for some $n, m \in \mathbb{N}$, all pair of indices $k, l \in \mathbb{N}$ and a constant $C_0 < \kappa_0(\bar\lambda, s)$, and*

- $T < \kappa_1(C_0, \bar{\lambda}, s)$,

then, a classic smooth solution f to equation (1.5) exists in $[0,T]\times\mathbb{T}\times\mathbb{R}$ which satisfies $\int_{\mathbb{R}} f(t,x,u)du = 1$ and $\partial_x \int_{\mathbb{R}^d} uf(t,x,u)du = 0$ at all $(t,x) \in [0,T]\times\mathbb{T}$.

This result will allow us to deduce the following one on the " incompressible Langevin SDE" (1.2):

Corollary 1.2. *Under the assumptions Theorem 1.1, in the case $\sigma > 0$, there exists in $[0,T]$ a solution (X_t, U_t) to the stochastic differential equation (1.2). Moreover, the drift $P(t,x)$ therein is the spatial gradient of the unique solution of the periodic elliptic equation (1.3).*

The remainder of the paper is organized as follows:

In Subsection 1.1 we briefly establish the validity of system (1.4) for any solution to equation (1.1) in arbitrary space dimension, and state additional conditions required in order that, reciprocally, a solution to the former also solves the latter. From Section 2 on, we restrain ourselves to the one-dimensional case. We recall therein the analytical norms and seminorms introduced in [12] and we state useful properties of them. Following their strategy, in the case $\beta = 0$ we then introduce an equivalent formulation of equation (1.5), in order to deal with the integrability problems posed by the first and second order velocity moments involved in the equation. We then show that solutions to (1.4) in these particular spaces of analytical functions actually do satisfy the conditions required to be solutions of (1.1). Using the fixed point argument of [12], we will then prove a local existence result in these analytical spaces, which indeed is a slightly more general version of Theorem 1.1 restricted to the case $\beta = 0$. In Section 3, we extend the previous result to the case $\beta \geq 0$. In Section 4 we deduce from the previous sections a local existence result for the stochastic differential equation (1.2). Finally, some technical results are proved in the Appendix section.

We fix some notation to be used throughout:

- $T > 0$ is a fixed time horizon.
- Functions $[0,T]\times\mathbb{T}^d\times\mathbb{R}^d \ni (t,x,v) \mapsto \phi(t,x,v) \in \mathbb{R}$ are identified with functions $[0,T]\times\mathbb{R}^d\times\mathbb{R}^d \ni (t,x,v) \mapsto \phi(t,x,v) \in \mathbb{R}$ that are 1- periodic in the variable x. Similar identification are made for functions defined on $[0,T]\times\mathbb{T}^d$ and $\mathbb{T}^d$.
- Given $T > 0$ and $d \in \mathbb{N}$, a function $\phi : [0,T]\times E \to \mathbb{R}$ with $E = \mathbb{R}^d\times\mathbb{R}^d, \mathbb{T}^d\times\mathbb{R}^d$ or $E = \mathbb{T}^d$ is said to be of class $\mathcal{C}^{k,l}$ for $k \in \{0,1\}$ and $l \in \mathbb{N}\cup\{\infty\}$ if it has continuos derivatives up to order k in $t \in [0,T]$ and up to order l in $y \in E$ (or of all order if $l = \infty$). For functions $\phi : E \to \mathbb{R}$ the notation $\mathcal{C}^l$ is used analogously.

1.1 The Lagrangian stochastic model coupled with a Poisson equation

We start by establishing connections between conditions related to the homogeneous mass distribution constraint, which are valid in arbitrary dimension:

Lemma 1.3. *Assume that f is a classical solution to equations (1.1a) and (1.1b) for some function $P : [0,T]\times\mathbb{T}^d \to \mathbb{R}^d$ of class $\mathcal{C}^{0,2}$. Moreover, assume that*

$$\rho(t,x) := \int_{\mathbb{R}^d} f(t,x,u)du, \quad V(t,x) := \int_{\mathbb{R}^d} uf(t,x,u)du$$

are functions of class $\mathcal{C}^{1,1}$ in $[0,T]\times\mathbb{T}^d$, that $\int_{\mathbb{R}^d} u^2|D^m f(t,x,u)|du < +\infty$ for each multiindex $|m| \leq 2$ and each $(t,x) \in [0,T]\times\mathbb{T}^d$ (where D is the derivative operator), and further that for all $t \in [0,T]$ the function

$$x \mapsto \int_{\mathbb{R}^d} v_i v_j f(t,x,v)dv$$

is of class $\mathcal{C}^2$. Then, the following system of equations is satisfied for $(t,x) \in (0,T]\times\mathbb{T}^d$:

$$\begin{cases} \partial_t \rho(t,x) + \nabla_x \cdot V(t,x) = 0, \\ \partial_t(\nabla_x \cdot V(t,x)) + \beta\nabla_x \cdot V(t,x) + \nabla_x \cdot \big(\rho(t,x)\left(\nabla_x P(t,x) - \beta\alpha V(t,x)\right)\big) + \displaystyle\sum_{i,j=1}^{d} \partial_{x_i x_j} \int_{\mathbb{R}^d} v_i v_j f(t,x,v)dv = 0 \end{cases}$$

We deduce:

a) $\rho(t,x) = \rho(0,x)$ *for every* $(t,x) \in [0,T] \times \mathbb{T}^d$ *if and only if* $\nabla_x \cdot V(t,x) = 0$ *for every* $(t,x) \in [0,T] \times \mathbb{T}^d$.

b) $\nabla_x \cdot V(t,x) = e^{-\beta t}\nabla_x \cdot V(0,x)$ *for every* $(t,x) \in [0,T] \times \mathbb{T}^d$ *if and only if* P *satisfies the equation of elliptic type:*

$$\nabla_x \cdot \big(\rho(t,x)\,(\nabla_x P(t,x) - \beta\alpha V(t,x))\,\big) = -\sum_{i,j=1}^{d} \partial_{x_i x_j} \int_{\mathbb{R}^d} v_i v_j f(t,x,v)dv\,, \quad (t,x) \in (0,T] \times \mathbb{T}^d.$$

c) If in addition to (1.1a) *and* (1.1b)*, condition* (1.1c) *is verified, then* $P(t,x)$ *is a solution to the Poisson equation*

$$\triangle_x P(t,x) = -\sum_{i,j=1}^{d} \partial_{x_i x_j} \int_{\mathbb{R}^d} v_i v_j f(t,x,v)dv, \quad (t,x) \in (0,T] \times \mathbb{T}^d.$$

d) Set $\bar\rho(t,x) := \rho(t,x) - 1$*. If in addition to* (1.1a) *and* (1.1b) *we assume that the Poisson equation in part c) holds, we have:*

when $\alpha = 1$, $\partial_t(\nabla_x \cdot V(t,x)) + \nabla_x \cdot \big(\bar\rho(t,x)\,(\nabla_x P(t,x) - \beta V(t,x))\,\big) = 0$;

when $\alpha = 0$, $\partial_t(\nabla_x \cdot V(t,x)) + \nabla_x \cdot \big(\bar\rho(t,x)\nabla_x P(t,x)\big) + \beta\nabla_x \cdot V(t,x) = 0$.

Proof. The first equation is obtained by integrating equation (1.1a) with respect to $u \in \mathbb{R}^d$, and using the assumptions in order to integrate by parts and get rid of integrals of divergence type terms. To get the second equation, we first take the derivative with respect to the variable x_i in equation (1.1a), then multiply it by u_i and sum over $i = 1, \dots, d$, before integrating and proceeding as before. Statements a),b),c) and d) are then easily deduced. □

Remark 1.4. *a) According to Lemma 1.3 part c), finding a solution to equation* (1.1) *requires in particular to find a solution to the highly singular Vlasov-Fokker-Planck equation* (1.4).

b) If conditions (1.1a) *and* (1.1b) *hold, and the Poisson equation in Lemma 1.3 part c) is satisfied, the equation obtained in Lemma 1.3 part d) together with the continuity equation*

$$\partial_t \bar\rho(t,x) + \nabla_x \cdot V(t,x) = 0$$

furnish a system of two equations that the pair $(\bar\rho, V(t,x))$ *must satisfy. Thus, a strategy to prove in that situation that* (1.1c) *also holds is to prove that such a system has the unique solution* $\bar\rho(t,x) = \nabla_x \cdot V(t,x) \equiv 0$ *when starting from* $(0,0)$*. We will be able to do this in the functional setting that we will consider, deducing thus a solution to* (1.1) *from a solution to* (1.4).

2 Local analytic well-posedness in the vanishing kinetic potential case ($\beta = 0$)

In this section, we construct an analytical solution to the nonlinear Vlasov-Fokker-Planck equation associated with the incompressible Lagrangian stochastic model up to some small time horizon T, in the case $\beta = 0$. Using the weighted analytical functional space introduced in [12] and a fixed point argument developed therein, we shall give in Theorem 2.5 below a local-in-time well-posedness result for the nonlinear Vlasov-Fokker-Planck equation:

$$\begin{cases} \partial_t f + u\partial_x f - \partial_x P \partial_u f - \dfrac{\sigma^2}{2}\partial_u^2 f = 0 \text{ on } (0,T] \times \mathbb{R}^2, \\ f(0,x,u) = f_0(x,u) \text{ on } \mathbb{R}^2, \end{cases} \tag{VFP}$$

where

$$P(t,x) = -\int_{\mathbb{R}} u^2 f(t,x,u)du, \;\; x \in \mathbb{R}$$

(In order to lighten the notations, we will write implicitly the dependency w.r.t. (t, x, u) of the function in the interior equation). Notice that periodicity is not yet imposed. Then, we will show in Corollary 2.6 that if the obtained local solution $f(t, x, u)$ of (VFP) satisfies at $t = 0$ the condition

$(\mathrm{H}_{\mathrm{unif}(t)})$:

$$
\begin{aligned}
&f(t,x,u) \text{ is } 1-\text{periodic in } x \text{ for all } u \in \mathbb{R},\\
&\int_{\mathbb{R}} f(t,x,u)du = 1 \text{ for all } x \in \mathbb{T} \ (\textit{Uniform mass repartition in } \mathbb{T}),\\
&\partial_x \textstyle\int_{\mathbb{R}} uf(t,x,u)du = 0 \text{ for all } x \in \mathbb{T} \ (\textit{Mean incompressibility in } \mathbb{T}),
\end{aligned}
$$

it then satisfies *a fortiori* the same properties for all $t \in [0, T]$. To establish the latter result, the choice of analytical functional spaces and the use of the analytic norms in [12] will also be fundamental.

2.1 The nonlinear Vlasov-Fokker-Planck equation in analytic spaces

We start by defining the functional spaces where an equivalent version of equation (VFP) will be studied.

A function $\mathbb{R}^2 \ni (x, u) \mapsto \psi(x, u) \in \mathbb{R}$ having bounded derivatives of all order is said to be analytic if there exists $C > 0$ and $\lambda > 0$ such that for all $k, l \in \mathbb{N}$ $\|\partial_x^k \partial_u^l \psi\|_\infty \le C\lambda^{k+l} k!l!$, for $\|\cdot\|_\infty$ the uniform norm on $\mathbb{R}^2$ and using the convention $0! = 1$. For such functions, we introduce the analytic norm:

$$
\|\psi\|_\lambda := \sum_{k,l\in\mathbb{N}} \frac{\lambda^{k+l}}{k!l!} \left\|\partial_x^k \partial_u^l \psi\right\|_\infty .
$$

We further introduce the λ-derivatives of these norms for each $a \in \mathbb{N}$,

$$
\|\psi\|_{\lambda,a} := \frac{d^a}{d\lambda^a}\|\psi\|_\lambda = \sum_{k+l\ge a} \frac{(k+l)!}{(k+l-a)!} \frac{\lambda^{k+l-a}}{k!l!} \left\|\partial_x^k \partial_u^l \psi\right\|_\infty .
$$

Notice that $\|\psi\|_{\lambda,0} = \|\psi\|_\lambda$. We then define respectively a norm and a seminorm by

$$
\|\psi\|_{\mathcal{H},\lambda} := \sum_{a\in\mathbb{N}} \frac{1}{(a!)^2}\|\psi\|_{\lambda,a}, \qquad \|\psi\|_{\widetilde{\mathcal{H}},\lambda} := \sum_{a\ge1} \frac{a^2}{(a!)^2}\|\psi\|_{\lambda,a}.
$$

Last, we define the functional spaces associated with the norm $\|\psi\|_{\mathcal{H},\lambda}$ and the seminorm $\|\psi\|_{\widetilde{\mathcal{H}},\lambda}$:

$$
\mathcal{H}(\lambda) := \left\{\psi \in \mathcal{C}^\infty(\mathbb{R}^2) \text{ such that } \|\psi\|_{\mathcal{H},\lambda} < +\infty\right\}, \tag{2.2a}
$$

$$
\widetilde{\mathcal{H}}(\lambda) := \left\{\psi \in \mathcal{C}^\infty(\mathbb{R}^2) \text{ such that } \|\psi\|_{\widetilde{\mathcal{H}},\lambda} < +\infty\right\}. \tag{2.2b}
$$

The next two lemmas giving some insight about these (semi)norms will be useful later on:

Lemma 2.1. *Let $v : \mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^\infty$ be such that $\|\partial_x^l \partial_u^k v\|_\infty \le \frac{C(k+m)!(l+n)!}{\overline{\lambda}^{k+l}}$ for some $C, \overline{\lambda} > 0$, some $m, n, j \in \mathbb{N}$ and all $k, l \ge j$.*

a) If the previous holds for $j = 0$, (in particular if $\|v\|_{\overline{\lambda},0} < +\infty$), then $v \in \mathcal{H}(\lambda)$ for all $\lambda \in [0, \overline{\lambda})$.

b) If the previous holds for $j = 1$, (in particular if $\|v\|_{\overline{\lambda},1} < +\infty$), then $v \in \widetilde{\mathcal{H}}(\lambda)$ for all $\lambda \in [0, \overline{\lambda})$.

Proof. For $a \ge j$ and $\lambda \in [0, \overline{\lambda})$ we obtain from the assumption that

$$
\begin{aligned}
\frac{d^a}{d\lambda^a}\|v\|_{\lambda,0} \le& \frac{C}{\overline{\lambda}^a} \sum_{k+l\ge a} \frac{(k+l)!(k+m)!(l+n)!}{k!l!(k+l-a)!} \left(\lambda/\overline{\lambda}\right)^{(k+l-a)}\\
=& \frac{C}{\overline{\lambda}^a} \sum_{k+l\ge a+m+n,k\ge m,l\ge n} \frac{(k+l-(m+n))!k!l!}{(k-m)!(l-n)!(k+l-(a+m+n))!} \left(\lambda/\overline{\lambda}\right)^{(k+l-(a+m+n))}
\end{aligned}
$$

changing indexes $k+m$ to k and $l+n$ to l. Since $\frac{(k+l-(m+n))!k!l!}{(k-m)!(l-n)!(k+l)!} \leq 1$ for $k \geq m, l \geq n$, we deduce that

$$\frac{d^a}{d\lambda^a}\|v\|_{\lambda,0} \leq \frac{C}{\overline{\lambda}^a} \sum_{k+l\geq a+m+n} \frac{(k+l)!}{(k+l-(a+m+n))!} (\lambda/\overline{\lambda})^{(k+l-(a+m+n))} = \frac{C}{\overline{\lambda}^a}\left[\frac{d^{a+m+n}}{dr^{a+m+n}}\left(\sum_{k,l\in\mathbb{N}} r^{k+l}\right)\right]\Big|_{r=\lambda/\overline{\lambda}}.$$

Observing that for $r \in [0,1)$, $\sum_{k,l\in\mathbb{N}} r^{k+l} = (\sum_{j\in\mathbb{N}} r^j)^2 = \frac{1}{(1-r)^2}$ and $|\frac{d^a}{dr^a}\frac{1}{(1-r)^2}| \leq (1+a)!$, we conclude that

$$\|v\|_{\mathcal{H},\lambda} = \sum_{a=0}^{\infty} \frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\|v\|_{\lambda,0} \leq C\sum_{a=0}^{\infty}\frac{1}{\overline{\lambda}^a}\frac{(a+1)\cdots(a+m+n+1)}{a!} < +\infty \text{ and}$$
$$\|v\|_{\widetilde{\mathcal{H}},\lambda} = \sum_{a=1}^{\infty} \frac{a^2}{(a!)^2}\frac{d^a}{d\lambda^a}\|v\|_{\lambda,0} \leq \frac{C}{\overline{\lambda}}\sum_{a=0}^{\infty}\frac{1}{\overline{\lambda}^a}\frac{(a+1)\cdots(a+m+n+2)}{a!} < +\infty.$$

□

Remark 2.2. *Notice that Lemma 2.1 in particular implies that .... usual analytical spaces..... are included in...... if*

Lemma 2.3. *Let ψ be an analytic function defined on $\mathbb{R}^2$. Then:*

(i) For each $a \in \mathbb{N}$ one has $\|\psi\|_{\lambda,a+1} = \|\partial_x\psi\|_{\lambda,a} + \|\partial_u\psi\|_{\lambda,a}$. We deduce that

$$\|\psi\|_{\widetilde{\mathcal{H}},\lambda} = \|\partial_x\psi\|_{\mathcal{H},\lambda} + \|\partial_u\psi\|_{\mathcal{H},\lambda}.$$

(ii) Moreover,

$$\frac{d}{d\lambda}\|\psi\|_{\mathcal{H},\lambda} = \|\psi\|_{\widetilde{\mathcal{H}},\lambda}.$$

(iii) Last, for any pair ψ_1, ψ_2 of analytic functions defined on $\mathbb{R}^2$

$$\|\psi_1\psi_2\|_\lambda \leq \|\psi_1\|_\lambda\|\psi_2\|_\lambda.$$

Proof. (i). The first identity follows from

$$\begin{aligned}\|\psi\|_{\lambda,a+1} = \frac{d^{a+1}}{d\lambda^{a+1}}\|\psi\|_{\lambda,0} &= \sum_{m+l\geq a+1} \frac{(m+l)\cdots(m+l-a-1)\lambda^{m+l-a-1}}{m!l!}\|\partial_x^m\partial_u^l\psi\|_\infty \\ &= \sum_{m+l\geq a+1, l\geq 1} \frac{(m+l-1)\cdots(m-a+l-1)\lambda^{m-a+l-1}}{m!(l-1)!}\|\partial_x^m\partial_u^l\psi\|_\infty \\ &\quad + \sum_{m+l\geq a+1, m\geq 1} \frac{(m-1+l)\cdots(l-a+m-1)\lambda^{l-a+m-1}}{(m-1)!l!}\|\partial_x^m\partial_u^l\psi\|_\infty,\end{aligned}$$

by respectively changing the indexes l to $l+1$ and m to $m+1$ in the first and second sums in the last expression. Multiplying by $\frac{a^2}{(a!)^2}$ both sides of the previously established identity and summing the resulting expressions over $a \geq 1$ yields the identity for $\|\psi\|_{\widetilde{\mathcal{H}},\lambda}$. (ii) readily follows from

$$\frac{d}{d\lambda}\|\psi\|_{\mathcal{H},\lambda} = \frac{d}{d\lambda}\sum_{a\in\mathbb{N}}\frac{1}{(a!)^2}\|\psi\|_{\lambda,0} = \sum_{a\in\mathbb{N}}\frac{1}{(a!)^2}\|\psi\|_{\lambda,a+1} = \sum_{a\geq 1}\frac{1}{((a-1)!)^2}\|\psi\|_{\lambda,a} = \sum_{a\geq 1}\frac{a^2}{(a!)^2}\|\psi\|_{\lambda,a} = \|\psi\|_{\widetilde{\mathcal{H}},\lambda}.$$

Finally, since $\|\partial_x^k\partial_u^l(\psi_1\psi_2)\|_\infty \leq \sum_{r=0}^k\sum_{n=0}^l C_k^r C_l^n \|\partial_x^r\partial_u^n\psi_1\|_\infty\|\partial_x^{k-r}\partial_u^{l-n}\psi_2\|_\infty$, we have

$$\begin{aligned}\|\psi_1\psi_2\|_{\lambda,0} = \sum_{k,l\in\mathbb{N}}\frac{\lambda^{k+l}}{k!l!}\|\partial_x^k\partial_u^l(\psi_1\psi_2)\|_\infty &\leq \sum_{r,n\in\mathbb{N}}\|\partial_x^r\partial_u^n\psi_1\|_\infty\sum_{k\geq r}\sum_{l\geq n}\frac{C_k^rC_l^n\lambda^{k+l}}{k!l!}\|\partial_x^{k-r}\partial_u^{l-n}\psi_2\|_\infty \\ &\leq \sum_{r,n\in\mathbb{N}}\frac{\lambda^{r+n}}{r!n!}\|\partial_x^r\partial_u^n\psi_1\|_\infty\sum_{k\geq r}\sum_{l\geq n}\frac{\lambda^{(k-r)+(l-n)}}{(k-r)!(l-n)!}\|\partial_x^{k-r}\partial_u^{l-n}\psi_2\|_\infty\end{aligned}$$

which provides (iii) by changing the indexes k to $k+r$ and l to $l+n$ in the inner sums. □

We now observe that finiteness of the analytical norm of a solution f to (VFP) is not enough to provide a control of the function $(t,x) \mapsto \int_{\mathbb{R}} u^2 f(t,x,u)du$. This is the reason why we introduce a weight function intended to truncate the velocity state space in a suitable sense. More precisely, assume that $f : [0,T] \times \mathbb{R}^2 \to \mathbb{R}$ is a $\mathcal{C}^{1,\infty}$ solution of equation (VFP) with bounded derivatives of all order, and set

$$g(t,x,u) := \omega(u) f(t,x,u), \tag{2.3}$$

where $\omega : \mathbb{R} \to (0,+\infty)$ is a weight function such that $\int_{\mathbb{R}} \frac{u^2}{\omega(u)} du < +\infty$. Then, the regularity of velocity moments of f is easily controlled in terms of the regularity of g:

$$\begin{aligned}\sup_{(t,x)\in[0,T]\times\mathbb{R}} \left|\partial_x^k \int_{\mathbb{R}} u^2 f(t,x,u)\,du\right| &= \sup_{(t,x)\in[0,T]\times\mathbb{R}} \left|\int_{\mathbb{R}} \frac{u^2}{\omega(u)} \partial_x^k g(t,x,u)\,du\right| \\ &\le \sup_{(t,x,u)\in[0,T]\times\mathbb{R}^2} \left|\partial_x^k g(t,x,u)\right| \int_{\mathbb{R}} \frac{u^2}{\omega(u)} du.\end{aligned}$$

Moreover, since $\partial_u f = \partial_u g - g(\partial_u \ln \omega)$ and $\omega \partial_u^2 f = \partial_u^2 g - 2(\partial_u \ln(\omega))(\partial_u g) + g(\frac{2|\partial_u \omega|^2}{\omega^2} - \frac{\partial_u^2 \omega}{\omega})$, the function g defined in (2.3) is seen to satisfy the equation

$$\begin{cases} \partial_t g + u\partial_x g - [\partial_x P - \partial_u \ln \omega]\,\partial_u g - \dfrac{\sigma^2}{2}\partial_u^2 g = g\partial_x P \partial_u \ln \omega - gh \text{ on } (0,T] \times \mathbb{R}^2, \\ P(t,x) = -\displaystyle\int_{\mathbb{R}} \frac{u^2}{\omega(u)} g(t,x,u)du, \\ g(0,x,u) = g_0(x,u) \text{ on } \mathbb{R}^2, \end{cases} \tag{VFPω}$$

where

$$h(u) := \frac{\partial_u^2 \omega(u)}{2\omega(u)} - |\partial_u \ln(\omega(u))|^2\,;$$

reciprocally, given a solution g to (VFPω), the function f defined by (2.3) is a solution to (VFP).

In all the sequel, we shall assume that $\omega : \mathbb{R} \to (0,+\infty)$ is a function of class $\mathcal{C}^\infty$ such that

(H$_\omega$)
$$\lim_{|u|\to+\infty} \frac{\omega(u)}{|u|} = +\infty \text{ and } \int_{\mathbb{R}} \frac{u^2}{\omega(u)} du = 1.$$
$$\limsup_{|u|\to\infty} \left|\frac{\omega'(u)}{\omega(u)}\right| < \infty,\ \limsup_{|u|\to\infty} \left|\frac{\omega''(u)}{\omega'(u)}\right| < \infty.$$
Moreover, for some $\lambda_0 > 0$ we have $\ln(\omega) \in \widetilde{\mathcal{H}}(\lambda_0)$ and $h \in \mathcal{H}(\lambda_0)$.

The following result provides examples of such functions ω, as well tractable conditions on the initial condition f_0 ensuring the type of bounds on g_0 required for our results on equation (VFPω). Its proof relies on Lemma 2.1 and is given in Appendix A.1.

Lemma 2.4. *i) Let $s \ge 4$ be a positive integer. Then, condition (H$_\omega$) holds for the weight function $\omega(u) := c(s)(1+u^2)^{\frac{s}{2}}$ for all value $\lambda_0 \in (0, \frac{1}{4})$, where $c(s) > 0$ is such that $\int_{\mathbb{R}} \frac{u^2}{\omega(u)}\,du = 1$.*

ii) Let $f_0 : \mathbb{R}^2 \to \mathbb{R}$ be a function of class $\mathcal{C}^\infty$ such that for some even integer $s \ge 4$, constants $C_0, \bar{\lambda} > 0$, some $m,n,j \in \mathbb{N}$ and all $k,l \ge j$, one has

$$\|(1+u^2)^{\frac{s}{2}} \partial_x^l \partial_u^k f_0\|_\infty \le \frac{C_0 (k+m)!(l+n)!}{\bar{\lambda}^{k+l}}. \tag{2.5}$$

Then, the function $g_0(x,u) := \omega(u) f_0(x,u)$ with $\omega(u)$ as in i) satisfies the assumptions of Lemma 2.1 with $C := C_0' = C_0 \kappa(s) e^{\bar{\lambda}}$ and $\kappa(s) > 0$ a bound for the absolute values of the coefficients of the polynomials $\omega, \partial_u \omega, \dots, \partial_u^s \omega$. In particular, if for some $n,m \in \mathbb{N}$ condition (2.5) holds all $k,l \ge 0$, then for all $\lambda \in [0, \bar{\lambda})$ one has

$$\begin{aligned}\|g\|_{\mathcal{H},\lambda} &\le C_0 \kappa(s) e^{\bar{\lambda}} \mu(\bar{\lambda}, m+n+1) \text{ and} \\ \|g\|_{\widetilde{\mathcal{H}},\lambda} &\le C_0 \kappa(s) \frac{e^{\bar{\lambda}}}{\bar{\lambda}} \mu(\bar{\lambda}, m+n+2)\end{aligned}$$

where $\mu(\bar{\lambda}, p) := \sum_{a=0}^{\infty} \frac{1}{\bar{\lambda}^a} \frac{(a+1)\cdots(a+p)}{a!} < +\infty$ for all $p \in \mathbb{N}, p \ge 1$.

2.2 Main results

Given K, T and λ_0 strictly positive real numbers such that $\lambda_0 > T(1+K)$, and the function

$$\lambda(t) := \lambda_0 - (1+K)t,$$

we now define the spaces

$$\mathcal{H}_{\lambda_0,K,T} := \left\{ \psi \in \mathcal{C}^{1,\infty}([0,T] \times \mathbb{R}^2) \text{ such that } \sup_{t\in[0,T]} \|\psi(t)\|_{\mathcal{H},\lambda(t)} < +\infty \right\},$$

$$\widetilde{\mathcal{H}}_{\lambda_0,K,T} := \left\{ \psi \in \mathcal{C}^{1,\infty}([0,T] \times \mathbb{R}^2) \text{ such that } \int_0^T \|\psi(t)\|_{\widetilde{\mathcal{H}},\lambda(t)} dt < +\infty \right\}$$

and their subsets defined for a positive constant M:

$$\mathcal{B}^M_{\lambda_0,K,T} := \left\{ \psi \in \mathcal{H}_{\lambda_0,K,T} \text{ such that } \sup_{t\in[0,T]} \|\psi(t)\|_{\mathcal{H},\lambda(t)} \le M \right\},$$

$$\widetilde{\mathcal{B}}^M_{\lambda_0,K,T} := \left\{ \psi \in \widetilde{\mathcal{H}}_{\lambda_0,K,T} \text{ such that } \int_0^T \|\psi(t)\|_{\widetilde{\mathcal{H}},\lambda(t)} dt \le M \right\}.$$

We are ready to state the main result of this section:

Theorem 2.5. *Let M, T be positive constants and $\omega : \mathbb{R} \to (0, +\infty)$ be a function of class $\mathcal{C}^\infty$ satisfying (H_ω) for some $\lambda_0 > 0$. Introduce the finite constants $\gamma_0 := \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda_0}$ and $\gamma_1 := \|h\|_{\mathcal{H},\lambda_0}$ and assume that*

a) $T < \frac{\lambda_0}{2+\lambda_0+4\gamma_0}$,

b) $M \le \frac{1}{16}(K - \lambda_0 - 4\gamma_0 - 1)$ for some K in the nonempty interval $(1 + \lambda_0 + 4\gamma_0, \frac{\lambda_0}{T} - 1)$ and

c) $M(1+\gamma_0)\exp\{(M\gamma_0 + \gamma_1)T\} < 1$.

Assume moreover that $f_0 : \mathbb{R}^2 \to \mathbb{R}$ is a function of class $\mathcal{C}^\infty$ and that $g_0(x,u) := \omega(u)f_0(x,u)$ satisfies

d) $\max\{\|g_0\|_{\mathcal{H},\lambda_0}, T\|g_0\|_{\widetilde{\mathcal{H}},\lambda_0}\} \le M$ and

e) $\|g_0\|_{\mathcal{H},\lambda_0} \exp(T(\gamma_1 + 16\gamma_0)) \le M \exp(-(16+\gamma_0)M)$.

Then, equation (VFPω) has a unique smooth solution $g \in \mathcal{B}^M_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^M_{\lambda_0,K,T}$. In particular, under the previous assumptions, a solution $f \in \mathcal{C}^{1,\infty}$ to (VFP) with initial condition f_0 exists.

Corollary 2.6. *Let f be the solution to (VFP) given in Theorem 2.5 and assume that $(\mathrm{H}_{\mathrm{unif}(0)})$ holds. Then, $f(t,x,u)$ satisfies $(\mathrm{H}_{\mathrm{unif}(t)})$, for all t in $[0,T]$. In particular, if the assumptions of Theorem 2.5 and condition $(\mathrm{H}_{\mathrm{unif}(0)})$ hold, then a solution to (1.1) with $\beta = 0$ exists.*

Remark 2.7. *For instance, let $f_0 : (\mathbb{R}^d)^2 \to \mathbb{R}$ be a function of class $\mathcal{C}^\infty$ and $C_0, \bar{\lambda} > 0$, $n, m \in \mathbb{N}$ be numbers satisfying condition (2.5) for every $k, l \ge 0$. Suppose moreover that for some $\lambda_0 < \min\{\bar{\lambda}, \frac{1}{4}\}$ one has*

$$C_0 < \kappa_0(\bar{\lambda}, s) := \frac{1}{2\kappa(s)e^{\bar{\lambda}}\mu(\bar{\lambda}, m+n+1)} \frac{\ln 2}{(16 + \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda_0})}$$

for ω as in Lemma 2.4 i). Setting $M := 2C_0\kappa(s)e^{\bar{\lambda}}\mu(\bar{\lambda}, m+n+1)$ and $\gamma_0 = \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda_0}$, we then have

$$\kappa_1(C_0, \bar{\lambda}, s) := \min\left\{ \frac{\lambda_0}{16M + \lambda_0 + 4\gamma_0 + 2}, \frac{2\bar{\lambda}\mu(\bar{\lambda}, m+n+1)}{\mu(\bar{\lambda}, m+n+2)}, -\frac{\ln(M(1+\gamma_0))}{M\gamma_0 + \gamma_1}, \frac{\ln 2 - M(16+\gamma_0)}{\gamma_1 + 16\gamma_0} \right\} > 0.$$

Taking $T < \kappa_1(C_0, \bar{\lambda}, s)$, conditions a) and c) of Theorem 2.5 are trivially satisfied, condition b) is satisfied (with equality) for $K := 16M + \lambda_0 + 4\gamma_0 + 1$, and conditions d) and e) hold because of the estimates in Lemma 2.4 ii).

The steps of the proof of Theorem 2.5 are the following: first we will establish in Section 2.3 the existence of an analytic solution to a suitable linear version of (VFP) in a small time interval, along with useful estimates. Then, under additional constraints we construct in Section 2.4 a solution to the nonlinear equation (VFP) by means of a fixed point argument.

Before proceeding, let us prove Corollary 2.6:

Proof of Corollary 2.6. Periodicity of the solution is an easy consequence of the fixed point method employed in the proof of Theorem 2.5 (see remark 2.16 in Section 2.4).

Now, thanks to the assumptions on ω and the fact that $f(t,x,u)\omega(u)$ belongs to $\mathcal{H}(\lambda(t))$ for each $t \in [0,T]$, the assumptions of Lemma 1.3 are satisfied (in particular the integrals $\int_{\mathbb{R}} u\partial_u^2 f(t,x,u)du = \int_{\mathbb{R}} \partial_u f(t,x,u)du = \int_{\mathbb{R}} \partial_u^2 f(t,x,u)du$ exist and vanish; moreover, we have $\int_{\mathbb{R}} u\partial_u f(t,x,u)du = -\int_{\mathbb{R}} f(t,x,u)du$). Therefore, thanks to Lemma 1.3 b), the functions $\overline{\rho}(t,x) := \rho(t,x) - 1 = \int_{\mathbb{R}} f(t,x,u)du - 1$, $V(t,x) := \int_{\mathbb{R}} uf(t,x,u)\,du$ and $P(t,x) = -\int_{\mathbb{R}} u^2 f(t,x,u)\,du$ satisfy the following system of equations: for all $(t,x) \in (0,T] \times \mathbb{R}$,

$$\begin{cases} \partial_t \overline{\rho}(t,x) = -\partial_x V(t,x), \\ \partial_t(\partial_x V(t,x)) = -\partial_x(\overline{\rho}(t,x)\partial_x P(t,x)). \end{cases}$$

From the latter and from Lemma 2.3-(iii), we obtain, for each $\lambda \in [0,\lambda_0)$,

$$\begin{cases} \partial_t \|\overline{\rho}(t)\|_\lambda \le \|\partial_x V(t)\|_\lambda, \\ \partial_t \|\partial_x V(t)\|_\lambda \le \|\partial_x P(t)\|_\lambda \|\partial_x \overline{\rho}(t)\|_\lambda + \|\partial_x^2 P(t)\|_\lambda \|\overline{\rho}(t)\|_\lambda. \end{cases} \tag{2.6}$$

Since $\|\partial_x \overline{\rho}(t)\|_\lambda = \frac{d}{d\lambda}\|\overline{\rho}(t)\|_\lambda$ by Lemma 2.3-(ii), (2.6) rewrites as

$$\begin{cases} \partial_t A(t,\lambda) \le B(t,\lambda), \\ \partial_t B(t,\lambda) \le \|\partial_x P(t)\|_\lambda \partial_\lambda A(t,\lambda) + \|\partial_x^2 P(t))\|_\lambda A(t,\lambda), \end{cases}$$

for $A(t,\lambda) := \|\overline{\rho}(t)\|_\lambda$ and $B(t,\lambda) := \|\partial_x V(t)\|_\lambda$. Since $t \mapsto \lambda(t)$ is decreasing and, by Theorem 2.5, $P \in \mathcal{B}^M_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^M_{\lambda_0,K,T}$, we have

$$\begin{aligned} \|\partial_x P(t)\|_{\lambda(T)} &\le \|\partial_x P(t)\|_{\lambda(t)} \le \max_{s\in[0,T]} \|P(s)\|_{\mathcal{H},\lambda(s)} \le M, \\ \|\partial_x^2 P(t)\|_{\lambda(T)} &\le \|\partial_x^2 P(t)\|_{\lambda(t)} \le 4 \max_{s\in[0,T]} \|P(s)\|_{\mathcal{H},\lambda(s)} \le 4M. \end{aligned}$$

We deduce that for all $\lambda \in [0,\lambda(T)]$, $t \in [0,T]$,

$$\begin{cases} \partial_t A(t,\lambda) \le B(t,\lambda), \\ \partial_t B(t,\lambda) \le M\partial_\lambda A(t,\lambda) + 4MA(t,\lambda) \end{cases} \tag{2.7}$$

because $\partial_\lambda A(t,\lambda) \ge 0$. Now set $\mathcal{Y}(t,\lambda) := A(t,\lambda) + bB(t,\lambda)$ where b is a positive constant that we will specify later. Since also $\partial_\lambda B(t,\lambda) \ge 0$, from (2.7) we obtain

$$\partial_t \mathcal{Y}(t,\lambda) \le B(t,\lambda) + 4bMA(t,\lambda) + bM\partial_\lambda A(t,\lambda) \le \left(\frac{1}{b} \vee 4bM\right)\mathcal{Y}(t,\lambda) + bM\partial_\lambda \mathcal{Y}(t,\lambda).$$

That is, with $b_2 := bM > 0$ and $b_1 := \left(\frac{1}{b} \vee 4bM\right) > 0$, it holds that

$$\partial_t \mathcal{Y}(t,\lambda) \le b_1 \mathcal{Y}(t,\lambda) + b_2 \partial_\lambda \mathcal{Y}(t,\lambda), \ \forall t \in [0,T], \ \forall \lambda \in [0,\lambda(T)).$$

We now observe that the function $t \mapsto \mathcal{Y}(t,\gamma(t))$ with $\gamma(t) := \lambda(T) - b_2 t$ is constant for all $t \in [0, \frac{\lambda(T)}{b_2})$. Indeed, we have

$$\partial_t(\mathcal{Y}(t,\gamma(t))) = (\partial_t \mathcal{Y})(t,\gamma(t)) - b_2 \partial_\lambda \mathcal{Y}(t,\gamma(t)) \le b_1 \mathcal{Y}(t,\gamma(t)),$$

and Gronwall's lemma, together with assumption $(\mathrm{H}_{\mathrm{unif}(0)})$ implying that $\mathcal{Y}(0,\lambda) = 0$ for all non negative λ, yield $\mathcal{Y}(t,\gamma(t)) = 0$ for all $t \in [0, \frac{\lambda(T)}{b_2})$. This shows that $\overline{\rho}(t,x) = |\partial_x V(t,x)| = 0$ for all $t \in [0, \frac{\lambda(T)}{b_2})$. Choosing $b = \lambda(T)/(MT)$, we conclude the result, using also the uniform bounds available up to time $t = T$. □

2.3 The linearized equation

Consider the linear equation

$$\begin{cases} \partial_t g + u\partial_x g - (\partial_x Q - \partial_u(\ln\omega))\,\partial_u g - \dfrac{\sigma^2}{2}\partial_u^2 g = g\partial_x Q\partial_u \ln\omega + gh \text{ on } (0,T)\times\mathbb{R}^2, \\ g(0,x,u) = g_0(x,u) := \omega(u) f_0(x,u) \text{ on } \mathbb{R}^2, \end{cases} \tag{FPω}$$

where $Q : [0,T]\times\mathbb{R} \to \mathbb{R}$ is a given function, with uniformly in $t \in [0,T]$ bounded derivatives of all order in $x \in \mathbb{R}$. Equation (FPω) is easily seen to be equivalent, through the relation (2.3), to the linear version of (VFP):

$$\begin{cases} \partial_t f + u\partial_x f - \partial_x Q\partial_u f - \dfrac{\sigma^2}{2}\partial_u^2 f = 0 \text{ on } (0,T)\times\mathbb{R}^2 \\ f(0,x,u) = f_0(x,u) \text{ on } \mathbb{R}^2. \end{cases} \tag{FP}$$

Existence and uniqueness of a $\mathcal{C}^\infty$-solution to the two previous equations is recalled in Theorem A.1 in Appendix A.2. We next prove that the solution g to (FPω) is indeed analytic whenever the inputs g_0 and Q have small enough analytic norms and the time horizon $T > 0$ is small enough:

Theorem 2.8. *Assume that for some $\lambda_0 > 0$ condition* (H_ω) *holds, and that $g_0 : \mathbb{R}^2 \to \mathbb{R}$ is a function of class $\mathcal{C}^\infty$ such that $\|g_0\|_{\mathcal{H},\lambda_0} < +\infty$. For γ_0 and γ_1 as in Theorem 2.5, let $T > 0$ and $M_1>0$ be a time horizon and a constant satisfying*

a) $T < \frac{\lambda_0}{2+\lambda_0+4\gamma_0}$ and

b) $M_1 \leq \frac{1}{16}(K - \lambda_0 - 4\gamma_0 - 1)$ for some K in the nonempty set $(1+\lambda_0+4\gamma_0, \frac{\lambda_0}{T} - 1)$.

Then, for any $M_2 > 0$ and $Q \in \mathcal{B}^{M_1}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{M_2}_{\lambda_0,K,T}$, equation (FP$\omega$) has a solution g of class $\mathcal{C}^{1,\infty}$ such that

$$g \in \mathcal{B}^{\hat{M}}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{\hat{M}}_{\lambda_0,K,T}$$

where $\hat{M} = \|g_0\|_{\mathcal{H},\lambda_0} \exp\{T(\gamma_1 + 16\gamma_0) + (16+\gamma_0)M_2\}$.

In the proof we need to deal with truncated versions of the analytic norms previously introduced. For an arbitrary function ψ of class $\mathcal{C}^\infty$ and a fixed $A \in \mathbb{N}$, set

$$\mathbb{A} := \{0,\cdots,A\}, \qquad \|\psi\|_{\lambda;A} := \sum_{k,l\in\mathbb{A}} \frac{\lambda^{k+l}}{k!l!}\left\|\partial_x^k\partial_u^l\psi\right\|_\infty,$$

$$\|\psi\|_{\lambda,a;A} := \frac{d^a}{d\lambda^a}\|\psi\|_{\lambda;A} = \sum_{k,l\in\mathbb{A};k+l\geq a} \frac{(k+l)!}{(k+l-a)!}\frac{\lambda^{k+l-a}}{k!l!}\left\|\partial_x^k\partial_u^l\psi\right\|_\infty,$$

$$\|\psi\|_{\mathcal{H},\lambda;A} := \sum_{a\in\mathbb{A}} \frac{1}{(a!)^2}\|\psi\|_{\lambda,a;A}, \qquad \|\psi\|_{\widetilde{\mathcal{H}},\lambda;A} := \sum_{a\in\mathbb{A}} \frac{a^2}{(a!)^2}\|\psi\|_{\lambda,a;A}.$$

Using a maximum principle for kinetic Fokker-Planck equation, stated in Appendix A.2, we start the proof by establishing estimates for the time evolution of the norms $\|g(t)\|_{\mathcal{H},\lambda(t);A}$ and $\|g(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A}$ along a solution g of the linear equation (FPω), in terms of $\|Q(t)\|_{\mathcal{H},\lambda(t)}$, $\|\partial_u\ln(\omega)\|_{\mathcal{H},\lambda(t)}$, $\|h\|_{\widetilde{\mathcal{H}},\lambda(t)}$ and $\|Q(t)\|_{\widetilde{\mathcal{H}},\lambda(t)}$.

2.3.1 Regularity estimates

Let g be a smooth solution to (FPω). Observe that, for all $(t,x,u) \in [0,T] \times \mathbb{R}^2$,

$$\partial_x^k \partial_u^l (u \partial_x g(t,x,u)) = \sum_{n=0}^{l} C_l^n (\partial_u^n u)(\partial_x^{k+1} \partial_u^{l-n} g(t,x,u)) = u \partial_x^{k+1} \partial_u^l g(t,x,u) + \mathbb{1}_{\{l\geq 1\}} l \partial_x^{k+1} \partial_u^{l-1} g(t,x,u),$$

$$\begin{aligned}
\partial_x^k \partial_u^l (\partial_x Q(t,x) \partial_u g(t,x,u)) &= \sum_{m=0}^{k} C_k^m (\partial_x^{m+1} Q(t,x))(\partial_x^{k-m} \partial_u^{l+1} g(t,x,u)) \\
&= \partial_x Q(t,x) \partial_x^k \partial_u^{l+1} g(t,x,u) + \mathbb{1}_{\{k\geq 1\}} \sum_{m=0}^{k-1} C_k^m \partial_x^{k-m+1} Q(t,x) \partial_x^m \partial_u^{l+1} g(t,x,u),
\end{aligned}$$

$$\begin{aligned}
\partial_x^k \partial_u^l \left(\partial_u \ln(\omega(u)) \partial_u g(t,x,u)\right) &= \sum_{n=0}^{l} C_l^n \partial_u^{l-n+1} \ln(\omega(u)) \partial_u^{n+1} \partial_x^k g(t,x,u) \\
&= \partial_u \ln \omega(u) \partial_u^{l+1} \partial_x^k g(t,x,u) + \mathbb{1}_{\{l\geq 1\}} \sum_{n=0}^{l-1} C_l^n \, \partial_u^{l-n+1} \ln \omega(u) \, \partial_u^{n+1} \partial_x^k g(t,x,u), \\
\partial_x^k \partial_u^l \left(\partial_x Q(t,x) \partial_u \ln(\omega(u)) g(t,x,u)\right) &= \sum_{n=0}^{l} \sum_{m=0}^{k} C_l^n C_k^m \partial_x^{k-m+1} Q(t,x) \, \partial_u^{l-n+1} \ln \omega(u) \, \partial_x^m \partial_u^n g(t,x,u), \\
\partial_x^k \partial_u^l (g(t,x,u) h(u)) &= \sum_{n=0}^{l} C_l^n \partial_x^k \partial_u^n g(t,x,u) \, \partial_u^{l-n} h(u).
\end{aligned}$$

By applying the differential operator $\partial_x^k \partial_u^l$ to (FPω), we deduce,that

$$\begin{aligned}
&\partial_t (\partial_x^k \partial_u^l g) + u \partial_x (\partial_x^k \partial_u^l g) - (\partial_x Q - \partial_u \ln \omega) \, \partial_u (\partial_x^k \partial_u^l g) - \frac{\sigma^2}{2} \partial_u^2 (\partial_x^k \partial_u^l g) \\
&= -l \partial_x^{k+1} \partial_u^{l-1} g \mathbb{1}_{\{l\geq 1\}} + \mathbb{1}_{\{k\geq 1\}} \sum_{m=0}^{k-1} C_k^m \partial_x^{k-m} \partial_x Q \, \partial_x^m \partial_u^{l+1} g - \mathbb{1}_{\{l\geq 1\}} \sum_{n=0}^{l-1} C_l^n \partial_u^{l-n+1} \ln \omega \partial_u^{n+1} \partial_x^k g \\
&\quad + \sum_{n=0}^{l} C_l^n \partial_x^k \partial_u^n g \, \partial_u^{l-n} h + \sum_{n=0}^{l} \sum_{m=0}^{k} C_l^n C_k^m \partial_x^{k-m+1} Q \partial_u^{l-n+1} \ln \omega \partial_x^m \partial_u^n g.
\end{aligned}$$

The function $\partial_x^k \partial_u^l g$ is thus a classical solution to a linear Fokker-Planck equation. Applying the maximum principle stated in Theorem A.1 in the appendix section A.2, it follows that

$$\begin{aligned}
\frac{d}{dt} \|\partial_x^k \partial_u^l g(t)\|_\infty \leq\ & \mathbb{1}_{\{l\geq 1\}} l \|\partial_x^{k+1} \partial_u^{l-1} g(t)\|_\infty + \mathbb{1}_{\{k\geq 1\}} \sum_{m=0}^{k-1} C_k^m \|\partial_x^m \partial_u^{l+1} g(t)\|_\infty \|\partial_x^{k-m+1} Q(t)\|_\infty \\
&+ \mathbb{1}_{\{l\geq 1\}} \sum_{n=0}^{l-1} C_l^n \|\partial_u^{n+1} \partial_x^k g(t)\|_\infty \|\partial_u^{l-n+1} \ln(\omega)\|_\infty + \sum_{n=0}^{l} C_l^n \|\partial_x^k \partial_u^n g(t)\|_\infty \|\partial_u^{l-n} h\|_\infty \\
&+ \sum_{n=0}^{l} \sum_{m=0}^{k} C_l^n C_k^m \|\partial_x^m \partial_u^n g(t)\|_\infty \|\partial_x^{k-m+1} Q(t)\|_\infty \|\partial_u^{l-n+1} \ln(\omega)\|_\infty.
\end{aligned} \tag{2.10}$$

We now obtain estimates for the function $t \mapsto \|g(t)\|_{\lambda,a;A}$ for fixed $\lambda > 0$ and $A \in \mathbb{N}$.

Lemma 2.9. *For each $A \in \mathbb{N}$, $a \in \mathbb{A} = \{0, ..., A\}$ and $\lambda > 0$, a smooth solution g to (FPω) satisfies:*

$$\begin{aligned}
\frac{d}{dt} \|g(t)\|_{\lambda,a;A} \leq\ & \lambda \|g(t)\|_{\lambda,a+1;A} + a \|g(t)\|_{\lambda,a;A} + \frac{d^a}{d\lambda^a} \Big(\|g(t)\|_{\lambda,1;A} \big\{ \|Q(t)\|_{\lambda,1;A} + \|\ln(\omega)\|_{\lambda,1;A} \big\} \Big) \\
&+ \frac{d^a}{d\lambda^a} \Big(\|g(t)\|_{\lambda,0;A} \big\{ \|h\|_{\lambda,0} + \|Q(t)\|_{\lambda,1;A} \|\ln(\omega)\|_{\lambda,1;A} \big\} \Big).
\end{aligned}$$

Proof. Multiplying both sides of the inequality (2.10) by $\frac{d^a}{d\lambda^a}\frac{\lambda^{k+l}}{k!l!} = \frac{(k+l)!\lambda^{k+l-a}}{(k+l-a)!k!l!}\mathbb{1}_{\{k+l\geq a\}}$ and summing over $k, l \in \mathbb{A}$ with $k+l \geq a$, we get

$$\begin{aligned}
&\frac{d}{dt}\|g(t)\|_{\lambda,a;A} = \sum_{k,l\in\mathbb{A}:k+l\geq a}\frac{d^a}{d\lambda^a}\frac{\lambda^{k+l}}{k!l!}\frac{d}{dt}\|\partial_x^k\partial_u^l g(t)\|_\infty \\
&\leq \sum_{k,l\in\mathbb{A}:k+l\geq a,l\geq 1}\frac{d^a}{d\lambda^a}\frac{l\lambda^{k+l}}{k!l!}\|\partial_x^{k+1}\partial_u^{l-1}g(t)\|_\infty + \sum_{k,l\in\mathbb{A}:k+l\geq a,k\geq 1}\frac{d^a}{d\lambda^a}\frac{\lambda^{k+l}}{k!l!}\sum_{m=0}^{k-1}C_k^m\|\partial_x^m\partial_u^{l+1}g(t)\|_\infty\|\partial_x^{k-m+1}Q(t)\|_\infty \\
&\quad + \sum_{k,l\in\mathbb{A}:k+l\geq a,l\geq 1}\frac{d^a}{d\lambda^a}\frac{\lambda^{k+l}}{k!l!}\sum_{n=0}^{l-1}C_l^n\|\partial_u^{n+1}\partial_x^k g(t)\|_\infty\|\partial_u^{l-n+1}\ln(\omega)\|_\infty \\
&\quad + \sum_{k,l\in\mathbb{A}:k+l\geq a}\frac{d^a}{d\lambda^a}\frac{\lambda^{k+l}}{k!l!}\sum_{n=0}^{l}C_l^n\|\partial_x^k\partial_u^n g(t)\|_\infty\|\partial_u^{l-n}h\|_\infty \\
&\quad + \sum_{k,l\in\mathbb{A}:k+l\geq a}\frac{d^a}{d\lambda^a}\frac{\lambda^{k+l}}{k!l!}\sum_{n=0}^{l}\sum_{m=0}^{k}C_l^nC_k^m\|\partial_x^m\partial_u^n g(t)\|_\infty\|\partial_x^{k-m+1}Q(t)\|_\infty\|\partial_u^{l-n+1}\ln(\omega)\|_\infty.
\end{aligned} \tag{2.11}$$

To bound from above the first sum on the r.h.s. of (2.11) we observe that

$$\sum_{k,l\in\mathbb{A};k+l\geq a,l\geq 1}\frac{d^a}{d\lambda^a}\frac{l\lambda^{k+l}}{k!l!}\|\partial_x^{k+1}\partial_u^{l-1}g(t)\|_\infty = \frac{d^a}{d\lambda^a}\sum_{k,l\in\mathbb{A};l\geq 1}\frac{l\lambda^{k+l}}{k!l!}\|\partial_x^{k+1}\partial_u^{l-1}g(t)\|_\infty$$

with

$$\begin{aligned}
\sum_{k,l\in\mathbb{A}:l\geq 1}\frac{l\lambda^{k+l}}{k!l!}\|\partial_x^{k+1}\partial_u^{l-1}g(t)\|_\infty &= \sum_{k,l\in\mathbb{A}:l\geq 1}\frac{\lambda^{k+l}}{k!(l-1)!}\|\partial_x^{k+1}\partial_u^{l-1}g(t)\|_\infty \\
&= \sum_{k,l\in\mathbb{A}}\frac{\lambda^{k+l+1}}{k!l!}\|\partial_x^{k+1}\partial_u^{l}g(t)\|_\infty = \lambda\|\partial_x g(t)\|_{\lambda,0;A}.
\end{aligned}$$

Since

$$\begin{aligned}
\frac{d^a}{d\lambda^a}\left(\lambda\|\partial_x g(t)\|_{\lambda,0;A}\right) = \sum_{r=0}^{a}C_a^r\left(\frac{d^r}{d\lambda^r}\lambda\right)\left(\frac{d^{a-r}}{d\lambda^{a-r}}\|\partial_x g(t)\|_{\lambda,0;A}\right) &= C_a^a\lambda\|\partial_x g(t)\|_{\lambda,a;A} + C_a^{a-1}\|\partial_x g(t)\|_{\lambda,a-1;A} \\
&= \lambda\|\partial_x g(t)\|_{\lambda,a;A} + a\|\partial_x g(t)\|_{\lambda,a-1;A},
\end{aligned}$$

it follows that

$$\sum_{k,l\in\mathbb{A}:k+l\geq a,l\geq 1}\frac{d^a}{d\lambda^a}\frac{l\lambda^{k+l}}{k!l!}\|\partial_x^{k+1}\partial_u^{l-1}g(t)\|_\infty = \lambda\|g(t)\|_{\lambda,a+1;A} + a\|g(t)\|_{\lambda,a;A}.$$

For the second sum, we notice that

$$\begin{aligned}
\sum_{k,l\in\mathbb{A}:k\geq 1}\frac{\lambda^{k+l}}{k!l!}\sum_{m=0}^{k-1}C_k^m\|\partial_x^{k-m+1}Q(t)\|_\infty\|\partial_x^m\partial_u^{l+1}g(t)\|_\infty &= \sum_{m,l\in\mathbb{A}}\|\partial_x^m\partial_u^{l+1}g(t)\|_\infty\left(\sum_{k=m+1}^{A}\frac{C_k^m\lambda^{k+l}}{k!l!}\|\partial_x^{k-m+1}Q(t)\|_\infty\right) \\
&= \sum_{m,l\in\mathbb{A}}\frac{\lambda^{m+l}}{m!l!}\|\partial_x^m\partial_u^{l+1}g(t)\|_\infty\left(\sum_{k=m+1}^{A}\frac{\lambda^{k-m}}{(k-m)!}\|\partial_x^{k-m+1}Q(t)\|_\infty\right) \\
&= \sum_{m,l\in\mathbb{A}}\frac{\lambda^{m+l}}{m!l!}\|\partial_x^m\partial_u^{l+1}g(t)\|_\infty\left(\sum_{k=1}^{A-m}\frac{\lambda^{k}}{k!}\|\partial_x^{k+1}Q(t)\|_\infty\right) \\
&= \sum_{m,l\in\mathbb{A}}\frac{\lambda^{m+l}}{m!l!}\|\partial_x^m\partial_u^{l+1}g(t)\|_\infty\|Q(t)\|_{\lambda,1;A-m}.
\end{aligned}$$

Taking the a-th derivative with respect to λ, and noting that $\sum_{m,l\in\mathbb{A}} \frac{\lambda^{m+l}}{m!l!}\|\partial_x^m\partial_u^{l+1}g(t)\|_\infty = \|\partial_u g(t)\|_{\lambda,0;A} = \|g(t)\|_{\lambda,1;A} - \|\partial_x g(t)\|_{\lambda,0;A}$ (by similar computations as proof of Lemma 2.3-(i)), we deduce that

$$\frac{d^a}{d\lambda^a}\left(\sum_{k,l\in\mathbb{A}:k\geq 1}\frac{\lambda^{k+l}}{k!l!}\sum_{m=0}^{k-1} C_k^m\|\partial_x^{k-m+1}Q(t)\|_\infty\|\partial_x^m\partial_u^{l+1}g(t)\|_\infty\right) \leq \frac{d^a}{d\lambda^a}\Big(\|Q(t)\|_{\lambda,1;A}\|g(t)\|_{\lambda,1;A}\Big)$$

using also the fact that $\frac{d^b}{d\lambda^b}\|\partial_x g(t)\|_{\lambda,0;A} \geq 0$ and $\|Q(t)\|_{\lambda,a-b+1;A-m} \leq \|Q(t)\|_{\lambda,a-b+1;A}$ for all $b \in \{0,\dots,a\}$. In the same way, we obtain the estimate

$$\frac{d^a}{d\lambda^a}\left(\sum_{k,l\in\mathbb{A}}\frac{\lambda^{k+l}}{k!l!}\sum_{n=0}^{l-1} C_l^n\|\partial_u^{l-n+1}\ln(\omega)\|_\infty\|\partial_u^{n+1}\partial_x^k g(t)\|_\infty\right) \leq \frac{d^a}{d\lambda^a}\Big(\|\ln(\omega)\|_{\lambda,1;A}\|g(t)\|_{\lambda,1;A}\Big).$$

For the fourth sum, one can directly check that

$$\begin{aligned}
\sum_{k,l\in\mathbb{A}}\frac{\lambda^{k+l}}{k!l!}\sum_{n=0}^{l} C_l^n\|\partial_x^k\partial_u^n g(t)\|_\infty\|\partial_u^{l-n}h\|_\infty &= \sum_{k,n\in\mathbb{A}}\|\partial_x^k\partial_u^n g(t)\|_\infty\sum_{l=n}^{A}\frac{C_l^n\lambda^{k+l}}{k!l!}\|\partial_u^{l-n}h\|_\infty \\
&= \sum_{k,n\in\mathbb{A}}\|\partial_x^k\partial_u^n g(t)\|_\infty\sum_{l=0}^{A}\frac{C_{l+n}^n\lambda^{k+l+n}}{k!(l+n)!}\|\partial_u^{l}h\|_\infty \\
&= \sum_{k,n\in\mathbb{A}}\frac{\lambda^{k+n}}{k!n!}\|\partial_x^k\partial_u^n g(t)\|_\infty\sum_{l=0}^{A}\frac{\lambda^l}{l!}\|\partial_u^{l-n}h\|_\infty \\
&= \|h\|_{\lambda,0;A}\|g(t)\|_{\lambda,0;A},
\end{aligned}$$

so that

$$\frac{d^a}{d\lambda^a}\left(\sum_{k,l\in\mathbb{A}}\frac{\lambda^{k+l}}{k!l!}\sum_{n=0}^{l} C_l^n\|\partial_x^k\partial_u^n g(t)\|_\infty\|\partial_u^{l-n}h\|_\infty\right) \leq \frac{d^a}{d\lambda^a}\left(\|h\|_{\lambda,0;A}\|g(t)\|_{\lambda,0;A}\right).$$

Finally, since

$$\begin{aligned}
&\sum_{k,l\in\mathbb{A}}\frac{\lambda^{k+l}}{k!l!}\sum_{n=0}^{l}\sum_{m=0}^{k} C_l^nC_k^m\|\partial_x^{k-m+1}Q(t)\|_\infty\|\partial_u^{l-n+1}\ln(\omega)\|_\infty\|\partial_x^m\partial_u^n g(t)\|_\infty \\
&= \sum_{m,n\in\mathbb{A}}\frac{\lambda^{m+n}}{m!n!}\|\partial_x^m\partial_u^n g(t)\|_\infty\left(\sum_{k=m}^{A}\frac{\lambda^{k-m}}{(k-m)!}\|\partial_x^{k-m+1}Q(t)\|_\infty\right)\left(\sum_{l=n}^{A}\frac{\lambda^{l-n}}{(l-n)!}\|\partial_u^{l-n+1}\ln(\omega)\|_\infty\right) \\
&\leq \|g(t)\|_{\lambda,0;A}\|Q(t)\|_{\lambda,1;A}\|\ln(\omega)\|_{\lambda,1;A},
\end{aligned}$$

the last sum in is bounded from above by

$$\frac{d^a}{d\lambda^a}\Big(\|g(t)\|_{\lambda,1;A}\left(\|Q(t)\|_{\lambda,1;A} + \|\ln(\omega)\|_{\lambda,1;A}\right)\Big).$$

Coming back to (2.11), the above estimates prove Lemma 2.9. □

2.3.2 Evolution and control of the time-inhomogeneous analytic norms

Next Lemmas 2.10 and 2.11 are preliminaries for the bounds of the time derivative of $\|g(t)\|_{\mathcal{H},\lambda(t);A}$ in Proposition 2.12 below. Their proof is given in Appendix A.3.

Lemma 2.10. *Let f, v, w be functions of class $\mathcal{C}^\infty$ with bounded derivatives at all order. Then, for all $\lambda > 0$ and $A \in \mathbb{N}$,*

$$\sum_{a\in\mathbb{A}}\frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|f\|_{\lambda,0;A}\|v\|_{\lambda,1;A}\|w\|_{\lambda,1;A}\right) \leq \|f\|_{\mathcal{H},\lambda;A}\|v\|_{\widetilde{\mathcal{H}},\lambda;A}\|w\|_{\widetilde{\mathcal{H}},\lambda;A}. \tag{2.12}$$

Suppose moreover that for some $\bar\lambda > 0$, one has $f \in \mathcal{H}(\bar\lambda)$ and $v, w \in \widetilde{\mathcal{H}}(\bar\lambda)$. Then, for all $\lambda \in [0, \bar\lambda)$,

$$\sum_{a\in\mathbb{N}} \frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|f\|_{\lambda,0}\|v\|_{\lambda,1}\|w\|_{\lambda,1}\right) \le \|f\|_{\mathcal{H},\lambda}\|v\|_{\widetilde{\mathcal{H}},\lambda}\|w\|_{\widetilde{\mathcal{H}},\lambda}.$$

Lemma 2.11. *Let f, w be functions of class $\mathcal{C}^\infty$ with bounded derivatives at all order.*

(i) For all $\lambda > 0$ and $A \in \mathbb{N}$, one has

$$\sum_{a\in\mathbb{A}} \frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|f\|_{\lambda,1;A}\|v\|_{\lambda,1;A}\right) \le 16(\|f\|_{\mathcal{H},\lambda;A}\|v\|_{\widetilde{\mathcal{H}},\lambda;A} + \|f\|_{\widetilde{\mathcal{H}},\lambda;A}\|v\|_{\mathcal{H},\lambda;A}). \tag{2.13}$$

Moreover if for some $\bar\lambda > 0$ we have $f, v \in \mathcal{H}(\bar\lambda) \cap \widetilde{\mathcal{H}}(\bar\lambda)$ then, for all $\lambda \in [0, \bar\lambda)$

$$\sum_{a\in\mathbb{N}} \frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|f\|_{\lambda,1}\|v\|_{\lambda,1}\right) \le 16(\|f\|_{\mathcal{H},\lambda}\|v\|_{\widetilde{\mathcal{H}},\lambda} + \|f\|_{\widetilde{\mathcal{H}},\lambda}\|v\|_{\mathcal{H},\lambda}).$$

(ii) For all $\lambda > 0$ and $A \in \mathbb{N}$, one has

$$\sum_{a\in\mathbb{A}} \frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|f\|_{\lambda,1;A}\|v\|_{\lambda,1;A}\right) \le 4\|v\|_{\widetilde{\mathcal{H}},\lambda;A}(4\|f\|_{\mathcal{H},\lambda;A} + \|f\|_{\widetilde{\mathcal{H}},\lambda;A}). \tag{2.14}$$

Moreover for some $\bar\lambda > 0$, $f \in \mathcal{H}(\bar\lambda) \cap \widetilde{\mathcal{H}}(\bar\lambda)$ and $v \in \widetilde{\mathcal{H}}(\bar\lambda)$, for all $\lambda \in [0, \bar\lambda)$,

$$\sum_{a\in\mathbb{N}} \frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|f\|_{\lambda,1}\|v\|_{\lambda,1}\right) \le 4\|v\|_{\widetilde{\mathcal{H}},\lambda}(4\|f\|_{\mathcal{H},\lambda} + \|f\|_{\widetilde{\mathcal{H}},\lambda}).$$

Proposition 2.12. *For each $A \in \mathbb{N}$, the $\mathcal{C}^{1,\infty}$ function g solution to (FPω) satisfies*

$$\begin{aligned}\frac{d}{dt}\|g(t)\|_{\mathcal{H},\lambda(t);A} \le{}& \big(\lambda(t) + 1 + \lambda'(t) + 4\gamma_0 + 16\|Q(t)\|_{\mathcal{H},\lambda(t)}\big)\,\|g(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A}\\ &+ \Big(\gamma_1 + 16\gamma_0 + (\gamma_0 + 16)\,\|Q(t)\|_{\widetilde{\mathcal{H}},\lambda(t)}\Big)\,\|g(t)\|_{\mathcal{H},\lambda(t);A},\end{aligned}$$

where $\gamma_0 := \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda_0}$ and $\gamma_1 := \|h\|_{\mathcal{H},\lambda_0}$.

Proof. Differentiating in time the norm $\|g(t)\|_{\mathcal{H},\lambda(t);A}$, we get

$$\begin{aligned}\frac{d}{dt}\|g(t)\|_{\mathcal{H},\lambda(t);A} &= \sum_{a\in\mathbb{A}} \frac{1}{(a!)^2}\left(\lambda'(t)\frac{d^{a+1}}{d\lambda^{a+1}}\|g(t)\|_{\lambda,0;A}\right) + \sum_{a=0}^{A}\frac{1}{(a!)^2}\left(\frac{d}{dt}\|g(t)\|_A\right)\Big|_{\lambda=\lambda(t)}\\ &= \lambda'(t)\sum_{a=0}^{A}\frac{1}{(a!)^2}\|g(t)\|_{\lambda(t),a+1;A} + \sum_{a=0}^{A}\frac{1}{(a!)^2}\left(\frac{d}{dt}\|g(t)\|_{\lambda,a;A}\right)\Big|_{\lambda=\lambda(t)},\end{aligned}$$

Dividing both sides of the inequality in Lemma 2.9 by $(a!)^2$ and summing the resulting expression over $a \in \mathbb{A}$, it follows that

$$\begin{aligned}\frac{d}{dt}\|g(t)\|_{\mathcal{H},\lambda(t);A} \le{}& \sum_{a\in\mathbb{A}}\frac{\lambda'(t)+\lambda(t)}{(a!)^2}\|g(t)\|_{\lambda(t),a+1;A} + \sum_{a=0}^{A}\frac{a}{(a!)^2}\|g(t)\|_{\lambda(t),a;A}\\ &+ \sum_{a=0}^{A}\frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\Big(\|g(t)\|_{\lambda,0;A}\left(\|h\|_{\lambda,0;A} + \|Q(t)\|_{\lambda,1}\|\ln(\omega)\|_{\lambda,1;A}\right)\Big)\Big|_{\lambda=\lambda(t)}\\ &+ \sum_{a=0}^{A}\frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\Big(\|g(t)\|_{\lambda,1;A}\left(\|Q(t)\|_{\lambda,1;A} + \|\ln(\omega)\|_{\lambda,1;A}\right)\Big)\Big|_{\lambda=\lambda(t)}.\end{aligned} \tag{2.15}$$

For the first term in (2.15), we have

$$\sum_{a=0}^{A}\frac{\lambda'(t)+\lambda(t)}{(a!)^2}\|g(t)\|_{\lambda(t),a+1;A} = (\lambda'(t)+\lambda(t))\sum_{a=0}^{A}\frac{(a+1)^2}{((a+1)!)^2}\|g(t)\|_{\lambda(t),a+1;A}$$
$$= (\lambda'(t)+\lambda(t))\sum_{a=0}^{A}\frac{(a)^2}{((a)!)^2}\|g(t)\|_{\lambda(t),a;A},$$

and, for the second term

$$\sum_{a=0}^{A}\frac{a}{(a!)^2}\|g(t)\|_{\lambda(t),a;A} \le \sum_{a\in\mathbb{A}}\frac{(a)^2}{(a!)^2}\|g(t)\|_{\lambda(t),a;A},$$

so that

$$\sum_{a=0}^{A}\frac{\lambda'(t)+\lambda(t)}{(a!)^2}\|g(t)\|_{\lambda(t),a+1;A} + \sum_{a=0}^{A}\frac{a}{(a!)^2}\|g(t)\|_{\lambda(t),a;A} \le (1+\lambda'(t)+\lambda(t))\,\|g(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A}. \tag{2.16}$$

For the third term, observe that on one hand

$$\sum_{a=0}^{A}\frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|g(t)\|_{\lambda,0;A}\|h\|_{\lambda,0;A}\right) = \sum_{a=0}^{A}\frac{1}{(a!)^2}\sum_{r=0}^{a}C_a^r\left(\frac{d^r}{d\lambda^r}\|g(t)\|_{\lambda,0;A}\right)\left(\frac{d^{a-r}}{d\lambda^{a-r}}\|h\|_{\lambda,0;A}\right)$$
$$= \sum_{a=0}^{A}\frac{1}{(a!)^2}\sum_{r=0}^{a}C_a^r\|g(t)\|_{\lambda,r;A}\|h\|_{\lambda,a-r;A}$$
$$= \sum_{r=0}^{A}\|g(t)\|_{\lambda,r;A}\sum_{a=r}^{A}\|h\|_{\lambda,a-r;A}\frac{C_a^r}{(a!)^2}$$
$$= \sum_{r=0}^{A}\|g(t)\|_{\lambda,r;A}\sum_{a=0}^{A}\|h\|_{\lambda,a;A}\frac{C_{a+r}^r}{((a+r)!)^2}$$
$$= \sum_{r=0}^{A}\frac{\|g(t)\|_{\lambda,r;A}}{(r!)^2}\sum_{a=0}^{A}\frac{\|h\|_{\lambda,a-r;A}}{(a!)^2}\frac{a!r!}{(a+r))!},$$

and since $\frac{a!r!}{(a+r)!}\le 1$, for all $a,r\in\mathbb{N}$, we get that

$$\sum_{a=0}^{A}\frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|g(t)\|_{\lambda,0;A}\|h\|_{\lambda,0;A}\right) \le \|g(t)\|_{\mathcal{H},\lambda;A}\|h\|_{\mathcal{H},\lambda;A}. \tag{2.17}$$

On the other hand, inequality 2.12 provides a bound for the remaining summand in the third term of (2.15):

$$\sum_{a\in\mathbb{A}}\frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|g(t)\|_{\lambda,0;A}\|Q(t)\|_{\lambda,1;A}\|\ln(\omega)\|_{\lambda,1;A}\right) \le \|g(t)\|_{\mathcal{H},\lambda(t);A}\|Q(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A}\|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda(t);A}. \tag{2.18}$$

For the fourth term in (2.15) we use (2.13) and (2.14) in order to get the estimate

$$\begin{aligned}&\sum_{a=0}^{A}\frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|g(t)\|_{\lambda,1;A}\left(\|Q(t)\|_{\lambda,1;A}+\|\ln(\omega)\|_{\lambda,1;A}\right)\right)\\ &\le 16\|g(t)\|_{\mathcal{H},\lambda(t);A}\left(\|Q(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A}+\|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda(t);A}\right)\\ &\quad+4\|g(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A}\left(4\|Q(t)\|_{\mathcal{H},\lambda(t);A}+\|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda(t);A}\right).\end{aligned} \tag{2.19}$$

Inserting (2.16), (2.17), (2.18) and (2.19) in (2.15), we conclude that

$$
\begin{aligned}
\frac{d}{dt}\|g(t)\|_{\mathcal{H},\lambda(t);A} \leq\ & (\lambda'(t)+\lambda(t)+1)\,\|g(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A} + \|g(t)\|_{\mathcal{H},\lambda(t);A}\|h\|_{\mathcal{H},\lambda(t);A} \\
& + \|g(t)\|_{\mathcal{H},\lambda(t);A}\|Q(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A}\|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda(t);A} + 16\|g(t)\|_{\mathcal{H},\lambda(t);A}\left(\|Q(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A} + \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda(t);A}\right) \\
& + \|g(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A}\left(16\|Q(t)\|_{\mathcal{H},\lambda(t);A} + 4\|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda(t);A}\right).
\end{aligned}
$$

We end the proof by using the obvious upper bounds for the truncated norms. □

2.3.3 Proof of Theorem 2.8

Applying Gronwall's lemma to the inequality in Proposition 2.12, we obtain that, for all $t \in [0,T]$ and $A \in \mathbb{N}$,

$$
\begin{aligned}
&\|g(t)\|_{\mathcal{H},\lambda(t);A} \leq \|g_0\|_{\mathcal{H},\lambda_0} \exp\left\{\int_0^t \left(\gamma_1 + 16\gamma_0 + (16+\gamma_0)\|Q(\theta)\|_{\widetilde{\mathcal{H}},\lambda(\theta)}\right) ds\right\} \\
&+ \int_0^t \left(\lambda(\theta)+1+\lambda'(\theta)+4\gamma_0+16\|Q(\theta)\|_{\mathcal{H},\lambda(\theta)}\right)\|g(\theta)\|_{\widetilde{\mathcal{H}},\lambda(\theta);A} \exp\left\{\int_\theta^t \left(\gamma_1+16\gamma_0+(16+\gamma_0)\|Q(\theta')\|_{\widetilde{\mathcal{H}},\lambda(\theta')}\right) d\theta'\right\} d\theta \\
&\leq \|g_0\|_{\mathcal{H},\lambda_0} \exp\left\{T(\gamma_1+16\gamma_0)+(16+\gamma_0)M_2\right\} \\
&\quad + \exp\left\{T(\gamma_1+16\gamma_0)+(16+\gamma_0)M_2\right\}(\lambda_0 - K + 4\gamma_0 + 16M_1)\int_0^t \|g(\theta)\|_{\widetilde{\mathcal{H}},\lambda(\theta);A} d\theta.
\end{aligned}
\tag{2.20}
$$

where in the second inequality we use the facts that $Q \in \mathcal{B}^{M_1}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{M_2}_{\lambda_0,K,T}$ and that

$$
\lambda(t)+1+\lambda'(t)+4\gamma_0+16\|Q(t)\|_{\mathcal{H},\lambda(t)} \leq \lambda_0 - K + 4\gamma_0 + 16\|Q(t)\|_{\mathcal{H},\lambda(t)} \leq \lambda_0 - K + 4\gamma_0 + 16M_1
$$

for all $t \in [0,T]$. From the assumptions we can choose $K > 0$ such that $K < \frac{\lambda_0}{T} - 1$ and

$$
K - \lambda_0 - 4\gamma_0 - 16M_1 \geq 1.
$$

Then we deduce with (2.20) and the latter inequality that

$$
\begin{aligned}
\|g(t)\|_{\mathcal{H},\lambda(t);A} + \int_0^t \|g(\theta)\|_{\widetilde{\mathcal{H}},\lambda(\theta);A} d\theta &\leq \|g(t)\|_{\mathcal{H},\lambda(t);A} + \exp\left\{T(\gamma_1+16\gamma_0)+(16+\gamma_0)M_2\right\}\int_0^t \|g(\theta)\|_{\widetilde{\mathcal{H}},\lambda(\theta);A} d\theta \\
&\leq \|g_0\|_{\mathcal{H},\lambda_0} \exp\left\{T(\gamma_1+16\gamma_0)+(16+\gamma_0)M_2\right\}.
\end{aligned}
$$

After letting $A \to \infty$ we conclude that

$$
\begin{aligned}
\max_{t\in[0,T]} \|g(t)\|_{\mathcal{H},\lambda(t)} &\leq \|g_0\|_{\mathcal{H},\lambda_0} \exp\left\{T(\gamma_1+16\gamma_0)+(16+\gamma_0)M_2\right\}, \\
\int_0^T \|g(s)\|_{\widetilde{\mathcal{H}},\lambda(t)}\, dt &\leq \|g_0\|_{\mathcal{H},\lambda_0} \exp\left\{T(\gamma_1+16\gamma_0)+(16+\gamma_0)M_2\right\}.
\end{aligned}
\tag{2.21}
$$

2.4 Proof of Theorem 2.5 : solving the Vlasov-Fokker-Planck equation (VFP)

Relying upon Theorem 2.8, we construct now, by means of a Banach's fixed point method, a solution to the nonlinear Vlasov-Fokker-Planck equation (VFP).

Remark 2.13. *Since we are assuming in* (H_ω) *that* $\int_{\mathbb{R}} \frac{u^2}{\omega(u)} du = 1$*, for all* $\lambda \geq 0$ *and* $a \in \mathbb{N}$ *it holds that*

$$
\left\| \int_{\mathbb{R}} \frac{u^2}{\omega(u)} \varphi(t,\cdot,u)du \right\|_{\lambda,a} \leq \|\varphi(t,\cdot,\cdot)\|_{\lambda,a}
$$

for any function $\varphi : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ *of class* $\mathcal{C}^{1,\infty}$ *and every* $t \in [0,T]$*. Therefore, if we denote by* Φ *the mapping associating to a function* φ *the solution* $\Phi(\varphi)$ *of the linear equation* (FPω) *with potential* $\partial_x Q(t,x)$ *given by*

$$
Q(t,x) := -\int_{\mathbb{R}} \frac{u^2}{\omega(u)} \varphi(t,x,u)\, du\,,
$$

the inclusion

$$\Phi\left(\mathcal{B}^{M_1}_{\lambda_0,K,T}\cap\widetilde{\mathcal{B}}^{M_2}_{\lambda_0,K,T}\right)\subseteq\mathcal{B}^{\hat{M}}_{\lambda_0,K,T}\cap\widetilde{\mathcal{B}}^{\hat{M}}_{\lambda_0,K,T}$$

holds under the conditions on the constants T,λ_0,K,M_1,M_2 and $\hat{M}$ established in Theorem 2.8.

Corollary 2.14. *If in addition to the assumptions of Theorem 2.8, the constants $M:=M_1$ and $T>0$ satisfy the constraint*

$$\|g_0\|_{\mathcal{H},\lambda_0}\exp(T(\gamma_1+16\gamma_0))\le M\exp(-(16+\gamma_0)M),$$

then $\Phi\left(\mathcal{B}^{M}_{\lambda_0,K,T}\cap\widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T}\right)\subseteq\mathcal{B}^{M}_{\lambda_0,K,T}\cap\widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T}$.

Proof. Taking $M_2=M=M_1$ in Theorem 2.8 we get that $\Phi\left(\mathcal{B}^{M}_{\lambda_0,K,T}\cap\widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T}\right)\subseteq\mathcal{B}^{\hat{M}}_{\lambda_0,K,T}\cap\widetilde{\mathcal{B}}^{\hat{M}}_{\lambda_0,K,T}$ for $\hat{M}=\|g_0\|_{\mathcal{H},\lambda_0}\exp\{T(\gamma_1+16\gamma_0)+(16+\gamma_0)M\}$. The additional constraint ensures that $\hat{M}\le M$. $\square$

Theorem 2.15. *Under the assumptions of Corollary 2.14 and, moreover, that*

$$M(1+\gamma_0)\exp\{(M\gamma_0+\gamma_1)T\}<1 \tag{2.22}$$

the mapping

$$\Phi:\mathcal{B}^{M}_{\lambda_0,K,T}\cap\widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T}\to\mathcal{B}^{M}_{\lambda_0,K,T}\cap\widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T}$$

is well defined and is a contraction for the norm

$$\max\left\{\max_{t\in[0,T]}\|\psi(t)\|_{\lambda(t),0}\,,\int_0^T\|\psi(t)\|_{\lambda(t),1}\,dt\right\}.$$

If in addition to all the previous assumptions, we have

$$\max\{\|g_0\|_{\mathcal{H},\lambda_0},T\|g_0\|_{\widetilde{\mathcal{H}},\lambda_0}\}\le M,$$

then the (constant in time) function $g_0(t,x)=g_0(x)$ satisfies $g_0\in\mathcal{B}^{M}_{\lambda_0,K,T}\cap\widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T}$ and a solution to the nonlinear Vlasov-Fokker-Planck equation (VFPω) exists in $\mathcal{B}^{M}_{\lambda_0,K,T}\cap\widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T}$.

Proof. Given $f_i\in\mathcal{B}^{M}_{\lambda_0,K,T}\cap\widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T}$, $i=1,2$, we set $P_i(t,x):=\int_{\mathbb{R}}\frac{u^2}{\omega(u)}f_i(t,x,u)\,du$ for $i=1,2$. The difference $\Phi(f_1)-\Phi(f_2)$ satisfies

$$\begin{aligned}&\partial_t\left(\Phi(f_1)-\Phi(f_2)\right)+\left(u\partial_x\left(\Phi(f_1)-\Phi(f_2)\right)\right)-\left[\left(\partial_xP_1-\partial_u\ln(\omega)\right)\partial_u\left(\Phi(f_1)-\Phi(f_2)\right)\right]-\frac{1}{2}\partial_u^2\left(\Phi(f_1)-\Phi(f_2)\right)\\&=\partial_u\Phi(f_2)\left(\partial_xP_1-\partial_xP_2\right)+\Phi(f_2)\partial_u\ln(\omega)\left(\partial_xP_1-\partial_xP_2\right)+\left(\partial_u\ln(\omega)\partial_xP_1+h\right)\left(\Phi(f_1)-\Phi(f_2)\right).\end{aligned}$$

Writing $\bar{\Phi}:=\Phi(f_1)-\Phi(f_2)$ and $\bar{P}:=P_1-P_2$, we get

$$\begin{aligned}&\partial_t\bar{\Phi}+\left(u\partial_x\bar{\Phi}\right)-\left(\left(\partial_xP_1-\partial_u\ln(\omega)\right)\partial_u\hat{\Phi}\right)-\frac{1}{2}\partial_u^2\bar{\Phi}\\&=\left(\Phi(f_2)\partial_u\ln(\omega)+\partial_u\Phi(f_2)\right)\partial_x\bar{P}+\left(\partial_xP_1\partial_u\ln(\omega)+h\right)\bar{\Phi}.\end{aligned}$$

Then, by similar computations as in the proof of Theorem 2.8, we successively obtain:
- by applying the operator $\partial_x^k\partial_u^l$,

$$\begin{aligned}&\partial_t(\partial_x^k\partial_u^l\bar{\Phi})+u\partial_x\left(\partial_x^k\partial_u^l\bar{\Phi}\right)-\left(\partial_xP_1-\partial_u\ln(\omega)\right)\partial_u\left(\partial_x^k\partial_u^l\bar{\Phi}\right)-\frac{1}{2}\partial_u^2(\partial_x^k\partial_u^l\bar{\Phi})\\&=-l\partial_x^{k+1}\partial_u^{l-1}\bar{\Phi}+\mathbb{1}_{\{k\ge1\}}\sum_{m=0}^{k-1}C_k^m\left(\partial_x^{k-m+1}P_1\right)\partial_x^m\partial_u^{l+1}\bar{\Phi}-\mathbb{1}_{\{l\ge1\}}\sum_{n=0}^{l-1}C_l^n\left(\partial_u^{l-n+1}\ln(\omega)\right)\partial_x^k\partial_u^{n+1}\bar{\Phi}\\&\quad+\sum_{n=0}^{l}\sum_{m=0}^{k}C_k^mC_l^n\partial_x^{k-m}\partial_u^n\Phi(f_2)\partial_u^{l-n+1}\ln\omega(u)\partial_x^{m+1}\bar{P}+\sum_{m=0}^{k}C_k^m\left(\partial_x^{k-m}\partial_u^{l+1}\Phi(f_2)\right)\left(\partial_x^{m+1}\bar{P}\right)\\&\quad+\sum_{n=0}^{l}\sum_{m=0}^{k}C_k^mC_l^n\left(\partial_x^{k-m+1}P_1\right)\left(\partial_u^{l-n+1}\ln(\omega)\right)\left(\partial_x^m\partial_u^n\bar{\Phi}\right)+\sum_{n=0}^{l}C_l^n\left(\partial_x^k\partial_u^n\Phi\right)\partial_u^{l-n}h;\end{aligned}$$

• by a maximum principle, and the fact that for all $m \in \mathbb{N}$: $\|\partial^m P_i\|_\infty \leq \|\partial^m f_i\|_\infty$, $i = 1, 2$, and $\|\partial^m \bar{P}\|_\infty \leq \|\partial^m \bar{f}\|_\infty$ for $\bar{f} := f_1 - f_2$, we get

$$\begin{aligned}
&\frac{d}{dt}\|\partial_x^k \partial_u^l \bar{\Phi}(t)\|_\infty \\
&\leq l\|\partial_x^{k+1}\partial_u^{l-1}\bar{\Phi}(t)\|_\infty + \mathbb{1}_{\{k\geq 1\}} \sum_{m=0}^{k-1} C_k^m \|\partial_x^{k-m+1} f_1(t)\|_\infty \|\partial_x^m \partial_u^{l+1}\bar{\Phi}(t)\|_\infty \\
&\quad + \mathbb{1}_{\{l\geq 1\}} \sum_{n=0}^{l-1} C_l^n \|\partial_u^{l-n+1}\ln(\omega)\|_\infty \|\partial_x^k \partial_u^{n+1}\bar{\Phi}(t)\|_\infty \\
&\quad + \sum_{n=0}^{l}\sum_{m=0}^{k} C_k^m C_l^n \|\partial_u^{l-n+1}\ln(\omega)\|_\infty \|\partial_x^{k-m}\partial_u^n \Phi(f_2)(t)\|_\infty \|\partial_x^{m+1}\bar{f}(t)\|_\infty + \sum_{m=0}^{k} C_k^m \|\partial_x^{k-m}\partial_u^{l+1}\Phi(f_2)(t)\|_\infty \|\partial_x^{m+1}\bar{f}(t)\|_\infty \\
&\quad + \sum_{n=0}^{l}\sum_{m=0}^{k} C_k^m C_l^n \|\partial_u^{l-n+1}\ln(\omega)\|_\infty \|\partial_x^{k-m+1} f_1(t)\|_\infty \|\partial_x^m \partial_u^n \bar{\Phi}(t)\|_\infty + \sum_{n=0}^{l} C_l^n \|\partial_x^k \partial_u^n \bar{\Phi}(t)\|_\infty \|\partial_u^{l-n} h\|_\infty.
\end{aligned}$$

• Replicating the computations in the proof of Lemma 2.9 for $a = 0$, $A = +\infty$, we then obtain

$$\begin{aligned}
\frac{d}{dt}\|\bar{\Phi}(t)\|_{\lambda,0} &\leq \lambda\|\bar{\Phi}(t)\|_{\lambda,1} + \|\bar{\Phi}(t)\|_{\lambda,1}\left(\|f_1(t)\|_{\lambda,1} + \|\ln(\omega)\|_{\lambda,1}\right) \\
&\quad + \|\bar{\Phi}(t)\|_{\lambda,0}\left(\|f_1(t)\|_{\lambda,1}\|\ln(\omega)\|_{\lambda,1} + \|h\|_{\lambda,0}\right) \\
&\quad + \|\bar{f}(t)\|_{\lambda,1}\left(\|\Phi(f_2)(t)\|_{\lambda,1} + \|\Phi(f_2)(t)\|_{\lambda,0}\|\ln(\omega)\|_{\lambda,1}\right).
\end{aligned} \tag{2.23}$$

Hence,

$$\begin{aligned}
\frac{d}{dt}\|\bar{\Phi}(t)\|_{\lambda(t),0} &\leq \left(\lambda'(t) + \lambda(t) + \|f_1(t)\|_{\lambda,1} + \|\ln(\omega)\|_{\lambda(t),1}\right)\|\bar{\Phi}(t)\|_{\lambda(t),1} \\
&\quad + \|\bar{\Phi}(t)\|_{\lambda(t),0}\left(\|f_1(t)\|_{\lambda(t),1}\|\ln(\omega)\|_{\lambda(t),1} + \|h\|_{\lambda(t),0}\right) \\
&\quad + \|\bar{f}(t)\|_{\lambda(t),1}\left(\|\Phi(f_2)(t)\|_{\lambda(t),1} + \|\Phi(f_2)(t)\|_{\lambda(t),0}\|\ln(\omega)\|_{\lambda(t),1}\right).
\end{aligned}$$

Since, by our assumptions,

$$\max_{t\in[0,T]}\|f_1(t)\|_{\lambda(t),1}\left(\leq \max_{t\in[0,T]}\|f_1(t)\|_{\mathcal{H},\lambda(t)}\right) \leq M, \quad \text{and} \quad \max_{t\in[0,T]}\|\Phi(f_2(t))\|_{\lambda(t),1} \leq M,$$

we deduce that for all $t \in [0, T]$,

$$\frac{d}{dt}\|\bar{\Phi}(t)\|_{\lambda(t),0} \leq (\lambda_0 - K + M + \gamma_0)\|\bar{\Phi}(t)\|_{\lambda(t),1} + M(1+\gamma_0)\|\bar{\Phi}(t)\|_{\lambda(t),0} + (M\gamma_0 + \gamma_1)\|\bar{f}(t)\|_{\lambda(t),1}$$

thanks also to the upper-bounds $\|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda(t)} \leq \gamma_0 = \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda_0}$, $\|h\|_{\mathcal{H},\lambda(\theta),} \leq \gamma_1 = \|h\|_{\mathcal{H},\lambda_0}$. It follows then by Gronwall's inequality that

$$\|\bar{\Phi}(t)\|_{\lambda(t),0} \leq \exp\left\{T\left(M\gamma_0 + \gamma_1\right)\right\}\int_0^t \left(\|\bar{\Phi}(\theta)\|_{\lambda(\theta),1}(\lambda_0 - K + M + \gamma_0) + \|\bar{f}(\theta)\|_{\lambda(\theta),1}M(1+\gamma_0)\right)d\theta.$$

Observe that the current assumptions of Theorem 2.8 ensure that we can choose $K \in \left(0, \frac{\lambda_0}{T} - 1\right)$ such that

$$K - \lambda_0 - M - \gamma_0 > 1.$$

We thus get from the previous that for each $t \in [0, T]$,

$$\begin{aligned}
\|\bar{\Phi}(t)\|_{\lambda(t),0} + \int_0^t \|\bar{\Phi}(\theta)\|_{\lambda(\theta),1}\,d\theta &\leq \|\bar{\Phi}(t)\|_{\lambda(t),0} + \exp\left\{T\left(M\gamma_0 + \gamma_1\right)\right\}\int_0^t \|\bar{\Phi}(\theta)\|_{\lambda(\theta),1}\,d\theta \\
&\leq M(1+\gamma_0)\exp\left\{(M\gamma_0 + \gamma_1)T\right\}\int_0^T \|\bar{f}(t)\|_{\lambda(t),1}dt.
\end{aligned}$$

In particular,

$$\max_{t\in[0,T]} \|\Phi(f_1)(t)-\Phi(f_2)(t)\|_{\lambda(t),0} \le M(1+\gamma_0)\exp\{(M\gamma_0+\gamma_1)T\}\int_0^T \|f_1(t)-f_2(t)\|_{\lambda(t),1}dt.$$

$$\int_0^T \|\Phi(f_1)(\theta)-\Phi(f_2)(\theta)\|_{\lambda(\theta),1}\,d\theta \le M(1+\gamma_0)\exp\{(M\gamma_0+\gamma_1)T\}\int_0^T \|f_1(t)-f_2(t)\|_{\lambda(t),1}dt.$$

The contractivity property is thus granted by (2.22). □

Proof of Theorem 2.5. Under the assumptions on λ_0, M and T, Theorem 2.15 holds and, moreover, the assumptions on f_0 imply that $g_0 \in \mathcal{B}^M_{\lambda_0,K,T}\cap\widetilde{\mathcal{B}}^M_{\lambda_0,K,T}$. Therefore, by Banach's fixed point theorem the sequence $\Phi^n(g_0)$ converges to a function $g\in\mathcal{B}^M_{\lambda_0,K,T}\cap\widetilde{\mathcal{B}}^M_{\lambda_0,K,T}$ which is a solution of (VFPω). □

Remark 2.16. *If $g_0(x,u)$ is 1-periodic in x, uniqueness of classical solutions to the linear equation (FPω) implies that $\Phi^n(g_0)$ too is 1-periodic in x for each $n\in\mathbb{N}$. Consequently, so is the limit g.*

3 The kinetic potential case

In this section we extend the previous results to the situation $\beta\ge 0$ and $\alpha=0$ (corresponding to the standard kinetic energy potential) or $\alpha=1$ (corresponding to the turbulent kinetic energy). We notice that the same proofs can be applied also to the car $\beta<0$ by replacing in all estimates β by $|\beta|$.

We consider the nonlinear Vlasov-Fokker-Planck equation with additional kinetic potential

$$\begin{cases} \partial_t g + u\partial_x g - [\partial_x P + \beta(u-\alpha V) - \partial_u \ln(\omega)]\,\partial_u g - \dfrac{\sigma^2}{2}\partial_u^2 g = g\,[\partial_x P - \alpha\beta V\partial_u \ln(\omega)] - g\hat{h} \quad \text{on } (0,T]\times\mathbb{R}^2,\\ P(t,x) = -\displaystyle\int_{\mathbb{R}} \frac{u^2}{\omega(u)} g(t,x,u)\,du, \quad V(t,x) = \int_{\mathbb{R}} \frac{u}{\omega(u)} g(t,x,u)du \\ g(0,x,u) = g_0(x,u) \text{ on } \mathbb{R}^2, \end{cases} \tag{VFPωK}$$

where

$$\hat{h}(u) := \frac{\partial_u^2\omega(u)}{2\omega(u)} - |\partial_u \ln(\omega(u))|^2 - \beta - \beta u\partial_u(\ln\omega(u))\,.$$

Through the relation $g(t,x,u)=\omega(u)f(t,x,u)$, equation (VFP$\omega$K) is seen to be equivalent to

$$\begin{cases} \partial_t f + u\partial_x f - (\partial_x P + \beta(u-\alpha V))\,\partial_u f - \beta f - \dfrac{\sigma^2}{2}\partial_u^2 f = 0 \text{ on } (0,T]\times\mathbb{R}^2,\\ P(t,x) = -\displaystyle\int_{\mathbb{R}} u^2 f(t,x,u)du, \quad V(t,x) = \int_{\mathbb{R}} uf(t,x,u)du \\ f(0,x,u) = f_0(x,u) \text{ on } \mathbb{R}^2 \end{cases} \tag{VFPK}$$

(and to equation (1.5) if f_0 and the searched solution are periodic in x). We next prove

Theorem 3.1. *Let M,T be positive constants and $\omega:\mathbb{R}\to(0,+\infty)$ be a function of class $\mathcal{C}^\infty$ satisfying* (H$_\omega$) *for some $\lambda_0>0$ and moreover that $u\partial_u(\ln\omega(u))\in\mathcal{H}(\lambda_0)$. Define the finite constants*

$$\gamma_0 := \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda_0}, \quad \hat{\gamma}_1 := \|\hat{h}\|_{\mathcal{H},\lambda_0} \quad \text{and } C_\omega := \int_{\mathbb{R}} \frac{|u|}{\omega(u)}du,$$

and assume that

a) $T < \frac{(1+\beta)\lambda_0}{1+4\gamma_0+(1+\beta)(1+\lambda_0)}$,

b) $M \le \frac{(1+\beta)(K-\lambda_0)-4\gamma_0-1}{16+\alpha\beta}$ *for some* $K\in(\frac{1+4\gamma_0}{1+\beta}+\lambda_0, \frac{\lambda_0}{T}-1)\,(\neq\emptyset)$ *and*

c) $M(1+\gamma_0)(1+TC_\omega\alpha\beta)\exp\{(M(1+C_\omega\alpha\beta)\gamma_0+\hat{\gamma}_1)T\}<1.$

Assume moreover that $f_0:\mathbb{R}^2\to\mathbb{R}$ is a function of class $\mathcal{C}^\infty$ and that $g_0(x,u):=\omega(u)f_0(x,u)$ satisfies *d)* $\max\{\|g_0\|_{\mathcal{H},\lambda_0}, T\|g_0\|_{\widetilde{\mathcal{H}},\lambda_0}\} \le M$ *and*

e) $\|g_0\|_{\mathcal{H},\lambda_0} \exp\left(T(\hat{\gamma}_1 + 16\gamma_0 + \alpha\beta\gamma_0 C_\omega M)\right) < M \exp\left(-(16+\gamma_0)M\right)$.

Then, equation (VFPωK) *has a unique smooth solution* $g \in \mathcal{B}^M_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^M_{\lambda_0,K,T}$. *In particular, under the previous assumptions, a solution* $f \in \mathcal{C}^{1,\infty}$ *to* (VFPK) *with initial condition* f_0 *exists.*

It is checked in Appendix A.1 that the function $u \mapsto \hat{h}(u)$ belongs to $\mathcal{H}(\lambda_0)$ for every $\lambda_0 \in (0, \frac{1}{4})$ when $\omega(u) := c(1+u^2)^{\frac{s}{2}}$ (so that $u\partial_u(\ln\omega(u)) \in \mathcal{H}(\lambda_0)$ as required).

Corollary 3.2. *Let f be the solution to* (VFPK) *given above and assume that* $(\mathrm{H}_{\mathrm{unif}(0)})$ *holds. Then, $f(t,x,u)$ satisfies* $(\mathrm{H}_{\mathrm{unif}(t)})$ *for all t in $[0,T]$. In particular, under the assumptions of Theorem 3.1 and* $(\mathrm{H}_{\mathrm{unif}(0)})$ *a solution to* (1.1) *for $\beta \in \mathbb{R}$ exists.*

Remark 3.3. *Let f_0 be a function of class $\mathcal{C}^\infty$, $C_0, \bar{\lambda} > 0$, $n, m \in \mathbb{N}$ be numbers satisfying condition (2.5) for every $k, l \ge 0$ and assume that, moreover, for some $\lambda_0 < \min\{\bar{\lambda}, \frac{1}{4}\}$ one has*

$$C_0 < \kappa_0'(\bar{\lambda}, s) := \frac{1}{2\kappa(s)e^{\bar{\lambda}}\mu(\bar{\lambda}, m+n+1)} \frac{\ln 2}{(16 + \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda_0})} \frac{1}{(1 + C_\omega\alpha\beta)}$$

for ω as in Lemma 2.4 i). Choosing M as in Remark 2.7, one similarly checks that

$$\kappa_1'(C_0, \bar{\lambda}, s) := \min\left\{ \frac{(1+\beta)\lambda_0}{1 + 4\gamma_0 + (1+\beta)(1+\lambda_0) + (16+\alpha\beta)M}, \frac{2\bar{\lambda}\mu(\bar{\lambda}, m+n+1)}{\mu(\bar{\lambda}, m+n+2)}, 1, \right.$$
$$\left. -\frac{\ln(M(1+\gamma_0)(1+C_\omega\alpha\beta))}{M(1+C_\omega\alpha\beta)\gamma_0 + \hat{\gamma}_1}, \frac{\ln 2 - M(16+\gamma_0)}{\hat{\gamma}_1 + 16\gamma_0 + \alpha\beta C_\omega M} \right\} > 0,$$

and that the assumptions of Theorem 3.1 hold for $T < \kappa_1'(C_0, \bar{\lambda}, s)$ and $K = \frac{1+4\gamma_0+M(16+\alpha\beta)}{1+\beta} + \lambda_0$.

Most of the computations required in the proofs are the same as in the previous section, so we only provide some details about the additional terms that the case $\beta > 0$ requires to deal with.

Proof of Theorem 3.1. Consider the linear equation obtained by respectively replacing in (VFPωK) the functions P and V by fixed given functions $Q, H : [0,T] \times \mathbb{R} \to \mathbb{R}$:

$$\begin{cases} \partial_t g(t,x,u) + u\partial_x g - \left[\partial_x Q + \beta(u - \alpha H) - \partial_u \ln(\omega)\right]\partial_u g - \dfrac{\sigma^2}{2}\partial_u^2 g = g\left[\partial_x Q - \alpha\beta H \partial_u \ln(\omega)\right] - g\hat{h}, \\ \qquad \text{on } (0,T] \times \mathbb{R}^2, \\ g(0,x,u) = g_0(x,u) \text{ on } \mathbb{R}^2, \end{cases} \tag{FPωK}$$

First we notice that

$$\partial_x^k \partial_u^l (u\partial_u g(t,x,u)) = \sum_{n=0}^{l} C_l^n (\partial_u^n u)(\partial_u \partial_x^k \partial_u^{l-n} g(t,x,u)) = u\partial_x^k \partial_u^{l+1} g(t,x,u) + l\partial_x^k \partial_u^l g(t,x,u).$$

Therefore, application of the differential operator $\partial_x^k \partial_u^l$ to the linear equation (FPωK) yields the identity

$$\partial_t \partial_x^k \partial_u^l g + u\partial_x(\partial_x^k \partial_u^l g) - \left(\partial_x Q(t,x) - \partial_u \ln\omega + \beta(u - \alpha V)\right)\partial_u(\partial_x^k \partial_u^l g) - \frac{\sigma^2}{2}\partial_u^2(\partial_x^k \partial_u^l g)$$
$$= \mathbb{1}_{\{l\ge1\}}\beta l \partial_x^k \partial_u^l g - l\partial_x^{k+1}\partial_u^{l-1} g + \mathbb{1}_{\{k\ge1\}} \sum_{m=0}^{k-1} C_k^m \partial_x^{k-m}(\partial_x Q - \alpha\beta H)\partial_u(\partial_x^m \partial_u^l g) - \mathbb{1}_{\{l\ge1\}} \sum_{n=0}^{l-1} C_l^n \left(\partial_u^{l-n+1}\ln\omega \partial_u^{n+1}\partial_x^k g\right)$$
$$+ \sum_{n=0}^{l}\sum_{m=0}^{k} C_l^n C_k^m \left(\partial_x^{k-m}\partial_x Q - \alpha\beta H\right)\partial_u^{l-n+1}\ln\omega\, \partial_x^m \partial_u^n g + \sum_{n=0}^{l} C_l^n \partial_x^k \partial_u^n g \partial_u^{l-n}\hat{h}.$$

By the maximum principle we deduce that for all $A \in \mathbb{N}$ and $a \in \{0, \dots, A\}$, a smooth solution g to equation (FPωK) must satisfy

$$
\begin{aligned}
\frac{d}{dt}\|g(t)\|_{\lambda,a;A} \leq{}& \lambda(1+\beta)\|g(t)\|_{\lambda,a+1;A} + a(1+\beta)\|g(t)\|_{\lambda,a;A} \\
&+ \frac{d^a}{d\lambda^a}\Big(\|g(t)\|_{\lambda,1;A}\left(\|Q(t)\|_{\lambda,1;A} + \alpha\beta\|H(t)\|_{\lambda,0;A} + \|\ln(\omega)\|_{\lambda,1;A}\right)\Big) \\
&+ \frac{d^a}{d\lambda^a}\Big(\|g(t)\|_{\lambda,0;A}\left[\|\hat{h}\|_{\lambda,0;A} + (\|Q(t)\|_{\lambda,1;A} + \alpha\beta\|H(t)\|_{\lambda,0;A})\|\ln(\omega)\|_{\lambda,1;A}\right]\Big).
\end{aligned}
$$

By similar computation as in the proof of Lemma 2.10, it is also possible to establish

Lemma 3.4. *(i) Suppose that for some $\bar{\lambda} > 0$ we have $f \in \mathcal{H}(\bar{\lambda})$ and $v \in \widetilde{\mathcal{H}}(\bar{\lambda})$. Then, for all $\lambda \in [0, \bar{\lambda})$ one has*

$$
\sum_{a\in\mathbb{N}} \frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|f\|_{\lambda,0}\|v\|_{\lambda,1}\right) \leq \|f\|_{\mathcal{H},\lambda}\|v\|_{\widetilde{\mathcal{H}},\lambda}.
$$

(ii) Suppose that for some $\bar{\lambda} > 0$, $f, w \in \mathcal{H}(\bar{\lambda})$ and $v \in \widetilde{\mathcal{H}}(\bar{\lambda})$. Then, for all $\lambda \in [0, \bar{\lambda})$ one has

$$
\sum_{a\in\mathbb{N}} \frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|f\|_{\lambda,0}\|w\|_{\lambda,0}\|v\|_{\lambda,1}\right) \leq \|f\|_{\mathcal{H},\lambda}\|w\|_{\mathcal{H},\lambda}\|v\|_{\widetilde{\mathcal{H}},\lambda}.
$$

Proof. To obtain the first estimate, it suffices to apply Lemma 2.10 for $w(x,u) = u$ so that, according to Lemma 2.3 (i), $\|w\|_{\lambda,1} = \|w\|_{\widetilde{\mathcal{H}},\lambda} = 1$. For the second estimate, let w belong to $\mathcal{H}(\bar{\lambda})$. Taking $W(x,u) = \int_0^u w(x,v)dv$, one checks that $W \in \widetilde{\mathcal{H}}(\bar{\lambda})$, $\|W\|_{\lambda,1} = \|w\|_{\lambda,0}$ and $\|W\|_{\widetilde{\mathcal{H}},\lambda} = \|w\|_{\mathcal{H},\lambda}$. Applying Lemma 2.10 (ii) to f, W and v implies we get the estimate. $\square$

Truncated version of these estimates, combined with the already obtained ones yield:

Proposition 3.5. *For each $A \in \mathbb{N}$, the $\mathcal{C}^{1,\infty}$ function g solution to (FPωK) satisfies the estimate*

$$
\begin{aligned}
\frac{d}{dt}\|g(t)\|_{\mathcal{H},\lambda(t);A} \leq{}& \left((1+\beta)[\lambda(t)+1+\lambda'(t)] + 4\gamma_0 + 16\|Q(t)\|_{\mathcal{H},\lambda(t)} + \alpha\beta\|H(t)\|_{\mathcal{H},\lambda(t)}\right)\|g(t)\|_{\widetilde{\mathcal{H}},\lambda(t);A} \\
&+ \Big(\gamma_1 + 16\gamma_0 + (\gamma_0+16)\|Q(t)\|_{\widetilde{\mathcal{H}},\lambda(t)} + \alpha\beta\gamma_0\|H(t)\|_{\mathcal{H},\lambda(t)}\Big)\|g(t)\|_{\mathcal{H},\lambda(t);A},
\end{aligned}
$$

where $\gamma_0 := \|\ln(\omega)\|_{\widetilde{\mathcal{H}},\lambda_0}$ and $\hat{\gamma}_1 := \|\hat{h}\|_{\mathcal{H},\lambda_0}$.

Applying Gronwall's lemma and using the fact that

$$
\lambda(t) + 1 + \lambda'(t) + 4\gamma_0 + 16\|P(t)\|_{\mathcal{H},\lambda(t)} + \alpha\beta\|V(t)\|_{\mathcal{H},\lambda(t)} \leq (1+\beta)[\lambda_0 - K] + 4\gamma_0 + (16+\alpha\beta)M_1
$$

we then obtain that, for all $t \in [0,T]$ and $A \in \mathbb{N}$,

$$
\begin{aligned}
&\|g(t)\|_{\mathcal{H},\lambda(t);A} \leq \|g_0\|_{\mathcal{H},\lambda_0}\exp\left\{T(\hat{\gamma}_1 + 16\gamma_0 + \alpha\beta\gamma_0 M_1) + (16+\gamma_0)M_2\right\} \\
&\quad + \exp\left\{T(\hat{\gamma}_1 + 16\gamma_0 + \alpha\beta\gamma_0 M_1) + (16+\gamma_0)M_2\right\}\left((1+\beta)[\lambda_0 - K] + 4\gamma_0 + (16+\alpha\beta)M_1\right)\int_0^t \|g(s)\|_{\widetilde{\mathcal{H}},\lambda(s);A}ds.
\end{aligned} \tag{3.4}
$$

From assumptions a) and b) of Theorem 3.1 we can choose $K > 0$ such that $K < \frac{\lambda_0}{T} - 1$ and

$$
(1+\beta)(K - \lambda_0) - 4\gamma_0 - (16+\alpha\beta)M_1 \geq 1
$$

in which case we obtain

$$
\|g(t)\|_{\mathcal{H},\lambda(t);A} + \int_0^t \|g(s)\|_{\widetilde{\mathcal{H}},\lambda(s);A}ds \leq \|g_0\|_{\mathcal{H},\lambda_0}\exp\left\{T(\hat{\gamma}_1 + 16\gamma_0 + \alpha\beta\gamma_0 M_1) + (16+\gamma_0)M_2\right\},
$$

and then

$$\|g(t)\|_{\mathcal{H},\lambda(t)} + \int_0^t \|g(s)\|_{\widetilde{\mathcal{H}},\lambda(s)} ds \le \|g_0\|_{\mathcal{H},\lambda_0} \exp\left\{T(\hat{\gamma}_1 + 16\gamma_0 + \alpha\beta\gamma_0 M_1) + (16+\gamma_0)M_2\right\}.$$

Therefore, since for any function $\varphi : [0,T]\times\mathbb{R}^2 \to \mathbb{R}$ of class $\mathcal{C}^{1,\infty}$ and every $t \in [0,T]$ we have

$$\left\|\int_{\mathbb{R}} \frac{u^2}{\omega(u)}\varphi(t,\cdot,u)du\right\|_{\lambda,a} \le \|\varphi(t,\cdot,\cdot)\|_{\lambda,a} \quad \text{and} \quad \left\|\int_{\mathbb{R}} \frac{|u|}{\omega(u)}\varphi(t,\cdot,u)du\right\|_{\lambda,a} \le C_\omega \|\varphi(t,\cdot,\cdot)\|_{\lambda,a}$$

for all $\lambda \ge 0$ and $a \in \mathbb{N}$, the mapping Φ associating with a function φ the solution $\Phi(\varphi)$ of equation (FPωK) with the data

$$Q(t,x) := -\int_{\mathbb{R}} \frac{u^2}{\omega(u)}\varphi(t,x,u)du \quad \text{and} \quad H(t,x) := \int_{\mathbb{R}} \frac{u}{\omega(u)}\varphi(t,x,u)du$$

satisfies the inclusion

$$\Phi\left(\mathcal{B}^{M_1}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{M_2}_{\lambda_0,K,T}\right) \subseteq \mathcal{B}^{\hat{M}}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{\hat{M}}_{\lambda_0,K,T}$$

if $M_1, T, \lambda_0 > 0$ are as previously, $M_2 > 0$ is arbitrary and

$$\hat{M} = \|g_0\|_{\mathcal{H},\lambda_0} \exp\left\{T(\hat{\gamma}_1 + 16\gamma_0 + \alpha\beta\gamma_0 C_\omega M_1) + (16+\gamma_0)M_2\right\}.$$

In particular, one has $\Phi\left(\mathcal{B}^{M}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T}\right) \subseteq \mathcal{B}^{M}_{\lambda_0,K,T} \cap \widetilde{\mathcal{B}}^{M}_{\lambda_0,K,T}$ if in addition to conditions a) and b) of Theorem 3.1, the constants $M > 0$ and $T > 0$ satisfy condition d). Now, writing $\bar{\Phi} := \Phi(f_1) - \Phi(f_2)$, $\bar{P} := P_1 - P_2$ and $\bar{V} := V_1 - V_2$ where

$$P_i(t,x) := -\int_{\mathbb{R}} \frac{u^2}{\omega(u)} f_i(t,\cdot,u)\, du \text{ and } V_i(t,x) := \int_{\mathbb{R}} \frac{u}{\omega(u)} f_i(t,\cdot,u)du \quad i = 1,2,$$

we have

$$\begin{aligned}
&\partial_t(\partial_x^k\partial_u^l\bar{\Phi}) + u\partial_x\left(\partial_x^k\partial_u^l\bar{\Phi}\right) - \left(\partial_x Q_1 - \partial_u \ln\omega(u) + \beta(u - \alpha V_1(t,x))\right)\partial_u\left(\partial_x^k\partial_u^l\bar{\Phi}\right) - \frac{\sigma^2}{2}\partial_u^2(\partial_x^k\partial_u^l\bar{\Phi}) \\
&= -l\partial_x^{k+1}\partial_u^{l-1}\bar{\Phi} + \mathbb{1}_{\{k\ge1\}}\sum_{m=0}^{k-1} C_k^m \partial_x^{k-m}\left(\partial_x Q_1(t,x) - \alpha\beta V_1(t,x)\right)\partial_x^m\partial_u^{l+1}\bar{\Phi} - \mathbb{1}_{\{l\ge1\}}\sum_{n=0}^{l-1} C_l^n\left(\partial_u^{l-n+1}\ln(\omega)\right)\partial_x^k\partial_u^{n+1}\bar{\Phi} \\
&\quad + \sum_{n=0}^{l}\sum_{m=0}^{k} C_k^m C_l^n\left(\partial_x^{k-m}\partial_u^n\Phi(f_2)\right)\left(\partial_u^{l-n+1}\ln(\omega)\right)\partial_x^m\left(\partial_x\bar{P} - \alpha\beta\bar{V}(t,x)\right) \\
&\quad + \sum_{m=0}^{k} C_k^m\left(\partial_x^{k-m}\partial_u^{l+1}\Phi(f_2)\right)\partial_x^m\left(\partial_x\bar{P} - \alpha\beta\bar{V}(t,x)\right)\Big) \\
&\quad + \sum_{n=0}^{l}\sum_{m=0}^{k} C_k^m C_l^n \partial_x^{k-m}\left(\partial_x P_1 - \alpha\beta V_1(t,x)\right)\partial_u^{l-n+1}\ln\omega(u)\partial_x^m\partial_u^n\bar{\Phi} + \sum_{n=0}^{l} C_l^n\left(\partial_x^k\partial_u^n\bar{\Phi}\right)\partial_u^{l-n}\hat{h}(u);
\end{aligned}$$

From this and the maximum principle we deduce that $\frac{d}{dt}\|\partial_x^k\partial_u^l\bar{\Phi}(t)\|_\infty$ is bounded above by

$$
\begin{aligned}
&l\|\partial_x^{k+1}\partial_u^{l-1}\bar{\Phi}(t)\|_\infty + \mathbb{1}_{\{k\geq 1\}}\sum_{m=0}^{k-1} C_k^m(\|\partial_x^{k-m+1}f_1(t)\|_\infty + C_\omega\alpha\beta\|\partial_x^{k-m}f_1(t)\|_\infty)\|\partial_x^m\partial_u^{l+1}\bar{\Phi}(t)\|_\infty\\
&\quad+\mathbb{1}_{\{l\geq 1\}}\sum_{n=0}^{l-1} C_l^n\|\partial_u^{l-n+1}\ln(\omega)\|_\infty\|\partial_x^k\partial_u^{n+1}\bar{\Phi}(t)\|_\infty\\
&\quad+\sum_{n=0}^{l}\sum_{m=0}^{k} C_k^m C_l^n\|\partial_u^{l-n+1}\ln(\omega)\|_\infty\|\partial_x^{k-m}\partial_u^n\Phi(f_2)(t)\|_\infty(\|\partial_x^{m+1}\bar{f}(t)\|_\infty + C_\omega\alpha\beta\|\partial_x^m\bar{f}(t)\|_\infty)\\
&\quad+\sum_{m=0}^{k} C_k^m\|\partial_x^{k-m}\partial_u^{l+1}\Phi(f_2)(t)\|_\infty(\|\partial_x^{m+1}\bar{f}(t)\|_\infty + C_\omega\alpha\beta\|\partial_x^m\bar{f}(t)\|_\infty)\\
&\quad+\sum_{n=0}^{l}\sum_{m=0}^{k} C_k^m C_l^n\|\partial_u^{l-n+1}\ln(\omega)\|_\infty(\|\partial_x^{k-m+1}f_1(t)\|_\infty + C_\omega\alpha\beta\|\partial_x^{k-m}f_1(t)\|_\infty)\|\partial_x^m\partial_u^n\bar{\Phi}(t)\|_\infty\\
&\quad+\sum_{n=0}^{l} C_l^n\|\partial_x^k\partial_u^n\bar{\Phi}(t)\|_\infty\|\partial_u^{l-n}\hat{h}(u)\|_\infty.
\end{aligned}
$$

This yields

$$
\begin{aligned}
\frac{d}{dt}\|\bar{\Phi}(t)\|_{\lambda(t),0} \leq{}& \left(\lambda(t)+\lambda'(t)+\|f_1(t)\|_{\lambda(t),1}+C_\omega\alpha\beta\|f_1(t)\|_{\lambda(t),0}+\|\ln(\omega)\|_{\lambda(t),1}\right)\|\bar{\Phi}(t)\|_{\lambda(t),1}\\
&+\|\bar{\Phi}(t)\|_{\lambda(t),0}\left[\left(\|f_1(t)\|_{\lambda(t),1}+C_\omega\alpha\beta\|f_1(t)\|_{\lambda(t),0}\right)\|\ln(\omega)\|_{\lambda(t),1}+\|\hat{h}\|_{\lambda(t),0}\right]\\
&+\left(\|\bar{f}(t)\|_{\lambda(t),1}+C_\omega\alpha\beta\|\bar{f}(t)\|_{\lambda(t),0}\right)\left(\|\Phi(f_2)(t)\|_{\lambda(t),1}+\|\Phi(f_2)(t)\|_{\lambda(t),0}\|\ln(\omega)\|_{\lambda(t),1}\right)
\end{aligned}
\tag{3.5}
$$

and therefore, for all $t\in[0,T]$,

$$
\begin{aligned}
\frac{d}{dt}\|\bar{\Phi}(t)\|_{\lambda(t),0} \leq{}& \left((1+\beta)(\lambda_0-K)+(1+C_\omega\alpha\beta)M+\gamma_0\right)\|\bar{\Phi}(t)\|_{\lambda(t),1}\\
&+\|\bar{\Phi}(t)\|_{\lambda(t),0}\left(M(1+C_\omega\alpha\beta)\gamma_0+\hat{\gamma}_1\right)\\
&+\left(\|\bar{f}(t)\|_{\lambda(t),1}+C_\omega\alpha\beta\|\bar{f}(t)\|_{\lambda(t),0}\right)M(1+\gamma_0).
\end{aligned}
\tag{3.6}
$$

Since our assumptions allow us to choose $K\in(0,\frac{\lambda_0}{T}-1)$ such that

$$(1+\beta)(K-\lambda_0)-(1+C_\omega\alpha\beta)M-\gamma_0>1,$$

we get from the previous after applying Gronwall's lemma that

$$
\begin{aligned}
\|\bar{\Phi}(t)\|_{\lambda(t),0}+\int_0^t\|\bar{\Phi}(s)\|_{\lambda(t),1}\,ds &\leq \|\bar{\Phi}(t)\|_{\lambda(t),0}+\exp\left\{T\left(M(1+C_\omega\alpha\beta)\gamma_0+\hat{\gamma}_1\right)\right\}\int_0^t\|\bar{\Phi}(s)\|_{\lambda(s),1}\,ds\\
&\leq M(1+\gamma_0)\exp\left\{\left(M(1+C_\omega\alpha\beta)\gamma_0+\hat{\gamma}_1\right)T\right\}\int_0^T\|\bar{f}(t)\|_{\lambda(t),1}+C_\omega\alpha\beta\|\bar{f}(t)\|_{\lambda(t),0}dt.
\end{aligned}
$$

This implies that $\Phi:\mathcal{B}^M_{\lambda_0,K,T}\cap\widetilde{\mathcal{B}}^M_{\lambda_0,K,T}\to\mathcal{B}^M_{\lambda_0,K,T}\cap\widetilde{\mathcal{B}}^M_{\lambda_0,K,T}$ is a contraction for the norm

$$\max\left\{\max_{t\in[0,T]}\|\psi(t)\|_{\lambda(t),0}\,,\int_0^T\|\psi(t)\|_{\lambda(t),1}\,dt\right\}$$

under condition c) of Theorem 3.1, and condition d) allows us to conclude the existence of a solution starting from g_0. □

Proof of Corollary 3.2. By Lemma 1.3 we now obtain in the case $\alpha=1$ that, for all $(t,x)\in(0,T]\times\mathbb{R}$,

$$
\begin{cases}
\partial_t\bar{\rho}(t,x)=-\partial_x V(t,x),\\
\partial_t(\partial_x V(t,x))=-\partial_x(\bar{\rho}(t,x)\partial_x P(t,x))+\beta\partial_x\left(V(t,x)\bar{\rho}(t,x)\right),
\end{cases}
$$

where $\overline{\rho}(t,x) := \rho(t,x) - 1 = \int_{\mathbb{R}} f(t,x,u)du - 1$, $V(t,x) := \int_{\mathbb{R}} uf(t,x,u)\,du$ and $\partial_x P(t,x) = -\partial_x \int_{\mathbb{R}} u^2 f(t,x,u)\,du$. Hence, for all $\lambda > 0$,

$$\begin{cases} \quad \partial_t \|\overline{\rho}(t)\|_\lambda \le \|\partial_x V(t)\|_\lambda, \\ \partial_t \|\partial_x V(t)\|_\lambda \le \|\partial_x P(t)\|_\lambda \|\partial_x \overline{\rho}(t)\|_\lambda + \|\partial_x^2 P(t)\|_\lambda \|\overline{\rho}(t)\|_\lambda + \beta \left(\|\partial_x V(t)\|_\lambda \|\overline{\rho}(t)\|_\lambda + \|V(t)\|_\lambda \|\partial_x \overline{\rho}(t)\|_\lambda\right). \end{cases} \tag{3.7}$$

With $A(t,\lambda) := \|\overline{\rho}(t)\|_\lambda$ and $B(t,\lambda) := \|\partial_x V(t)\|_\lambda$ we have

$$\begin{cases} \partial_t A(t,\lambda) \le B(t,\lambda), \\ \partial_t B(t,\lambda) \le (\|\partial_x P(t)\|_\lambda + \beta \|V(t)\|_\lambda) \partial_\lambda A(t,\lambda) + (\|\partial_x^2 P(t)\|_\lambda + \beta B(t,\lambda)) A(t,\lambda). \end{cases}$$

From these inequalities, since the terms in parentheses are bounded, the conclusion is obtained by similar arguments as in the case $\beta = 0$. If now $\alpha = 0$, we obtain the equations, for all $(t,x) \in (0,T] \times \mathbb{R}$,

$$\begin{cases} \quad \partial_t \overline{\rho}(t,x) = -\partial_x V(t,x), \\ \partial_t(\partial_x V(t,x)) = -\partial_x(\overline{\rho}(t,x)\partial_x P(t,x)) + \beta \partial_x V(t,x) \end{cases}$$

whit the same notation as before. This yields

$$\begin{cases} \partial_t A(t,\lambda) \le B(t,\lambda), \\ \partial_t B(t,\lambda) \le \|\partial_x P(t)\|_\lambda \partial_\lambda A(t,\lambda) + \|\partial_x^2 P(t)\|_\lambda A(t,\lambda) + \beta B(t,\lambda). \end{cases}$$

Since the remainder of the proof in the case $\beta = 0$ relies on the inequality satisfied by the sum $\mathcal{Y}(t,\lambda) := A(t,\lambda) + bB(t,\lambda)$, by suitably modifying the constants therein one can conclude in a similar way. □

4 Local solutions for the incompressible Langevin SDE

We finally briefly state the main consequence of the previous results for the SDE (1.2).

Corollary 4.1. *Let $T > 0$ be a time horizon and $p_0 : \mathbb{T} \times \mathbb{R} \to \mathbb{R}_+$ a probability density such that*

- $\int_{\mathbb{R}} p_0(x,u)du = 1$ *for all* $x \in \mathbb{T}$,
- $\partial_x \int_{\mathbb{R}} u p_0(x,u)du = 0$ *for all* $x \in \mathbb{T}$,
- p_0 *or equivalently its periodic extension* f_0 *to* $\mathbb{R}^2$ *and the constant* $T > 0$ *satisfy the assumptions of Theorem 2.5 (resp. Theorem 3.1)).*

Then, there exists in $[0,T]$ *a solution to the stochastic differential equation* (1.2) *in the case* $\beta = 0$ *(resp. for each* $\beta \in \mathbb{R}$*).*

Proof. We write the proof for general β and $\sigma > 0$. Let f be the solution to equation (VFPK) given by Theorem 3.1 for f_0 equal to the periodic extension of p_0 to $\mathbb{R}^2$. On one hand, we know from Corollary 3.2 that this solution f is 1- periodic, and so are also the functions $P(t,y) = -\int_{\mathbb{R}} u^2 f(t,y,u)du$ and $V(t,y) = \int_{\mathbb{R}} uf(t,y,u)du$. On the other hand, since $P(t,y)$ and $V(t,y)$ have derivatives of all order in $y \in \mathbb{R}$ which are bounded in $[0,T] \times \mathbb{R}^2$, the following stochastic differential equation, where W_t is a standard one dimensional Brownian motion independent of the random variable (Y_0, U_0), has a pathwise unique solution (Y_t, U_t):

$$\begin{aligned} dY_t = U_t\,dt, \quad dU_t = \sigma dW_t - \partial_x P(t,Y_t)dt - \beta(U_t - \alpha V(t,Y_t))dt \\ law(Y_0,U_0) = f_0(y,u)\mathbf{1}_{[0,1]}(y)dy\,du. \end{aligned} \tag{4.1}$$

One can observe that the vector field associated to the coefficients of the equation (4.1) at time $t = 0$ satisfies the usual Hormander's condition. Indeed, introducing the vector fields

$$\begin{aligned} V_0(x,u) &= u\partial_x - (\partial_x P(0,x) + \beta(u - \alpha V(0,x)))\,\partial_u, \\ V_1(x,u) &= \sigma \partial_u \end{aligned}$$

one observe that the Lie bracket between V_0 and V_1 is given by

$$\begin{aligned}
[V_0, V_1](x,u) &= V_0 \circ V_1(x,u) - V_1 \circ V_0(x,u) \\
&= u\partial_x(\sigma\partial_u) + (\partial_x P(0,x) + \beta(u - \alpha V(0,x))\,\partial_u(\sigma\partial_u) - \sigma\partial_u(u\partial_x) - \sigma\partial_u\left((\partial_x P(0,x) + \beta(u - \alpha V(0,x)))\,\partial_u\right) \\
&= -\sigma\partial_x + \sigma\partial_u\left(\partial_x P(0,x) + \beta(u - \alpha V(0,x))\,\partial_u\right. \\
&= -\sigma\partial_x - \beta\sigma\partial_u.
\end{aligned}$$

This shows that $Span\{V_1, [V_0, V_1]\} = \mathbb{R}^2$. Owing to the time regularity of the coefficients, Malliavin calculus ensures that (see [9]), (Y_t, U_t) admits a $W^{\infty,1}(\mathbb{R}^2)$–density q_t with respect to Lebesgue measure for each $t \in [0,T]$. Therefore the density $p_t(x,u) = \sum_{k\in\mathbb{Z}} q_t(x+k,u)$ of the random variable $(X_t, U_t) = ([Y_t], U_t)$ in $\mathbb{T}\times\mathbb{R}$ is itself $W^{\infty,1}(\mathbb{T}\times\mathbb{R})$. So $\rho_t \in \mathcal{C}^\infty(\mathbb{T}\times\mathbb{R})$ by Sobolev embedding (see e.g. [5]). We further notice that ρ is also a classical $\mathcal{C}^{1,\infty}$–solution to the linear PDE:

$$\begin{aligned}
&\partial_t\rho + u\partial_x\rho - \partial_x P\partial_u\rho - \beta\,(u - \alpha V)\,\partial_u\rho - \frac{\sigma^2}{2}\partial_u^2\rho = \beta\rho \text{ on } (0,T)\times\mathbb{T}\times\mathbb{R} \\
&\rho_{t=0} = f_0 \text{ on } \mathbb{T}\times\mathbb{R},
\end{aligned}$$

where P and V depend only on f. Due to the smoothness of ρ and f, the maximum principle (A.3) in Theorem A.1 applies to the difference $\rho - f$ and implies that $\|\rho_t - f(t)\|_\infty = 0$ for all t. Hence, defining the process $X_t := [Y_t]$, the law of $(X_t, \mathcal{U}_t)$ corresponds to the solution to (1.1) and in particular satisfies the uniform mass repartition constraint (1.1c). □

A Appendix

A.1 A weight function of analytic type

In this section, we study some properties of the weight function $\omega(u) = c(1+u^2)^{\frac{s}{2}}$ where $c, s > 0$ are fixed constants. Notice that if $s > 3$ one has $\int_{\mathbb{R}} \frac{u^2}{\omega(u)}du < +\infty$ and the growth conditions on ω and its derivatives required in (H_ω) are satisfied. We will now show that the functions $u \mapsto \partial_u \ln(\omega(u)) = \frac{su}{(1+u^2)}$, $u \mapsto h(u) = \frac{\partial_u^2\omega(u)}{2\omega(u)} - |\partial_u \ln(\omega(u))|^2 = \frac{s-(s+s^2)u^2}{2(1+u^2)^2}$ and $u \mapsto \hat{h}(u) = h(u) - \beta(1 + u\partial_u(\ln\omega(u)))$ satisfy $\ln(\omega) \in \widetilde{\mathcal{H}}(\lambda_0)$ and $\hat{h}, h \in \mathcal{H}(\lambda_0)$ (H_ω) for all $\lambda_0 \in [0, 1/4)$. In particular this will prove part i) of Lemma 2.4.

Let us first consider $\partial_u \ln(\omega)$. We are going to identify $\partial_u^l \ln(\omega)$ for $l \geq 1$ with a function of the form $\frac{q_l(u)}{(1+u^2)^l}$ where q_l is a polynomial function of order l satisfying $q_1(u) = su$ and, for all $l \geq 1$,

$$\frac{q_{l+1}(u)}{(1+u^2)^{l+1}} = \partial_u\left(\frac{q_l(u)}{(1+u^2)^l}\right) = \frac{(1+u^2)\partial_u q_l(u) - 2luq_l(u)}{(1+u^2)^{l+1}},$$

or, equivalently, $q_{l+1}(u) = (1+u^2)\partial_u q_l(u) - 2luq_l(u)$. We can now determine the coefficients $\{a_n^{(l)}\}_{0\leq n\leq l}$ such that $q_l(u) = \sum_{n=0}^{l} a_n^{(l)}u^n$ observing that, for $l \geq 1$,

$$\begin{aligned}
(1+u^2)\partial_u q_l(u) - 2luq_l(u) &= (1+u^2)\sum_{n=1}^{l} na_n^{(l)}u^{n-1} - 2lu\sum_{n=0}^{l} a_n^{(l)}u^n \\
&= \sum_{n=1}^{l} na_n^{(l)}u^{n-1} + \sum_{n=1}^{l} na_n^{(l)}u^{n+1} - 2l\sum_{n=0}^{l} a_n^{(l)}u^{n+1} \\
&= \sum_{n=0}^{l-1}(n+1)a_{n+1}^{(l)}u^n + \sum_{n=2}^{l+1}(n-1)a_{n-1}^{(l)}u^n - 2l\sum_{n=1}^{l+1} a_{n-1}^{(l)}u^n.
\end{aligned}$$

Therefore, we have $a_0^{(1)} = 0$, $a_1^{(1)} = s$, $a_0^{(2)} = s$, $a_1^{(2)} = 0$, $a_2^{(2)} = -s$ and, for $l \geq 2$,

$$\begin{aligned}
&a_0^{(l+1)} = a_1^{(l)}, \quad a_1^{(l+1)} = 2a_2^{(l)} - 2la_0^{(l)}, \\
&a_n^{(l+1)} = (n+1)a_{n+1}^{(l)} + (n-1)a_{n-1}^{(l)} - 2la_{n-1}^{(l)}, \text{ if } 2 \leq n \leq l-1, \\
&\text{and } a_l^{(l+1)} = -(l+1)a_{l-1}^{(l)}, \quad a_{(l+1)}^{l+1} = -la_l^{(l)}.
\end{aligned} \tag{A.1}$$

Setting $a^{(l)} := \max_{n\in\{0,\dots,l\}} a_n^{(l)}$, we deduce the rough estimates: $a^{(l+1)} \le 4(l+1)a^{(l)}$ for $l \ge 2$, and then: $a^{(l)} \le \frac{s}{4}4^l l!$ for $l \ge 1$. Thus, for $l \ge 1$

$$|\partial_u^l(\partial_u(\ln(\omega))| \le \frac{\sum_{n=0}^{l+1} a_n^{(l+1)}|u|^n}{(1+u^2)^{l+1}} \le \frac{s}{4}\sum_{n=0}^{l+1}\frac{4^{l+1}(l+1)!|u|^n}{(1+u^2)^{l+1}} \le s4^l(l+2)!. \tag{A.2}$$

Consequently, by Lemma 2.1 we have $\partial_u(\ln(\omega)) \in \mathcal{H}(\lambda)$ for all $\lambda \in [0,1/4)$, and from Lemma 2.3-(i) we conclude that $\ln(\omega) \in \widetilde{\mathcal{H}}(\lambda)$ for all $\lambda \in [0,1/4)$.

As for the function h, it is similarly checked in this case that for all $l \ge 0$

$$\partial_u^l h(u) = \frac{r_{l+2}(u)}{2(1+u^2)^{l+2}},$$

where r_l is a polynomial function of order l, defined for $l \ge 2$. The coefficients $\{b_n^{(l)}\}_{0\le n\le l}$ such that $r_l(u) = \sum_{n=0}^l b_n^{(l)}u^n$ satisfy $b_0^{(2)} = s/2$, $b_1^{(2)} = 0$, $b_2^{(2)} = -(s+s^2)/2$ and moreover, for all $l \ge 2$, the recurrence relations (A.1) with $a_n^{(l)}$ replaced by $b_n^{(l)}$. It follows in a similar way as before that

$$|\partial_u^l h(u)| \le \frac{s+s^2}{4}4^l(l+3)!$$

and we conclude as well by Lemma 2.1 that $h \in \mathcal{H}(\lambda)$ for all $\lambda \in [0,1/4)$.

Finally, we have $u\partial_u(\ln\omega(u)) = s - \frac{s}{1+u^2}$ so that we only need to check that the function $\frac{s}{1+u^2}$ belongs to the space $\mathcal{H}(\lambda)$ for $\lambda \in [0,1/4)$. Plainly, for each $l \ge 0$, $\partial_u^l\left(\frac{s}{1+u^2}\right) = \frac{j_l(u)}{(1+u^2)^{l+1}}$ for some polynomial j_l of order l. This yields a recurrence relation for the coefficients that only differs from (A.1) in that the factors $-2l$ are replaced by $-2(l+1)$. The conclusion thus follows as previously.

We end this technical section verifying claim ii) of Lemma 2.4. Since $\partial_u^j\omega = 0$ for $j > s$ and $|\partial_u^j\omega| \le \kappa(s)(1+u^2)^{\frac{s}{2}}$ for $j \le s$, we have $\partial_u^l\partial_x^k(f_0(x,u)\omega(u)) = \sum_{j=0}^{s\wedge l}\frac{l!}{(l-j)!j!}\partial_u^j\omega(u)\partial_u^{l-j}(\partial_x^k f_0(x,u))$ and we deduce from the assumptions that

$$\|\partial_x^l\partial_u^k g_0\|_\infty \le \kappa(s)\frac{C_0(k+m)!}{\overline{\lambda}^{k+l}}\sum_{j=0}^{l}\frac{l!(l-j+n)!}{(l-j)!}\frac{\overline{\lambda}^j}{j!} \le \kappa(s)\frac{C_0(k+m)!(l+n)!}{\overline{\lambda}^{k+l}}e^{\overline{\lambda}}$$

using also the fact that $\frac{l!(l-j+n)!}{(l-j)!} = l!(l-j+n)\cdots(l-j+1) \le (l+n)!$ for all $j \le l$. The last assertion of Lemma 2.4 is an immediate consequence of the previous and of the proof of Lemma 2.1.

A.2 A maximum principle for kinetic Fokker-Planck equations

We next give for completeness a brief proof of the version of the maximum principle that has been used throughout.

Theorem A.1. *Let $d \ge 1$ and $\sigma \ge 0$. Consider bounded functions $f_0 : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ and $F, c : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}$ and a function $\phi : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d \to \mathbb{R}^d$ that grows linearly in (x,u) uniformly in $t \in [0,T]$. Assume moreover that all these functions are of class $\mathcal{C}^{0,\infty}$ and have bounded derivatives of all order. Then, there exists a unique solution f of class $\mathcal{C}^{1,\infty}$ to the linear Fokker-Planck equation in $Q_T := [0,T]\times\mathbb{R}^{2d}$:*

$$\begin{cases} \partial_t f(t,x,u) + u\cdot\nabla_x f(t,x,u) - \phi(t,x,u)\cdot\nabla_u f(t,x,u) - \dfrac{\sigma^2}{2}\triangle_u f(t,x,u) + c(t,x,u)f(t,x,u) = F(t,x,u), \\ f(0,x,u) = f_0(x,u), \ \text{on } \mathbb{R}^{2d} \end{cases}$$

which is bounded in that domain. Moreover, the function $t \mapsto \|f(t)\|_\infty$ is absolutely continuous, and for almost every $t \in [0,T]$ one has

$$\frac{d}{dt}\|f(t)\|_\infty \le \|c(t)\|_\infty\|f(t)\|_\infty + \|F(t)\|_\infty. \tag{A.3}$$

Proof. In the case $\sigma > 0$, existence of a bounded solution of class $\mathcal{C}^{1,\infty}$ can be obtained by probabilistic methods, considering the unique pathwise solution $(X_s^{t,x,u}, U_s^{t,x,u})_{t\le s\le T}$ to the stochastic differential equation in $\mathbb{R}^d \times \mathbb{R}^d$:

$$X_s^{t,x,u} = x - \int_t^s U_r^{t,x,u} dr$$
$$U_s^{t,x,u} = u + \int_t^s \phi(T-r, X_r^{t,x,u}, U_r^{t,x,u})dr + \sigma(W_s - W_t)$$

where W is a standard d-dimensional Brownian motion defined in some filtered probability space. Following Friedman [10] p.124, one shows that

$$\begin{aligned}(t,x,u) \mapsto f(t,x,u) := \mathbb{E}\left[f_0(X_T^{T-t,x,u}, U_T^{T-t;x,u}) \exp\left\{ \int_{T-t}^T c(T-\theta, X_\theta^{T-t,x,u}, U_\theta^{T-t,x,u})\, d\theta \right\}\right] \\ + \mathbb{E}\left[\int_{T-t}^T F(T-\theta, X_\theta^{T-t,x,u}, U_\theta^{T-t,x,u}) \exp\left\{ \int_{T-t}^\theta c(T-s, X_s^{T-t,x,u}, U_s^{T-t,x,u}) ds \right\} d\theta \right]\end{aligned}$$

is a solution to the Fokker-Planck equation. Moreover, using Itô' s formula one shows that any bounded solution has the previous Feynman-Kac representation and is therefore unique because of uniqueness in law for the previous SDE. Furthermore, the Jacobian matrix of the flow $(x,u) \mapsto (X_s^{t,x,u}, U_s^{t,x,u})$ satisfies a linear matrix ODE with bounded coefficient given for each $r \in [0,T]$ by the Jacobian matrix of the function $(x,u) \mapsto (u, \phi(T-r,x,u)$. This implies (by Gronwall's lemma) that $(x,u) \mapsto (X_s^{t,x,u}, U_s^{t,x,u})$ has bounded derivatives of first order, and then of all order by applying inductively a similar argument. Taking derivatives under the expectation sign in the above representation and using moreover the regularity of ρ_0, ϕ, c, F, one then deduces that $f(t,x,u)$ has bounded derivatives of all order in (x,u). Now set $\bar{f}(t,x,u) := f(T-t,x,u)$ and apply Itô's formula to get

$$\bar{f}(s, X_s^{t,x,u}, U_s^{t,x,u}) = \bar{f}(t,x,u) + \int_t^s (c\rho - F)(T-r, X_r^{t,x,u}, U_r^{t,x,u})dr + \sigma \int_t^s \nabla \bar{f}(r, X_r^{t,x,u}, U_r^{t,x,u})dW_r$$

for all $t \le s \le T$. Taking expectations we deduce that

$$\mathbb{E} f(T-s, X_s^{t,x,u}, U_s^{t,x,u}) = f(T-t,x,u) + \int_t^s \mathbb{E}\left[(cf - F)(T-r, X_r^{t,x,u}, U_r^{t,x,u})) \right] dr,$$

which implies (taking $\theta = T - s \le \theta' = T - t$) that

$$\|f(\theta')\|_\infty - \|f(\theta)\|_\infty \le \int_\theta^{\theta'} \|c(r)\|_\infty \|f(r)\|_\infty + \|F(r)\|_\infty \, dr \tag{A.4}$$

for all $0 \le \theta \le \theta' \le T$. Notice now that $\bar{f}$ defined above is a classic solution of the equation

$$\begin{cases} \partial_t \bar{f}(t,x,u) - u \cdot \nabla_x \bar{f}(t,x,u) + \bar{\phi}(t,x,u) \cdot \nabla_u \bar{f}(t,x,u) - \dfrac{\sigma^2}{2} \triangle_u \bar{f}(t,x,u) - \bar{c}(t,x,u)\bar{f}(t,x,u) = \hat{F}(t,x,u), \\ \bar{f}(0,x,u) = f(T,x,u), \text{ on } \mathbb{R}^{2d} \end{cases}$$

where $\bar{\phi}(t,x,u) = \phi(T-t,x,u)$, $\bar{c}(t,x,u) = c(T-t,x,u)$ and $\hat{F}(t,x,u) = -F(T-t,x,u) - \sigma^2 \triangle_u \bar{f}(t,x,u)$ is a bounded function. We can thus apply inequality (A.4) to the function $\bar{f}$ and deduce that

$$-\int_\theta^{\theta'} \|c(r)\|_\infty \|f(r)\|_\infty + \|F(r)\|_\infty + \sigma^2 \|\triangle_u \rho(r)\|_\infty \, dr \le \|f(\theta')\|_\infty - \|f(\theta)\|_\infty$$

for all $0 \le \theta \le \theta' \le T$. This inequality and (A.4) imply that $t \mapsto \|f(t)\|_\infty$ is absolutely continuous, and the upper bound (A.4) then yields the asserted bound (A.3) on the a.e. derivative.

Finally, in the case $\sigma = 0$ the same arguments go through by considering the limit $\sigma \to 0$ to obtain the corresponding ordinary differential equation. $\square$

A.3 Proof of Lemmas 2.10 and 2.11

Here, we provide the proofs of Lemmas 2.10 and 2.11 following arguments of [12]. Their truncated versions used in the proof of Proposition 2.12 are obtained in a similar way, namely replacing in the next proofs the norms $\|\cdot\|_{\lambda,a}$ by their truncated versions $\|\cdot\|_{\lambda,a,A}$ for each $A \in \mathbb{N}$, and the sums over $\mathbb{N}$ by sums over the set $\{0,\dots,A\}$.

Proof of Lemma 2.10. By definition, we have

$$\sum_{a\in\mathbb{N}} \frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|f\|_{\lambda,0}\|v\|_{\lambda,1}\|w\|_{\lambda,1}\right) = \sum_{a\in\mathbb{N}} \frac{1}{(a!)^2}\sum_{r=0}^{a} C_a^r \frac{d^r}{d\lambda^r}\|f\|_{\lambda,0}\frac{d^{a-r}}{d\lambda^{a-r}}\left(\|v\|_{\lambda,1}\|w\|_{\lambda,1}\right).$$

Then we see that

$$\begin{aligned}
&\sum_{a\in\mathbb{N}} \frac{1}{(a!)^2}\sum_{r=0}^{a} C_a^r \frac{d^r}{d\lambda^r}\|f\|_{\lambda,0}\frac{d^{a-r}}{d\lambda^{a-r}}\left(\|v\|_{\lambda,1}\|w\|_{\lambda,1}\right)\\
&= \sum_{r\in\mathbb{N}}\|f\|_{\lambda,r}\sum_{a=r}^{+\infty}\frac{C_a^r}{(a!)^2}\frac{d^{a-r}}{d\lambda^{a-r}}\left(\|v\|_{\lambda,1}\|w\|_{\lambda,1}\right)\quad \left(\text{since } \frac{d^p}{d\lambda^p}\|\psi\|_{\lambda,0} = \|\psi\|_{\lambda,p} \text{ by definition}\right)\\
&= \sum_{r\in\mathbb{N}}\|f\|_{\lambda,r}\sum_{a=r}^{+\infty}\frac{C_a^r}{(a!)^2}\sum_{q=0}^{a-r} C_{a-r}^q\left(\|v\|_{\lambda,q+1}\|w\|_{\lambda,a-q-r+1}\right)\\
&= \sum_{r\in\mathbb{N}}\|f\|_{\lambda,r}\sum_{a=0}^{+\infty}\sum_{q=0}^{a}\frac{C_{a+r}^r C_a^q}{((a+r)!)^2}\left(\|v\|_{\lambda,q+1}\|w\|_{\lambda,a-q+1}\right)\quad (\text{by a change of variables})\\
&= \sum_{r\in\mathbb{N}}\|f\|_{\lambda,r}\sum_{q=0}^{+\infty}\left(\|v\|_{\lambda,q+1}\sum_{a=q}^{+\infty}\|w\|_{\lambda,a-q+1}\frac{C_{a+r}^r C_a^q}{((a+r)!)^2}\right)\\
&= \sum_{r\in\mathbb{N}}\|f\|_{\lambda,r}\sum_{q=0}^{+\infty}\left(\|v\|_{\lambda,q+1}\sum_{a=0}^{+\infty}\|w\|_{\lambda,a+1}\frac{C_{a+r+q}^r C_{a+q}^q}{((a+r+q)!)^2}\right).
\end{aligned}$$

Thus,

$$\begin{aligned}
&\sum_{a\in\mathbb{N}} \frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|f\|_{\lambda,0}\|v\|_{\lambda,1}\|w\|_{\lambda,1}\right)\\
&= \sum_{r\in\mathbb{N}}\frac{\|f\|_{\lambda,r}}{(r!)^2}\sum_{q=0}^{+\infty}\frac{(q+1)^2}{((q+1)!)^2}\|v\|_{\lambda,q+1}\sum_{a=0}^{+\infty}\frac{(a+1)^2}{((a+1)!)^2}\|w\|_{\lambda,a+1}\frac{(r!)^2((q+1)!)^2((a+1)!)^2C_{a+r+q}^r C_{a+q}^q}{(a+1)^2(q+1)^2((a+r+q)!)^2},
\end{aligned}$$

where

$$\frac{(r!)^2((q+1)!)^2((a+1)!)^2C_{a+r+q}^r C_{a+q}^q}{(a+1)^2(q+1)^2((a+r+q)!)^2} = \frac{a!q!r!}{(a+q+r)!}.$$

The claim follows since $\frac{a!q!r!}{(a+q+r)!} \le 1,\ \forall\, a,q,r \in \mathbb{N}$. ∎

Proof of Lemma 2.11. One has

$$\sum_{a\in\mathbb{N}} \frac{1}{(a!)^2}\frac{d^a}{d\lambda^a}\left(\|f\|_{\lambda,1}\|v\|_{\lambda,1}\right) = \sum_{a\in\mathbb{N}} \frac{1}{(a!)^2}\left(\sum_{r=0}^{a} C_a^r \frac{d^r}{d\lambda^r}\|f\|_{\lambda,1}\frac{d^{a-r}}{d\lambda^{a-r}}\|v\|_{\lambda,1}\right)$$
$$= \sum_{a\in\mathbb{N}} \frac{1}{(a!)^2}\left(\sum_{r=0}^{a} C_a^r \|f\|_{\lambda,r+1}\|v\|_{\lambda,a-r+1}\right) \quad \left(\text{since } \frac{d^p}{d\lambda^p}\|\psi\|_{\lambda,0} = \|\psi\|_{\lambda,p} \text{ by definition}\right)$$
$$= \sum_{r\in\mathbb{N}} \|f\|_{\lambda,r+1}\left(\sum_{a=r}^{+\infty} \frac{C_a^r}{(a!)^2}\|v\|_{\lambda,a-r+1}\right)$$
$$= \sum_{r\in\mathbb{N}} \|f\|_{\lambda,r+1}\left(\sum_{a=0}^{+\infty} \frac{C_{a+r}^r}{((a+r)!)^2}\|v\|_{\lambda,a+1}\right)$$
$$= \sum_{r\in\mathbb{N}} \frac{\|f\|_{\lambda,r+1}}{((r+1)!)^2}\sum_{a\in\mathbb{N}} \frac{\|v\|_{\lambda,a+1}}{((a+1)!)^2}\left(\frac{C_{a+r}^r((a+1)!)^2((r+1)!)^2}{((a+r)!)^2}\right),$$

where

$$\frac{C_{a+r}^r((a+1)!)^2((r+1)!)^2}{((a+r)!)^2} = \frac{(a+1)(r+1)(a+1)!(r+1)!}{(a+r)!}.$$

As in [12], we observe that when $a \geq 2$ and $r \geq 2$ one has

$$\frac{(r+1)(a+1)(a+1)!(r+1)!}{(a+r)!} \leq 24.$$

Indeed, for $r \geq 2$ and $a \geq 3$,

$$\frac{(r+1)(a+1)(a+1)!(r+1)!}{(a+r)!} = \frac{1\times\cdots\times(r+1)\times(r+1)}{1\times\cdots\times r\times(r+1)\times(r+2)}\times 1\times 2\times 3\times 4\times\frac{5\times\cdots\times a\times(a+1)\times(a+1)}{(r+3)\cdots\times(a+r-1)\times(a+r)}$$
$$\leq 4!\times\frac{5\times\cdots\times(a+1)\times(a+1)}{(r+3)\times\cdots\times(a+r-1)\times(a+r)}$$
$$\leq 24,$$

where $a \geq 3$ was used in the first expansion and $r \geq 2$ in the second inequality. If $r \geq 2$ and $a = 2$ then

$$\frac{(r+1)(a+1)(a+1)!(r+1)!}{(a+r)!} = 18\times\frac{(r+1)(r+1)!}{(r+2)!} \leq 18.$$

If $a \leq 1$ or $r \leq 1$ we have to separate the corresponding terms in the estimation. Then, we get

$$\sum_{a\in\mathbb{N}} \frac{1}{(a!)^2}\sum_{r=0}^{a} C_a^r\|f\|_{\lambda,r+1}\|v\|_{\lambda,a-r+1} = \sum_{r\in\mathbb{N}} \frac{\|f\|_{\lambda,r+1}}{((r+1)!)^2}\sum_{a=0}^{+\infty}\frac{\|v\|_{\lambda,a+1}}{((a+1)!)^2}\left(\frac{(r+1)(a+1)(a+1)!(r+1)!}{(a+r)!}\right)$$
$$= \|v\|_{\lambda,1}\sum_{r\in\mathbb{N}}\frac{\|f\|_{\lambda,r+1}}{((r+1)!)^2}(r+1)^2 + 4\|v\|_{\lambda,2}\sum_{r\in\mathbb{N}}\frac{\|f\|_{\lambda,r+1}}{((r+1)!)^2}(r+1)$$
$$+ \|f\|_{\lambda,1}\sum_{a\geq 2}\frac{\|v\|_{\lambda,a+1}}{((a+1)!)^2}(a+1)^2 + 4\|f\|_{\lambda,2}\sum_{a\geq 2}\frac{\|v\|_{\lambda,a+1}}{((a+1)!)^2}(a+1)$$
$$+ 24\sum_{r\geq 2}\frac{\|f\|_{\lambda,r+1}}{((r+1)!)^2}\sum_{a\geq 2}\frac{\|v\|_{\lambda,a+1}}{((a+1)!)^2}$$
$$\leq \left(\|v\|_{\lambda,1} + 4\|v\|_{\lambda,2}\right)\|f\|_{\widetilde{\mathcal{H}},\lambda} + \left(\|f\|_{\lambda,1} + 4\|f\|_{\lambda,2}\right)\|v\|_{\widetilde{\mathcal{H}},\lambda} + 24\sum_{r\geq 2}\frac{\|f\|_{\lambda,r+1}}{((r+1)!)^2}\sum_{a\geq 2}\frac{\|v\|_{\lambda,a+1}}{((a+1)!)^2}$$

Since $\sum_{a\geq 3}\frac{\|v\|_{\lambda,a}}{(a!)^2}\leq \|v\|_{\widetilde{\mathcal{H}},\lambda}\wedge\|v\|_{\mathcal{H},\lambda}$ and $\sum_{r\geq 3}\frac{\|f\|_{\lambda,r}}{(r!)^2}\leq \|f\|_{\widetilde{\mathcal{H}},\lambda}\wedge\|f\|_{\mathcal{H},\lambda}$, the latter expression can be bounded above by

$$\left(\|v\|_{\lambda,1}+4\|v\|_{\lambda,2}+12\sum_{a\geq 3}\frac{\|v\|_{\lambda,a}}{(a!)^2}\right)\|f\|_{\widetilde{\mathcal{H}},\lambda}+\left(\|f\|_{\lambda,1}+4\|f\|_{\lambda,2}+12\sum_{r\geq 3}\frac{\|f\|_{\lambda,r}}{(r!)^2}\right)\|v\|_{\widetilde{\mathcal{H}},\lambda}$$
$$\leq 16(\|f\|_{\widetilde{\mathcal{H}},\lambda}\|v\|_{\mathcal{H},\lambda}+\|f\|_{\mathcal{H},\lambda}\|v\|_{\widetilde{\mathcal{H}},\lambda})$$

and with the bound $\|w\|_{\lambda,1}+4\|w\|_{\lambda,2}+12\sum_{a\geq 3}\frac{\|w\|_{\lambda,a}}{(a!)^2}\leq 16\|w\|_{\mathcal{H},\lambda}$ for $w=f,v$ we establish (i). Using the bound $\|v\|_{\lambda,1}+4\|v\|_{\lambda,2}+12\sum_{a\geq 3}\frac{\|v\|_{\lambda,a}}{(a!)^2}\leq 4\|v\|_{\widetilde{\mathcal{H}},\lambda}$ we alternatively obtain (ii). □

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