Real and complex rank for real symmetric tensors with low complex symmetric rank

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REAL AND COMPLEX RANK FOR REAL SYMMETRIC TENSORS WITH LOW COMPLEX SYMMETRIC RANK

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ABSTRACT. We study the case of real homogeneous polynomial P whose minimal real and complex decompositions in terms of powers of linear forms are different. In particularly we will show that, if the sum of the complex and the real ranks of P is smaller or equal than $3\deg(P)-1$, then the difference of the two decompositions is completely determined either on a line or on a conic.

INTRODUCTION

The tensor decomposition problem into a minimal sum of rank-1 terms, is raising interest and attention from many applied areas as signal processing for telecommunications [13], independent component analysis [9], complexity of matrix multiplication [18], complexity problem of P versus NP [19], quantum physics [14] and phylogenetic [1]. The particular instance in which the tensor is symmetric and hence representable by a homogeneous polynomial, is one of the most studied and developed one (cfr. [15] and references therein). In this last case we say that the rank of a homogeneous polynomial P of degree d is the minimum integer r needed to write it as a linear combination of pure powers of linear forms $L_1, \ldots, L_r$:

$$P = L_1^d + \cdots + L_r^d \tag{1}$$

Most of the papers concerning the abstract theory of the symmetric tensor rank, require that the base field is algebraically closed. However, for the applications, it is very important to consider the case of real polynomials and look at their real decomposition. Namely, one can study separately the case in which the linear forms appearing in (1) are complex or reals. When we look for a minimal complex (resp. real) decomposition as in (1) we say that we are computing the *complex symmetric rank* (resp. *real symmetric rank*) of P and we will indicate it $r_{\mathbb{C}}(P)$ (resp. $r_{\mathbb{R}}(P)$). Obviously

$$r_{\mathbb{C}}(P) \leq r_{\mathbb{R}}(P),$$

and in many cases, such an equality is strict.

In [10] P. Comon and G. Ottaviani studied the real case for bivariate symmetric tensors. Even in this case there are many open conjectures and, up to now, few cases completely settled ([10], [7], [2]).

In this paper we want to study the relation among $r_{\mathbb{C}}(P)$ and $r_{\mathbb{R}}(P)$ in the special circumstance in which $r_{\mathbb{C}}(P) < r_{\mathbb{R}}(P)$. In particular we will show that, in a certain

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range (say, $r_{\mathbb{C}}(P)+r_{\mathbb{R}}(P)\le 3\deg(P)-1$), all homogeneous polynomials P of that degree with $r_{\mathbb{R}}(P)\ne r_{\mathbb{C}}(P)$ are characterized by the existence of a curve with the property that the sets evincing the real and the complex ranks coincide out of it (see Theorem 1 and Proposition 1 for the precise statement). Let us make an example: let $P\in S^d\mathbb{R}^{m+1}$ be a real homogeneouspolynomial of degree d in $m+1$ variables such that $r_{\mathbb{C}}(P)<r_{\mathbb{R}}(P)$ and $r_{\mathbb{C}}(P)+r_{\mathbb{R}}(P)\le 3\deg(P)-1$; therefore its real and complex decomposition are

$$P = L_1^d+\cdots+L_k^d+M_1^d+\cdots+M_s^d,$$
$$P = N_1^d+\cdots+N_h^d+M_1^d+\cdots+M_s^d$$

respectively, with $k>h$, $M_1,\dots,M_s\in S^1\mathbb{R}^{m+1}$, $h+k>d+2$ and there exist either a line or a conic $C\subset\mathbb{P}(S^1\mathbb{R}^{m+1})$ such that $[L_1],\dots,[L_k],[N_1],\dots,[N_h]\in C\subset\mathbb{P}(S^1\mathbb{R}^{m+1})$. If C is a line (item (a) in Theorem 1) then both the L_i's and the N_i's are bilinear forms in the same two "variables". If C is a reduced conic (item (b) in Theorem 1) then the L_i's and the N_i's are both bilinear forms but in a set of 3 "variables" (for example $L_i\in\mathbb{R}[x,y]$ while $N_i\in\mathbb{C}[y,z]$). If C is a smooth conic, then L_i's and N_i's depends on 3 "variables" and their projectivizations lie on C.

1. Notation and statements

Before giving the precise statement of Theorem 1 we need to introduce the main algebraic geometric tools that we will use all along the paper.

Let $\nu_d:\mathbb{P}^m\to\mathbb{P}^N$, $N:=\binom{m+d}{d}-1$, denote the degree d Veronese embedding of $\mathbb{P}^m$ (say, defined over $\mathbb{C}$). Set $X_{m,d}:=\nu_d(\mathbb{P}^m)$. For any $P\in\mathbb{P}^N$, the *symmetric rank* or *symmetric tensor rank* or, just, the *rank* $r_{\mathbb{C}}(P)$ of P is the minimal cardinality of a finite set $S\subset X_{m,d}$ such that $P\in\langle S\rangle$, where $\langle\ \ \rangle$ denote the linear span (here the linear span is with respect to complex coefficients), and we will say that S *evince* $r_{\mathbb{C}}(P)$. Notice that the Veronese embedding ν_d is defined over $\mathbb{R}$, i.e. $\nu_d(\mathbb{P}^m(\mathbb{R}))\subset\mathbb{P}^N(\mathbb{R})$. For each $P\in\mathbb{P}^N(\mathbb{R})$ the *real symmetric rank* $r_{\mathbb{R}}(P)$ of P is the minimal cardinality of a finite set $S\subset\mathbb{P}^m(\mathbb{R})$ such that $P\in\langle\nu_d(S)\rangle_{\mathbb{R}}$, where $\langle\ \ \rangle_{\mathbb{R}}$ means the linear span with real coefficients, and we will say that S evince $r_{\mathbb{C}}(P)$. The integers $r_{\mathbb{R}}(P)$ is well-defined, because $\nu_d(\mathbb{P}^m(\mathbb{R}))$ spans $\mathbb{P}^N(\mathbb{R})$.

Let us fix some notation: If $C,S\subset\mathbb{P}^m$ is either a curve or a subspace and $S\subset\mathbb{P}^m$ is a set such that $\nu_d(S)$ evinces the rank (complex or real) of a point $P\in\mathbb{P}^N$, we will use the following abbreviations:

$$S_C := S\cap C,$$
$$S_{\hat{C}} := S\setminus(S\cap C).$$

Theorem 1. *Let $P\in\mathbb{P}^N(\mathbb{R})$ be such that $r_{\mathbb{C}}(P)+r_{\mathbb{R}}(P)\le 3d-1$ and $r_{\mathbb{C}}(P)\ne r_{\mathbb{R}}(P)$. Fix any set $S_{\mathbb{C}}\subset\mathbb{P}^m(\mathbb{C})$ and $S_{\mathbb{R}}\subset\mathbb{P}^m(\mathbb{R})$ such that $\nu_d(S_{\mathbb{C}})$ and $\nu_d(S_{\mathbb{R}})$ evince $r_{\mathbb{C}}(P)$ and $r_{\mathbb{R}}(P)$ respectively. Then one of the following cases (a), (b), (c) occurs:*

(a) There is a line $l\subset\mathbb{P}^m$ defined over $\mathbb{R}$ and with the following properties:

(i) $S_{\mathbb{C}}$ and $S_{\mathbb{R}}$ coincide out of the line l in a set $S_{\hat{l}}$:

$$S_{\mathbb{C}}\setminus S_{\mathbb{C}}\cap l = S_{\mathbb{R}}\setminus S_{\mathbb{R}}\cap l =: S_{\hat{l}};$$

(ii) Then there is a point $P_l\in\langle\nu_d(S_{\mathbb{C},l})\rangle\cap\langle\nu_d(S_{\mathbb{R},l})\rangle$ such that $\nu_d(S_{\mathbb{C},l})$ evinces $r_{\mathbb{C}}(P_l)$, $\nu_d(S_{\mathbb{R},l})$ evinces $r_{\mathbb{R}}(P_l)$;

(iii) $\sharp(S_{\mathbb{C},l}\cup S_{\mathbb{R},l})\ge d+2$ and $\sharp(S_{\mathbb{C},l})<\sharp(S_{\mathbb{R},l})$.

(b) There is a conic $C \subset \mathbb{P}^m$ defined over $\mathbb{R}$ and with the following properties:
  (i) $S_{\mathbb{C}}$ and $S_{\mathbb{R}}$ coincide out of the conic C in a set $S_{\hat{C}}$:

$$S_{\mathbb{C}} \setminus S_{\mathbb{C},C} = S_{\mathbb{R}} \setminus S_{\mathbb{R}} \cap C =: S_{\hat{C}};$$

  (ii) There is a point $P_C \in \langle \nu_d(S_{\mathbb{C},C}) \rangle \cap \langle \nu_d(S_{\mathbb{R},C}) \rangle$ such that $\nu_d(S_{\mathbb{C},C})$ evinces $r_{\mathbb{C}}(P_C)$, $\nu_d(S_{\mathbb{R},C})$ evinces $r_{\mathbb{R}}(P_C)$;
  (iii) $\sharp(S_{\mathbb{C},C} \cup S_{\mathbb{R},C}) \geq 2d+2$ and $\sharp(S_{\mathbb{C},C}) < \sharp(S_{\mathbb{R},C})$;
  (iv) If C is reducible, say $C = l_1 \cup l_2$ with $Q = l_1 \cap l_2$, then $\sharp((S_{\mathbb{C}} \cup S_{\mathbb{R}}) \cap (l_i \setminus Q)) \geq d+1$ for $i \in \{1,2\}$.

(c) $m \geq 3$ and there are 2 disjoint lines $l, r \subset \mathbb{P}^m$ defined over $\mathbb{R}$ with the following properties:
  (i) $S_{\mathbb{C}}$ and $S_{\mathbb{R}}$ coincide out of the union $\Gamma := l \cap r$ in a set $S_{\hat{\Gamma}}$:

$$S_{\mathbb{C}} \setminus S_{\mathbb{C}} \cap (l \cup r) = S_{\mathbb{R}} \setminus S_{\mathbb{R}} \cap (l \cup r) := S_{\hat{\Gamma}};$$

  (ii) $\sharp(S_{\mathbb{C},l} \cup S_{\mathbb{R},r}) \geq d+2$ and $\sharp(S_{\mathbb{C},r} \cup S_{\mathbb{R},r}) \geq d+2$.

Proposition 1. *Take $(P, S_{\mathbb{C}}, S_{\mathbb{R}})$ as in Theorem 1.*

(A) If $(P, S_{\mathbb{C}}, S_{\mathbb{R}})$ is as in case (a), then $\nu_d(S_{\mathbb{C},l})$ (resp. $\nu_d(S_{\mathbb{R},l})$) evinces the complex (resp. real) symmetric rank of P_l.

(B) If $(P, S_{\mathbb{C}}, S_{\mathbb{R}})$ is as in (b) with C a smooth conic, then $\nu_d(S_{\mathbb{C},C})$ (resp. $\nu_d(S_{\mathbb{R},C})$) evinces the complex (resp. real) symmetric rank of P_C.

(C) If $(P, S_{\mathbb{C}}, S_{\mathbb{R}})$ is as in case (c) with respect to the disjoint lines l, r, then:
  (I) The set $\langle \nu_d(S_{\hat{\Gamma}}) \rangle \cap \langle \nu_d(\Gamma) \rangle$ is a single point, $O_\Gamma \in \mathbb{P}^N(\mathbb{R})$;
  $\nu_d(S_{\mathbb{C},\Gamma})$ evinces $r_{\mathbb{C}}(O_\Gamma)$ and $\nu_d(S_{\mathbb{R},\Gamma})$ evinces O_Γ.
  (II) The set $\langle \{O_\Gamma\} \cup \nu_d(l) \rangle \cap \langle \nu_d(l) \rangle$ (resp. $\langle \{O_\Gamma\} \cup \nu_d(r) \rangle \cap \langle \nu_d(r) \rangle$) is formed by a unique point $O_l \in \mathbb{P}^N(\mathbb{R}$ (resp. $O_r \in \mathbb{P}^N(\mathbb{R})$);
  $S_{\mathbb{C},l}$ (resp. $S_{\mathbb{C},r}$) evinces $r_{\mathbb{C}}(O_l)$ (resp. $r_{\mathbb{C}}(O_r)$);
  $S_{\mathbb{R},l}$ (resp. $S_{\mathbb{R},r}$) evinces $r_{\mathbb{R}}(O_l)$ (resp $r_{\mathbb{R}}(O_r)$).

2. The proofs

Remark 1. Let $S \subset \mathbb{P}^N(\mathbb{R})$. It is noteworthy that S can be used to spann both a real space $\langle S \rangle_{\mathbb{R}} \subset \mathbb{P}^N(\mathbb{R})$ and a complex space $\langle S \rangle_{\mathbb{C}} \subset \mathbb{P}^N(\mathbb{C})$ of the same dimension and $\langle S \rangle_{\mathbb{C}} \cap \mathbb{P}^N(\mathbb{R}) = \langle S \rangle_{\mathbb{R}}$. In the following we will always use $\langle\ \rangle$ to denote $\langle S \rangle_{\mathbb{C}}$

Remark 2. Fix $P \in \mathbb{P}^N$ and a finite set $S \subset \mathbb{P}^N$ such that S evinces P. Fix any $E \subsetneq S$. Then the set $\langle \{P\} \cup E \rangle \cap \langle S \setminus E \rangle$ is a single point (call it P_1). If S evinces $r_{\mathbb{C}}(P)$, then $S \setminus E$ evinces $r_{\mathbb{C}}(P_1)$. Now assume $P \in \mathbb{P}^N(\mathbb{R})$ and $S \subset \mathbb{P}^N(\mathbb{R})$. Then $P_1 \in \mathbb{P}^N(\mathbb{R})$. If S evinces $r_{\mathbb{R}}(P)$, then $S \setminus E$ evinces $r_{\mathbb{R}}(P_1)$.

Lemma 1. *Let $C \subset \mathbb{P}^m$ be a reduced curve of degree t with $t = 1, 2$. Consider the finite sets $A, B \subset \mathbb{P}^m$ with B reduced. Indicate $A \cup B \setminus (A \cup B) \cap C)$ with $(A \cup B)_{\hat{C}}$. Assume that for $d > t$ we have that:*

$$h^1(\mathcal{I}_{(A \cup B)_{\hat{C}}}(d-t)) = 0.$$

Assume the existence of $P \in \langle \nu_d(A) \rangle \cap \langle \nu_d(B) \rangle$ and $P \notin \langle \nu_d(S') \rangle$ for any $S' \subsetneq A$ and any $S' \subsetneq B$. Then

$$A_{\hat{C}} = B_{\hat{C}}$$

where $A_{\hat{C}} := A \setminus A \cap C$ and $B_{\hat{C}} := B \setminus B \cap C$.

Proof. The case $t = 1$ is [4, Lemma 8]. If $t = 2$, then either C is a conic or $m \ge 3$ and C is a disjoint union of 2 lines. In both cases we have $h^0(\mathcal{I}_C(t)) > 0$ and the linear system $|\mathcal{I}_C(t)|$ has no base points outside C. Since $A \cup B$ is a finite set, there is $M \in |\mathcal{I}_C(t)|$ such that $M \cap (A \cup B) = C \cap (A \cup B)$. Look at the residual exact sequence (also called the Castelnuovo's exact sequence):

$$0 \to \mathcal{I}_{(A\cup B)_{\hat{C}}}(d-t) \to \mathcal{I}_{A\cup B}(d) \to \mathcal{I}_{(A\cup B)\cap M, M}(d) \to 0 \tag{2}$$

We can now repeat the same proof of [4, Lemma 8] but starting with (2) instead of the exact sequence used there (cfr. first displayed formula in the proof of [4, Lemma 8]). We will therefore get $A_{\hat{M}} = B_{\hat{M}}$. Now, since $M \cap (A \cup B) = C \cap (A \cup B)$, we are done. □

We are now going to prove Theorem 1 together with Proposition 1. Since the proof is long and hinged, we put margin right notes to help the read in keeping track of where we prove each item of Theorem 1 and Proposition 1. We will use the acronym "tbc" to mean that the proof of that item has "to be completed".

Proofs of Theorem 1 and of Proposition 1

Fix $P \in \mathbb{P}^N(\mathbb{R})$ such that $r_{\mathbb{C}}(P) + r_{\mathbb{R}}(P) \le 3d - 1$ and $r_{\mathbb{C}}(P) \ne r_{\mathbb{R}}(P)$.
Fix any set $S_{\mathbb{C}} \subset \mathbb{P}^m(\mathbb{C})$ such that $\nu_d(S_{\mathbb{C}})$ evinces $r_{\mathbb{C}}(P)$ and any $S_{\mathbb{R}} \subset \mathbb{P}^m(\mathbb{R})$ such that $\nu_d(S_{\mathbb{R}})$ evinces $r_{\mathbb{R}}(P)$.
By applying [3], Lemma 1, we immediately get that

$$h^1(\mathcal{I}_{S_{\mathbb{C}}\cup S_{\mathbb{R}}}(d)) > 0.$$

Since $\sharp(S_{\mathbb{C}}) + \sharp(S_{\mathbb{R}}) \le 3d-1$, either there is a line $l \subset \mathbb{P}^m$ such that $\sharp(S_{\mathbb{C},} \cup S_{\mathbb{R},l}) \ge d+2$ or there is a conic C such that $\sharp(S_{\mathbb{C},C} \cup S_{\mathbb{R},C}) \ge 2d+2$ ([12], Theorem 3.8). We are going to study separately these two cases in items (1) and (2) below.

(1) In this step we assume the existence of a line $l \subset \mathbb{P}^m$ such that

$$\sharp(S_{\mathbb{C},l} \cup S_{\mathbb{R},l}) \ge d+2.$$

This hypothesis, together with $r_{\mathbb{R}}(P) \ne r_{\mathbb{C}}(P)$, immediately implies case (aiii) of the statement of the theorem. [Thm.1, aiii]

We are now going to distinguish the case $h^1(\mathcal{I}_{S_{\mathbb{C}}\cup S_{\mathbb{R}}}(d-1)) = 0$ (item (1.1) below) from the case $h^1(\mathcal{I}_{S_{\mathbb{C}}\cup S_{\mathbb{R}}}(d-1)) > 0$ (item (1.2) below).

(1.1) Assume $h^1(\mathcal{I}_{S_{\mathbb{C}}\cup S_{\mathbb{R}}}(d-1)) = 0$.

First of all, observe that the line $l \subset \mathbb{P}^m$ is well defined over $\mathbb{R}$ since it contains at least 2 points of $S_{\mathbb{R}}$ (Remark 1). Then, by Lemma 1, we have that $S_{\mathbb{C}}$ and $S_{\mathbb{R}}$ have to coincide out of the line l:

$$S_{\mathbb{C}} \setminus S_{\mathbb{C},l} = S_{\mathbb{R}} \setminus S_{\mathbb{R},l} := S_{\hat{l}},$$

and this proves (ai) of the statement of the theorem in this case (1.1). [Thm.1, ai (tbc)]

The fact that $\sharp(S_{\mathbb{C}}) < \sharp(S_{\mathbb{R}})$ implies that $\sharp(S_{\mathbb{R},l}) > (d+2)/2$ and $\sharp(S_{\hat{l}}) \le d$, hence $h^1(\mathcal{I}_{S_{\hat{l}}\cup l}(d)) = 0$. Therefore we have that $\dim\langle \nu_d(S_{\hat{l}} \cup l)\rangle = \sharp(S_{\hat{l}}) + d + 1$, $\dim(\langle \nu_d(S_{\hat{l}})\rangle) = \sharp(S_{\hat{l}}) - 1$ and $\dim(\langle \nu_d(l)\rangle) = d+1$, and Grassmann's formula gives $\langle \nu_d(S_{\hat{l}})\rangle \cap \langle \nu_d(l)\rangle = \emptyset$.
Since $P \in \langle \nu_d(S_{\hat{l}} \cup S_{\mathbb{C}})\rangle$ and $S_{\mathbb{C},l} \subset l$, the set $\langle \nu_d(S_{\hat{l}}) \cup \{P\}\rangle \cap \langle \nu_d(l)\rangle$ is a single point, $P_l \in \mathbb{P}^N(\mathbb{R})$.
Since $P \in \langle \nu_d(S_{\mathbb{C}})\rangle$ and $P \notin \langle \nu_d(S'_{\mathbb{C}})\rangle$ for any $S' \subsetneq S_{\mathbb{C}}$, the set $\langle \nu_d(S_{\hat{l}}) \cup \{P\}\rangle \cap$

$\langle \nu_d(S_{\mathbb{C},l})\rangle$ is a single point, P_C (Remark 2). Then obviously:

$$P_C = P_l \in \mathbb{P}^N(\mathbb{R}).$$

Since $\nu_d(S_{\mathbb{C}})$ evinces $r_{\mathbb{C}}(P)$, then $\nu_d(S_{\mathbb{C},l})$ evinces P_l (Remark 2). In the same way we see that $\langle \nu_d(S_{\hat{l}}) \cup \{P\}\rangle \cap \langle \nu_d(S_{\mathbb{R},l})\rangle = \{P_l\}$ and that $\nu_d(S_{\mathbb{R},l})$ evinces $r_{\mathbb{R}}(P_l)$. This proves (aii) of Theorem 1 and (A) of Proposition 1 in this case (1.1). [Thm.1, aii (tbc)]

(1.2) Assume $h^1(\mathcal{I}_{S_{\mathbb{C}}\cup S_{\mathbb{R}}}(d-1)) > 0$. [Prop.1, A (tbc)]

First of all, observe that there exists a line $r \subset \mathbb{P}^m$ such that $\sharp(r \cap (S_{\mathbb{C}} \cup S_{\mathbb{R}})_{\hat{l}}) \geq d+1$, because $\sharp(S_{\mathbb{C}} \cup S_{\mathbb{R}})_{\hat{l}} \leq 3d-1-d-2 \leq 2(d-1)+1$.
By the same reason, if we write: $C := l \cup r$, we get that $\sharp(S_{\mathbb{C}} \cup S_{\mathbb{R}})_{\hat{C}} \leq 3d-1-d-2-d-1 \leq d-2$ and hence $h^1(\mathcal{I}_{(S_{\mathbb{C}}\cup S_{\mathbb{R}})_{\hat{C}}}(d-2)) = 0$ (e.g. by [5], Lemma 34, or by [12], Theorem 3.8). Lemma 1 gives:

$$S_{\mathbb{C},\hat{C}} = S_{\mathbb{R},\hat{C}}. \tag{3}$$

– Assume for the moment $l \cap r \neq \emptyset$. In this case, Remark 2 shows that we are in case (b) of the statement of the theorem. Therefore (3) proves (bi) in the case that the conic C in (b) in the statement of the theorem is reduced. Moreover, condition (biv) is satisfied, because $\sharp(r \cap (S_{\mathbb{C}} \cup S_{\mathbb{R}})_{\hat{l}}) \geq d+1)$. [Thm.1, bi (tbc)] [Thm.1, biv]

– Now assume $l \cap r = \emptyset$. We will check that we are in case (a) with respect to the line l if $\sharp(S_{\mathbb{C},r} \cup S_{\mathbb{R},r}) = d+1$, while we are in case (c) with respect to the lines l and r if $\sharp(S_{\mathbb{C},r} \cup S_{\mathbb{R},r}) \geq d+2$, and the case $\sharp(S_{\mathbb{C},l} \cup S_{\mathbb{R},l}) = \sharp(S_{\mathbb{C},r} \cup S_{\mathbb{R},r}) = d+1$ is not occuring.

Set $\Gamma := l \cup r$. Since $r \cap l = \emptyset$, we have $\dim\langle\Gamma\rangle = 3$ and hence $m \geq 3$.

— Assume for the moment $m \geq 4$. Hence $\sharp(S_{\mathbb{C}} \cup S_{\mathbb{R}})_{\widehat{\langle\Gamma\rangle}} \leq d$ and $h^1(\mathcal{I}_{(S_{\mathbb{C}}\cup S_{\mathbb{R}})_{\widehat{\langle\Gamma\rangle}}}(d-1)) = 0$. Therefore:

$$S_{\mathbb{C},\widehat{\langle\Gamma\rangle}} = S_{\mathbb{R},\widehat{\langle\Gamma\rangle}} \tag{4}$$

and the set $\langle \{P\} \cup \nu_d(S_{\mathbb{C},\widehat{\langle\Gamma\rangle}})\rangle \cap \langle \nu_d(\langle\Gamma\rangle)\rangle$ is a single real point:

$$O := \langle \{P\} \cup \nu_d(S_{\mathbb{C},\widehat{\langle\Gamma\rangle}})\rangle \cap \langle \nu_d(\langle\Gamma\rangle)\rangle \in \mathbb{P}^N(\mathbb{R}), \tag{5}$$

$\nu_d(S_{\mathbb{C},\Gamma})$ evinces $r_{\mathbb{C}}(O)$ and $\nu_d(S_{\mathbb{R},\Gamma})$ evinces $r_{\mathbb{R}}(O)$. Now, (4) implies that if we are either in case (a) or in case (c) of the theorem, we can simply study what happens at $S_{\mathbb{C},\langle\Gamma\rangle}$ and at $S_{\mathbb{R},\langle\Gamma\rangle}$, which means that we can reduce our study to the case $m = 3$, since $\langle\Gamma\rangle = \mathbb{P}^3$.

— Until step (2) below, we will assume $m = 3$.

The linear system $|\mathcal{I}_\Gamma(2)|$ on $\langle\Gamma\rangle$ has no base points outside Γ itself. Since $S_{\mathbb{C}} \cup S_{\mathbb{R}}$ is finite, there is a smooth quadric surface W containing Γ such that

$$S_{\mathbb{C},W} \cup S_{\mathbb{R},W} = S_{\mathbb{C},\Gamma} \cup S_{\mathbb{R},\Gamma}.$$

Moreover, such a W can be found among the real smooth quadrics, since l and r are real lines.
Since $\sharp(S_{\mathbb{C},\langle\Gamma\rangle} \cup S_{\mathbb{R},\langle\Gamma\rangle})_{\hat{W}} \leq d-1$, we have $h^1(\mathcal{I}_{(S_{\mathbb{C}}\cup S_{\mathbb{R}})_{\hat{W}}}(d-2)) = 0$. Hence Lemma 1 applied to the point O defined in (5), gives:

$$(S_{\mathbb{C},\langle\Gamma\rangle})_{\hat{W}} = (S_{\mathbb{R},\langle\Gamma\rangle})_{\hat{W}},$$

$\langle\{O\}\cup\nu_d(S_{\mathbb{C},\langle\Gamma,\rangle})_{\hat{W}}\rangle\cap\langle\nu_d(W)\rangle$ is a single real point

$$O'=\langle\{O\}\cup\nu_d(S_{\mathbb{C},\langle\Gamma,\rangle})_{\hat{W}}\rangle\cap\langle\nu_d(W)\rangle\in\mathbb{P}^N(\mathbb{R}),\tag{6}$$

and $\nu_d(S_{\mathbb{C},W})$ evinces $r_{\mathbb{C}}(O')$. If $(O',S_{\mathbb{C},W},S_{\mathbb{R},W})$ is either as in case (a) or in case (c) of the statement of the theorem, then $(O,S_{\mathbb{C},\langle\Gamma\rangle},S_{\mathbb{R},\langle\Gamma\rangle})$ is in the same case. Consider the the system $|(1,0)|$ of lines on the smooth quadric surface W containing Γ. In this contest we have that $h^1(W,\mathcal{O}_W(d-2,d))=0$, and hence the restriction map $H^0(W,\mathcal{O}_W(d))\to H^0(\Gamma,\mathcal{O}_\Gamma(d))$ is surjective. Therefore

$$h^1(W,\mathcal{I}_{W\cap(S_{\mathbb{C}}\cup S_{\mathbb{R}})}(d))=h^1(l,\mathcal{I}_{l\cap(S_{\mathbb{C}}\cup S_{\mathbb{R}}),l}(d))+h^1(r,\mathcal{I}_{r\cap(S_{\mathbb{C}}\cup S_{\mathbb{R}}),r}(d)).$$

Now, this last equality, together with the facts that $\nu_d(S_{\mathbb{C},W})$ and $\nu_d(S_{\mathbb{R},W})$ are linearly independent and $(S_{\mathbb{C}}\cup S_{\mathbb{R}})_W\subset\Gamma$, gives:

$$\begin{aligned}&\dim(\langle\nu_d(S_{\mathbb{C},W})\rangle\cap\langle\nu_d(S_{\mathbb{R},W})\rangle)=\sharp(S_{\mathbb{C}}\cap S_{\mathbb{R}})_l+\sharp(S_{\mathbb{C}}\cap S_{\mathbb{R}})_r+\\&h^1(l,\mathcal{I}_{l\cap(S_{\mathbb{C}}\cup S_{\mathbb{R}}),l}(d))+h^1(r,\mathcal{I}_{r\cap(S_{\mathbb{C}}\cup S_{\mathbb{R}}),r}(d)).\end{aligned}\tag{7}$$

(1.2.1) Observe that (7) implies that the case $\sharp(S_{\mathbb{C}}\cup S_{\mathbb{R}})_r=\sharp(S_{\mathbb{C}}\cup S_{\mathbb{R}})_l=d+1$ cannot happen because there is no contribution from $h^1(l,\mathcal{I}_{l\cap(S_{\mathbb{C}}\cup S_{\mathbb{R}}),l}(d))+h^1(r,\mathcal{I}_{r\cap(S_{\mathbb{C}}\cup S_{\mathbb{R}}),r}(d))$ since both terms, in this case, are equal to 0. So, we can assume that at least $\sharp(S_{\mathbb{C}}\cup S_{\mathbb{R}})_l>d+1$.

(1.2.2) Assume $\sharp(S_{\mathbb{C}}\cup S_{\mathbb{R}})_r=d+1$ and $\sharp(S_{\mathbb{C}}\cup S_{\mathbb{R}})_l>d+1$.
To prove that we are in case (a) with respect to l it is sufficient to prove $S_{\mathbb{C},r}=S_{\mathbb{R},r}$. We have $\dim\langle\nu_d(S_{\mathbb{C}}\cup S_{\mathbb{R}})\rangle=\sharp(S_{\mathbb{C}}\cup S_{\mathbb{R}})-1-h^1(\mathcal{I}_{S_{\mathbb{C}}\cup S_{\mathbb{R}}}(d))$. Since $\nu_d(S_{\mathbb{C}})$ and $\nu_d(S_{\mathbb{R}})$ are linearly independent, $\sharp(S_{\mathbb{C}}\cup S_{\mathbb{R}})=\sharp(S_{\mathbb{C}})+\sharp(S_{\mathbb{R}})-\sharp(S_{\mathbb{C}}\cap S_{\mathbb{R}})$, $\sharp(S_{\mathbb{C},l}\cup S_{\mathbb{R},l})=\sharp(S_{\mathbb{C},l})+\sharp(S_{\mathbb{R},l})-\sharp(S_{\mathbb{C},l}\cap S_{\mathbb{R},l})$ and $h^1(\mathcal{I}_{S_{\mathbb{C}}\cup S_{\mathbb{R}}}(d))=h^1(l,\mathcal{I}_{S_{\mathbb{C},l}\cup S_{\mathbb{R},l}})$, Grassmann's formula gives that $\langle\nu_d(S_{\mathbb{C}})\rangle\cap\langle\nu_d(S_{\mathbb{R}})\rangle$ is generated by $\langle\nu_d(S_{\mathbb{C},l})\rangle\cap\langle\nu_d(S_{\mathbb{R},l})\rangle$. Since $P\notin\langle\nu_d(S')\rangle$ for any $S'\subsetneq S_{\mathbb{C}}$, we get $S_{\mathbb{C}}=S_{\mathbb{C},l}\cup(S_{\mathbb{C}}\cap S_{\mathbb{R}})_{\hat{l}}$, i.e. $S_{\hat{l}}=S_{\mathbb{R}}\setminus S_{\mathbb{C},l}$. Hence we are in case (a). Therefore (3) proves the case (ai) of the statement of
[Thm.1, ai] the theorem also for the case (1.2) that we are treating now. The point P_l of the statement of the theorem can here be identified with the point O' defined in (6).

[Thm.1, aii] This concludes the proofs of cases (a) of Theorem 1, and (A) of Proposition 1.

[Prop.1, A] (1.2.3) Assume that both $\sharp(S_{\mathbb{C}}\cup S_{\mathbb{R}})_r\geq d+2$ and $\sharp(S_{\mathbb{C}}\cup S_{\mathbb{R}})_l\geq d+2$.
We need to prove that we are in case (c). Recall that $S_{\mathbb{C},\hat{\Gamma}}=S_{\mathbb{R},\hat{\Gamma}}$ and that $h^1(\mathcal{I}_{S_{\mathbb{C},\hat{\Gamma}}\cup\Gamma}(d))=0$. The latter equality implies, as in Remark 2, that $\langle\{P\}\cup\nu_d(S_{\mathbb{C},\Gamma})\rangle\cap\langle\nu_d(\Gamma)\rangle$ is a single real point O_1, that $\nu_d(S_{\mathbb{C},\Gamma})$ evinces $r_{\mathbb{C}}(O_1)$ and that $\nu_d(S_{\mathbb{R},\Gamma})$ evinces O_1. Now O_1 plays the role of O_Γ of case (CI) in Proposition 1.

Since $\langle\nu_d(l)\rangle\cap\langle\nu_d(r)\rangle=\emptyset$ and $O_1\in\langle\nu_d(\Gamma)\rangle$ the sets $\langle\{O_1\}\cup\nu_d(l)\rangle\cap\langle\nu_d(l)\rangle$ (resp. $\langle\{O_1\}\cup\nu_d(r)\rangle\cap\langle\nu_d(r)\rangle$) are formed by a unique point O_2 (resp. O_3). Remark 2 gives that $O_i\in\mathbb{P}^N(\mathbb{R})$, $i=1,2$, $S_{\mathbb{C},l}$ evinces $r_{\mathbb{C}}(O_2)$, $S_{\mathbb{R},l}$ evinces $r_{\mathbb{R}}(O_2)$, $S_{\mathbb{C},r}$ evinces $r_{\mathbb{C}}(O_3)$ and $S_{\mathbb{R},r}$ evinces $r_{\mathbb{R}}(O_3)$. Then we are actually in case (c) and the hypothesis of the case (1.2.3) that we are treating coincides with case (cii) of the
[Thm.1, cii] statement of the theorem. Moreover (3) shows also case (ci).

[Thm.1, ci] This concludes the proof of the case (c) of Theorem 1.

Moreover O_2 and O_3 defined above coincide with O_l and O_r in (CII) of Propo-
[Prop.1, C] sition 1, therefore we have also proved case (C) Proposition 1.

(2) Now assume the existence of a conic $C\subset\mathbb{P}^m$ such that

$$\deg(S_{\mathbb{C}}\cup S_{\mathbb{R}})_C\geq 2d+2.\tag{8}$$

Since $\sharp(S_{\mathbb{C}} \cup S_{\mathbb{R}})_C \le 3d - 1 - 2d - 2 \le d - 1$, we have $h^1(\mathcal{I}_{(S_{\mathbb{C}} \cup S_{\mathbb{R}})_{\hat{C}}}(d-2)) = 0$. By Lemma, 1 we have

$$S_{\mathbb{C},\hat{C}} = S_{\mathbb{R},\hat{C}}, \tag{9}$$

the set $\langle \{P\} \cup \nu_d(S_{\mathbb{C},\hat{C}}) \rangle \cap \langle \nu_d(\langle C \rangle) \rangle$, is a single point:

$$P' := \langle \{P\} \cup \nu_d(S_{\mathbb{C},\hat{C}}) \rangle \cap \langle \nu_d(\langle C \rangle) \rangle \tag{10}$$

and $\nu_d(S_{\mathbb{C},C})$ evinces $r_{\mathbb{C}}(P')$. Moreover, if C is defined over $\mathbb{R}$, then $P' \in \mathbb{P}^N(\mathbb{R})$ and $\nu_d(S_{\mathbb{R},C})$ evinces $r_{\mathbb{R}}(P')$. Hence $\sharp(S_{\mathbb{C},C}) < \sharp(S_{\mathbb{C}} \cap S_{\mathbb{R}})$.

(2.1) Assume that C is smooth. Therefore (9) proves (bi) of the statement of the theorem in the case where C is smooth. Since the reduced case is proved above (immediately after the displayed formula (3)), we have concluded the proof of (bi). [Thm.1, bi]

Moreover the hypothesis (8) coincides with (biii) of the statement of the theorem since $\sharp(S_{\mathbb{C},C})$ is obviously strictly smaller than $\sharp(S_{\mathbb{R},C})$. This concludes (biii). [Thm.1, biii]

The fact that $\sharp(S_{\mathbb{C},C}) < \sharp(S_{\mathbb{R},C})$ also implies that $\sharp(S_{\mathbb{R},C}) \ge 5$. Since each point of $S_{\mathbb{R}}$ is real, C is real. Remark 2 gives that $\nu_d(S_{\mathbb{R},C})$ evinces $r_{\mathbb{R}}(P')$. Since $S_{\mathbb{R},C} \subset C$, $\nu_d(S_{\mathbb{R},C})$ also evinces the real symmetric tensor rank of P' with respect to the degree $2d$ rational normal curve $\nu_d(C)$. The point P' defined in (10) plays the role of the point P_C appearing in (bii) of the statement of the theorem. Therefore, we have just proved (bii) of Theorem 1 and (B) of Proposition 1. [Thm.1, bii]

We treat the case (2.2) below for sake of completeness, but we can observe that [Prop.1, B] this concluds the proof of Theorem 1 and Proposition 1.

(2.2) Assume that C is reducible, say $C = L_1 \cup L_2$ with L_1 and L_2 lines and $\sharp((S_{\mathbb{C}} \cup S_{\mathbb{R}})_{L_1}) \ge \sharp((S_{\mathbb{C}} \cap S_{\mathbb{R}})_{L_2})$. If $\sharp((S_{\mathbb{C}} \cup S_{\mathbb{R}}) \cap (L_2 \setminus L_2 \cap L_1)) \le d$, then we proved in step (1) that we are in case (a) with respect to the line L_1. Hence we may assume $\sharp((S_{\mathbb{C}} \cup S_{\mathbb{R}}) \cap (L_2 \setminus L_2 \cap L_1)) \ge d+1$. Thus even condition (biv) is satisfied as already remarked above after the displayed formula (3). $\square$

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