GKW representation theorem and linear BSDEs under restricted information. An application to risk-minimization.

Claudia Ceci, Alessandra Cretarola, Francesco Russo

GKW representation theorem and linear BSDEs under restricted information. An application to risk-minimization.

Claudia Ceci* Alessandra Cretarola† Francesco Russo‡

Abstract

In this paper we provide Galtchouk-Kunita-Watanabe representation results in the case where there are restrictions on the available information. This allows to prove existence and uniqueness for linear backward stochastic differential equations driven by a general càdlàg martingale under partial information. Furthermore, we discuss an application to risk-minimization where we extend the results of Föllmer and Sondermann (1986) to the partial information framework and we show how our result fits in the approach of Schweizer (1994).

Mathematics Subject Classification (2000): 60H10, 60H30, 91B28.
Key words and phrases: Backward stochastic differential equations, partial information, Galtchouk-Kunita-Watanabe decomposition, predictable dual projection, risk-minimization.

1 Introduction

This paper provides two main contributions. First, we prove Galtchouk-Kunita-Watanabe representation results in the case where there are restrictions on the available information and we show an application to risk-minimization. Second, as an important consequence, we prove existence and uniqueness for linear backward stochastic differential equations (in short BSDEs) driven by a general càdlàg martingale under partial information.
For BSDEs driven by a general càdlàg martingale beyond the Brownian setting, there exist very few results in literature (see [5] and more recently [1] and [2], as far as we are aware). Here we study for the first time such a general case in the situation where there

*Dipartimento di Economia, Università degli Studi "G. D'Annunzio" di Chieti-Pescara, Viale Pindaro 42, I-65127 Pescara, Italy. E-mail: `ceci@sci.unich.it`.
†Dipartimento di Matematica e Informatica, Università degli Studi di Perugia, via Vanvitelli, 1, I-06123 Perugia, Italy. E-mail: `alessandra.cretarola@dmi.unipg.it`.
‡Ecole Nationale Supérieure des Techniques Avancées, ENSTA-ParisTech Unité de Mathématiques appliquées, 32 Boulevard Victor, F-75739 Paris Cedex 15, France and INRIA Rocquencourt & Cermics Ecole des Ponts, Projet MathFi. E-mail: `francesco.russo@ensta-paristech.fr`.

are restrictions on the available information, that represents an interesting issue arising in many financial problems. Mathematically, this means to consider an additional filtration $\mathbb{H}$ smaller than the full information flow $\mathbb{F}$. A typical example arises when $\mathcal{H}_t = \mathcal{F}_{(t-\tau)^+}$ where $\tau \in (0,T)$ is a fixed delay and $(t-\tau)^+ := \max\{0, t-\tau\}$ and T denotes a time horizon.
We start our investigation by considering BSDEs of the form

$$Y_t = \xi - \int_t^T Z_s \mathrm{d}M_s - (O_T - O_t), \quad 0 \le t \le T, \tag{1.1}$$

driven by a square-integrable (càdlàg) martingale $M = (M_t)_{0 \le t \le T}$, where $T > 0$ is a fixed time horizon, $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$[1] denotes the terminal condition and $O = (O_t)_{0 \le t \le T}$ is a square-integrable $\mathbb{F}$-martingale with $O_0 = 0$, satisfying a suitable orthogonality condition that we will make more precise in the next section.
We look for a solution (Y, Z) to equation (1.1) under partial information, where $Y = (Y_t)_{0 \le t \le T}$ is a càdlàg $\mathbb{F}$-adapted process and $Z = (Z_t)_{0 \le t \le T}$ is an $\mathbb{H}$-predictable process such that $\mathbb{E}\left[\int_0^T |Z_t|^2 \mathrm{d}\langle M \rangle_t\right] < \infty$.
To this aim, we prove a Galtchouk-Kunita-Watanabe decomposition in the case where there are restrictions on the available information. More precisely, we obtain that every random variable $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ can be uniquely written as

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} \mathrm{d}M_t + O_T, \quad \mathbb{P} - \text{a.s.}, \tag{1.2}$$

where $H^{\mathcal{H}} = (H^{\mathcal{H}})_{0 \le t \le T}$ is an $\mathbb{H}$-predictable process such that $\mathbb{E}\left[\int_0^T |H_t^{\mathcal{H}}|^2 \mathrm{d}\langle M \rangle_t\right] < \infty$.
To the authors' knowledge such a decomposition has not been proved yet in the existing literature. We will see that decomposition (1.2) allows to construct a solution to the BSDE (1.1) and ensures its uniqueness in this setting.
Moreover, we are able to provide an explicit characterization of the integrand process $H^{\mathcal{H}}$ given in decomposition (1.2) in terms of the one appearing in the classical Galtchouk-Kunita-Watanabe decomposition, by using $\mathbb{H}$-predictable (dual) projections.
Finally, we discuss a financial application. More precisely, we study the problem of hedging a contingent claim in the case where investors acting in the market have partial information. Since the market is incomplete we choose the risk-minimization approach, a quadratic hedging method which keeps the replicability constraint and relaxes the self financing condition, see [6] and [14] for further details. As in [6] and [13], we consider the case where the price process is a martingale under the real world probability measure. In [6], under the case of full information, the authors provide the risk-minimizing hedging strategy in terms of the classical Galtchouk-Kunita-Watanabe decomposition. Here, by using the Galtchouk-Kunita-Watanabe decomposition under partial information, we extend this result to the case where there are restrictions on the available information. Finally, thanks to the explicit representation of the integrand process $H^{\mathcal{H}}$ appearing in

[1]The space $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ denotes the set of all $\mathcal{F}_T$-measurable real-valued random variables H such that $\mathbb{E}\left[|H|^2\right] = \int_\Omega |H|^2 \mathrm{d}\mathbb{P} < \infty$.

decomposition (1.2), we find the same expression for the optimal strategy in terms of the Radon-Nikodym derivative of two $\mathbb{H}$-predictable dual projections, that is proved in [13]. The paper is organized as follows. In Section 2 we give the definition of solution to BSDEs under partial information. Section 3 is devoted to prove existence and uniqueness results for the solutions, which are obtained by applying the Galtchouk-Kunita-Watanabe decomposition adapted to the restricted information setting. The explicit representation of the integrand process $H^{\mathcal{H}}$ appearing in (1.2) can be found in Section 4. Finally, an application to risk-minimization is given in Section 5.

2 Setting

Let us fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ endowed with a filtration $\mathbb{F} := (\mathcal{F}_t)_{0\le t\le T}$, where $\mathcal{F}_t$ represents the full information at time t. We assume that $\mathcal{F}_T = \mathcal{F}$. Then we consider a subfiltration $\mathbb{H} := (\mathcal{H}_t)_{0\le t\le T}$ of $\mathbb{F}$, i.e. $\mathcal{H}_t \subseteq \mathcal{F}_t$, for each $t \in [0, T]$, corresponding to the available information level. We remark that both filtrations are assumed to satisfy the usual hypotheses of completeness and right-continuity, see e.g. [12].
For simplicity we only consider the one-dimensional case. Extensions to several dimensions are straightforward and left to the reader. The data of the problem are:

- an $\mathbb{R}$-valued square-integrable (càdlàg) $\mathbb{F}$-martingale $M = (M_t)_{0\le t\le T}$ with $\mathbb{F}$-predictable quadratic variation process denoted by $\langle M\rangle = (\langle M, M\rangle)_{0\le t\le T}$;
- a terminal condition $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$.

Definition 2.1. *A solution of the BSDE*

$$Y_t = \xi - \int_t^T Z_s \mathrm{d}M_s - (O_T - O_t), \quad 0 \le t \le T, \tag{2.1}$$

with data $(\xi, \mathbb{H})$ under partial information, where $O = (O_t)_{0\le t\le T}$ is a square-integrable $\mathbb{F}$-martingale with $O_0 = 0$, satisfying the orthogonality condition

$$\mathbb{E}\left[O_T \int_0^T \varphi_t \mathrm{d}M_t\right] = 0, \tag{2.2}$$

for all $\mathbb{H}$-predictable processes $\varphi = (\varphi_t)_{0\le t\le T}$ such that $\mathbb{E}\left[\int_0^T |\varphi_t|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$, is a couple (Y, Z) of processes with values in $\mathbb{R} \times \mathbb{R}$, satisfying (2.1) such that

- *$Y = (Y_t)_{0\le t\le T}$ is a càdlàg $\mathbb{F}$-adapted process;*
- *$Z = (Z_t)_{0\le t\le T}$ is an $\mathbb{H}$-predictable process such that $\mathbb{E}\left[\int_0^T |Z_t|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$.*

Remark 2.2. *The orthogonality condition given in (2.2) is weaker than the classical strong orthogonality condition, see e.g [11] or [12]. Indeed, set $N_t = \int_0^t \varphi_s \mathrm{d}M_s$, for each $t \in [0, T]$, where φ is an $\mathbb{H}$-predictable process such that $\mathbb{E}\left[\int_0^T |\varphi_t|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$. If*

$$\langle O, M\rangle_t = 0 \quad \mathbb{P}-\text{a.s.}, \quad \forall t \in [0, T],$$

then

$$\langle O, N\rangle_t = \int_0^t \varphi_s \mathrm{d}\langle O, M\rangle_s = 0 \quad \mathbb{P}-\text{a.s.}, \quad \forall t \in [0,T].$$

Consequently, $O \cdot N$ is an $\mathbb{F}$-martingale null at zero, that implies

$$\mathbb{E}\left[O_t N_t\right] = 0, \quad \forall t \in [0,T],$$

and in particular condition (2.2).

Remark 2.3. *Since for any $\mathbb{H}$-predictable process φ, the process $\mathbf{1}_{(0,t]}(s)\varphi_s$, with $t \leq T$, is $\mathbb{H}$-predictable, condition* (2.2) *implies that for every $t \in [0,T]$*

$$\mathbb{E}\left[O_T \int_0^t \varphi_s \mathrm{d}M_s\right] = 0,$$

and by conditioning with respect to $\mathcal{F}_t$ (note that O is an $\mathbb{F}$-martingale), we have

$$\mathbb{E}\left[O_t \int_0^t \varphi_s \mathrm{d}M_s\right] = \mathbb{E}\left[\int_0^t \varphi_s \mathrm{d}\langle M, O\rangle_s\right] = 0 \quad \forall t \in [0,T].$$

From this last equality, we can argue that in the case of full information, i.e., $\mathcal{H}_t = \mathcal{F}_t$, for each $t \in [0,T]$, condition (2.2) *is equivalent to the strong orthogonality condition between O and M (see e.g. Lemma 2 and Theorem 36, Chapter IV, page 180 of [12] for a rigorous proof).*

3 Existence and uniqueness for linear BSDEs under partial information

Our aim is to investigate existence and uniqueness of a solution to the BSDE (2.1) with data $(\xi, \mathbb{H})$ driven by the general martingale M in the sense of Definition 2.1. This requires to prove a Galtchouk-Kunita-Watanabe representation result under restricted information.
We introduce the linear subspace $\mathcal{L}_T^{\mathcal{H}}$ of $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ given by all random variables η of the form

$$\left\{U_0 + \int_0^T \varphi_t \mathrm{d}M_t \,\middle|\, U_0 \in \mathcal{H}_0,\ \varphi \text{ is } \mathbb{H}-\text{predictable with } \mathbb{E}\left[\int_0^T |\varphi_t|^2 \mathrm{d}\langle M\rangle_t\right] < \infty\right\}. \tag{3.1}$$

Lemma 3.1. *The set $\mathcal{L}_T^{\mathcal{H}}$ is a closed subspace of $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$.*

Proof. Let $U_0^n \in \mathcal{H}_0$ and $(\varphi^n)_{n\in\mathbb{N}}$, with $\varphi^n = (\varphi_t^n)_{0\leq t\leq T}$, be a sequence of $\mathbb{H}$-predictable processes satisfying $\mathbb{E}\left[\int_0^T |\varphi_t^n|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$ such that the sequence

$$\eta^n = U_0^n + \int_0^T \varphi_t^n \mathrm{d}M_t, \quad n \in \mathbb{N},$$

converges to some random variable $\eta \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, as n goes to infinity. By taking the conditional expectation with respect to $\mathcal{H}_0$, we have

$$U_0^n = \mathbb{E}\left[\eta^n | \mathcal{H}_0\right] \longrightarrow \mathbb{E}\left[\eta | \mathcal{H}_0\right], \quad \text{as } n \to \infty.$$

We set $U_0 = \mathbb{E}\left[\eta | \mathcal{H}_0\right]$. Since $(\eta^n - U_0^n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, it follows that

$$\mathbb{E}\left[\int_0^T (\varphi_t^n - \varphi_t^m)^2 \mathrm{d}\langle M\rangle_t\right] \longrightarrow 0, \quad \text{as } n, m \to \infty.$$

Consequently, $(\varphi^n)_{n\in\mathbb{N}}$ converges in $L^2(\Omega, \mathrm{d}\langle M\rangle \otimes \mathrm{d}\mathbb{P})$[2] to some process $\varphi = (\varphi_t)_{0\le t\le T} \in L^2(\Omega, \mathrm{d}\langle M\rangle \otimes \mathrm{d}\mathbb{P})$. Finally, since there is a subsequence converging $\mathrm{d}\langle M\rangle \otimes \mathrm{d}\mathbb{P}$-a.e., the limit φ is necessarily an $\mathbb{H}$-predictable process. □

Proposition 3.2. *Let $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. There exists a unique decomposition of the form*

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} \mathrm{d}M_t + O_T, \quad \mathbb{P}-\text{a.s.}, \tag{3.2}$$

where $U_0 \in \mathcal{H}_0$, $H^{\mathcal{H}}$ is an $\mathbb{H}$-predictable process such that $\mathbb{E}\left[\int_0^T |H_t^{\mathcal{H}}|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$ and O is a square-integrable $\mathbb{F}$-martingale with $O_0 = 0$ such that $\mathbb{E}\left[O_T \cdot \eta\right] = 0$, for every $\eta \in \mathcal{L}_T^{\mathcal{H}}$. Moreover $U_0 = \mathbb{E}\left[\xi | \mathcal{H}_0\right]$ and $\mathbb{E}\left[O_T | \mathcal{H}_0\right] = 0$.

Proof. The existence and uniqueness property of decomposition (3.2) is clearly ensured by the orthogonal projection of the random variable ξ onto the space $\mathcal{L}_T^{\mathcal{H}}$, that is closed in virtue of Lemma 3.1. Since $(U_0 + \int_0^\cdot H_t^{\mathcal{H}} \mathrm{d}M_t)$ is an $\mathbb{F}$-martingale, by taking the conditional expectation of ξ with respect to $\mathcal{H}_0$ in (3.2), we have

$$\begin{aligned}
\mathbb{E}\left[\xi | \mathcal{H}_0\right] &= \mathbb{E}\left[U_0 + \int_0^T H_t^{\mathcal{H}} \mathrm{d}M_t \middle| \mathcal{H}_0\right] + \mathbb{E}\left[O_T | \mathcal{H}_0\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[U_0 + \int_0^T H_t^{\mathcal{H}} \mathrm{d}M_t \middle| \mathcal{F}_0\right] \middle| \mathcal{H}_0\right] + \mathbb{E}\left[\mathbb{E}\left[O_T | \mathcal{F}_0\right] | \mathcal{H}_0\right] \\
&= \mathbb{E}\left[U_0 | \mathcal{H}_0\right],
\end{aligned}$$

where in the last equality we have used the fact that $\mathbb{E}\left[O_T | \mathcal{F}_0\right] = O_0 = 0$. Consequently $\mathbb{E}\left[O_T | \mathcal{H}_0\right] = 0$ and $U_0 = \mathbb{E}\left[U_0 | \mathcal{H}_0\right] = \mathbb{E}\left[\xi | \mathcal{H}_0\right]$. This concludes the proof. □

Theorem 3.3. *Given data $(\xi, \mathbb{H})$, there exists a unique couple (Y, Z) which solves the BSDE (2.1) according to Definition 2.1.*

[2] The space $L^2(\Omega, \mathrm{d}\langle M\rangle \otimes \mathrm{d}\mathbb{P})$ denotes the set of all $\mathbb{F}$-adapted processes $\varphi = (\varphi_t)_{0\le t\le T}$ such that

$$\|\varphi\|_{L^2(\Omega, \mathrm{d}\langle M\rangle \otimes \mathrm{d}\mathbb{P})} := \left(\mathbb{E}\left[\int_0^T |\varphi_t|^2 \mathrm{d}\langle M\rangle_t\right]\right)^{\frac{1}{2}} < \infty.$$

Proof. **Existence.** Let $\mathcal{L}_T^{\mathcal{H}}$ be the linear subspace of $L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ introduced in (3.1). Given $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, we know by Proposition 3.2 that there exists a unique decomposition of the form

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} \mathrm{d}M_t + A_T, \quad \mathbb{P} - \text{a.s.},$$

where in particular A is a square-integrable $\mathbb{F}$-martingale with $A_0 = 0$ orthogonal to all the elements in $\mathcal{L}_T^{\mathcal{H}}$. We use this result to construct a solution to the BSDE (2.1). We consider the orthogonal projection of $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$ onto this space:

$$P_{\mathcal{L}_T^{\mathcal{H}}}(\xi) := U_0 + \int_0^T H_t^{\mathcal{H}} \mathrm{d}M_t.$$

The couple (U_0, H), where $U_0 \in \mathcal{H}_0$ and $H^{\mathcal{H}}$ is an $\mathbb{H}$-predictable process in $L^2(\Omega, \mathrm{d}\langle M \rangle \otimes \mathrm{d}\mathbb{P})$, uniquely identifies the projection, that exists and it is well-defined since $\mathcal{L}_T^{\mathcal{H}}$ is closed. We set

$$A_T := \xi - P_{\mathcal{L}_T^{\mathcal{H}}}(\xi) \in (\mathcal{L}_T^{\mathcal{H}})^{\perp},$$

where $(\mathcal{L}_T^{\mathcal{H}})^{\perp}$ denotes the orthogonal subspace of $\mathcal{L}_T^{\mathcal{H}}$. Here A_T corresponds to the final value of a square-integrable $\mathbb{F}$-martingale A with zero initial value, that implies $\mathbb{E}\left[\xi - U_0 \middle| \mathcal{F}_0\right] = 0$. Clearly, we have

$$\mathcal{L}_T^{\mathcal{H}} \oplus (\mathcal{L}_T^{\mathcal{H}})^{\perp} = L^2(\Omega, \mathcal{F}_T, \mathbb{P}).$$

Now we define the process Y as follows:

$$\begin{aligned} Y_t &:= \mathbb{E}\left[\xi \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[U_0 + \int_0^T H_t^{\mathcal{H}} \mathrm{d}M_t + A_T \middle| \mathcal{F}_t\right] \\ &= \mathbb{E}\left[U_0 \middle| \mathcal{F}_0\right] + \int_0^t H_s^{\mathcal{H}} \mathrm{d}M_s + A_t \\ &= Y_0 + \int_0^t H_s^{\mathcal{H}} \mathrm{d}M_s + A_t, \quad 0 \le t \le T, \end{aligned}$$

and we set $Z_t := H_t^{\mathcal{H}}$ and $O_t := A_t$, for every $t \in [0, T]$. Then we get

$$Y_t = \xi - \int_t^T Z_s \mathrm{d}M_s - (O_T - O_t), \quad 0 \le t \le T.$$

Uniqueness. Let (Y, Z), (Y', Z') be two solutions to the BSDE (2.1) under partial information associated to the terminal condition $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. We set $(\bar{Y}, \bar{Z}) = (Y - Y', Z - Z')$. Then $(\bar{Y}, \bar{Z})$ satisfies the BSDE

$$\bar{Y}_t = -\int_t^T \bar{Z}_s \mathrm{d}M_s - (\bar{O}_T - \bar{O}_t), \quad 0 \le t \le T, \tag{3.3}$$

with final condition $\bar{Y}_T = 0$. In addition, we have set $\bar{O} := O - O'$ in (3.3), where O and O' denote the square-integrable $\mathbb{F}$-martingales with $O_0 = O_0^{'} = 0$ satisfying the orthogonality condition

$$\mathbb{E}\left[O_T \int_0^T \varphi_t \mathrm{d}M_t\right] = \mathbb{E}\left[O_T^{'} \int_0^T \varphi_t \mathrm{d}M_t\right] = 0,$$

for all $\mathbb{H}$-predictable processes φ such that $\mathbb{E}\left[\int_0^T |\varphi_t|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$. Since $(\bar{Y}, \bar{Z})$ is a solution of (3.3), then

$$\bar{Y}_t = \bar{Y}_0 + \int_0^t \bar{Z}_s \mathrm{d}M_s + \bar{O}_t, \quad 0 \le t \le T. \tag{3.4}$$

Since the process $\bar{Y}$ is an $\mathbb{F}$-martingale such that $\bar{Y}_T = 0$, we have

$$\bar{Y}_t = \mathbb{E}\left[\bar{Y}_T \middle| \mathcal{F}_t\right] = 0, \quad \text{for all } t \in [0, T].$$

Thus $Y_t = Y_t^{'}$ $\mathbb{P}$-a.s., for every $t \in [0, T]$. Then we can rewrite (3.4) as follows

$$0 = \int_0^t \bar{Z}_s \mathrm{d}M_s + \bar{O}_t, \quad 0 \le t \le T.$$

By computing the predictable covariation of $\int_0^\cdot \bar{Z}_s \mathrm{d}M_s + \bar{O}$ and $\bar{O}$ and by taking the expectation of both sides in the equality, for each $t \in [0, T]$, we obtain

$$\begin{aligned} 0 &= \int_0^t \bar{Z}_s \mathrm{d}\langle M, \bar{O}\rangle_s + \langle \bar{O}\rangle_t \\ &= \mathbb{E}\left[\int_0^t \bar{Z}_s \mathrm{d}\langle M, \bar{O}\rangle_s\right] + \mathbb{E}\left[\langle \bar{O}\rangle_t\right]. \end{aligned}$$

Since $\bar{Z}$ and $\bar{O}$ are differences of solutions to the BSDE (2.1), then $\mathbb{E}\left[\int_0^t \bar{Z}_s \mathrm{d}\langle M, \bar{O}\rangle_s\right] = 0$ for $t \in [0, T]$, and it follows

$$\mathbb{E}\left[\langle \bar{O}\rangle_t\right] = 0, \quad 0 \le t \le T. \tag{3.5}$$

By Theorem 4.2 of [11], since $\bar{O}$ is a square-integrable $\mathbb{F}$-martingale null at zero, we have that $\bar{O}^2 - \langle \bar{O}\rangle$ is an $\mathbb{F}$-martingale null at zero. Then by (3.5)

$$\mathbb{E}\left[\bar{O}_t^2\right] = \mathbb{E}\left[\langle \bar{O}\rangle_t\right] = 0, \quad 0 \le t \le T,$$

that implies $\bar{O}_t^2 = 0$ $\mathbb{P}$-a.s. for every $t \in [0, T]$ and then $O_t = O_t^{'}$ $\mathbb{P}$-a.s. for every $t \in [0, T]$. Now, let Y be the unique solution of (2.1) for a certain $\mathbb{H}$-predictable Z such that $\mathbb{E}\left[\int_0^T |Z_t|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$, i.e.

$$Y_t = Y_0 + \int_0^t Z_s \mathrm{d}M_s + O_t, \quad 0 \le t \le T. \tag{3.6}$$

It only remains to prove that Z is unique. For $t = T$ equation (3.6) becomes

$$Y_T = \xi = Y_0 + \int_0^T Z_s \mathrm{d}M_s + O_T.$$

By Proposition 3.2, $Z_t = H_t^{\mathcal{H}}$ $\mathbb{P}$-a.s., for each $t \in [0,T]$ and then Z is univocally determined. This concludes the proof. $\square$

4 Galtchouk-Kunita-Watanabe representation under partial information

We now wish to provide an explicit characterization of the integrand process $H^{\mathcal{H}}$ appearing in the representation (3.2) in terms of the one given in the classical Galtchouk-Kunita-Watanabe decomposition, by means in particular of the concept of $\mathbb{H}$-predictable dual projection.

Let $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. We consider the well-known Galtchouk-Kunita-Watanabe decomposition of ξ with respect to M:

$$\xi = \tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} \mathrm{d}M_t + \tilde{O}_T, \quad \mathbb{P}-\text{a.s.}, \tag{4.1}$$

where $\tilde{U}_0 \in \mathcal{F}_0$, the integrand $H^{\mathcal{F}} = (H_t^{\mathcal{F}})_{0 \le t \le T}$ is an $\mathbb{F}$-predictable process such that $\mathbb{E}\left[\int_0^T |H_t^{\mathcal{F}}|^2 \mathrm{d}\langle M \rangle_t\right] < \infty$ and $\tilde{O} = (\tilde{O}_t)_{0 \le t \le T}$ is a square-integrable $\mathbb{F}$-martingale with $\tilde{O}_0 = 0$ such that $\langle \tilde{O}, M \rangle_t = 0$, for every $t \in [0,T]$. Moreover, let us observe that $\tilde{U}_0 = \mathbb{E}\left[\xi | \mathcal{F}_0\right]$.

In the sequel we will denote by ${}^pX$ the $\mathbb{H}$-predictable projection of a (generic) integrable process $X = (X_t)_{0 \le t \le T}$, defined as the unique $\mathbb{H}$-predictable process such that

$$\mathbb{E}\left[X_\tau \mathbf{1}_{\{\tau < \infty\}} \middle| \mathcal{H}_{\tau-}\right] = {}^pX_\tau \mathbf{1}_{\{\tau < \infty\}} \quad \mathbb{P}-\text{a.s.}$$

for every $\mathbb{H}$-predictable stopping time τ.

First we give a preliminary result under the additional assumption that the predictable quadratic variation $\langle M \rangle$ of the $\mathbb{F}$-martingale M is an $\mathbb{H}$-predictable process. In Theorem 4.7 we extend such result to the general case.

Proposition 4.1. *Let $(\tilde{U}_0, H^{\mathcal{F}}, \tilde{O}_T)$ be the triplet corresponding to decomposition* (4.1) *of $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. Suppose that the predictable quadratic variation $\langle M \rangle$ of the $\mathbb{F}$-martingale M is an $\mathbb{H}$-predictable process. Then*

$$\xi = U_0 + \int_0^T H_t^{\mathcal{H}} \mathrm{d}M_t + O_T, \quad \mathbb{P}-\text{a.s.},$$

with

$$U_0 = \mathbb{E}\left[\tilde{U}_0 \middle| \mathcal{H}_0\right], \tag{4.2}$$

$$H_t^{\mathcal{H}} = {}^p(H_t^{\mathcal{F}}), \quad 0 \le t \le T, \tag{4.3}$$

and O is a square-integrable $\mathbb{F}$-martingale with $O_0 = 0$ such that $\mathbb{E}[O_T \cdot \eta] = 0$, for every $\eta \in \mathcal{L}_T^{\mathcal{H}}$.

Proof. Let

$$\xi = \tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} \mathrm{d}M_t + \tilde{O}_T, \quad \mathbb{P} - \text{a.s.}$$

be the classical Galtchouk-Kunita-Watanabe decomposition of $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. By taking the expectation of ξ with respect to $\mathcal{H}_0$, we have:

$$\mathbb{E}\left[\xi \middle| \mathcal{H}_0\right] = \mathbb{E}\left[\tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} \mathrm{d}M_t \middle| \mathcal{H}_0\right] + \mathbb{E}\left[\tilde{O}_T \middle| \mathcal{H}_0\right]. \tag{4.4}$$

Since $(\tilde{U}_0 + \int_0^{\cdot} H_t^{\mathcal{F}} \mathrm{d}M_t)$ is an $\mathbb{F}$-martingale, it follows:

$$\begin{aligned}
\mathbb{E}\left[\tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} \mathrm{d}M_t \middle| \mathcal{H}_0\right] &= \mathbb{E}\left[\mathbb{E}\left[\tilde{U}_0 + \int_0^T H_t^{\mathcal{F}} \mathrm{d}M_t \middle| \mathcal{F}_0\right] \middle| \mathcal{H}_0\right] \\
&= \mathbb{E}\left[\tilde{U}_0 \middle| \mathcal{H}_0\right],
\end{aligned}$$

so that we can rewrite (4.4) as follows:

$$\mathbb{E}\left[\xi \middle| \mathcal{H}_0\right] = \mathbb{E}\left[\tilde{U}_0 \middle| \mathcal{H}_0\right] + \mathbb{E}\left[\tilde{O}_T \middle| \mathcal{H}_0\right].$$

Moreover, since $\tilde{O}$ is an $\mathbb{F}$-martingale null at zero, we have

$$\mathbb{E}\left[\tilde{O}_T \middle| \mathcal{H}_0\right] = \mathbb{E}\left[\mathbb{E}\left[\tilde{O}_T \middle| \mathcal{F}_0\right] \middle| \mathcal{H}_0\right] = 0.$$

This implies equality (4.2). To prove equality (4.3), we need to calculate the orthogonal projection of ξ onto the space $\mathcal{L}_T^{\mathcal{H}}$, see (3.1). For the sake of brevity, we suppose that $\tilde{U}_0 = 0$. Thanks to Proposition 3.2, this means we need to check the following condition:

$$\mathbb{E}\left[\xi \int_0^T \varphi_t \mathrm{d}M_t\right] = \mathbb{E}\left[\int_0^T {}^p(H_t^{\mathcal{F}}) \mathrm{d}M_t \int_0^T \varphi_t \mathrm{d}M_t\right],$$

for every $\mathbb{H}$-predictable process φ such that $\mathbb{E}\left[\int_0^T |\varphi_t|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$. Taking decomposition (4.1) into account, this corresponds to the following equality:

$$\mathbb{E}\left[\int_0^T H_t^{\mathcal{F}} \varphi_t \mathrm{d}\langle M\rangle_t\right] = \mathbb{E}\left[\int_0^T {}^p(H_t^{\mathcal{F}}) \varphi_t \mathrm{d}\langle M\rangle_t\right], \tag{4.5}$$

for every $\mathbb{H}$-predictable process φ such that $\mathbb{E}\left[\int_0^T |\varphi_t|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$. If we write the process φ as follows

$$\varphi = \varphi^+ - \varphi^-,$$

where φ^+ and φ^- denote the positive and the negative part of φ respectively, and define the $\mathbb{F}$-martingales

$$R_t^+ = \int_0^t \sqrt{\varphi_s^+} \mathrm{d}M_s, \quad R_t^- = \int_0^t \sqrt{\varphi_s^-} \mathrm{d}M_s, \quad 0 \le t \le T,$$

equality (4.5) is equivalent to the following relationships:

$$\mathbb{E}\left[\int_0^T H_t^{\mathcal{F}} \mathrm{d}\langle R^+\rangle_t\right] = \mathbb{E}\left[\int_0^T {}^p(H_t^{\mathcal{F}}) \mathrm{d}\langle R^+\rangle_t\right]$$
$$\mathbb{E}\left[\int_0^T H_t^{\mathcal{F}} \mathrm{d}\langle R^-\rangle_t\right] = \mathbb{E}\left[\int_0^T {}^p(H_t^{\mathcal{F}}) \mathrm{d}\langle R^-\rangle_t\right].$$

Hence, we can reduce the problem by assuming directly $\varphi_t = 1$ in (4.5), for each $t \in [0, T]$. Then, it is enough to prove the equality

$$\mathbb{E}\left[\int_0^T H_t^{\mathcal{F}} \mathrm{d}\langle M\rangle_t\right] = \mathbb{E}\left[\int_0^T {}^p(H_t^{\mathcal{F}}) \mathrm{d}\langle M\rangle_t\right]. \tag{4.6}$$

Since $\langle M\rangle$ is $\mathbb{H}$-predictable, Theorem VI.57 in [4] guarantees that equality (4.6) holds, once we have the positivity of the process $H^{\mathcal{F}}$. By writing

$$H^{\mathcal{F}} = (H^{\mathcal{F}})^+ - (H^{\mathcal{F}})^-,$$

and applying the above theorem to the positive and negative parts of $H^{\mathcal{F}}$, $(H^{\mathcal{F}})^+$ and $(H^{\mathcal{F}})^-$ respectively, and to the associated $\mathbb{H}$-predictable projections, we can get the result by setting

$$H^{\mathcal{H}} := {}^p(H^{\mathcal{F}}) = {}^p((H^{\mathcal{F}})^+) - {}^p((H^{\mathcal{F}})^-).$$

□

Example 4.2. *Let us consider the particular case where M is a square-integrable $\mathbb{F}$-martingale that is in addition a Lévy process, $\mathcal{F}_t = \mathcal{F}_t^M$ and $\mathcal{H}_t = \mathcal{F}_{(t-\tau)^+}^M$, with $\tau \in (0, T)$ a fixed delay. We assume $\xi = h(M_T) \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, for some measurable function $h : \mathbb{R} \longrightarrow \mathbb{R}$.*
In this framework, by Lemma A.1 (see Appendix), we know that the integrand appearing in the Galtchouk-Kunita-Watanabe decomposition (4.1) *can be written as*

$$H_t^{\mathcal{F}} = F(t, M_{t^-}), \quad t \in [0, T],$$

where the function F is such that the condition $\mathbb{E}\left[\int_0^T |F(t, M_{t^-})|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$ is satisfied. Since in this case $\langle M\rangle$ is a deterministic process, we can apply Proposition 4.1 and get

$$H_t^{\mathcal{H}} = {}^pF(t, M_{t^-}) = \mathbb{E}\left[F(t, M_{t^-})|\mathcal{H}_{t^-}\right], \quad t \in [0, T].$$

Then, it is easy to derive the following:

$$H_t^{\mathcal{H}} = \begin{cases} c(t, M_{(t-\tau)^-}) & \text{if t} > \tau \\ c(t, M_0) & \text{if t} \le \tau, \end{cases}$$

where the function c is given by

$$c(t,y)=\int_{\mathbb{R}} F(t,y+z)\mathrm{d}\rho_{t\wedge\tau}(z),$$

with ρ_t denoting the law of M_t, for every $t\in[0,T]$.

4.1 The $\mathbb{H}$-predictable dual projection

It is possible to extend the result of Proposition 4.1 by using the concept of $\mathbb{H}$-predictable dual projection. For reasons of clarity, we provide a self-contained discussion about this kind of projection in presence of more than one filtration. Let $G=(G_t)_{0\le t\le T}$ be a càdlàg $\mathbb{F}$-adapted process of integrable variation, that is, $\mathbb{E}\left[\|G\|_T\right]<\infty$. Here the process $\|G\|=(\|G\|)_{0\le t\le T}$ defined, for each $t\in[0,T]$, by

$$\|G\|_t(\omega)=\sup_{\Delta}\sum_{i=0}^{n(\Delta)-1}|G_{t_i+1}(\omega)-G_{t_i}(\omega)|,$$

where $\Delta=\{t_0=0<t_1<\ldots<t_{n(\Delta)}=t\}$ is a partition of $[0,t]$, denotes the total variation of the function $t\rightsquigarrow G_t(\omega)$.

Proposition 4.3. *Let $G=(G_t)_{0\le t\le T}$ be a càdlàg $\mathbb{F}$-adapted process of integrable variation. Then there exists a unique $\mathbb{H}$-predictable process $G^{\mathbb{H}}=(G_t^{\mathbb{H}})_{0\le t\le T}$ of integrable variation, such that*

$$\mathbb{E}\left[\int_0^T\varphi_t\mathrm{d}G_t^{\mathbb{H}}\right]=\mathbb{E}\left[\int_0^T\varphi_t\mathrm{d}G_t\right],$$

for every $\mathbb{H}$-predictable (bounded) process φ. The process $G^{\mathbb{H}}$ is called the $\mathbb{H}$-predictable dual projection *of G.*

Proof. Without loss of generality, we can restrict our attention to the case where G is an increasing process and prove the statement on the generators φ of the form $\varphi_u=\mathbf{1}_{(s,t]}(u)\mathbf{1}_B$, with $B\in\mathcal{H}_s$ and $s,t\in[0,T]$ with $s<t$. Indeed, decomposing the process G as $G=G^+-G^-$, where both the positive and negative parts of G are assumed to be increasing integrable processes, we can suppose G to be increasing such that

$$\mathbb{E}\left[G_T\right]=\mathbb{E}\left[\|G\|_T\right]<\infty.$$

If G is a càdlàg, increasing, integrable $\mathbb{F}$-adapted process, we will prove that there exists a unique increasing, integrable $\mathbb{H}$-predictable process $G^{\mathbb{H}}$ such that for every $s,t\in[0,T]$ with $s<t$ and $B\in\mathcal{H}_s$, the following relationship holds

$$\mathbb{E}\left[\mathbf{1}_B(G_t-G_s)\right]=\mathbb{E}\left[\mathbf{1}_B(G_t^{\mathbb{H}}-G_s^{\mathbb{H}})\right].$$

Let $\tilde{G}=(\tilde{G}_t)_{0\le t\le T}$ be the $\mathbb{H}$-optional projection of G, such that for fixed times $t\in(0,T]$

$$\tilde{G}_t=\mathbb{E}\left[G_t|\,\mathcal{H}_t\right]\ \mathbb{P}-\text{a.s.}.$$

We observe that for every $s, t \in [0,T]$ with $s < t$ and $B \in \mathcal{H}_s$, we have

$$\mathbb{E}\left[\mathbf{1}_B(G_t - G_s)\right] = \mathbb{E}\left[\mathbf{1}_B(\tilde{G}_t - \tilde{G}_s)\right].$$

Indeed,

$$\begin{aligned}
\mathbb{E}\left[\mathbf{1}_B(G_t - G_s)\right] &= \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_B(G_t - G_s)\middle|\,\mathcal{H}_s\right]\right] = \mathbb{E}\left[\mathbf{1}_B\left(\mathbb{E}\left[G_t\middle|\,\mathcal{H}_s\right] - \tilde{G}_s\right)\right] \\
&= \mathbb{E}\left[\mathbf{1}_B\left(\mathbb{E}\left[\tilde{G}_t\middle|\,\mathcal{H}_s\right] - \tilde{G}_s\right)\right] = \mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_B(\tilde{G}_t - \tilde{G}_s)\middle|\,\mathcal{H}_s\right]\right] \\
&= \mathbb{E}\left[\mathbf{1}_B(\tilde{G}_t - \tilde{G}_s)\right].
\end{aligned}$$

Furthermore, since G is increasing, then $\tilde{G}$ is an $\mathbb{H}$-submartingale, that is

$$\mathbb{E}\left[\tilde{G}_t\middle|\,\mathcal{H}_s\right] = \mathbb{E}\left[\mathbb{E}\left[G_t\middle|\,\mathcal{H}_t\right]\middle|\,\mathcal{H}_s\right] = \mathbb{E}\left[G_t\middle|\,\mathcal{H}_s\right] \geq \mathbb{E}\left[G_s\middle|\,\mathcal{H}_s\right] = \tilde{G}_s, \quad 0 \leq s \leq t \leq T.$$

Thanks to Doob-Meyer Theorem on decomposition of submartingales, see e.g. Theorem 3.15 of [11], there exists a unique increasing, integrable $\mathbb{H}$-predictable process $G^{\mathbb{H}}$ such that $\tilde{G} - G^{\mathbb{H}}$ is an $\mathbb{H}$-martingale, that is, for every $s, t \in [0,T]$ with $s < t$ and $B \in \mathcal{H}_s$, we have

$$\mathbb{E}\left[\mathbf{1}_B(\tilde{G}_t - \tilde{G}_s)\right] = \mathbb{E}\left[\mathbf{1}_B(G^{\mathbb{H}}_t - G^{\mathbb{H}}_s)\right].$$

□

Remark 4.4. *If G is an $\mathbb{H}$-predictable process of integrable variation and X is an $\mathbb{F}$-adapted process satisfying $\mathbb{E}\left[\int_0^T X_t \mathrm{d}G_t\right] < \infty$, then*

$$(X_t \mathrm{d}G_t)^{\mathbb{H}} = {}^p X_t \mathrm{d}G_t, \quad \mathbb{P}-\text{a.s.}, \ \textit{for every } t \in [0,T].$$

Indeed, by Theorem VI.57 in [4], for any $\mathbb{H}$-predictable (bounded) process φ we can prove that

$$\mathbb{E}\left[\int_0^T \varphi_t X_t \mathrm{d}G_t\right] = \mathbb{E}\left[\int_0^T \varphi_t {}^p X_t \mathrm{d}G_t\right].$$

4.2 Explicit representation results

We now can apply the results of Subsection 4.1 to extend Proposition 4.1. Let $\mathcal{P}^{\mathcal{H}}$ and $\mathcal{P}$ be the $\mathbb{H}$-predictable and $\mathbb{F}$-predictable σ-field respectively. We consider the measures $\mu^{\mathcal{H}}$ (respectively μ) defined on $\mathcal{P}^{\mathcal{H}}$ (respectively $\mathcal{P}$) such that

$$\mu^{\mathcal{H}}((s,t] \times B) = \mathbb{E}\left[\mathbf{1}_B(A^{\mathbb{H}}_t - A^{\mathbb{H}}_s)\right], \quad B \in \mathcal{H}_s, \ \ s,t \in [0,T], \ s < t, \tag{4.7}$$

where $A^{\mathbb{H}}$ is the $\mathbb{H}$-predictable dual projection of $A := (\int_0^t H^{\mathcal{F}}_s \mathrm{d}\langle M\rangle_s)_{0 \leq t \leq T}$, that exists thanks to Theorem 4.3, and

$$\mu((u,v] \times F) = \mathbb{E}\left[\mathbf{1}_F(\langle M\rangle^{\mathbb{H}}_v - \langle M\rangle^{\mathbb{H}}_u)\right], \quad F \in \mathcal{F}_u, \ u,v \in [0,T], \ u < v. \tag{4.8}$$

Here $H^{\mathcal{F}}$ is the integrand appearing in the Galtchouk-Kunita-Watanabe decomposition (4.1).

Lemma 4.5. *Let $\mu^{\mathcal{H}}$ and μ measures satisfying conditions* (4.7) *and* (4.8) *respectively. Then $\mu^{\mathcal{H}} \ll \mu$ on $\mathcal{P}^{\mathcal{H}}$, that is, $\mu^{\mathcal{H}}$ is absolutely continuous with respect to the restriction of μ on $\mathcal{P}^{\mathcal{H}}$.*

Proof. By using the definition of absolute continuity, we wish to show that if whenever $\mu(E) = 0$ for $E \in \mathcal{P}^{\mathcal{H}}$, then $\mu^{\mathcal{H}}(E) = 0$. Let $\psi = (\psi_t)_{0 \le t \le T}$ be a nonnegative $\mathbb{H}$-predictable process such that

$$\mathbb{E}\left[\int_0^T \psi_t \mathrm{d}\langle M\rangle_t^{\mathbb{H}}\right] = 0.$$

Then

$$\mathbb{E}\left[\int_0^T \psi_t \mathrm{d}\langle M\rangle_t\right] = 0,$$

that implies that $\psi = 0$ $\mathrm{d}\langle M\rangle \otimes \mathrm{d}\mathbb{P}$ a.e., since ψ is nonnegative. Finally

$$\mathbb{E}\left[\int_0^T \psi_t \mathrm{d}A_t^{\mathbb{H}}\right] = \mathbb{E}\left[\int_0^T \psi_t \mathrm{d}A_t\right] = \mathbb{E}\left[\int_0^T \psi_t H_t^{\mathcal{F}} \mathrm{d}\langle M\rangle_t\right] = 0.$$

□

Since $\mu^{\mathcal{H}} \ll \mu$ on $\mathcal{P}^{\mathcal{H}}$, thanks to Lemma 4.5, by the Radon-Nikodym theorem there exists a $\mathcal{P}^{\mathcal{H}}$-measurable function g on $[0,T] \times \Omega$ such that

$$\mu^{\mathcal{H}}(E) = \int_E g(t,\omega)\mathrm{d}\mu(t,\omega), \quad \forall E \in \mathcal{P}^{\mathcal{H}}.$$

This allows to identify the process $H^{\mathcal{H}}$ as the Radon-Nikodym derivative:

$$H_t^{\mathcal{H}}(\omega) := \left.\frac{\mathrm{d}\mu^{\mathcal{H}}(t,\omega)}{\mathrm{d}\mu(t,\omega)}\right|_{\mathcal{P}^{\mathcal{H}}}, \quad (t,\omega) \in [0,T] \times \Omega. \tag{4.9}$$

Finally, we are ready to state the following theorem.

Theorem 4.6. *For any nonnegative $\mathbb{F}$-measurable process $H^{\mathcal{F}}$, the following equality holds*

$$\mathbb{E}\left[\int_0^T \varphi_t H_t^{\mathcal{F}} \mathrm{d}\langle M\rangle_t\right] = \mathbb{E}\left[\int_0^T \varphi_t H_t^{\mathcal{H}} \mathrm{d}\langle M\rangle_t\right], \tag{4.10}$$

for every $\mathbb{H}$-predictable process φ such that $\mathbb{E}\left[\int_0^T |\varphi_t|^2 \mathrm{d}\langle M\rangle_t\right] < \infty$. Here $H^{\mathcal{H}}$ is given by (4.9).

Proof. By relationship (4.9) and definition of the measure μ, see (4.8), we have for every $s,t \in [0,T]$ with $s < t$ and $B \in \mathcal{H}_s$

$$\mu^{\mathcal{H}}((s,t] \times B) = \int_s^t \int_B H_u^{\mathcal{H}}(\omega)\mathrm{d}\mu(u,\omega) = \mathbb{E}\left[\mathbf{1}_B \int_s^t H_u^{\mathcal{H}} \mathrm{d}\langle M\rangle_u^{\mathbb{H}}\right] = \mathbb{E}\left[\mathbf{1}_B \int_s^t H_u^{\mathcal{H}} \mathrm{d}\langle M\rangle_u\right].$$

On the other hand, by (4.7)

$$\mu^{\mathcal{H}}((s,t]\times B) = \mathbb{E}\left[\mathbf{1}_B(A^{\mathbb{H}}_t - A^{\mathbb{H}}_s)\right] = \mathbb{E}\left[\mathbf{1}_B \int_s^t H^{\mathcal{F}}_u \mathrm{d}\langle M\rangle_u\right].$$

If φ is of the form $\varphi_u = \mathbf{1}_{(s,t]}(u)\mathbf{1}_B$, with $B \in \mathcal{H}_s$ and $s,t \in [0,T]$ with $s < t$, then the statement is proved since relationship (4.10) is verified on the generators of $\mathcal{P}^{\mathcal{H}}$. □

We now give the analogous of Proposition 4.1, without assuming that the process $\langle M\rangle$ is $\mathbb{H}$-predictable.

Theorem 4.7. *Let $(\tilde{U}_0, H^{\mathcal{F}}, \tilde{O}_T)$ be the triplet corresponding to decomposition* (4.1) *of $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$. Then*

$$\xi = U_0 + \int_0^T H^{\mathcal{H}}_t \mathrm{d}M_t + O_T, \quad \mathbb{P}-\text{a.s.}, \tag{4.11}$$

with

$$U_0 = \mathbb{E}\left[\tilde{U}_0 \middle| \mathcal{H}_0\right],$$
$$H^{\mathcal{H}}_t = \left.\frac{\mathrm{d}\mu^{\mathcal{H}}(t,\omega)}{\mathrm{d}\mu(t,\omega)}\right|_{\mathcal{P}^{\mathcal{H}}}, \quad (t,\omega) \in [0,T]\times\Omega,$$

where $\mu^{\mathcal{H}}$ and μ are given in (4.7) *and* (4.8) *respectively, and O is a square-integrable $\mathbb{F}$-martingale with $O_0 = 0$ such that $\mathbb{E}[O_T \cdot \eta] = 0$, for every $\eta \in \mathcal{L}^{\mathcal{H}}_T$.*

Proof. We proceed as in the proof of Proposition 4.1 by observing that condition (4.10) plays the same role of condition (4.5). □

In the next proposition we give a useful result which allows us to compute $H^{\mathcal{H}}$ as the Radon-Nikodym derivative of the $\mathbb{H}$-predictable dual projection $A^{\mathbb{H}}$ of the process $A = (\int_0^t H^{\mathcal{F}}_s \mathrm{d}\langle M\rangle_s)_{0\le t\le T}$ with respect to the $\mathbb{H}$-predictable dual projection $\langle M\rangle^{\mathbb{H}}$ of the $\mathbb{F}$-predictable quadratic variation $\langle M\rangle$.

Proposition 4.8. *The process $A^{\mathbb{H}} = (\int_0^{\cdot} H^{\mathcal{F}}_s \mathrm{d}\langle M\rangle_s)^{\mathbb{H}}$ is absolutely continuous with respect to $\langle M\rangle^{\mathbb{H}}$ and it is given by*

$$A^{\mathbb{H}}_t = \int_0^t H^{\mathcal{H}}_s \mathrm{d}\langle M\rangle^{\mathbb{H}}_s, \quad 0 \le t \le T.$$

As a consequence

$$H^{\mathcal{H}}_t = \frac{\mathrm{d}A^{\mathbb{H}}_t}{\mathrm{d}\langle M\rangle^{\mathbb{H}}_t}, \quad 0 \le t \le T. \tag{4.12}$$

Proof. Set $\widetilde{A}_t := \int_0^t H^{\mathcal{H}}_s \mathrm{d}\langle M\rangle^{\mathbb{H}}_s$, for each $t \in [0,T]$. It is sufficient to prove that

$$\mathbb{E}\left[\int_0^T \varphi_u \mathrm{d}A_u\right] = \mathbb{E}\left[\int_0^T \varphi_u \mathrm{d}\widetilde{A}_u\right]$$

for every $\mathbb{H}$-predictable (bounded) process φ. As before, we can consider φ of the form $\varphi_u = \mathbf{1}_{(s,t]}(u)\mathbf{1}_B$, with $B \in \mathcal{H}_s$ and $s < t \in [0,T]$.
Then by the definitions of the measure μ and $\mu^{\mathcal{H}}$, see (4.8) and (4.7), and recalling (4.9) we get

$$\mathbb{E}\left[\int_0^T \varphi_u \mathrm{d}\widetilde{A}_u\right] = \mathbb{E}\left[\mathbf{1}_B \int_s^t H_u^{\mathcal{H}} \mathrm{d}\langle M\rangle_u^{\mathbb{H}}\right] = \int_s^t \int_B H_u^{\mathcal{H}}(\omega)\mathrm{d}\mu(u,\omega) = \mu^{\mathcal{H}}((s,t] \times B) =$$

$$\mathbb{E}\left[\mathbf{1}_B(A_t^{\mathbb{H}} - A_s^{\mathbb{H}})\right] = \mathbb{E}\left[\mathbf{1}_B \int_s^t H_u^{\mathcal{F}} \mathrm{d}\langle M\rangle_u\right] = \mathbb{E}\left[\int_0^T \varphi_u \mathrm{d}A_u\right]$$

which concludes the proof. □

Example 4.9. *Suppose that the process $\langle M\rangle$ is of the form*

$$\langle M\rangle_t = \int_0^t a_s \mathrm{d}G_s, \quad 0 \le t \le T$$

for some $\mathbb{F}$-predictable process $a = (a_t)_{0 \le t \le T}$ and an increasing deterministic function G. Then by Remark 4.4

$$\langle M\rangle_t^{\mathbb{H}} = \int_0^t {}^p a_s \mathrm{d}G_s, \quad A_t^{\mathbb{H}} = \int_0^t {}^p(H_s^{\mathcal{F}} a_s)\mathrm{d}G_s, \quad 0 \le t \le T,$$

and as a consequence of Proposition 4.8 we get

$$H_t^{\mathcal{H}} = \frac{{}^p(H_t^{\mathcal{F}} a_t)}{{}^p a_t}, \quad 0 \le t \le T.$$

Remark 4.10. *Let us observe that if the process $\langle M\rangle$ is $\mathbb{H}$-predictable, then again by Remark 4.4*

$$\langle M\rangle_t^{\mathbb{H}} = \langle M\rangle_t, \quad A_t^{\mathbb{H}} = \int_0^t {}^p(H_s^{\mathcal{F}})\mathrm{d}\langle M\rangle_s, \quad 0 \le t \le T,$$

and by applying Proposition 4.8 we obtain that

$$H_t^{\mathcal{H}} = {}^p(H_t^{\mathcal{F}}), \quad 0 \le t \le T.$$

5 Risk-minimization under restricted information

In relation to the connection between risk-minimization under full and partial information respectively, we now show how our result obtained in Proposition 4.8 fits in the approach of [13] of risk-minimization under restricted information.
On a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ we consider a financial market with one riskless asset with (discounted) price 1 and one risky asset whose (discounted) price is given by a square-integrable (càdlàg) martingale $M = (M_t)_{0 \le t \le T}$ adapted to a (large) filtration $\mathbb{F} := (\mathcal{F}_t)_{0 \le t \le T}$.
We will study the problem of hedging a contingent claim, whose final payoff is given by

a random variable $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, in the case where investors acting in the market can access only to the information flow $\mathbb{H} := (\mathcal{H}_t)_{0 \leq t \leq T}$ with $\mathcal{H}_t \subseteq \mathcal{F}_t$, for each $t \in [0, T]$. We choose the risk-minimization approach to solve the above hedging problem. In the case of full information, in [6] the authors proved that there exists a unique $\mathbb{F}$-risk-minimizing hedging strategy $\phi^* = (\theta^*, \eta^*)$, where $\theta^* = (\theta^*_t)_{0 \leq t \leq T}$ is given by the integrand with respect to M in the classical Galtchouk-Kunita-Watanabe decomposition of ξ, i.e. $\theta^* = H^{\mathcal{F}}$ (see equation (4.1)).
In this section we extend this result to the case where there are restrictions on the available information, by using the Galtchouk-Kunita-Watanabe decomposition under partial information (see equation (3.2)). More precisely, we prove that the $\mathbb{H}$-risk-minimizing hedging strategy $\phi^{\mathcal{H}} = (\theta^{\mathcal{H}}, \eta^{\mathcal{H}})$ (see Definition 5.2 below) is such that $\theta^{\mathcal{H}} = H^{\mathcal{H}}$.

Risk-minimization under restricted information was studied in [13] by using a different approach. We obtain the same explicit representation given in Theorem 3.1 of [13] by applying Proposition 4.8. About risk-minimization under partial information for defaultable markets via nonlinear filtering, we refer to [8]. In particular, they consider the case where the contingent claim ξ is $\mathcal{H}_T$-measurable, in which we can solve the risk-minimization problem by using the classical Galtchouk-Kunita-Watanabe decomposition.
We now assume that the agent has at her/his disposal the information flow $\mathbb{H}$ about trading in stocks while a complete information about trading in the riskless asset.

Definition 5.1. *An $\mathbb{H}$-strategy is a pair $\phi = (\theta, \eta)$ (θ_t is the number of shares of the risky asset to be held at time t, while η_t is the amount invested in the riskless asset at time t) where θ is $\mathbb{H}$-predictable and η is $\mathbb{F}$-adapted and such that*

$$\mathbb{E}\left[\int_0^T \theta_s^2 \mathrm{d}\langle M\rangle_s\right] < \infty$$

and the value process $V(\phi) := \theta M + \eta$ satisfies

$$\mathbb{E}\left[\left(\sup_{t\in[0,T]} |V_t(\phi)|\right)^2\right] < \infty.$$

For any $\mathbb{H}$-strategy ϕ, the associated cost process $C(\phi)$ is given by

$$C_t(\phi) := V_t(\phi) - \int_0^t \theta_r \mathrm{d}M_r, \quad 0 \leq t \leq T.$$

Finally the $\mathbb{H}$-risk process of ϕ is defined by

$$R_t(\phi) := \mathbb{E}\left[(C_T(\phi) - C_t(\phi))^2 \middle| \mathcal{H}_t\right], \quad 0 \leq t \leq T.$$

Definition 5.2. *An $\mathbb{H}$-strategy $\phi = (\theta, \eta)$ is called $\mathbb{H}$-risk-minimizing if $V_T(\phi) = \xi$ $\mathbb{P}$-a.s. and if for any $\mathbb{H}$-strategy ψ such that $V_T(\psi) = \xi$ $\mathbb{P}$-a.s., we have $R_t(\phi) \leq R_t(\psi)$ $\mathbb{P}$-a.s. for every $t \in [0, T]$.*

Remark 5.3. *By Corollary 3.1 in [13] we have that if $\phi = (\theta, \eta)$ is an $\mathbb{H}$-risk-minimizing strategy then ϕ is mean-self-financing, i.e. the cost process $C(\phi)$ is an $\mathbb{F}$-martingale. Moreover, if $\phi = (\theta, \eta)$ is a mean-self-financing $\mathbb{H}$-strategy, then $V(\phi)$ is an $\mathbb{F}$-martingale, hence for a given $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, we have that $V_t(\phi) = \mathbb{E}\left[\xi | \mathcal{F}_t\right]$, for every $t \in [0, T]$.*

To prove the main result of this section, we need the following Lemma.

Lemma 5.4. *Let $O = (O_t)_{0 \le t \le T}$ be a square-integrable $\mathbb{F}$-martingale with $O_0 = 0$, satisfying the orthogonality condition*

$$\mathbb{E}\left[O_T \int_0^T \varphi_t \mathrm{d}M_t\right] = 0$$

for all $\mathbb{H}$-predictable processes $\varphi = (\varphi_t)_{0 \le t \le T}$ such that $\mathbb{E}\left[\int_0^T |\varphi_t|^2 \mathrm{d}\langle M \rangle_t\right] < \infty$. Then for any $t \in [0, T]$

$$\mathbb{E}\left[(O_T - O_t) \int_t^T \varphi_s \mathrm{d}M_s \middle| \mathcal{H}_t\right] = 0 \quad \mathbb{P}-\text{a.s.}.$$

Proof. Since for any $\mathbb{H}$-predictable process φ

$$\mathbf{1}_{(t,T]}(s)\mathbf{1}_B \varphi_s, \quad B \in \mathcal{H}_t, \ t \in [0, T),$$

is $\mathbb{H}$-predictable, we get

$$\mathbb{E}\left[O_T \mathbf{1}_B \int_t^T \varphi_s \mathrm{d}M_s\right] = \mathbb{E}\left[\mathbf{1}_B \mathbb{E}\left[O_T \int_t^T \varphi_s \mathrm{d}M_s \middle| \mathcal{H}_t\right]\right] = 0, \quad \forall B \in \mathcal{H}_t,$$

and then

$$\mathbb{E}\left[O_T \int_t^T \varphi_s \mathrm{d}M_s \middle| \mathcal{H}_t\right] = 0 \quad \mathbb{P}-\text{a.s.}.$$

Finally, let us observe that

$$\mathbb{E}\left[O_t \int_t^T \varphi_s \mathrm{d}M_s \middle| \mathcal{H}_t\right] = \mathbb{E}\left[O_t \int_0^T \varphi_s \mathrm{d}M_s \middle| \mathcal{H}_t\right] - \mathbb{E}\left[O_t \int_0^t \varphi_s \mathrm{d}M_s \middle| \mathcal{H}_t\right] = 0$$

since

$$\mathbb{E}\left[O_t \int_0^T \varphi_s \mathrm{d}M_s \middle| \mathcal{H}_t\right] = \mathbb{E}\left[\mathbb{E}\left[O_t \int_0^T \varphi_s \mathrm{d}M_s \middle| \mathcal{F}_t\right] \middle| \mathcal{H}_t\right] = \mathbb{E}\left[O_t \int_0^t \varphi_s \mathrm{d}M_s \middle| \mathcal{H}_t\right],$$

and this concludes the proof. □

We are now in the position to provide an alternative proof to that given in [13], concerning the explicit representation for an $\mathbb{H}$-risk-minimizing strategy, by applying the Galtchouk-Kunita-Watanabe decomposition under partial information and the representation result given in Proposition 4.8.

Theorem 5.5. *For every $\xi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P}; \mathbb{R})$, there exists a unique $\mathbb{H}$-risk-minimizing strategy $\phi^{\mathcal{H}} = (\theta^{\mathcal{H}}, \eta^{\mathcal{H}})$ such that $\theta^{\mathcal{H}} = H^{\mathcal{H}}$, where $H^{\mathcal{H}}$ is given by (4.12) and $\eta^{\mathcal{H}}_t = \mathbb{E}\left[\xi \middle| \mathcal{F}_t\right] - \theta^{\mathcal{H}}_t M_t$, for every $t \in [0,T]$.*

Proof. The proof is similar to that of Theorem 2.4 of [14] performed in the full information case. Let $\phi = (\theta, \eta)$ be a mean-self-financing $\mathbb{H}$-strategy such that $V_T(\phi) = \xi$ $\mathbb{P}$-a.s.. Hence, by computing the Galtchouk-Kunita-Watanabe decomposition under partial information, see (4.11), we have

$$
\begin{aligned}
C_T(\phi) - C_t(\phi) &= V_T(\phi) - V_t(\phi) - \int_t^T \theta_s \mathrm{d}M_s = \xi - V_t(\phi) - \int_t^T \theta_s \mathrm{d}M_s \\
&= U_0 + \int_0^T H^{\mathcal{H}}_t \mathrm{d}M_t + O_T - \int_t^T \theta_s \mathrm{d}M_s - V_t(\phi),
\end{aligned}
$$

where $H^{\mathcal{H}}$ is given by (4.12). Since $V_t(\phi) = E[\xi|\mathcal{F}_t]$, for every $t \in [0,T]$, see Remark 5.3, we get that

$$V_t(\phi) = U_0 + \int_0^t H^{\mathcal{H}}_s \mathrm{d}M_s + O_t$$

and

$$C_T(\phi) - C_t(\phi) = \int_t^T \{H^{\mathcal{H}}_s - \ \theta_s\} \mathrm{d}M_s + O_T - O_t.$$

By similar computation we get that

$$C_T(\phi^{\mathcal{H}}) - C_t(\phi^{\mathcal{H}}) = O_T - O_t.$$

Finally

$$
\begin{aligned}
(C_T(\phi) - C_t(\phi))^2 = \Big(C_T(\phi^{\mathcal{H}}) - C_t(\phi^{\mathcal{H}})\Big)^2 + \Big(\int_t^T \{H^{\mathcal{H}}_s - \theta_s\} \mathrm{d}M_s\Big)^2 \\
+ 2(O_T - O_t) \int_t^T \{H^{\mathcal{H}}_s - \ \theta_s\} \mathrm{d}M_s
\end{aligned}
$$

and by Lemma 5.4 we obtain that

$$R_t(\phi) = R_t(\phi^{\mathcal{H}}) + \mathbb{E}\left[\left(\int_t^T \{H^{\mathcal{H}}_s - \theta_s\} \mathrm{d}M_s\right)^2 \middle| \mathcal{H}_t\right] \geq R_t(\phi^{\mathcal{H}}).$$

Hence $\phi^{\mathcal{H}}$ is $\mathbb{H}$-risk-minimizing. If some other ϕ is also $\mathbb{H}$-risk-minimizing then

$$\mathbb{E}\left[\left(\int_0^T \{H^{\mathcal{H}}_s - \theta_s\} \mathrm{d}M_s\right)^2 \middle| \mathcal{H}_0\right] = \mathbb{E}\left[\int_0^T \{H^{\mathcal{H}}_s - \theta_s\}^2 \mathrm{d}\langle M\rangle_s \middle| \mathcal{H}_0\right] = 0,$$

which implies $H^{\mathcal{H}} = \theta$. Since $V_t(\phi) = V_t(\phi^{\mathcal{H}}) = \mathbb{E}\left[\xi \middle| \mathcal{F}_t\right]$ for each $t \in [0,T]$, we also obtain $\phi = \phi^{\mathcal{H}}$. □

In the rest of the section we investigate the case where there is a relationship between the information flow $\mathbb{H}$ and the filtration generated by the stock price M, that we denote by $\mathbb{F}^M$. A possible choice is the assumption that investors acting in the market have access only to the information contained in past asset prices, that is $\mathbb{H} = \mathbb{F}^M$. Such a situation has been studied for instance in [7] and [3] for stock price dynamics with jumps. In the sequel we will assume $\mathbb{H} \subseteq \mathbb{F}^M$, which takes also into account, for instance, the case where the asset price is only observed at discrete times or with a fixed delay $\tau \in (0, T)$, i.e. $\mathcal{H}_t = \mathcal{F}^M_{(t-\tau)^+}$, for every $t \in (0, T)$.

In a such particular case, when in addition $\xi \in L^2(\Omega, \mathcal{F}^M_T, \mathbb{P}; \mathbb{R})$, we can find an $\mathbb{H}$-risk minimizing strategy, $\phi^{\mathcal{H}} = (\theta^{\mathcal{H}}, \eta^{\mathcal{H}})$, where θ^H is $\mathbb{H}$-predictable and η^H is $\mathbb{F}^M$-adapted, while in the general case $\eta^{\mathcal{H}}$ has been taken $\mathbb{F}$-adapted. This means that we study the situation where the agent has at her/his disposal the information flow $\mathbb{H} \subseteq \mathbb{F}^M$ about trading in stocks and the filtration $\mathbb{F}^M$ about trading in the riskless asset, and when $\mathbb{H} = \mathbb{F}^M$ the same information flow.

More precisely, from now on we restrict ourself to consider $\mathbb{H}$-strategies $\phi = (\theta, \eta)$ as in Definition 5.1 where η is chosen $\mathbb{F}^M$-adapted.

Remark 5.6. *Let us observe that given an $\mathbb{H}$-strategy $\phi = (\theta, \eta)$, the associated value process $V(\phi) := \theta M + \eta$ turns out to be $\mathbb{F}^M$-adapted. By Corollary 3.1 in [13], we have that if $\phi = (\theta, \eta)$ is an $\mathbb{H}$-risk-minimizing strategy according to this new definition, then ϕ is mean-self-financing, i.e. the cost process $C(\phi)$ is an $\mathbb{F}^M$-martingale. Moreover, if $\phi = (\theta, \eta)$ is a mean-self-financing $\mathbb{H}$-strategy, then $V(\phi)$ is an $\mathbb{F}^M$-martingale, hence $V_t(\phi) = E[\xi|\mathcal{F}^M_t]$, for every $t \in [0, T]$.*

We are now ready to give the following result.

Theorem 5.7. *For every $\xi \in L^2(\Omega, \mathcal{F}^M_T, \mathbb{P}; \mathbb{R})$, there exists a unique $\mathbb{H}$-risk-minimizing strategy $\phi^{\mathcal{H}} = (\theta^{\mathcal{H}}, \eta^{\mathcal{H}})$ such that $\theta^{\mathcal{H}} = H^{\mathcal{H}}$, where $H^{\mathcal{H}}$ is given by (4.12) and $\eta^{\mathcal{H}} = E[\xi|\mathcal{F}^M_t] - \theta^{\mathcal{H}}_t M_t$, for every $t \in [0, T]$.*

Proof. Since ξ is $\mathcal{F}^M_T$-measurable, by decomposition (4.11) we obtain that

$$\xi = \mathbb{E}[\xi|\mathcal{F}^M_T] = U_0 + \int_0^T H^{\mathcal{H}}_t \mathrm{d}M_t + \mathbb{E}[O_T|\mathcal{F}^M_T],$$

where $H^{\mathcal{H}}$ is given by (4.12). Set $\hat{O}_t := \mathbb{E}[O_t|\mathcal{F}^M_t]$, for each $t \in [0, T]$. It is known that $\hat{O}$ is an $\mathbb{F}^M$-martingale and

$$\mathbb{E}\left[\hat{O}_T \int_0^T H^{\mathcal{H}}_t \mathrm{d}M_t\right] = \mathbb{E}\left[\mathbb{E}\left[\hat{O}_T|\mathcal{F}^M_T\right] \int_0^T H^{\mathcal{H}}_t \mathrm{d}M_t\right] = \mathbb{E}\left[O_T \int_0^T H^{\mathcal{H}}_t \mathrm{d}M_t\right] = 0.$$

Therefore we obtain the Galtchouk-Kunita-Watanabe decomposition of ξ under restricted information with respect to the filtration $\mathbb{F}^M$, that is

$$\xi = U_0 + \int_0^T H^{\mathcal{H}}_t \mathrm{d}M_t + \hat{O}_T, \quad \mathbb{P} - \text{a.s.}.$$

The rest of the proof follows from Theorem 5.5 by replacing the filtration $\mathbb{F}$ by $\mathbb{F}^M$ and the $\mathbb{F}$-martingale O by the $\mathbb{F}^M$-martingale $\hat{O}$. □

Acknowledgements

Part of the paper was written during the stay of the second named author at INRIA Paris-Rocquencourt (Projet MathFi). The second and third named authors would like to thank Prof. Agnès Sulem for stimulating discussions. The work of the first and second named authors was partially supported by PRIN 2008 Project "Probability and Finance". The work of the third named author was partially supported by the ANR Project MASTERIE 2010 BLAN-0121-01 and it was partially carried out during his visit to the Bernoulli Center (EPFL Lausanne).

APPENDIX

A Technical results

Recall that $M = (M_t)_{0\le t\le T}$ is a square-integrable $\mathbb{F}$-martingale and assume that $\mathbb{F} = \mathbb{F}^M := (\mathcal{F}_t^M)_{0\le t\le T}$, i.e. the information flow $\mathbb{F}$ coincides with the canonical filtration $\mathbb{F}^M$ of M.

Lemma A.1. *Let M be a Lévy process and $\xi = h(M_T) \in L^2(\Omega, \mathcal{F}_T^M, \mathbb{P}; \mathbb{R})$ for some measurable function $h : \mathbb{R} \to \mathbb{R}$. Then, there exists a measurable function $F : [0,T]\times\mathbb{R} \to \mathbb{R}$ such that*

$$\xi = \mathbb{E}[\xi] + \int_0^T F(s, M_{s-})\mathrm{d}M_s + \tilde{O}_T, \quad \mathbb{P} - a.s.,$$

where $\tilde{O} = (\tilde{O}_t)_{0\le t\le T}$ is a square-integrable $\mathbb{F}^M$-martingale null at zero such that $\langle \tilde{O}, M\rangle_t = 0$, for every $t \in [0,T]$. Moreover, the following integrability condition is satisfied

$$\mathbb{E}\left[\int_0^T |F(s, M_{s-})|^2 \mathrm{d}\langle M\rangle_s\right] < \infty.$$

Proof. If ξ is given as a Fourier transform of M_T, that is, the function h is of the form

$$h(x) = \int_{\mathbb{R}} e^{iax}\mathrm{d}\nu(a), \quad \text{for all } x \in \mathbb{R}, \tag{A.1}$$

where ν is a finite measure, the result is contained in Proposition 4.3 of [9], which was an adaptation of [10].
As a consequence, the thesis follows once we show the existence of a sequence $(h_n)_{n\in\mathbb{N}}$ of functions of the kind (A.1) such that

$$\mathbb{E}\left[|h_n(M_T) - h(M_T)|^2\right] \xrightarrow[n\to\infty]{} 0. \tag{A.2}$$

To see that, denoting by $F_n(t, M_{t-})$ the integrand in the Galtchouk-Kunita-Watanabe decomposition of $h_n(M_T)$, $n \in \mathbb{N}$, we can proceed as in the proof of Lemma 3.1 and we get that the sequence $(F_n(t, M_{t-}))_{n\in\mathbb{N}}$ converges in $L^2(\Omega, \mathrm{d}\langle M\rangle \otimes \mathrm{d}\mathbb{P})$ to the integrand $H^{\mathcal{F}}$ in the Galtchouk-Kunita-Watanabe decomposition of $h(M_T)$. Now, there is a subsequence converging $\mathrm{d}\langle M\rangle \otimes \mathrm{d}\mathbb{P}$-a.e. to the $\mathbb{F}$-predictable process $(H_t^{\mathcal{F}})_{0\leq t\leq T}$ and for almost all $t \in [0, T]$, $H_t^{\mathcal{F}}$ is $\sigma(M_{t-})$-measurable. Finally this implies the existence of a measurable function $F : [0, T] \times \mathbb{R} \to \mathbb{R}$ such that $H_t^{\mathcal{F}} = F(t, M_{t-})$.
It remains to show the existence of a sequence $(h_n)_{n\in\mathbb{N}}$ of functions of type (A.1) verifying (A.2). If ρ_T is the law of M_T, (A.2), translates into

$$\int_{\mathbb{R}} (h_n(y) - h(y))^2 \mathrm{d}\rho_T(y) \xrightarrow[n\to\infty]{} 0. \tag{A.3}$$

Since ρ_T is a finite non-negative measure, it is well-known that the space of smooth functions with compact support is dense in $L^2(\rho_T)$. This implies that the Schwartz space $\mathcal{S}(\mathbb{R})$ of the fast decreasing functions is dense in $L^2(\rho_T)$. Let $(h_n)_{n\in\mathbb{N}}$ belong to $\mathcal{S}(\mathbb{R})$ such that (A.3), and consequently, (A.2) holds. Since the inverse Fourier transform $\mathcal{F}^{-1}$ maps $\mathcal{S}(\mathbb{R})$ into itself, then we observe that h_n are of the type (A.1) with $\nu(\mathrm{d}a) = \mathcal{F}^{-1}h_n(a)\mathrm{d}a$.
□

References

[1] P. Briand, B. Delyon, and J. Mémin. On the robustness of backward stochastic differential equations. *Stochastic Processes and their Applications*, 97(2):229–253, 2002.

[2] R. Carbone, B. Ferrario, and M. Santacroce. Backward stochastic differential equations driven by càdlàg martingales. *Theory of Probability and its Applications*, 52(2):304–314, 2008.

[3] C. Ceci. Risk-minimizing hedging for a partially observed high frequency data model. *Stochastics: An International Journal of Probability and Stochastic Processes*, 78(1):13–31, 2006.

[4] C. Dellacherie and P. A. Meyer. *Probabilities and Potential B*. North Holland, Amsterdam, 1982.

[5] N. El Karoui and S. J. Huang. A general result of existence and uniqueness of backward stochastic differential equations. In N. El Karoui and L. Mazliak, editors, *Backward Stochastic Differential Equations*, pages 27–36. Longman, Harlow, 1997.

[6] H. Föllmer and D. Sondermann. Hedging of non-redundant contingent claims. In W. Hildenbrand and A. Mas-Colell, editors, *Contributions to Mathematical Economics*, pages 203–233. 1986.

[7] R. Frey. Risk minimization with incomplete information in a model for high-frequency data. *Mathematical Finance*, 10(2):215–225, 2000.

[8] R. Frey and T. Schmidt. Pricing and hedging of credit derivatives via the innovations approach to nonlinear filtering. *Finance and Stochastics*, 16(1):105–133, 2012.

[9] N. Goutte, N. Oudjane, and F. Russo. On some expectation and derivative operators related to integral representations of random variables with respect to a PII process. *Arxiv:1202.0619v1 [math.PR]*, 2012.

[10] F. Hubalek, J. Kallsen, and L. Krawczyk. Variance-optimal hedging for processes with stationary independent increments. *The Annals of Applied Probability*, 16(2):853–885, 2006.

[11] J. Jacod and A.N. Shiryaev. *Limit Theorems for Stochastic Processes*. Springer, 2nd edition, 2003.

[12] P. Protter. *Stochastic Integration and Differential Equations*, volume 21 of *Applications of Mathematics*. Springer-Verlag Berlin Heidelberg, 2nd edition, 2004.

[13] M. Schweizer. Risk-minimizing hedging strategies under restricted information. *Mathematical Finance*, 4(4):327–342, 1994.

[14] M. Schweizer. A guided tour through quadratic hedging approaches. In E. Jouini, J. Cvitanic, and M. Musiela, editors, *Option Pricing, Interest Rates and Risk Management*, pages 538–574. Cambridge University Press, Cambridge, 2001.