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Abstract: This paper deals with optimal control problems of integral equations, with initial-final and running state constraints. The order of a running state constraint is defined in the setting of integral dynamics, and we work here with constraints of arbitrary high orders. First and second-order necessary conditions of optimality are obtained, as well as second-order sufficient conditions.

Key-words: optimal control, integral equations, state constraints, second-order optimality conditions

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Conditions d'optimalité du premier et second ordre pour des problèmes de commande optimale d'équations intégrales avec contraintes sur l'état

Résumé : On s'intéresse dans cet article à des problèmes de commande optimale d'équations intégrales, avec contraintes sur l'état initial-final ainsi que sur l'état à chaque instant. L'ordre d'une contrainte sur l'état à chaque instant est défini dans le cadre d'une dynamique intégrale, et on considère ici des contraintes d'ordre quelconque. On obtient des conditions d'optimalité nécessaires du premier et second ordre, ainsi que des conditions suffisantes du second ordre.

Mots-clés : commande optimale, équations intégrales, contraintes sur l'état, conditions d'optimalité du second ordre

1 Introduction

The dynamics in the optimal control problems we consider in this paper is given by an integral equation. Such equations, sometimes called nonlinear Volterra integral equations, belong to the family of equations with memory and thus are found in many models. Among the fields of application of these equations are population dynamics in biology and growth theory in economy: see [25] or its translation in [21] for one of the first use of integral equations in ecology in 1927 by Volterra, who contributed earlier to their theoretical study [24]; in 1976, Kamien and Muller model the capital replacement problem by an optimal control problem with an integral state equation [16]. First-order optimality conditions for such problems were known under the form of a maximum principle since Vinokurov's paper [22] in 1967, translated in 1969 [23] and whose proof has been questionned by Neustadt and Warga [18] in 1970. Maximum principles have then been provided by Bakke [2], Carlson [9], or more recently de la Vega [12] for an optimal terminal time control problem. First-order optimality conditions for control problems of the more general family of equations with memory are obtained by Carlier and Tahraoui [8].

None of the previously cited articles consider what we will call 'running state constraints'. That is what Bonnans and de la Vega did in [3], where they provide Pontryagin's principle, i.e. first-order optimality conditions. In this work we are particularly interested in second-order necessary conditions, in presence of running state constraints. Such constraints drive to optimization problems with inequality constraints in the infinite-dimensional space of continuous functions. Thus second-order necessary conditions on a so-called *critical cone* will contain an extra term, as it has been discovered in 1988 by Kawasaki [17] and generalized in 1990 by Cominetti [11], in an abstract setting. It is possible to compute this extra term in the case of state constrained optimal control problems; this is what is done by Páles and Zeidan [19] or Bonnans and Hermant [4, 6] in the framework of ODEs.

Our strategy here is different and follows [5], with the differences that we work with integral equations and that we add initial-final state constraints which lead to nonunique Lagrange multipliers. The idea was already present in [17] and is closely related to the concept of extended polyhedricity [7]: the extra term mentioned above vanishes if we write second-order necessary conditions on a subset of the critical cone, the so-called *radial critical cone*. This motivates to introduce an auxiliary optimization problem, the *reduced problem*, for which under some assumptions the radial critical cone is dense in the critical cone. Optimality conditions for the reduced problem are relevant for the original problem and the extra term now appears as the derivative of a new constraint in the reduced problem. We will devote a lot of effort to the proof of the density result and we will mention a flaw in [5] concerning this proof.

The paper is organized as follows. We set the optimal control problem, define Lagrange multipliers and work on the notion of order of a running state constraint in our setting in section 2. The reduced problem is introduced in section 3, followed by first-order necessary conditions and second-order necessary conditions on the radial critical cone. The main results are presented in section 4. After some specific assumptions, we state and prove the technical lemma 23 which is then used to strengthen the first-order necessary conditions already obtained and to get the density result that we need. With this density result, we obtain second-order necessary conditions on the critical cone. Second-order suficient conditions are also given in this section. Some of the technical aspects are postponed in the appendix.

Notations We denote by h_t the value of a function h at time t if h depends only on t, and by $h_{i,t}$ its ith component if h is vector-valued. To avoid confusion we denote partial derivatives of a function h of (t, x) by $D_t h$ and $D_x h$. We identify the dual space of $\mathbb{R}^n$ with the space $\mathbb{R}^{n*}$ of n-dimensional horizontal vectors. Generally, we denote by X^* the dual space of a topological vector space X. We use $|\cdot|$ for both the Euclidean norm on finite-dimensional vector spaces and for the cardinal of finite sets, $\|\cdot\|_s$ and $\|\cdot\|_{q,s}$ for the standard norms on the Lesbesgue spaces L^s and the Sobolev spaces $W^{q,s}$, respectively.

2 Optimal control of state constrained integral equations

2.1 Setting

We consider an optimal control problem with running and initial-final state constraints, of the following type:

$$(P) \qquad \min_{(u,y)\in\mathcal{U}\times\mathcal{Y}} \int_0^T \ell(u_t,y_t)\mathrm{d}t + \phi(y_0,y_T) \tag{2.1}$$

$$\text{subject to} \qquad y_t = y_0 + \int_0^t f(t,s,u_s,y_s)\mathrm{d}s, \quad t\in[0,T], \tag{2.2}$$

$$g(y_t) \le 0, \quad t\in[0,T], \tag{2.3}$$

$$\Phi^E(y_0,y_T) = 0, \tag{2.4}$$

$$\Phi^I(y_0,y_T) \le 0, \tag{2.5}$$

where

$$\mathcal{U} := L^\infty([0,T];\mathbb{R}^m), \quad \mathcal{Y} := W^{1,\infty}([0,T];\mathbb{R}^n)$$

are the *control space* and the *state space*, respectively.

The data are $\ell\colon \mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}$, $\phi\colon \mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, $f\colon \mathbb{R}\times\mathbb{R}\times\mathbb{R}^m\times\mathbb{R}^n\to\mathbb{R}^n$, $g\colon\mathbb{R}^n\to\mathbb{R}^r$, $\Phi^E\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{s_E}$, $\Phi^I\colon\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^{s_I}$ and $T>0$. We set τ as the symbol for the first variable of f. Observe that if $D_\tau f = 0$, we recover an optimal control problem of a state constrained ODE. We make the following assumption:

(A0) $\ell,\phi,f,g,\Phi^E,\Phi^I$ are of class C^∞ and f is Lipschitz.

We call *trajectory* a pair $(u,y)\in\mathcal{U}\times\mathcal{Y}$ which satisfies the *state equation* (2.2). Under assumption **(A0)** it can be shown by standard contraction arguments that for any $(u,y_0)\in\mathcal{U}\times\mathbb{R}^n$, the state equation (2.2) has a unique solution y in $\mathcal{Y}$, denoted by $y[u,y_0]$. Moreover, the map $\Gamma\colon\mathcal{U}\times\mathbb{R}^n\to\mathcal{Y}$ defined by $\Gamma(u,y_0) := y[u,y_0]$ is of class C^∞.

2.2 Lagrange multipliers

The dual space of the space of vector-valued continuous functions $C([0,T];\mathbb{R}^r)$ is the space of finite vector-valued Radon measures $\mathcal{M}([0,T];\mathbb{R}^{r*})$, under the pairing

$$\langle\mu,h\rangle := \int_{[0,T]} \mathrm{d}\mu_t h_t = \sum_{1\le i\le r}\int_{[0,T]} h_{i,t}\mathrm{d}\mu_{i,t}.$$

We define $BV([0,T];\mathbb{R}^{n*})$, the space of vector-valued functions of bounded variations, as follows: let I be an open set which contains $[0,T]$; then

$$BV([0,T];\mathbb{R}^{n*}) := \left\{h\in L^1(I;\mathbb{R}^{n*}) \,:\, Dh\in\mathcal{M}(I;\mathbb{R}^{n*}),\, \mathrm{supp}(Dh)\subset[0,T]\right\},$$

where Dh is the distributional derivative of h; if h is of bounded variations, we denote it by $\mathrm{d}h$. For $h \in BV([0,T];\mathbb{R}^{n*})$, there exists $h_{0_-}, h_{T_+} \in \mathbb{R}^{n*}$ such that

$$\begin{aligned} h &= h_{0_-} \quad \text{a.e. on } (-\infty, 0) \cap I, \\ h &= h_{T_+} \quad \text{a.e. on } (T, +\infty) \cap I. \end{aligned} \tag{2.6}$$

Conversely, we can identify any measure $\mu \in \mathcal{M}([0,T];\mathbb{R}^{r*})$ with the derivative of a function of bounded variations, denoted again by μ, such that $\mu_{T_+} = 0$. This motivates the notation $\mathrm{d}\mu$ for any measure in the sequel, setting implicitly $\mu_{T_+} = 0$. See appendix A.1 for more details.

Let

$$\mathcal{M} := \mathcal{M}([0,T];\mathbb{R}^{r*}), \quad \mathcal{P} := BV([0,T];\mathbb{R}^{n*}).$$

We define the *Hamiltonian* $H\colon [\mathcal{P}] \times \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}$ by

$$H[p](t,u,y) := \ell(u,y) + p_t f(t,t,u,y) + \int_t^T p_s D_\tau f(s,t,u,y)\mathrm{d}s \tag{2.7}$$

and the *end points Lagrangian* $\Phi\colon [\mathbb{R}^{s*}] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ by

$$\Phi[\Psi](y_1,y_2) := \phi(y_1,y_2) + \Psi\Phi(y_1,y_2) \tag{2.8}$$

where $s := s_E + s_I$ and $\Phi := (\Phi^E, \Phi^I)$. We also denote $K := \{0\}_{s_E} \times (\mathbb{R}_-)^{s_I}$, so that (2.4)-(2.5) can be rewritten as $\Phi(y_0, y_T) \in K$. Given a trajectory (u,y) and $(\mathrm{d}\eta, \Psi) \in \mathcal{M} \times \mathbb{R}^{s*}$, the *adjoint state* p, whenever it exists, is defined as the solution in $\mathcal{P}$ of

$$\left\{ \begin{array}{l} -\mathrm{d}p_t = D_y H[p](t,u_t,y_t)\mathrm{d}t + \mathrm{d}\eta_t g'(y_t), \\ (-p_{0_-}, p_{T_+}) = D\Phi[\Psi](y_0,y_T). \end{array} \right. \tag{2.9}$$

Note that $\mathrm{d}\eta_t g'(y_t) = \sum_{i=1}^r \mathrm{d}\eta_{i,t} g_i'(y_t)$. The adjoint state does not exist in general, but when it does it is unique. More precisely, we have:

Lemma 1. *There exists a unique solution in $\mathcal{P}$ of the adjoint state equation with final condition only (i.e. without initial condition):*

$$\left\{ \begin{array}{l} -\mathrm{d}p_t = D_y H[p](t,u_t,y_t)\mathrm{d}t + \mathrm{d}\eta_t g'(y_t), \\ p_{T_+} = D_{y_2}\Phi[\Psi](y_0,y_T). \end{array} \right. \tag{2.10}$$

Proof. The contraction argument is given in appendix A.1. □

We can now define Lagrange multipliers for optimal control problems in our setting:

Definition 2. $(\mathrm{d}\eta, \Psi, p) \in \mathcal{M} \times \mathbb{R}^{s*} \times \mathcal{P}$ is a *Lagrange multiplier* associated with $(\bar{u}, \bar{y})$ if

$$p \text{ is the adjoint state associated with } (\bar{u}, \bar{y}, \mathrm{d}\eta, \Psi), \tag{2.11}$$

$$\mathrm{d}\eta \geq 0, \quad g(\bar{y}) \leq 0, \quad \int_{[0,T]} \mathrm{d}\eta_t g(\bar{y}_t) = 0, \tag{2.12}$$

$$\Psi \in N_K(\Phi(\bar{y}_0, \bar{y}_T)), \tag{2.13}$$

$$D_u H[p](t, \bar{u}_t, \bar{y}_t) = 0 \text{ for a.a. } t \in [0,T]. \tag{2.14}$$

2.3 Linearized state equation

For $s \in [1, \infty]$, let

$$\mathcal{V}_s := L^s([0,T];\mathbb{R}^m), \quad \mathcal{Z}_s := W^{1,s}([0,T];\mathbb{R}^n).$$

Given a trajectory (u, y) and $(v, z_0) \in \mathcal{V}_s \times \mathbb{R}^n$, we consider the *linearized state equation* in $\mathcal{Z}_s$:

$$z_t = z_0 + \int_0^t D_{(u,y)} f(t, s, u_s, y_s)(v_s, z_s) \mathrm{d}s. \tag{2.15}$$

It is easily shown that there exists a unique solution $z \in \mathcal{Z}_s$ of (2.15), called the *linearized state* associated with the trajectory (u, y) and the direction (v, z_0), and denoted by $z[v, z_0]$ (keeping in mind the nominal trajectory).

Lemma 3. *There exists $C > 0$ and $C_s > 0$ for any $s \in [1, \infty]$ (depending on (u, y)) such that, for all $(v, z_0) \in \mathcal{V}_s \times \mathbb{R}^n$ and all $t \in [0, T]$,*

$$|z[v, z_0]_t| \leq C \left(|z_0| + \int_0^t |v_s| \mathrm{d}s \right), \tag{2.16}$$

$$\|z[v, z_0]\|_{1,s} \leq C_s \left(|z_0| + \|v\|_s \right). \tag{2.17}$$

Proof. (2.16) is an application of Gronwall's lemma and (2.17) is a consequence of (2.16). □

Observe that for $s = \infty$, the linearized state equation arises naturally: let $(u, y_0) \in \mathcal{U} \times \mathbb{R}^n$, $y := \Gamma(u, y_0) \in \mathcal{Y}$. We consider the linearized state associated with the trajectory (u, y) and a direction $(v, z_0) \in \mathcal{U} \times \mathbb{R}^n$. Then

$$z[v, z_0] = D\Gamma(u, y_0)(v, z_0). \tag{2.18}$$

Similarly we can define the *second-order linearized state*:

$$z^2[v, z_0] := D^2\Gamma(u, y_0)(v, z_0)^2. \tag{2.19}$$

Note that $z^2[v, z_0]$ is the unique solution in $\mathcal{Y}$ of

$$z_t^2 = \int_0^t \left(D_y f(t, s, u_s, y_s) z_s^2 + D^2_{(u,y)^2} f(t, s, u_s, y_s)(v_s, z[v, z_0]_s)^2 \right) \mathrm{d}s. \tag{2.20}$$

2.4 Running state constraints

The running state constraints g_i, $i = 1, \ldots, r$, are considered along trajectories (u, y). They produce functions of one variable, $t \mapsto g_i(y_t)$, which belong *a priori* to $W^{1,\infty}([0, T])$ and satisfy

$$\frac{\mathrm{d}}{\mathrm{d}t} g_i(y_t) = g_i'(y_t) \left(f(t, t, u_t, y_t) + \int_0^t D_\tau f(t, s, u_s, y_s) \mathrm{d}s \right). \tag{2.21}$$

There are two parts in this derivative:

- ▷ $t \mapsto g_i'(y_t) f(t, t, u_t, y_t)$, where u appears pointwisely.
- ▷ $t \mapsto g_i'(y_t) \int_0^t D_\tau f(t, s, u_s, y_s) \mathrm{d}s$, where u appears in an integral.

Below we will distinguish these two behaviors and set $\tilde{u}$ as the symbol for the pointwise variable, u for the integral variable (similarly for y). If there is no dependance on $\tilde{u}$, one can differentiate again (2.21) w.r.t. t. This motivates the definition of a notion of total derivative that always "forget" the dependence on $\tilde{u}$. Let us do that formally.

First we need a set which is stable by operations such as in (2.21), so that it will contain the derivatives of any order. It is also of interest to know how the functions we consider depend on $(u,y) \in \mathcal{U} \times \mathcal{Y}$. To answer this double issue, we define the following commutative ring:

$$\mathcal{S} := \left\{ h \,:\, h(t,\tilde{u},\tilde{y},u,y) = \sum_{\alpha} a_\alpha(t,\tilde{u},\tilde{y}) \prod_{\beta} \int_0^t b_{\alpha,\beta}(t,s,u_s,y_s)\mathrm{d}s \right\}, \tag{2.22}$$

where $(t,\tilde{u},\tilde{y},u,y) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathcal{U} \times \mathcal{Y}$, the a_α, $b_{\alpha,\beta}$ are real functions of class C^∞, the sum and the products are finite and an empty product is equal to 1. The following is straightforward:

Lemma 4. *Let $h \in \mathcal{S}$, $(u,y) \in \mathcal{U} \times \mathcal{Y}$. There exists $C > 0$ such that, for a.a. $t \in [0,T]$ and for all $(\tilde{v},\tilde{z},v,z) \in \mathbb{R}^m \times \mathbb{R}^n \times \mathcal{U} \times \mathcal{Y}$,*

$$\left| D_{(\tilde{u},\tilde{y},u,y)} h(t,u_t,y_t,u,y)(\tilde{v},\tilde{z},v,z) \right| \leq C \left(|\tilde{v}| + |\tilde{z}| + \int_0^t (|v_s| + |z_s|)\,\mathrm{d}s \right). \tag{2.23}$$

Next we define the derivation $D^{(1)} \colon \mathcal{S} \longrightarrow \mathcal{S}$ as follows (recall that we set τ as the symbol for the first variable of f or b):

1. for $h \colon (t,\tilde{u},\tilde{y}) \in \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \mapsto a(t,\tilde{u},\tilde{y}) \in \mathbb{R}$,

$$\begin{aligned} \left(D^{(1)} h\right)(t,\tilde{u},\tilde{y},u,y) := {} & D_t a(t,\tilde{u},\tilde{y}) \\ & + D_{\tilde{y}} a(t,\tilde{u},\tilde{y}) \left(f(t,t,\tilde{u},\tilde{y}) + \int_0^t D_\tau f(t,s,u_s,y_s)\mathrm{d}s \right). \end{aligned}$$

2. for $h \colon (t,u,y) \in \mathbb{R} \times \mathcal{U} \times \mathcal{Y} \mapsto \int_0^t b(t,s,u_s,y_s)\mathrm{d}s \in \mathbb{R}$,

$$\left(D^{(1)} h\right)(t,\tilde{u},\tilde{y},u,y) := b(t,t,\tilde{u},\tilde{y}) + \int_0^t D_\tau b(t,s,u_s,y_s)\mathrm{d}s.$$

3. for any $h_1, h_2 \in \mathcal{S}$,

$$\begin{aligned} \left(D^{(1)}(h_1 + h_2)\right) &= \left(D^{(1)} h_1\right) + \left(D^{(1)} h_2\right), \\ \left(D^{(1)}(h_1 h_2)\right) &= \left(D^{(1)} h_1\right) h_2 + h_1 \left(D^{(1)} h_2\right). \end{aligned}$$

It is clear that $D^{(1)} h \in \mathcal{S}$ for any $h \in \mathcal{S}$. The following formula, which is easily checked on $h = a(t,\tilde{u},\tilde{y})$ and $h = \int_0^t b(t,s,u_s,y_s)\mathrm{d}s$, will be used for any $h \in \mathcal{S}$:

$$\begin{aligned} \left(D^{(1)} h\right)(t,u_t,y_t,u,y) = {} & D_t h(t,u_t,y_t,u,y) + D_{\tilde{y}} h(t,u_t,y_t,u,y) f(t,t,u_t,y_t) \\ & + D_{\tilde{y}} h(t,u_t,y_t,u,y) \int_0^t D_\tau f(t,s,u_s,y_s)\mathrm{d}s. \end{aligned} \tag{2.24}$$

Let us now highlight two important properties of $D^{(1)}$. First, it is a notion of total derivative:

Lemma 5. *Let $h \in \mathcal{S}$ be such that $D_{\tilde{u}} h \equiv 0$, $(u, y) \in \mathcal{U} \times \mathcal{Y}$ be a trajectory and*

$$\varphi \colon t \mapsto h(t, u_t, y_t, u, y). \tag{2.25}$$

Then $\varphi \in W^{1,\infty}([0, T])$ and

$$\frac{\mathrm{d}\varphi}{\mathrm{d}t}(t) = \left(D^{(1)} h\right)(t, u_t, y_t, u, y). \tag{2.26}$$

Proof. We write h as in (2.22). If $D_{\tilde{u}} h \equiv 0$, then for any $u_0 \in \mathbb{R}^m$,

$$\varphi(t) = h(t, u_0, y_t, u, y) \tag{2.27}$$

$$= \sum_{\alpha} a_\alpha(t, u_0, y_t) \prod_{\beta} \int_0^t b_{\alpha,\beta}(t, s, u_s, y_s)\mathrm{d}s. \tag{2.28}$$

By (2.28), $\varphi \in W^{1,\infty}([0, T])$. And by (2.27),

$$\begin{aligned} \frac{\mathrm{d}\varphi}{\mathrm{d}t}(t) &= D_t h(t, u_0, y_t, u, y) + D_{\tilde{y}} h(t, u_0, y_t, u, y)\dot{y}_t \\ &= D_t h(t, u_t, y_t, u, y) + D_{\tilde{y}} h(t, u_t, y_t, u, y)\dot{y}_t \end{aligned}$$

since $D_{\tilde{u}} D_t h \equiv D_t D_{\tilde{u}} h \equiv 0$ and $D_{\tilde{u}} D_{\tilde{y}} h \equiv 0$. Using the expression of $\dot{y}_t$ and (2.24), we recognize (2.26). □

Second, it satisfies a principle of commutation with the linearization:

Lemma 6. *Let $h, (u, y)$ be as in lemma 5, $(v, z_0) \in \mathcal{V}_s \times \mathbb{R}^n$, $z := z[v, z_0] \in \mathcal{Z}_s$ for some $s \in [1, \infty]$ and*

$$\psi \colon t \mapsto D_{(\tilde{y},u,y)} h(t, u_t, y_t, u, y)(z_t, v, z). \tag{2.29}$$

Then $\psi \in W^{1,s}([0, T])$ and

$$\frac{\mathrm{d}\psi}{\mathrm{d}t}(t) = D_{(\tilde{u},\tilde{y},u,y)} \left[\left(D^{(1)} h\right)(t, u_t, y_t, u, y)\right](v_t, z_t, v, z). \tag{2.30}$$

Proof. Using $D_{\tilde{u}} D_{(\tilde{y},u,y)} h \equiv 0$, we have

$$\begin{aligned} \psi(t) &= D_{(\tilde{y},u,y)} h(t, u_0, y_t, u, y)(z_t, v, z) \\ &= \sum_{\alpha} D_{\tilde{y}} a_\alpha(t, u_0, y_t) z_t \prod_{\beta} \int_0^t b_{\alpha,\beta} \mathrm{d}s \\ &\quad + \sum_{\alpha,\beta} a_\alpha(t, u_0, y_t) \int_0^t D_{(u,y)} b_{\alpha,\beta}(t, s, u_s, y_s)(v_s, z_s)\mathrm{d}s \prod_{\beta' \neq \beta} \int_0^t b_{\alpha,\beta'} \mathrm{d}s. \end{aligned}$$

It implies that $\psi \in W^{1,s}([0, T])$ and that

$$\begin{aligned} \frac{\mathrm{d}\psi}{\mathrm{d}t}(t) = D^2_{t,(\tilde{y},u,y)} h(t, u_t, y_t, u, y)(z_t, v, z) & \\ + D^2_{\tilde{y},(\tilde{y},u,y)} h(t, u_t, y_t, u, y)\left(\dot{y}_t, (z_t, v, z)\right) &+ D_{\tilde{y}} h(t, u_t, y_t, u, y)\dot{z}_t. \end{aligned}$$

On the other hand, we differentiate $D^{(1)} h$ w.r.t. $(\tilde{u}, \tilde{y}, u, y)$ using (2.24). Then with the expressions of $\dot{y}_t$ and $\dot{z}_t$, we get the relation (2.30). □

Finally we define the order of a running state constraint g_i. We denote $g_i^{(j+1)} := D^{(1)} g_i^{(j)}$ (with $g_i^{(0)} := g_i$). Note that $g_i \in \mathcal{S}$, so $g_i^{(j)} \in \mathcal{S}$ for all $j \geq 0$. Moreover, if we write $g_i^{(j)}$ as in (2.22), the a_α and $b_{\alpha,\beta}$ are combinations of derivatives of f and g_i.

Definition 7. The *order* of the constraint g_i is the greatest positive integer q_i such that

$$D_{\tilde{u}} g_i^{(j)} \equiv 0 \quad \text{for all } j = 0, \ldots, q_i - 1.$$

We have a result similar to Lemma 9 in [4], but now for integral dynamics. Let $(u, y) \in \mathcal{U} \times \mathcal{Y}$ be a trajectory, $(v, z_0) \in \mathcal{V}_s \times \mathbb{R}^n$, and $z := z[v, z_0] \in \mathcal{Z}_s$ for some $s \in [1, \infty]$.

Lemma 8. *Let g_i be of order at least $q_i \in \mathbb{N}^*$. Then*

$$\begin{aligned} t &\mapsto g_i(y_t) && \in W^{q_i,\infty}([0,T]), \\ t &\mapsto g_i'(y_t) z_t && \in W^{q_i,s}([0,T]), \end{aligned}$$

and

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j} g_i(y)|_t = g_i^{(j)}(t, y_t, u, y), \quad j = 1, \ldots, q_i - 1, \tag{2.31}$$

$$\frac{\mathrm{d}^{q_i}}{\mathrm{d}t^{q_i}} g_i(y)|_t = g_i^{(q_i)}(t, u_t, y_t, u, y), \tag{2.32}$$

$$\frac{\mathrm{d}^j}{\mathrm{d}t^j} g_i'(y) z|_t = \widehat{D} g_i^{(j)}(t, y_t, u, y)(z_t, v, z), \quad j = 1, \ldots, q_i - 1, \tag{2.33}$$

$$\frac{\mathrm{d}^{q_i}}{\mathrm{d}t^{q_i}} g_i'(y) z|_t = D_{\tilde{u}} g_i^{(q_i)}(t, u_t, y_t, u, y) v_t + \widehat{D} g_i^{(q_i)}(t, u_t, y_t, u, y)(z_t, v, z), \tag{2.34}$$

where we denote by $\widehat{D}$ the differentiation w.r.t. $(\tilde{y}, u, y)$.

Proof. It is straightforward with lemmas 5 and 6, definition 7 and an induction on j. □

3 Weak results

3.1 A first abstract formulation

The optimal control problem (P) can be rewritten as an abstract optimization problem on (u, y_0). The most naive way to do that is the following equivalent formulation:

$$(P) \qquad \min_{(u,y_0) \in \mathcal{U} \times \mathbb{R}^n} J(u, y_0) \tag{3.1}$$

$$\text{subject to} \qquad g(y[u, y_0]) \in C_-\left([0,T]; \mathbb{R}^r\right), \tag{3.2}$$

$$\Phi(y_0, y[u, y_0]_T) \in K, \tag{3.3}$$

where

$$J(u, y_0) := \int_0^T \ell(u_t, y[u, y_0]_t) \mathrm{d}t + \phi(y_0, y[u, y_0]_T) \tag{3.4}$$

and $\Phi = (\Phi^E, \Phi^I)$, $K = \{0\}_{s_E} \times (\mathbb{R}_-)^{s_I}$. In order to write optimality conditions for this problem, we first compute its Lagrangian

$$L(u, y_0, \mathrm{d}\eta, \Psi) := J(u, y_0) + \int_{[0,T]} \mathrm{d}\eta_t g(y[u, y_0]_t) + \Psi \Phi(y_0, y[u, y_0]_T) \tag{3.5}$$

where $(u, y_0, \mathrm{d}\eta, \Psi) \in \mathcal{U} \times \mathbb{R}^n \times \mathcal{M} \times \mathbb{R}^{s*}$ (see the beginning of section 2.2). A *Lagrange multiplier* at (u, y_0) in this setting is any $(\mathrm{d}\eta, \Psi)$ such that

$$D_{(u,y_0)} L(u, y_0, \mathrm{d}\eta, \Psi) \equiv 0, \tag{3.6}$$

$$(\mathrm{d}\eta, \Psi) \in N_{C_-([0,T];\mathbb{R}^r)\times K} \left(g(y), \Phi(y_0, y_T)\right). \tag{3.7}$$

This definition has to be compared to definition 2:

Lemma 9. *We have that* $(\mathrm{d}\eta, \Psi)$ *is a Lagrange multiplier of the abstract problem* (3.1)-(3.3) *at* $(\bar{u}, \bar{y}_0)$ iff $(\mathrm{d}\eta, \Psi, p)$ *is a Lagrange multiplier of the optimal control problem* (2.1)-(2.5) *associated with* $(\bar{u}, y[\bar{u}, \bar{y}_0])$*, where* p *is the unique solution of* (2.10).

Proof. Using the Hamiltonian (2.7), the end points Lagrangian (2.8) and the formula (A.10) of integration by parts for functions of bounded variations (see appendix A.1), we get

$$L(u, y_0, \mathrm{d}\eta, \Psi) = \int_0^T H[p](t, u_t, y_t)\mathrm{d}t + \int_{[0,T]} (\mathrm{d}p_t y_t + \mathrm{d}\eta_t g(y_t)) + p_{0_-} y_0 - p_{T_+} y_T + \Phi[\Psi](y_0, y_T)$$

for any $p \in \mathcal{P}$ and $y = y[u, y_0]$. We fix $(\bar{u}, \bar{y}_0, \mathrm{d}\eta, \Psi)$, we differentiate L w.r.t. (u, y_0) at this point, and we choose p as the unique solution of (2.10). Then

$$D_{(u,y_0)} L(\bar{u}, \bar{y}_0, \mathrm{d}\eta, \Psi)(v, z_0) = \int_0^T D_u H[p](t, \bar{u}_t, \bar{y}_t) v_t \mathrm{d}t + \left(p_{0_-} + D_{y_1} \Phi[\Psi](\bar{y}_0, \bar{y}_T)\right) z_0$$

for all $(v, z_0) \in \mathcal{U} \times \mathbb{R}^n$. It follows that (3.6) is equivalent to (2.11) and (2.14). And it is obvious that (3.7) is equivalent to (2.12)-(2.13). □

Second we need a qualification condition.

Definition 10. We say that $(\bar{u}, \bar{y})$ is *qualified* if

(i) $\left\{ \begin{array}{ccc} (v, z_0) & \mapsto & D\Phi^E(\bar{y}_0, \bar{y}_T)(z_0, z[v, z_0]_T) \\ \mathcal{U} \times \mathbb{R}^n & \to & \mathbb{R}^{s_E} \end{array} \right.$ is onto,

(ii) there exists $(\bar{v}, \bar{z}_0) \in \mathcal{U} \times \mathbb{R}^n$ such that, with $\bar{z} = z[\bar{v}, \bar{z}_0]$,

$$\left\{ \begin{array}{ll} D\Phi^E(\bar{y}_0, \bar{y}_T)(\bar{z}_0, \bar{z}_T) = 0, & \\ D\Phi_i^I(\bar{y}_0, \bar{y}_T)(\bar{z}_0, \bar{z}_T) < 0, & i \in \left\{ i \,:\, \Phi_i^I(\bar{y}_0, \bar{y}_T) = 0 \right\}, \\ g_i'(\bar{y}_t)\bar{z}_t < 0 \text{ on } \left\{ t \,:\, g_i(\bar{y}_t) = 0 \right\}, & i = 1, \ldots, r. \end{array} \right.$$

Remark 11. 1. This condition is equivalent to Robinson's constraint qualification (introduced in [20], Definition 2) for the abstract problem (3.1)-(3.3) at $(\bar{u}, \bar{y}_0)$; see the discussion that follows Definition 3.4 and Definition 3.5 in [17] for a proof of the equivalence.

2. It is sometimes possible to give optimality conditions without qualification condition by considering an auxiliary optimization problem (see e.g. the proof of Theorem 3.50 in [7]). Nevertheless, observe that if $(\bar{u}, \bar{y})$ is feasible but not qualified because (i) does not hold, then there exists a *singular Langrange multiplier* of the form $(0, \Phi^E, 0)$. One can see that second-order necessary conditions become pointless since $-(0, \Phi^E, 0)$ is a singular Lagrange multiplier too.

Finally we derive the following first-order necessary optimality conditions:

Theorem 12. *Let $(\bar{u}, \bar{y})$ be a qualified local solution of (P). Then the set of associated Lagrange multipliers is nonempty, convex, bounded and weakly $*$ compact.*

Proof. Since the abstract problem (3.1)-(3.3) is qualified, we get the result for the set $\{(\mathrm{d}\eta, \Psi)\}$ of Lagrange multipliers in this setting (Theorem 4.1 in [26]). We conclude with lemma 9 and the fact that

$$\begin{aligned} \mathcal{M} \times \mathbb{R}^{s*} &\longrightarrow \mathcal{M} \times \mathbb{R}^{s*} \times \mathcal{P} \\ (\mathrm{d}\eta, \Psi) &\longmapsto (\mathrm{d}\eta, \Psi, p) \end{aligned}$$

is affine continuous (it is obvious from the proof of lemma 1). □

We will prove a stronger result in section 4, relying on another abstract formulation, the so-called *reduced problem*. The main motivation for the reduced problem, as mentioned in the introduction, is actually to satisfy an *extended polyhedricity condition* (see Definition 3.52 in [7]), in order to easily get second-order necessary conditions (see Remark 3.47 in the same reference).

3.2 The reduced problem

In the sequel we fix a feasible trajectory $(\bar{u}, \bar{y})$, i.e. which satisfies (2.2)-(2.5), and denote by Λ the set of associated Lagrange multipliers (definition 2). We need some definitions:

Definition 13. An *arc* is a maximal interval, relatively open in $[0, T]$, denoted by (τ_1, τ_2), such that the set of active running state constraints at time t is constant for all $t \in (\tau_1, \tau_2)$. It includes intervals of the form $[0, \tau)$ or $(\tau, T]$. If τ does not belong to any arc, we say that τ is a *junction time*.

Consider an arc (τ_1, τ_2). It is a *boundary arc* for the constraint g_i if the latter is active on (τ_1, τ_2); otherwise it is an *interior arc* for g_i.

Consider an interior arc (τ_1, τ_2) for g_i. If $g_i(\tau_2) = 0$, then τ_2 is an *entry point* for g_i; if $g_i(\tau_1) = 0$, then τ_1 is an *exit point* for g_i. If τ is an entry point and an exit point, then it is a *touch point* for g_i.

Consider a touch point τ for g_i. We say that τ is *reducible* if $\frac{\mathrm{d}^2}{\mathrm{d}t^2} g_i(\bar{y}_t)$, defined in a weak sense, is a function for t close to τ, continuous at τ, and

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} g_i(\bar{y}_t)|_{t=\tau} < 0.$$

Remark 14. Let τ be a touch point for g_i. By lemma 8, if g_i is of order at least 2, then τ is reducible if $t \mapsto g_i^{(2)}(t, \bar{u}_t, \bar{y}_t, \bar{u}, \bar{y})$ is continuous at τ and $g_i^{(2)}(\tau, \bar{u}_\tau, \bar{y}_\tau, \bar{u}, \bar{y}) < 0$. Note that the continuity holds either if u is continuous at τ or if g_i is of order at least 3.

The interest of reducibility will appear with the next lemma. For $\tau \in [0, T]$ and $\varepsilon > 0$ (to be fixed), we define $\mu_\tau \colon W^{2,\infty}([0, T]) \to \mathbb{R}$ by

$$\mu_\tau(x) := \max \{ x_t \, : \, t \in [\tau - \varepsilon, \tau + \varepsilon] \cap [0, T] \} . \tag{3.8}$$

Lemma 15. *Let g_i not be of order 1 (i.e. $D_{\bar{u}} g_i^{(1)} \equiv 0$) and τ be a reducible touch point for g_i. Then for $\varepsilon > 0$ small enough, μ_τ is C^1 in a neighbourhood of $g_i(\bar{y}) \in W^{2,\infty}([0, T])$ and twice*

Fréchet differentiable at $g_i(\bar{y})$, with first and second derivatives at $g_i(\bar{y})$ given by

$$D\mu_\tau(g_i(\bar{y}))x = x_\tau, \tag{3.9}$$

$$D^2\mu_\tau(g_i(\bar{y}))(x)^2 = -\frac{\left(\frac{\mathrm{d}}{\mathrm{d}t}x_t|_\tau\right)^2}{\frac{\mathrm{d}^2}{\mathrm{d}t^2}g_i(\bar{y}_t)|_\tau}, \tag{3.10}$$

for any $x \in W^{2,\infty}([0,T])$.

Proof. We apply Lemma 23 of [4] to $g_i(\bar{y})$, which belongs to $W^{2,\infty}([0,T])$ by lemma 8 and satisfies the required hypotheses at τ by definition of a reducible touch point. □

Remark 16. We can write (3.9) and (3.10) for $x = g_i'(\bar{y})z[v, z_0]$ (since g_i is not of order 1). By lemma 8, (3.10) becomes

$$D^2\mu_\tau(g_i(\bar{y}))(g_i'(\bar{y})z)^2 = -\frac{\left(\widehat{D}g_i^{(1)}(\tau, \bar{y}_\tau, \bar{u}, \bar{y})(z_\tau, v, z)\right)^2}{g_i^{(2)}(\tau, \bar{u}_\tau, \bar{y}_\tau, \bar{u}, \bar{y})} \tag{3.11}$$

where $z = z[v, z_0]$, $(v, z_0) \in \mathcal{U} \times \mathbb{R}^n$ and $\widehat{D}$ is the differentiation w.r.t. $(\tilde{y}, u, y)$. We will also use (3.9) for $x = g_i''(\bar{y})(z[v, z_0])^2 + g_i'(\bar{y})z^2[v, z_0]$, $z^2[v, z_0]$ being defined by (2.19). It can indeed be shown that it belongs to $W^{2,\infty}([0,T])$.

In view of these results we distinguish running state constraints of order 1. Without loss of generality, we suppose that

- ▷ g_i is of order 1 for $i = 1, \dots, r_1$,
- ▷ g_i is not of order 1 for $i = r_1 + 1, \dots, r$,

where $0 \le r_1 \le r$. We make now the following assumption:

(A1) There are finitely many junction times, and for $i = r_1 + 1, \dots, r$ all touch points for g_i are reducible.

For $i = 1, \dots, r_1$ we consider the contact sets of the constraints

$$\mathcal{I}_i := \{t \in [0,T] \,:\, g_i(\bar{y}_t) = 0\}. \tag{3.12}$$

For $i = r_1 + 1, \dots, r$ we remove the touch points from the contact sets:

$$\mathcal{T}_i := \text{the set of (reducible) touch points for } g_i, \tag{3.13}$$

$$\mathcal{I}_i := \{t \in [0,T] \,:\, g_i(\bar{y}_t) = 0\} \setminus \mathcal{T}_i. \tag{3.14}$$

For $i = 1, \dots, r$ and $\varepsilon \ge 0$ we denote

$$\mathcal{I}_i^\varepsilon := \{t \in [0,T] \,:\, \mathrm{dist}(t, \mathcal{I}_i) \le \varepsilon\}. \tag{3.15}$$

Assumption **(A1)** implies that $\mathcal{I}_i^\varepsilon$ has finitely many connected components for any $\varepsilon \ge 0$ ($1 \le i \le r$) and that $\mathcal{T}_i$ is finite ($1 \le i \le r_1$). Let $N := \sum_{r_1 < i \le r} |\mathcal{T}_i|$.

Now we fix $\varepsilon > 0$ small enough (so that lemma 15 holds) and we define

$$G_1(u,y_0) := \big(g_i\,(y[u,y_0])\,|_{\mathcal{I}_i^\varepsilon}\big)_{1\le i\le r}, \qquad K_1 := \prod_{i=1}^{r} C_-\left(\mathcal{I}_i^\varepsilon\right), \tag{3.16}$$

$$G_2(u,y_0) := \big(\mu_\tau\,(g_i\,(y[u,y_0]))\,\big)_{\tau\in\mathcal{T}_i,\ r_1<i\le r}, \qquad K_2 := \left(\mathbb{R}_-\right)^N, \tag{3.17}$$

$$G_3(u,y_0) := \Phi\left(y_0, y[u,y_0]_T\right), \qquad K_3 := K. \tag{3.18}$$

Recall that J has been defined by (3.4).

The **reduced problem** is the following abstract optimization problem:

$$(P_R) \qquad \min_{(u,y_0)\in\mathcal{U}\times\mathbb{R}^n} J(u,y_0), \quad \text{subject to} \quad \begin{cases} G_1(u,y_0)\in K_1 \\ G_2(u,y_0)\in K_2 \\ G_3(u,y_0)\in K_3 \end{cases}.$$

Remark 17. We had fixed $(\bar{u},\bar{y})$ as a feasible trajectory; then $(\bar{u},\bar{y}_0)$ is feasible for (P_R). Moreover, $(\bar{u},\bar{y})$ is a local solution of (P) *iff* $(\bar{u},\bar{y}_0)$ is a local solution of (P_R), and the qualification condition at $(\bar{u},\bar{y})$ (definition 10) is equivalent to Robinson's constraints qualification for (P_R) at $(\bar{u},\bar{y}_0)$ (using lemma 15).

Thus it is of interest for us to write optimality conditions for (P_R).

3.3 Optimality conditions for the reduced problem

The *Lagrangian* of (P_R) is

$$L_R(u,y_0,\mathrm{d}\rho,\nu,\Psi) := J(u,y_0) + \sum_{1\le i\le r}\int_{\mathcal{I}_i^\varepsilon} g_i(y[u,y_0]_t)\mathrm{d}\rho_{i,t} + \sum_{\substack{\tau\in\mathcal{T}_i\\ r_1<i\le r}} \nu_{i,\tau}\mu_\tau\left(g_i\left(y[u,y_0]\right)\right) + \Psi\Phi(y_0,y[u,y_0]_T) \tag{3.19}$$

where $\quad u\in\mathcal{U}, \quad y_0\in\mathbb{R}^n, \quad \mathrm{d}\rho\in\prod_{i=1}^{r}\mathcal{M}\left(\mathcal{I}_i^\varepsilon\right), \quad \nu\in\mathbb{R}^{N*}, \quad \Psi\in\mathbb{R}^{s*}.$

As before, a measure on a closed interval is denoted by $\mathrm{d}\mu$ and is identified with the derivative of a function of bounded variations which is null on the right of the interval.

A *Lagrange multiplier* of (P_R) at $(\bar{u},\bar{y}_0)$ is any $(\mathrm{d}\rho,\nu,\Psi)$ such that

$$D_{(u,y_0)}L_R(\bar{u},\bar{y}_0,\mathrm{d}\rho,\nu,\Psi) = 0, \tag{3.20}$$

$$\mathrm{d}\rho_i \ge 0,\ g_i(\bar{y})|_{\mathcal{I}_i^\varepsilon} \le 0,\ \int_{\mathcal{I}_i^\varepsilon} g_i(\bar{y}_t)\mathrm{d}\rho_{i,t} = 0, \quad i=1,\dots,r, \tag{3.21}$$

$$\nu_{i,\tau}\ge 0,\ \mu_\tau\left(g_i(\bar{y})\right)\le 0,\ \nu_{i,\tau}\mu_\tau\left(g_i(\bar{y})\right) = 0, \quad \tau\in\mathcal{T}_i,\ i = r_1+1,\dots,r, \tag{3.22}$$

$$\Psi\in N_K\left(\Phi(\bar{y}_0,\bar{y}_T)\right). \tag{3.23}$$

We denote by Λ_R the set of Lagrange multipliers of (P_R) at $(\bar{u},\bar{y}_0)$. The first-order necessary conditions for (P_R) are the same as in theorem 12:

Lemma 18. *Let $(\bar{u},\bar{y}_0)$ be a qualified local solution of (P_R). Then Λ_R is nonempty, convex, bounded and weakly $*$ compact.*

Given $(\mathrm{d}\rho,\nu) \in \prod_{i=1}^{r} \mathcal{M}(\mathcal{I}_i^{\varepsilon}) \times \mathbb{R}^{N*}$, we define $\mathrm{d}\eta \in \mathcal{M}$ by

$$\mathrm{d}\eta_i := \begin{cases} \mathrm{d}\rho_i & \text{on } \mathcal{I}_i^{\varepsilon}, \quad i = 1,\dots,r, \\ \sum_{\tau\in\mathcal{T}_i} \nu_{i,\tau}\delta_\tau & \text{elsewhere}, \quad i = r_1+1,\dots,r. \end{cases} \tag{3.24}$$

Conversely, given $\mathrm{d}\eta \in \mathcal{M}$, we define $(\mathrm{d}\rho,\nu) \in \prod_{i=1}^{r} \mathcal{M}(\mathcal{I}_i^{\varepsilon}) \times \mathbb{R}^{N*}$ by

$$\begin{cases} \mathrm{d}\rho_i := \mathrm{d}\eta_i|_{\mathcal{I}_i^{\varepsilon}} & i = 1,\dots,r, \\ \nu_{i,\tau} := \mathrm{d}\eta_i(\{\tau\}) & \tau\in\mathcal{T}_i, \quad i = r_1+1,\dots,r. \end{cases} \tag{3.25}$$

In the sequel we use these definitions to identify $(\mathrm{d}\rho,\nu)$ and $\mathrm{d}\eta$, and we denote

$$[\eta_{i,\tau}] := \mathrm{d}\eta_i(\{\tau\}). \tag{3.26}$$

Recall that Λ is the set of Lagrange multipliers associated with $(\bar{u},\bar{y})$ (definition 2). We have a result similar to lemma 9:

Lemma 19. *We have that* $(\mathrm{d}\rho,\nu,\Psi) \in \Lambda_R$ iff $(\mathrm{d}\eta,\Psi,p) \in \Lambda$, *with* p *the unique solution of* (2.10).

Proof. With the identification between $(\mathrm{d}\rho,\nu)$ and $\mathrm{d}\eta$ given by (3.24) and (3.25), it is clear that (3.21)-(3.22) are equivalent to (2.12). Let these relations be satisfied by $(\mathrm{d}\rho,\nu,\Psi)$ and $(\mathrm{d}\eta,\Psi)$. Then in particular

$$\begin{array}{ll} \mathrm{supp}(\mathrm{d}\eta_i) = \mathrm{supp}(\mathrm{d}\rho_i) \subset \mathcal{I}_i & i = 1,\dots,r_1, \\ \mathrm{supp}(\mathrm{d}\eta_i) = \mathrm{supp}(\mathrm{d}\rho_i) \cup \mathrm{supp}(\sum \nu_{i,\tau}\delta_\tau) \subset \mathcal{I}_i \cup \mathcal{T}_i & i = r_1+1,\dots,r. \end{array} \tag{3.27}$$

We claim that in this case (3.20) is equivalent to (2.11) and (2.14). Indeed, using $H[p]$ defined by (2.7), $\Phi[\Psi]$ by (2.8), the integration by parts formula (A.10) and (3.27), we have

$$\begin{aligned} L_R(u,y_0,\mathrm{d}\rho,\nu,\Psi) = &\int_{[0,T]} \left(H[p](t,u_t,y_t)\mathrm{d}t + \mathrm{d}p_t y_t\right) + p_{0_-}y_0 - p_{T_+}y_T \\ &+ \sum_{1\le i\le r} \int_{\mathcal{I}_i} g_i(y_t)\mathrm{d}\eta_{i,t} + \sum_{\substack{\tau\in\mathcal{T}_i \\ r_1<i\le r}} [\eta_{i,\tau}]\mu_\tau\left(g_i(y)\right) + \Phi[\Psi](y_0,y_T) \end{aligned} \tag{3.28}$$

for any $p \in \mathcal{P}$ and $y = y[u,y_0]$. Let us differentiate (say for $i > r_1$)

$$\int_{\mathcal{I}_i} g_i(y_t)\mathrm{d}\eta_{i,t} + \sum_{\tau\in\mathcal{T}_i} [\eta_{i,\tau}]\mu_\tau\left(g_i(y)\right) \tag{3.29}$$

w.r.t. (u,y_0) at $(\bar{u},\bar{y}_0)$ in the direction (v,z_0) and use (3.9) and (3.27); we get

$$\int_{\mathcal{I}_i} g_i'(\bar{y}_t)z_t\mathrm{d}\eta_{i,t} + \sum_{\tau\in\mathcal{T}_i} [\eta_{i,\tau}]D\mu_\tau\left(g_i(\bar{y})\right)\left(g_i'(\bar{y})z\right) = \int_{[0,T]} g_i'(\bar{y}_t)z_t\mathrm{d}\eta_{i,t}$$

where $z = z[v,z_0]$. Let us now differentiate similarly the whole expression (3.28) of L_R; we get

$$\begin{aligned} &\int_0^T D_uH[p](t,\bar{u}_t,\bar{y}_t)v_t\mathrm{d}t + \int_{[0,T]} \big(D_yH[p](t,\bar{u}_t,\bar{y}_t)\mathrm{d}t + \mathrm{d}p_t + \mathrm{d}\eta_t g'(\bar{y}_t)\big)z_t \\ &\quad + \big(p_{0_-} + D_{y_1}\Phi[\Psi](\bar{y}_0,\bar{y}_T)\big)z_0 + \big(-p_{T_+} + D_{y_2}\Phi[\Psi](\bar{y}_0,\bar{y}_T)\big)z_T. \end{aligned} \tag{3.30}$$

Fixing p as the unique solution of (2.10) in (3.30) gives

$$\begin{aligned} D_{(u,y_0)}L_R(\bar{u},\bar{y}_0,\mathrm{d}\rho,\nu,\Psi)(v,z_0) = &\int_0^T D_uH[p](t,\bar{u}_t,\bar{y}_t)v_t\mathrm{d}t \\ &+ \big(p_{0_-} + D_{y_1}\Phi[\Psi](\bar{y}_0,\bar{y}_T)\big)z_0. \end{aligned}$$

It is now clear that (3.20) is equivalent to (2.11) and (2.14). □

For the second-order optimality conditions, we need to evaluate the Hessian of L_R. For $\lambda = (\mathrm{d}\eta, \Psi, p) \in \Lambda$, $(v, z_0) \in \mathcal{U} \times \mathbb{R}^n$ and $z = z[v, z_0] \in \mathcal{Y}$, we denote

$$\begin{aligned}\mathcal{J}[\lambda](v, z_0) := \int_0^T D^2_{(u,y)^2} H[p](t, \bar{u}_t, \bar{y}_t)(v_t, z_t)^2 \mathrm{d}t + D^2\Phi[\Psi](\bar{y}_0, \bar{y}_T)(z_0, z_T)^2 \\ + \sum_{1 \le i \le r} \int_{\mathcal{I}_i} g_i''(\bar{y}_t)(z_t)^2 \mathrm{d}\eta_{i,t} \\ + \sum_{\substack{\tau \in \mathcal{T}_i \\ r_1 < i \le r}} [\eta_{i,\tau}] \left[g_i''(\bar{y}_\tau)(z_\tau)^2 + D^2\mu_\tau\left(g_i(\bar{y})\right)\left(g_i'(\bar{y})z\right)^2 \right]. \end{aligned} \tag{3.31}$$

In view of (3.11) and (3.27), we could also write

$$\begin{aligned}\mathcal{J}[\lambda](v, z_0) = \int_0^T D^2_{(u,y)^2} H[p](t, \bar{u}_t, \bar{y}_t)(v_t, z_t)^2 \mathrm{d}t + D^2\Phi[\Psi](\bar{y}_0, \bar{y}_T)(z_0, z_T)^2 \\ + \int_{[0,T]} \mathrm{d}\eta_t g''(\bar{y}_t)(z_t)^2 - \sum_{\substack{\tau \in \mathcal{T}_i \\ r_1 < i \le r}} [\eta_{i,\tau}] \frac{\left(\widehat{D} g_i^{(1)}(\tau, \bar{y}_\tau, \bar{u}, \bar{y})(z_\tau, v, z)\right)^2}{g_i^{(2)}(\tau, \bar{u}_\tau, \bar{y}_\tau, \bar{u}, \bar{y})}. \end{aligned} \tag{3.32}$$

Lemma 20. *Let* $(\mathrm{d}\rho, \nu, \Psi) \in \Lambda_R$. *Let* $\lambda = (\mathrm{d}\eta, \Psi, p) \in \Lambda$ *be as in lemma 19. Then for all* $(v, z_0) \in \mathcal{U} \times \mathbb{R}^n$,

$$D^2_{(u,y_0)^2} L_R(\bar{u}, \bar{y}_0, \mathrm{d}\rho, \nu, \Psi)(v, z_0)^2 = \mathcal{J}[\lambda](v, z_0). \tag{3.33}$$

Proof. We will use (3.28) and (3.29) from the previous proof. First we differentiate (3.29) twice w.r.t. (u, y_0) at $(\bar{u}, \bar{y}_0)$ in the direction (v, z_0). Denoting $z = z[v, z_0]$ and $z^2 = z^2[v, z_0]$, defined by (2.19), we get

$$\begin{aligned}\int_{\mathcal{I}_i} \left(g_i''(\bar{y}_t)(z_t)^2 + g_i'(\bar{y}_t)z_t^2\right) \mathrm{d}\eta_{i,t} \\ + \sum_{\tau \in \mathcal{T}_i} [\eta_{i,\tau}] \left[D^2\mu_\tau\left(g_i(\bar{y})\right)\left(g_i'(\bar{y})z\right)^2 + D\mu_\tau\left(g_i(\bar{y})\right)\left(g_i''(\bar{y})(z)^2 + g_i'(\bar{y})z^2\right) \right] \\ = \int_{\mathcal{I}_i} g_i''(\bar{y}_t)(z_t)^2 \mathrm{d}\eta_{i,t} + \int_{[0,T]} g_i'(\bar{y}_t) z_t^2 \mathrm{d}\eta_{i,t} \\ + \sum_{\tau \in \mathcal{T}_i} [\eta_{i,\tau}] \left[D^2\mu_\tau\left(g_i(\bar{y})\right)\left(g_i'(\bar{y})z\right)^2 + g_i''(\bar{y}_\tau)(z_\tau)^2 \right],\end{aligned}$$

where we have used remark 16, (3.9) and (3.27). Second we differentiate L_R twice using (3.28) and then we fix p as the unique solution of (2.10). The result follows as in the proof of lemma 19. □

Suppose that $\Lambda \neq \emptyset$ and let $\bar{\lambda} = (\mathrm{d}\bar{\eta}, \bar{\Psi}, \bar{p}) \in \Lambda$. We define the *critical* L^2 *cone* as the set C_2 of $(v, z_0) \in \mathcal{V}_2 \times \mathbb{R}^n$ such that

$$\left\{ \begin{array}{ll} g_i'(\bar{y})z \le 0 & \text{on } \mathcal{I}_i, \\ g_i'(\bar{y})z = 0 & \text{on supp}\,(\mathrm{d}\bar{\eta}_i) \cap \mathcal{I}_i, \end{array} \right. \quad i = 1, \ldots, r, \tag{3.34}$$

$$\left\{ \begin{array}{l} g_i'(\bar{y}_\tau)z_\tau \le 0, \\ [\bar{\eta}_{i,\tau}]\, g_i'(\bar{y}_\tau)z_\tau = 0, \end{array} \right. \quad \tau \in \mathcal{T}_i, \; i = r_1 + 1, \ldots, r, \tag{3.35}$$

$$\left\{ \begin{array}{l} D\Phi(\bar{y}_0, \bar{y}_T)(z_0, z_T) \in T_K\left(\Phi(\bar{y}_0, \bar{y}_T)\right), \\ \bar{\Psi} D\Phi(\bar{y}_0, \bar{y}_T)(z_0, z_T) = 0, \end{array} \right. \tag{3.36}$$

where $z = z[v, z_0] \in \mathcal{Z}_2$. Then the *critical cone* for (P_R) (see Proposition 3.10 in [7]) is the set

$$C_\infty := C_2 \cap (\mathcal{U} \times \mathbb{R}^n)\,,$$

and the *cone of radial critical directions* for (P_R) (see Definition 3.52 in [7]) is the set

$$C_\infty^R := \{(v, z_0) \in C_\infty \,:\, \exists \bar\sigma > 0 \,:\, g_i(\bar y) + \bar\sigma g_i'(\bar y) z \le 0 \text{ on } \mathcal{I}_i^\varepsilon,\ i = 1, \dots, r\}\,,$$

where $z = z[v, z_0] \in \mathcal{Y}$. These three cones do not depend on the choice of $\bar\lambda$. In view of lemma 20, the second-order necessary conditions for (P_R) can be written as follows:

Lemma 21. *Let $(\bar u, \bar y_0)$ be a qualified local solution of (P_R). Then for any $(v, z_0) \in C_\infty^R$, there exists $\lambda \in \Lambda$ such that*

$$\mathcal{J}[\lambda](v, z_0) \ge 0. \tag{3.37}$$

Proof. Corollary 5.1 in [17]. □

4 Strong results

Recall that $(\bar u, \bar y)$ is a feasible trajectory that has been fixed to define the reduced problem at the beginning of section 3.2.

4.1 Extra assumptions and consequences

We were so far under the assumptions **(A0)-(A1)**. We make now some extra assumptions, which will imply a partial qualification of the running state constraints, as well as the density of C_∞^R in a larger critical cone.

(A2) Each running state constraint $g_i, i = 1, \dots, r$ is of finite order q_i.

Notations Given a subset $J \subset \{1, \dots, r\}$, say $J = \{i_1 < \cdots < i_l\}$, we define $G_J^{(q)} : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^n \times \mathcal{U} \times \mathcal{Y} \to \mathbb{R}^{|J|}$ by

$$G_J^{(q)}(t, \tilde u, \tilde y, u, y) := \begin{pmatrix} \bar g_{i_1}^{(q_{i_1})}(t, \tilde u, \tilde y, u, y) \\ \vdots \\ \bar g_{i_l}^{(q_{i_l})}(t, \tilde u, \tilde y, u, y) \end{pmatrix}. \tag{4.1}$$

For $\varepsilon_0 \ge 0$ and $t \in [0, T]$, let

$$I_t^{\varepsilon_0} := \{1 \le i \le r \,:\, t \in \mathcal{I}_i^{\varepsilon_0}\}\,, \tag{4.2}$$

$$M_t^{\varepsilon_0} := D_{\tilde u} G_{I_t^{\varepsilon_0}}^{(q)}(t, \bar u_t, \bar y_t, \bar u, \bar y) \in \mathbb{R}^{|I_t^{\varepsilon_0}|} \times \mathbb{R}^{m*}. \tag{4.3}$$

(A3) There exists $\varepsilon_0, \gamma > 0$ such that, for all $t \in [0, T]$,

$$\left|(M_t^{\varepsilon_0})^T \xi\right| \ge \gamma\,|\xi| \quad \forall \xi \in \mathbb{R}^{|I_t^{\varepsilon_0}|}. \tag{4.4}$$

(A4) The initial condition satisfies $g(\bar y_0) < 0$ and the final time T is not an entry point (i.e. there exists $\tau < T$ such that the set I_t^0 of active constraints at time t is constant for $t \in (\tau, T]$).

Remark 22. 1. We do not assume that $\bar{u}$ is continuous, as was done in [5].

2. Recall that ε has been fixed to define the reduced problem. Without loss of generality we suppose that $\varepsilon_0 > \varepsilon$, $\varepsilon_0 < \min\{\tau : \tau \text{ junction times}\}$ and $2\varepsilon_0 < \min\{|\tau - \tau'| : \tau, \tau' \text{ distinct junction times}\}$. We omit it in the notation $M_t^{\varepsilon_0}$.

3. In some cases, we can treat the case where T is an entry point, say for the constraint g_i:

   ▷ if $1 \le i \le r_1$ (i.e. if $q_i = 1$), then what follows works similarly.

   ▷ if $r_1 < i \le r$ (i.e. if $q_i > 1$) and $\frac{\mathrm{d}}{\mathrm{d}t} g_i(\bar{y}_t)|_{t=T} > 0$, then we can replace in the reduced problem $g_i(y[u,y_0])|_{[T-\varepsilon,T]} \le 0$ by the final constraint $g_i(y[u,y_0]_T) \le 0$.

4. By **(A1)**, we can write
$$[0,T] = J_0 \cup \cdots \cup J_\kappa \tag{4.5}$$
where J_l ($l = 0, \ldots, \kappa$) are the maximal intervals in $[0,T]$ such that $I_t^{\varepsilon_0}$ is constant (say equal to I_l) for $t \in J_l$. We order $J_0, \ldots, J_\kappa$ in $[0,T]$. Observe that for any $l \ge 1$, $\overline{J_{l-1}} \cap \overline{J_l} = \{\tau \pm \varepsilon_0\}$ with τ a junction time.

For $s \in [1,\infty]$, we denote

$$W^{(q),s}([0,T]) := \prod_{i=1}^{r} W^{q_i,s}([0,T]), \quad W^{(q),s}(\mathcal{I}^\varepsilon) := \prod_{i=1}^{r} W^{q_i,s}(\mathcal{I}_i^\varepsilon), \tag{4.6}$$

and for $\varphi = \begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_r \end{pmatrix} \in W^{(q),s}([0,T])$, $\varphi|_{\mathcal{I}^\varepsilon} := \begin{pmatrix} \varphi_1|_{\mathcal{I}_1^\varepsilon} \\ \vdots \\ \varphi_r|_{\mathcal{I}_r^\varepsilon} \end{pmatrix} \in W^{(q),s}(\mathcal{I}^\varepsilon)$.

Using lemma 8 we define, for $s \in [1,\infty]$ and $z_0 \in \mathbb{R}^n$,

$$\begin{aligned} \mathcal{A}_{s,z_0} \colon \mathcal{V}_s &\longrightarrow W^{(q),s}([0,T]) \\ v &\longmapsto g'(\bar{y}) z[v, z_0]. \end{aligned} \tag{4.7}$$

We give now the statement of a lemma in two parts, which will be of great interest for us (particularly in section 4.3.3). The proof is technical and can be skipped at a first reading. It is given in the next section.

Lemma 23. *a) Let $s \in [1,\infty]$ and $z_0 \in \mathbb{R}^n$. Let $\bar{b} \in W^{(q),s}(\mathcal{I}^\varepsilon)$. Then there exists $v \in \mathcal{V}_s$ such that*
$$(\mathcal{A}_{s,z_0} v)\,|_{\mathcal{I}^\varepsilon} = \bar{b}. \tag{4.8}$$

b) Let $z_0 \in \mathbb{R}^n$. Let $(\bar{b}, \bar{v}) \in W^{(q),2}(\mathcal{I}^\varepsilon) \times \mathcal{V}_2$ be such that

$$(\mathcal{A}_{2,z_0} \bar{v})\,|_{\mathcal{I}^\varepsilon} = \bar{b}. \tag{4.9}$$

Let $b^k \in W^{(q),\infty}(\mathcal{I}^\varepsilon)$, $k \in \mathbb{N}$, be such that $b^k \xrightarrow{W^{(q),2}(\mathcal{I}^\varepsilon)} \bar{b}$. Then there exists $v^k \in \mathcal{U}$, $k \in \mathbb{N}$, such that $v^k \xrightarrow{L^2} \bar{v}$ and
$$\left(\mathcal{A}_{\infty,z_0} v^k\right)|_{\mathcal{I}^\varepsilon} = b^k. \tag{4.10}$$

4.2 A technical proof

In this section we prove lemma 23. The proofs of a) and b) are very similar; in both cases we proceed in $\kappa + 1$ steps using the decomposition (4.5) of $[0, T]$. At each step, we will use the following two lemmas, proved in appendixes A.3 and A.2, respectively.

The first one uses only **(A1)** and the definitions that follow.

Lemma 24. *Let $t_0 := \tau \pm \varepsilon_0$ where τ is a junction time.*

a) Let $s \in [1, \infty]$ and $z_0 \in \mathbb{R}^n$. Let $(\bar{b}, v) \in W^{(q),s}(\mathcal{I}^\varepsilon) \times \mathcal{V}_s$ be such that

$$(\mathcal{A}_{s,z_0} v)\,|_{\mathcal{I}^\varepsilon} = \bar{b} \text{ on } [0, t_0]. \tag{4.11}$$

Then we can extend $\bar{b}$ to $\tilde{b} \in W^{(q),s}([0, T])$ in such a way that

$$\tilde{b} = \mathcal{A}_{s,z_0} v \text{ on } [0, t_0]. \tag{4.12}$$

b) Let $z_0 \in \mathbb{R}^n$. Let $(\bar{b}, \bar{v}) \in W^{(q),2}(\mathcal{I}^\varepsilon) \times \mathcal{V}_2$ be such that

$$(\mathcal{A}_{2,z_0} \bar{v})\,|_{\mathcal{I}^\varepsilon} = \bar{b}. \tag{4.13}$$

Let $(b^k, v^k) \in W^{(q),\infty}(\mathcal{I}^\varepsilon) \times \mathcal{U}$, $k \in \mathbb{N}$, be such that $(b^k, v^k) \xrightarrow{W^{(q),2} \times L^2} (\bar{b}, \bar{v})$ and

$$\left(\mathcal{A}_{\infty,z_0} v^k\right)|_{\mathcal{I}^\varepsilon} = b^k \text{ on } [0, t_0]. \tag{4.14}$$

Then we can extend b^k to $\tilde{b}^k \in W^{(q),\infty}([0, T])$, $k \in \mathbb{N}$, in such a way that $\tilde{b}^k \xrightarrow{W^{(q),2}([0,T])} \mathcal{A}_{2,z_0}\bar{v}$ and

$$\tilde{b}^k = \mathcal{A}_{\infty,z_0} v^k \text{ on } [0, t_0]. \tag{4.15}$$

The second lemma relies on **(A3)**.

Lemma 25. *Let $s \in [1, \infty]$ and $z_0 \in \mathbb{R}^n$. Let l be such that $I_l \neq \emptyset$. For $t \in J_l$, we denote (recall that $\widehat{D}$ is the differentiation w.r.t. $(\tilde{y}, u, y)$)*

$$\begin{cases} M_t := D_{\tilde{u}} G^{(q)}_{I_l}(t, \bar{u}_t, \bar{y}_t, \bar{u}, \bar{y}) \in \mathbb{R}^{|I_l|} \times \mathbb{R}^{m*}, \\ N_t := \widehat{D} G^{(q)}_{I_l}(t, \bar{u}_t, \bar{y}_t, \bar{u}, \bar{y}) \in \mathbb{R}^{|I_l|} \times \mathbb{R}^{n*} \times \mathcal{U}^* \times \mathcal{Y}^*. \end{cases} \tag{4.16}$$

a) Let $(\bar{h}, v) \in L^s(J_l; \mathbb{R}^{|I_l|}) \times \mathcal{V}_s$. Then there exists $\tilde{v} \in \mathcal{V}_s$ such that

$$\begin{cases} \tilde{v} = v \text{ on } J_0 \cup \cdots \cup J_{l-1}, \\ M_t \tilde{v}_t + N_t \left(z[\tilde{v}, z_0]_t, \tilde{v}, z[\tilde{v}, z_0]\right) = \bar{h}_t \text{ for a.a. } t \in J_l. \end{cases} \tag{4.17}$$

b) Let $(\bar{h}, \bar{v}) \in L^s(J_l; \mathbb{R}^{|I_l|}) \times \mathcal{V}_s$ be such that

$$M_t \bar{v}_t + N_t \left(z[\bar{v}, z_0]_t, \bar{v}, z[\bar{v}, z_0]\right) = \bar{h}_t \text{ for a.a. } t \in J_l. \tag{4.18}$$

Let $(h^k, v^k) \in L^\infty(J_l; \mathbb{R}^{|I_l|}) \times \mathcal{U}$, $k \in \mathbb{N}$, be such that $(h^k, v^k) \xrightarrow{L^s \times L^s} (\bar{h}, \bar{v})$. Then there exists $\tilde{v}^k \in \mathcal{U}$, $k \in \mathbb{N}$, such that $\tilde{v}^k \xrightarrow{L^s} \bar{v}$ and

$$\begin{cases} \tilde{v}^k = v^k \text{ on } J_0 \cup \cdots \cup J_{l-1}, \\ M_t \tilde{v}^k_t + N_t \left(z[\tilde{v}^k, z_0]_t, \tilde{v}^k, z[\tilde{v}^k, z_0]\right) = h^k_t \text{ for a.a. } t \in J_l. \end{cases} \tag{4.19}$$

Proof of lemma 23. In the sequel we omit z_0 in the notations.

a) Let $\bar{b} \in W^{(q),s}(\mathcal{I}^\varepsilon)$. We need to find $v \in \mathcal{V}_s$ such that

$$g_i'(\bar{y})z[v] = \bar{b}_i \text{ on } \mathcal{I}_i^\varepsilon, \ i = 1, \dots, r. \tag{4.20}$$

Since

$$v = v' \text{ on } [0,t] \Longrightarrow z[v] = z[v'] \text{ on } [0,t],$$

let us construct $v^0, \dots, v^\kappa \in \mathcal{V}_s$ such that, for all l,

$$\begin{cases} v^l = v^{l-1} \text{ on } J_0 \cup \dots \cup J_{l-1}, \\ g_i'(\bar{y})z[v^l] = \bar{b}_i \text{ on } \mathcal{I}_i^\varepsilon \cap J_l, \ i = 1, \dots, r \end{cases}$$

and $v := v^\kappa$ will satisfy (4.20).

By **(A4)**, $J_0 = [0, \tau_1 - \varepsilon_0)$ where τ_1 is the first junction time and $\mathcal{I}_i^\varepsilon \cap J_0 = \emptyset$ for all i. Then we can choose $v^0 := 0$.

Suppose we have $v^0, \dots v^{l-1}$ for some $l \geq 1$ and let us construct v^l. Applying lemma 24 a) to $(\bar{b}, v^{l-1})$ with $\{t_0\} = \overline{J_{l-1}} \cap \overline{J_l}$, we get $\tilde{b} \in W^{(q),s}([0,T])$. Since $\mathcal{I}_i^\varepsilon \cap J_l = \emptyset$ if $i \notin I_l$, it is now enough to find v^l such that

$$\begin{cases} v^l = v^{l-1} \text{ on } J_0 \cup \dots \cup J_{l-1}, \\ g_i'(\bar{y})z[v^l] = \tilde{b}_i \text{ on } J_l, \ i \in I_l. \end{cases} \tag{4.21}$$

Suppose that $v^l = v^{l-1}$ on $J_0 \cup \dots \cup J_{l-1}$. Then $g_i'(\bar{y})z[v^l] = \tilde{b}_i$ on J_{l-1}, and it follows that

$$g_i'(\bar{y})z[v^l] = \tilde{b}_i \text{ on } J_l \tag{4.22}$$

$$\Updownarrow$$

$$\frac{d^{q_i}}{dt^{q_i}} g_i'(\bar{y})z[v^l] = \frac{d^{q_i}}{dt^{q_i}} \tilde{b}_i \text{ on } J_l. \tag{4.23}$$

And by lemma 8, (4.23) is equivalent to

$$D_{\tilde{u}} g_i^{(q_i)}(t, \bar{u}_t, \bar{y}_t, \bar{u}, \bar{y})v_t^l + \widehat{D}g_i^{(q_i)}(t, \bar{u}_t, \bar{y}_t, \bar{u}, \bar{y})(z[v^l]_t, v^l, z[v^l]) = \tilde{b}_i^{(q_i)}(t) \tag{4.24}$$

for a.a. $t \in J_l$.

If $I_l = \emptyset$, we choose $v^l := v^{l-1}$. Otherwise, say $I_l = \{i_1 < \dots < i_p\}$ and define on J_l

$$\bar{h} := \begin{pmatrix} \tilde{b}_{i_1}^{(q_{i_1})} \\ \vdots \\ \tilde{b}_{i_p}^{(q_{i_p})} \end{pmatrix} \in L^s(J_l; \mathbb{R}^{|I_l|}).$$

Then (4.21) is equivalent to

$$\begin{cases} v^l = v^{l-1} \text{ on } J_0 \cup \dots \cup J_{l-1}, \\ M_t v_t^l + N_t(z[v^l]_t, v^l, z[v^l]) = \bar{h}_t \text{ for a.a. } t \in J_l. \end{cases} \tag{4.25}$$

Applying lemma 25 a) to (h, v^{l-1}), we get $\tilde{v}$ such that (4.25) holds; we choose $v^l := \tilde{v}$.

b) We follow a similar scheme to the one of the proof of a).

Let $(\bar{b}, \bar{v}) \in W^{(q),2}(\mathcal{I}^\varepsilon) \times \mathcal{V}_2$ be such that

$$g_i'(\bar{y})z[\bar{v}] = \bar{b}_i \text{ on } \mathcal{I}^\varepsilon, \ i = 1, \ldots, r.$$

Let $b^k \in W^{(q),\infty}(\mathcal{I}^\varepsilon)$, $k \in \mathbb{N}$, be such that $b^k \xrightarrow{W^{(q),2}} \bar{b}$. Let us construct $v^{k,0}, \ldots, v^{k,\kappa} \in \mathcal{U}$, $k \in \mathbb{N}$, such that for all l, $v^{k,l} \xrightarrow[k\to\infty]{L^2} \bar{v}$ and

$$\begin{cases} v^{k,l} = v^{k,l-1} \text{ on } J_0 \cup \cdots \cup J_{l-1}, \\ g_i'(\bar{y})z[v^{k,l}] = b_i^k \text{ on } \mathcal{I}_i^\varepsilon \cap J_l, \ i \in I_l. \end{cases}$$

We will conclude the proof by defining $v^k := v^{k,\kappa}$, $k \in \mathbb{N}$.

We choose for $v^{k,0}$ the truncation of $\bar{v}$, $k \in \mathbb{N}$ (see definition 41 in appendix A.2).

Suppose we have $v^{k,0}, \ldots, v^{k,l-1}$, $k \in \mathbb{N}$, for some $l \geq 1$ and let us construct $v^{k,l}$, $k \in \mathbb{N}$. Applying lemma 24 b) to $(b^k, v^{k,l-1})$ with $\{t_0\} = \overline{J_{l-1}} \cap \overline{J_l}$, we get $\tilde{b}^k \in W^{(q),\infty}([0,T])$, $k \in \mathbb{N}$. In particular,

$$\tilde{b}^k \xrightarrow{W^{(q),2}} \tilde{b} \tag{4.26}$$

where $\tilde{b} := g'(\bar{y})z[\bar{v}] \in W^{(q),2}([0,T])$. And it is now enough to find $v^{k,l}$, $k \in \mathbb{N}$, such that $v^{k,l} \xrightarrow[k\to\infty]{L^2} \bar{v}$ and

$$\begin{cases} v^{k,l} = v^{k,l-1} \text{ on } J_0 \cup \cdots \cup J_{l-1}, \\ g_i'(\bar{y})z[v^{k,l}] = \tilde{b}_i^k \text{ on } J_l, \ i \in I_l. \end{cases} \tag{4.27}$$

If $I_l = \emptyset$, we choose $v^{k,l} = v^{k,l-1}$, $k \in \mathbb{N}$. Otherwise, say $I_l = \{i_1 < \cdots < i_p\}$ and define on J_l

$$\bar{h} := \begin{pmatrix} \tilde{b}_{i_1}^{(q_{i_1})} \\ \vdots \\ \tilde{b}_{i_p}^{(q_{i_p})} \end{pmatrix} \in L^2(J_l; \mathbb{R}^{|I_l|}), \quad h^k := \begin{pmatrix} (\tilde{b}_{i_1}^k)^{(q_{i_1})} \\ \vdots \\ (\tilde{b}_{i_p}^k)^{(q_{i_p})} \end{pmatrix} \in L^\infty(J_l; \mathbb{R}^{|I_l|}).$$

We have

$$M_t \bar{v}_t + N_t(z[\bar{v}]_t, \bar{v}, z[\bar{v}]) = \bar{h}_t \text{ for a.a } t \in J_l$$

and (4.27) is equivalent to

$$\begin{cases} v^{k,l} = v^{k,l-1} \text{ on } J_0 \cup \cdots \cup J_{l-1}, \\ M_t v_t^{k,l} + N_t(z[v^{k,l}]_t, v^{k,l}, z[v^{k,l}]) = h_t^k \text{ for a.a. } t \in J_l. \end{cases} \tag{4.28}$$

By (4.26), $h^k \xrightarrow{L^2} \bar{h}$, and by assumption, $v^{k,l-1} \xrightarrow[k\to\infty]{L^2} \bar{v}$. Applying lemma 25 b) to $(h^k, v^{k,l-1})$, we get $\tilde{v}^k$, $k \in \mathbb{N}$, such that $\tilde{v}^k \xrightarrow{L^2} \bar{v}$ and (4.28) holds; we choose $v^{k,l} = \tilde{v}^k$, $k \in \mathbb{N}$.

□

4.3 Necessary conditions

Recall that we are under the assumptions **(A0)-(A4)**.

4.3.1 Structure of the set of Lagrange multipliers

Recall that we denote by Λ the set of Lagrange multipliers associated with $(\bar{u}, \bar{y})$ (definition 2). We consider the projection map

$$\begin{array}{rccc} \pi: & \mathcal{M} \times \mathbb{R}^{s*} \times \mathcal{P} & \longrightarrow & \mathbb{R}^{N*} \times \mathbb{R}^{s*} \\ & (\mathrm{d}\eta, \Psi, p) & \longmapsto & \left(([\eta_{i,\tau}])_{\tau,i}, \Psi \right) \end{array}$$

where $\tau \in \mathcal{T}_i$, $i = r_1 + 1, \ldots, r$. A consequence of lemma 23 a) is the following:

Lemma 26. $\pi|_\Lambda$ *is injective.*

Proof. We will use the fact that one of the constraint, namely G_1, has a surjective derivative. For $\mathrm{d}\rho \in \prod_{i=1}^r \mathcal{M}(\mathcal{I}_i^\varepsilon)$, we define $F_\rho \in \left(W^{(q),\infty}(\mathcal{I}^\varepsilon)\right)^*$ by

$$F_\rho(\varphi) := \sum_{1 \le i \le r} \int_{\mathcal{I}_i^\varepsilon} \varphi_{i,t} \mathrm{d}\rho_{i,t} \quad \text{for all } \varphi \in W^{(q),\infty}(\mathcal{I}^\varepsilon).$$

Since by lemma 8, $DG_1(\bar{u}, \bar{y}_0)(v, z_0) \in W^{(q),\infty}(\mathcal{I}^\varepsilon)$ for all $(v, z_0) \in \mathcal{U} \times \mathbb{R}^n$, we have

$$\begin{aligned} \langle \mathrm{d}\rho, DG_1(\bar{u}, \bar{y}_0)(v, z_0) \rangle &= \langle F_\rho, DG_1(\bar{u}, \bar{y}_0)(v, z_0) \rangle \\ &= \left\langle (DG_1(\bar{u}, \bar{y}_0))^* F_\rho, (v, z_0) \right\rangle. \end{aligned}$$

Then differentiating L_R, defined by (3.19), w.r.t. (u, y_0) we get

$$\begin{aligned} D_{(u,y_0)} & L_R(\bar{u}, \bar{y}_0, \mathrm{d}\rho, \nu, \Psi) \\ &= DJ(\bar{u}, \bar{y}_0) + DG_1(\bar{u}, \bar{y}_0)^* F_\rho + DG_2(\bar{u}, \bar{y}_0)^* \nu + DG_3(\bar{u}, \bar{y}_0)^* \Psi. \end{aligned} \tag{4.29}$$

Let $(\mathrm{d}\eta, \Psi, p), (\mathrm{d}\eta', \Psi', p') \in \Lambda$ and suppose that $\pi((\mathrm{d}\eta, \Psi, p)) = \pi((\mathrm{d}\eta', \Psi', p'))$. By lemma 19, let $(\mathrm{d}\rho, \nu), (\mathrm{d}\rho', \nu')$ be such that $(\mathrm{d}\rho, \nu, \Psi), (\mathrm{d}\rho', \nu', \Psi') \in \Lambda_R$. Then $(\nu, \Psi) = (\nu', \Psi')$, and by definition of Λ_R,

$$D_{(u,y_0)} L_R(\bar{u}, \bar{y}_0, \mathrm{d}\rho, \nu, \Psi) = D_{(u,y_0)} L_R(\bar{u}, \bar{y}_0, \mathrm{d}\rho', \nu, \Psi) = 0.$$

Then by (4.29), $DG_1(\bar{u}, \bar{y}_0)^* F_\rho = DG_1(\bar{u}, \bar{y}_0)^* F_{\rho'}$. And it is a consequence of lemma 23 a) that $DG_1(\bar{u}, \bar{y}_0)^*$ is injective on $\left(W^{(q),\infty}(\mathcal{I}^\varepsilon)\right)^*$. Then $F_\rho = F_{\rho'}$, and by density of $W^{(q),\infty}(\mathcal{I}^\varepsilon)$ in $\prod C(\mathcal{I}_i^\varepsilon)$, we get $\mathrm{d}\rho = \mathrm{d}\rho'$. Together with $\nu = \nu'$, it implies $\mathrm{d}\eta = \mathrm{d}\eta'$ and then $(\mathrm{d}\eta, \Psi, p) = (\mathrm{d}\eta', \Psi', p')$. ◻

As a corollary, we get a refinement of theorem 12:

Theorem 27. *Let $(\bar{u}, \bar{y})$ be a qualified local solution of (P). Then Λ is nonempty, convex, of finite dimension and compact.*

Proof. Let $\Lambda_\pi := \pi(\Lambda)$. By theorem 12, Λ is nonempty, convex, weakly $*$ compact and Λ_π is nonempty, convex, of finite dimension and compact (π is linear continuous and its values lie in a finite-dimensional vector space). By lemma 26, $\pi|_\Lambda \colon \Lambda \to \Lambda_\pi$ is a bijection. We claim that its inverse

$$\begin{array}{rccc} m: & \Lambda_\pi & \longrightarrow & \Lambda \\ & (([\eta_{i,\tau}])_{\tau,i}, \Psi) & \longmapsto & (\mathrm{d}\eta, \Psi, p) \end{array}$$

is the restriction of a continuous affine map. Since $\Lambda = m(\Lambda_\pi)$, the result follows. For the claim, using the convexity of both Λ_π and Λ, the linearity of π and its injectivity when restricted to Λ, we get that m preserves convex combinations of elements from Λ_π. Thus we can extend it to an affine map on the affine subspace of $\mathbb{R}^{N*} \times \mathbb{R}^{s*}$ spanned by Λ_π. Since this subspace is of finite dimension, the extension of m is continuous. ◻

4.3.2 Second-order conditions on a large critical cone

Recall that for $\lambda \in \Lambda$, $\mathcal{J}[\lambda]$ has been defined on $\mathcal{U} \times \mathbb{R}^n$ by (3.31) or (3.32).

Remark 28. $\mathcal{J}$ is quadratic w.r.t. (v, z_0) and affine w.r.t. λ. By lemmas 3, 4 and 8, $\mathcal{J}[\lambda]$ can be extended continuously to $\mathcal{V}_2 \times \mathbb{R}^n$ for any $\lambda \in \Lambda$. We obtain the so-called *Hessian of Lagrangian*

$$\mathcal{J} \colon\ [\Lambda] \times \mathcal{V}_2 \times \mathbb{R}^n \longrightarrow \mathbb{R} \tag{4.30}$$

which is jointly continuous w.r.t. λ and (v, z_0).

The critical L^2 cone C_2 has been defined by (3.34)-(3.36). Let the *strict critical L^2 cone* be the set

$$C_2^S := \{(v, z_0) \in C_2 \,:\, g_i'(\bar{y})z = 0 \text{ on } \mathcal{I}_i,\ i = 1, \dots, r\},$$

where $z = z[v, z_0] \in \mathcal{Z}_2$.

Theorem 29. *Let $(\bar{u}, \bar{y})$ be a qualified local solution of (P). Then for any $(v, z_0) \in C_2^S$, there exists $\lambda \in \Lambda$ such that*

$$\mathcal{J}[\lambda](v, z_0) \geq 0. \tag{4.31}$$

The proof is based on the following density lemma, announced in the introduction and proved in the next section:

Lemma 30. *$C_\infty^R \cap C_2^S$ is dense in C_2^S for the $L^2 \times \mathbb{R}^n$ norm.*

Proof of theorem 29. Let $(v, z_0) \in C_2^S$. By lemma 30, there exists a sequence $(v^k, z_0^k) \in C_\infty^R \cap C_2^S$, $k \in \mathbb{N}$, such that

$$(v^k, z_0^k) \longrightarrow (v, z_0).$$

By lemma 21, there exists a sequence $\lambda^k \in \Lambda$, $k \in \mathbb{N}$, such that

$$\mathcal{J}[\lambda^k](v^k, z_0^k) \geq 0. \tag{4.32}$$

By theorem 27, Λ is strongly compact; then there exists $\lambda \in \Lambda$ such that, up to a subsequence,

$$\lambda^k \longrightarrow \lambda.$$

We conclude by passing to the limit in (4.32), thanks to remark 28. □

4.3.3 A density result

In this section we prove lemma 30, using lemma 23 b). A result similar to lemma 30 is stated, in the framework of ODEs, as Lemma 5 in [5], but the proof given there is wrong. Indeed, the costates in the optimal control problems of steps a) and c) are actually not of bounded variations and thus the solutions are not essentially bounded. It has to be highlighted that in lemma 23 b) we get a sequence of essentially bounded v^k.

Proof of lemma 30. We define one more cone:

$$C_\infty^{R+} = \left\{(v, z_0) \in C_\infty^R \cap C_2^S \,:\, \exists \delta > 0 \,:\, g_i'(\bar{y})z[v, z_0] = 0 \text{ on } \mathcal{I}_i^\delta,\ i = 1, \dots, r\right\},$$

and we show actually that C_∞^{R+} is dense in C_2^S.

To do so, we consider the following two normed vector spaces:

$$X_\infty^+ := \left\{(v,z_0)\in\mathcal{U}\times\mathbb{R}^n \,:\, \exists\delta>0 \,:\, g_i'(\bar y)z[v,z_0]=0 \text{ on } \mathcal{I}_i^\delta,\ i=1,\dots,r\right\},$$
$$X_2 := \left\{(v,z_0)\in\mathcal{V}_2\times\mathbb{R}^n \,:\, g_i'(\bar y)z[v,z_0]=0 \text{ on } \mathcal{I}_i,\ i=1,\dots,r\right\}.$$

Observe that C_∞^{R+} and C_2^S are defined as the same polyhedral cone by (3.35)-(3.36), respectively in X_∞^+ and X_2. In view of Lemma 1 in [13], it is then enough to show that X_∞^+ is dense in X_2.

We will need the following lemma, proved in appendix A.3:

Lemma 31. *Let $\bar b_i \in W^{(q_i),2}(\mathcal{I}_i^\varepsilon)$ be such that*

$$\bar b_i = 0 \text{ on } \mathcal{I}_i. \tag{4.33}$$

Then there exists $b_i^\delta\in W^{(q_i),\infty}(\mathcal{I}_i^\varepsilon)$, $\delta\in(0,\varepsilon)$, such that $b_i^\delta \xrightarrow[\delta\to 0]{W^{(q_i),2}} \bar b_i$ and

$$b_i^\delta = 0 \text{ on } \mathcal{I}_i^\delta. \tag{4.34}$$

Going back to the proof of lemma 30, let $(\bar v,\bar z_0)\in X_2$ and $\bar b := (\mathcal{A}_{2,\bar z_0}\bar v)\,|_{\mathcal{I}^\varepsilon}$. We consider a sequence $\delta_k \searrow 0$ and for $i=1,\dots,r$, $b_i^k := b_i^{\delta_k}\in W^{(q_i),\infty}(\mathcal{I}_i^\varepsilon)$ given by lemma 31. Applying lemma 23 b) to b^k, we get v^k, $k\in\mathbb{N}$. We have $(v^k,\bar z_0)\in X_\infty^+$ and $(v^k,\bar z_0)\longrightarrow(\bar v,\bar z_0)$. The proof is completed. □

4.4 Sufficient conditions

We still are under the assumptions **(A0)-(A4)**.

Definition 32. A quadratic form Q over a Hilbert space X is a *Legendre form* if it is weakly lower semi-continuous and if it satisfies the following property: if $x^k \rightharpoonup x$ weakly in X and $Q(x^k)\to Q(x)$, then $x^k\to x$ strongly in X.

Theorem 33. *Suppose that for any $(v,z_0)\in C_2$, there exists $\lambda\in\Lambda$ such that $\mathcal{J}[\lambda]$ is a Legendre form and*

$$\mathcal{J}[\lambda](v,z_0) > 0 \quad \text{if } (v,z_0)\neq 0. \tag{4.35}$$

Then $(\bar u,\bar y)$ is a local solution of (P) satisfying the following quadratic growth condition: there exists $\beta>0$ and $\alpha>0$ such that

$$J(u,y_0)\geq J(\bar u,\bar y_0)+\frac{1}{2}\beta\left(\|u-\bar u\|_2+|y_0-\bar y_0|\right)^2 \tag{4.36}$$

for any trajectory (u,y) feasible for (P) and such that $\|u-\bar u\|_\infty+|y_0-\bar y_0|\leq\alpha$.

Remark 34. Let $\lambda=(d\eta,\Psi,p)\in\Lambda$. The *strengthened Legendre-Clebsch condition*

$$\exists\bar\alpha>0 \,:\, D^2_{uu}H[p](t,\bar u_t,\bar y_t)\geq\bar\alpha I_m \text{ for a.a. } t\in[0,T] \tag{4.37}$$

is satisfied *iff* $\mathcal{J}[\lambda]$ is a Legendre form (it can be porved by combining Theorem 11.6 and Theorem 3.3 in [15]).

Proof of theorem 33. (i) Let us assume that (4.35) holds but that (4.36) does not. Then there exists a sequence of feasible trajectories (u^k,y^k) such that

$$\begin{cases} (u^k,y_0^k)\xrightarrow{L^\infty\times\mathbb{R}^n}(\bar u,\bar y_0),\ (u^k,y_0^k)\neq(\bar u,\bar y_0), \\ J(u^k,y_0^k)\leq J(\bar u,\bar y_0)+o\left(\|u^k-\bar u\|_2+|y_0^k-\bar y_0|\right)^2. \end{cases} \tag{4.38}$$

Let $\sigma_k := \|u^k - \bar{u}\|_2 + |y_0^k - \bar{y}_0|$ and $(v^k, z_0^k) := \sigma_k^{-1}\left(u^k - \bar{u}, y_0^k - \bar{y}_0\right) \in \mathcal{U} \times \mathbb{R}^n$. There exists $(\bar{v}, \bar{z}_0) \in \mathcal{V}_2 \times \mathbb{R}^n$ such that, up to a subsequence,

$$(v^k, z_0^k) \rightharpoonup (\bar{v}, \bar{z}_0) \text{ weakly in } \mathcal{V}_2 \times \mathbb{R}^n.$$

(ii) We claim that $(\bar{v}, \bar{z}_0) \in C_2$.

Let $z^k := z[v^k, z_0^k] \in \mathcal{Y}$ and $\bar{z} := z[\bar{v}, \bar{z}_0] \in \mathcal{Z}_2$. We derive from the compact embedding $\mathcal{Z}_2 \subset C([0,T];\mathbb{R}^n)$ that, up to a subsequence,

$$z^k \to \bar{z} \text{ in } C([0,T];\mathbb{R}^n). \tag{4.39}$$

Moreover, it is classical (see e.g. the proof of Lemma 20 in [4]) that

$$J(u^k, y_0^k) = J(\bar{u}, \bar{y}_0) + \sigma_k DJ(\bar{u}, \bar{y}_0)(v^k, z_0^k) + o(\sigma_k), \tag{4.40}$$

$$g(y^k) = g(\bar{y}) + \sigma_k g'(\bar{y}) z^k + o(\sigma_k), \tag{4.41}$$

$$\Phi(y_0^k, y_T^k) = \Phi(\bar{y}_0, \bar{y}_T) + \sigma_k D\Phi(\bar{y}_0, \bar{y}_T)(z_0^k, z_T^k) + o(\sigma_k). \tag{4.42}$$

It follows that

$$DJ(\bar{u}, \bar{y}_0)(\bar{v}, \bar{z}_0) \leq 0, \tag{4.43}$$

$$\begin{cases} g_i'(\bar{y})\bar{z} \leq 0 \text{ on } \mathcal{I}_i & i = 1, \dots, r_1, \\ g_i'(\bar{y})\bar{z} \leq 0 \text{ on } \mathcal{I}_i \cup \mathcal{T}_i & i = r_1 + 1, \dots, r. \end{cases} \tag{4.44}$$

$$D\Phi(\bar{y}_0, \bar{y}_T)(\bar{z}_0, z[\bar{v}, \bar{z}_0]_T) \in T_K\left(\Phi(\bar{y}_0, \bar{y}_T)\right), \tag{4.45}$$

using (4.38) for (4.43) and the fact that $(\bar{u}, \bar{y})$, (u^k, y^k) are feasible for (4.44) and (4.45). By lemma 9, given $\bar{\lambda} = (\mathrm{d}\bar{\eta}, \bar{\Psi}, \bar{p}) \in \Lambda$, we have

$$DJ(\bar{u}, \bar{y}_0)(\bar{v}, \bar{z}_0) + \int_{[0,T]} \mathrm{d}\bar{\eta}_t g'(\bar{y}_t) + \bar{\Psi} D\Phi(\bar{y}_0, \bar{y}_T)(\bar{z}_0, \bar{z}_T) = 0.$$

Together with definition 2 and (4.43)-(4.45), it implies that each of the three terms is null, i.e. $(\bar{v}, \bar{z}_0) \in C_2$.

(iii) Then by (4.35) there exists $\bar{\lambda} \in \Lambda$ such that $\mathcal{J}[\bar{\lambda}]$ is a Legendre form and

$$0 \leq \mathcal{J}[\bar{\lambda}](\bar{v}, \bar{z}_0). \tag{4.46}$$

In particular, $\mathcal{J}[\bar{\lambda}]$ is weakly lower semi continuous. Then

$$\mathcal{J}[\bar{\lambda}](\bar{v}, \bar{z}_0) \leq \liminf_k \mathcal{J}[\bar{\lambda}](v^k, z_0^k) \leq \limsup_k \mathcal{J}[\bar{\lambda}](v^k, z_0^k). \tag{4.47}$$

And we claim that

$$\limsup_k \mathcal{J}[\bar{\lambda}](v^k, z_0^k) \leq 0. \tag{4.48}$$

Indeed, similarly to (4.40)-(4.42), one can show that, $\bar{\lambda}$ being a multiplier,

$$L_R(u^k, y_0^k, \bar{\lambda}) - L_R(\bar{u}, \bar{y}_0, \bar{\lambda}) = \frac{1}{2}\sigma_k^2 D^2_{(u,y_0)^2} L_R(\bar{u}, \bar{y}_0, \bar{\lambda})(v^k, z_0^k)^2 + o(\sigma_k^2). \tag{4.49}$$

Since $L_R(u^k, y_0^k, \bar{\lambda}) - L_R(\bar{u}, \bar{y}_0, \bar{\lambda}) \leq J(u^k, y_0^k) - J(\bar{u}, \bar{y}_0)$, we derive from (4.38), (4.49) and lemma 20 that

$$\mathcal{J}[\bar{\lambda}](v^k, z_0^k) \leq o(1). \tag{4.50}$$

(iv) We derive from (4.46), (4.47) and (4.48) that

$$\mathcal{J}[\bar{\lambda}](v^k, z_0^k) \longrightarrow 0 = \mathcal{J}[\bar{\lambda}](\bar{v}, \bar{z}_0).$$

By (4.35), $(\bar{v}, \bar{z}_0) = 0$, and by definition of a Legendre form, $(v^k, z_0^k) \longrightarrow (\bar{v}, \bar{z}_0)$ strongly in $\mathcal{V}_2 \times \mathbb{R}^n$. We get a contradiction with the fact that $\|v^k\|_2 + |z_0^k| = 1$ for all k.

□

In view of theorems 29 and 33 it appears that under an extra assumption, of the type of strict complementarity on the running state constraints, we can state no-gap second-order optimality conditions. We denote by $\mathrm{ri}\,(\Lambda)$ the relative interior of Λ (see Definition 2.16 in [7]).

Corollary 35. *Let $(\bar{u}, \bar{y})$ be a qualified feasible trajectory for (P). We assume that $C_2^S = C_2$ and that for any $\lambda \in \mathrm{ri}\,(\Lambda)$, the strengthened Legendre-Clebsch condition* (4.37) *holds. Then $(\bar{u}, \bar{y})$ is a local solution of (P) satisfying the quadratic growth condition* (4.36) iff *for any $(v, z_0) \in C_2 \setminus \{0\}$, there exists $\lambda \in \Lambda$ such that*

$$\mathcal{J}[\lambda](v, z_0) > 0. \tag{4.51}$$

Proof. Suppose (4.51) holds for some $\lambda \in \Lambda$; then it holds for some $\lambda \in \mathrm{ri}\,(\Lambda)$ too and now $\mathcal{J}[\lambda]$ is a Legendre form. By theorem 33, there is locally quadratic growth.

Conversely, suppose (4.36) holds for some $\beta > 0$ and let

$$J_\beta(u, y_0) := J(u, y_0) - \frac{1}{2}\beta \left(\|u - \bar{u}\|_2 + |y_0 - \bar{y}_0|\right)^2.$$

Then $(\bar{u}, \bar{y}_0)$ is a local solution of the following optimization problem:

$$\min_{(u,y_0)\in\mathcal{U}\times\mathbb{R}^n} J_\beta(u, y_0), \quad \text{subject to} \quad G_i(u, y_0) \in K_i,\ i = 1, 2, 3.$$

This problem has the same Lagrange multipliers as the reduced problem (write that the respective Lagrangian is stationary at $(\bar{u}, \bar{y}_0)$), the same critical cones and its Hessian of Lagrangian is

$$\mathcal{J}_\beta[\lambda](v, z_0) = \mathcal{J}[\lambda](v, z_0) - \beta \left(\|v\|_2 + |z_0|\right)^2.$$

Theorem 29 applied to this problem gives (4.51). □

Remark 36. A sufficient condition (not necessary *a priori*) to have $C_2^S = C_2$ is the existence of $(\mathrm{d}\bar{\eta}, \bar{\Psi}, \bar{p}) \in \Lambda$ such that

$$\mathrm{supp}(\mathrm{d}\bar{\eta}_i) = \mathcal{I}_i,\ i = 1, \ldots, r.$$

A Appendix

A.1 Functions of bounded variations

The main reference here is [1], Section 3.2. Recall that with the definition of $BV\,([0,T];\mathbb{R}^{n*})$ given at the beginning of section 2.2, for $h \in BV\,([0,T];\mathbb{R}^{n*})$ there exist $h_{0_-}, h_{T_+} \in \mathbb{R}^{n*}$ such that (2.6) holds.

Lemma 37. *Let $h \in BV\,([0,T];\mathbb{R}^{n*})$. Let h^l, h^r be defined for all $t \in [0,T]$ by*

$$h_t^l := h_{0_-} + \mathrm{d}h\big([0,t)\big), \tag{A.1}$$

$$h_t^r := h_{0_-} + \mathrm{d}h\big([0,t]\big). \tag{A.2}$$

Then they are both in the same equivalence class of h, h^l *is left continuous,* h^r *is right continuous and, for all* $t \in [0, T]$,

$$h^l_t = h_{T_+} - \mathrm{d}h\big([t, T]\big), \tag{A.3}$$
$$h^r_t = h_{T_+} - \mathrm{d}h\big((t, T]\big). \tag{A.4}$$

Proof. Theorem 3.28 in [1]. □

The identification between measures and functions of bounded variations that we mention at the beginning of section 2.2 relies on the following:

Lemma 38. *The linear map*

$$(c, \mu) \longmapsto \Big(h \colon t \mapsto c - \mu\left([t, T]\right)\Big) \tag{A.5}$$

is an isomorphism between $\mathbb{R}^{r*} \times \mathcal{M}\left([0, T]; \mathbb{R}^{r*}\right)$ *and* $BV\left([0, T]; \mathbb{R}^{r*}\right)$, *whose inverse is*

$$h \longmapsto \Big(h_{T_+}, \mathrm{d}h\Big). \tag{A.6}$$

Proof. Theorem 3.30 in [1]. □

Let us now prove lemma 1:

Proof of lemma 1. By (A.3), a solution in $\mathcal{P}$ of (2.10) is any $p \in L^1(0, T; \mathbb{R}^{n*})$ such that, for a.e. $t \in [0, T]$,

$$p_t = D_{y_2}\Phi[\Psi](y_0, y_T) + \int_t^T D_y H[p](s, u_s, y_s)\mathrm{d}s + \int_{[t,T]} \mathrm{d}\eta_s g'(y_s). \tag{A.7}$$

We define $\Theta \colon L^1(0, T; \mathbb{R}^{n*}) \to L^1(0, T; \mathbb{R}^{n*})$ by

$$\Theta(p)_t := D_{y_2}\Phi[\Psi](y_0, y_T) + \int_t^T D_y H[p](s, u_s, y_s)\mathrm{d}s + \int_{[t,T]} \mathrm{d}\eta_s g'(y_s) \tag{A.8}$$

for a.e. $t \in [0, T]$, and we show that Θ has a unique fixed point. Let $C > 0$ such that $\|D_y f\|_\infty, \|D^2_{y,\tau} f\|_\infty \leq C$ along (u, y).

$$\begin{aligned}
|\Theta(p_1)_t - \Theta(p_2)_t| &= \left| \int_t^T \left(D_y H[p_1](s, u_s, y_s) - D_y H[p_2](s, u_s, y_s)\right)\mathrm{d}s \right| \\
&\leq C \int_t^T \left[|p_1(s) - p_2(s)| + \int_s^T |p_1(\theta) - p_2(\theta)|\mathrm{d}\theta \right] \mathrm{d}s \\
&= C \int_t^T \left[|p_1(s) - p_2(s)| + \int_t^s |p_1(s) - p_2(s)|\mathrm{d}\theta \right] \mathrm{d}s \\
&\leq C(1 + T) \int_t^T |p_1(s) - p_2(s)|\mathrm{d}s.
\end{aligned}$$

We consider the family of equivalent norms on $L^1(0, T; \mathbb{R}^{n*})$

$$\|v\|_{1,K} := \|t \mapsto e^{-K(T-t)} v(t)\|_1 \quad (K \geq 0). \tag{A.9}$$

$$\begin{aligned}\|\Theta(p_1)-\Theta(p_2)\|_{1,K} &\le C(1+T)\int_0^T\int_t^T e^{-K(T-t)}|p_1(s)-p_2(s)|\mathrm{d}s\mathrm{d}t\\ &= C(1+T)\int_0^T e^{-K(T-s)}|p_1(s)-p_2(s)|\left[\int_0^s e^{K(t-s)}\mathrm{d}t\right]\mathrm{d}s\\ &\le \frac{C(1+T)}{K}\|p_1-p_2\|_{1,K}.\end{aligned}$$

For K big enough Θ is a contraction on $L^1(0,T;\mathbb{R}^{n*})$ for $\|\cdot\|_{1,K}$; its unique fixed point is the unique solution of (2.10). □

Another useful result is the following integration by parts formula:

Lemma 39. *Let $h,k\in BV([0,T])$. Then $h^l\in L^1(\mathrm{d}k)$, $k^r\in L^1(\mathrm{d}h)$ and*

$$\int_{[0,T]} h^l\mathrm{d}k+\int_{[0,T]} k^r\mathrm{d}h = h_{T_+}k_{T_+}-h_{0_-}k_{0_-}. \tag{A.10}$$

Proof. Let $\Omega := \{0\le y\le x\le T\}$. Since $\chi_\Omega\in L^1(\mathrm{d}h\otimes\mathrm{d}k)$, we have by Fubini's Theorem (Theorem 7.27 in [14]) and lemma 37 that $h^l\in L^1(\mathrm{d}k)$, $k^r\in L^1(\mathrm{d}h)$ and we can compute $\mathrm{d}h\otimes\mathrm{d}k(\Omega)$ in two different ways:

$$\begin{aligned}\mathrm{d}h\otimes\mathrm{d}k(\Omega) &= \int_{[0,T]}\int_{[y,T]}\mathrm{d}h_x\mathrm{d}k_y\\ &= \int_{[0,T]}\left(h_{T_+}-h^l_y\right)\mathrm{d}k_y\\ &= h_{T_+}\left(k_{T_+}-k_{0_-}\right)-\int_{[0,T]}h^l_y\mathrm{d}k_y,\\ \mathrm{d}h\otimes\mathrm{d}k(\Omega) &= \int_{[0,T]}\int_{[0,x]}\mathrm{d}k_y\mathrm{d}h_x\\ &= \int_{[0,T]}k^r_x\mathrm{d}h_x-k_{0_-}\left(h_{T_+}-h_{0_-}\right).\end{aligned}$$

□

A.2 The hidden use of assumption 3

We use **(A3)** to prove lemma 25 (and then lemma 23, and then ...) through the following:

Lemma 40. *Recall that $M_t := D_{\tilde u}G^{(q)}_{I^{\varepsilon_0}(t)}(t,\bar u_t,\bar y_t,\bar u,\bar y)\in\mathbb{R}^{|I^{\varepsilon_0}(t)|}\times\mathbb{R}^{m*}$. Then for all $t\in[0,T]$, $M_tM_t^T$ is invertible and $\left|\left(M_tM_t^T\right)^{-1}\right|\le\gamma^{-2}$.*

Proof. For any $x\in\mathbb{R}^{|I^{\varepsilon_0}(t)|}$,

$$\left\langle M_tM_t^Tx,x\right\rangle = |M_t^Tx|^2\ge\gamma^2|x|^2.$$

Then $M_tM_t^Tx=0$ implies $x=0$ and the invertibility follows.

Let $y\in\mathbb{R}^{|I^{\varepsilon_0}(t)|}$ and $x:=\left(M_tM_t^T\right)^{-1}y$.

$$|y||x|\ge\langle y,x\rangle=\left\langle M_tM_t^Tx,x\right\rangle=|M_t^Tx|^2\ge\gamma^2|x|^2.$$

For $y \neq 0$, we have $x \neq 0$; dividing the previous inequality by $|x|$, we get

$$\gamma^2 \left| \left(M_t M_t^T\right)^{-1} y \right| \leq |y|.$$

The result follows. □

Before we prove lemma 25, we define the truncation of an integrable function:

Definition 41. Given any $\phi \in L^s(J)$ ($s \in [1, \infty)$ and J interval), we will call *truncation of* ϕ the sequence $\phi^k \in L^\infty(J)$ defined for $k \in \mathbb{N}$ and a.a. $t \in J$ by

$$\phi_t^k := \begin{cases} \phi_t & \text{if } |\phi_t| \leq k, \\ k\dfrac{\phi_t}{|\phi_t|} & \text{otherwise.} \end{cases}$$

Observe that $\phi^k \xrightarrow[k\to\infty]{L^s} \phi$.

Proof of lemma 25. In the sequel we omit z_0 in the notations.
(i) Let $v \in \mathcal{V}_s$. We claim that v satisfies

$$M_t v_t + N_t\left(z[v]_t, v, z[v]\right) = h_t \text{ for a.a. } t \in J_l \tag{A.11}$$

iff there exists $w \in L^s(J_l; \mathbb{R}^m)$ such that (v, w) satisfies

$$\begin{cases} M_t w_t = 0, \\ v_t = M_t^T \left(M_t M_t^T\right)^{-1} \left(h_t - N_t(z[v]_t, v, z[v])\right) + w_t, \end{cases} \quad \text{for a.a. } t \in J_l. \tag{A.12}$$

Clearly, if (v, w) satisfies (A.12), then v satisfies (A.11). Conversly, suppose that v satisfies (A.11). With lemma 40 in mind, we define $\alpha \in L^s(J_l; \mathbb{R}^{|I_l|})$ and $w \in L^s(J_l; \mathbb{R}^m)$ by

$$\alpha := \left(MM^T\right)^{-1} Mv,$$
$$w := \left(I_m - M^T \left(MM^T\right)^{-1} M\right) v.$$

Then

$$\begin{cases} Mw = 0, \\ v = M^T\alpha + w, \end{cases} \quad \text{on } J_l. \tag{A.13}$$

We derive from (A.11) and (A.13) that

$$M_t M_t^T \alpha_t + N_t\left(z[v]_t, v, z[v]\right) = h_t \text{ for a.a. } t \in J_l.$$

Using again lemma 40 and (A.13), we get (A.12).

(ii) Given $(v, h, w) \in \mathcal{V}_s \times L^s(J_l; \mathbb{R}^{|I_l|}) \times L^s(J_l; \mathbb{R}^m)$, there exists a unique $\tilde{v} \in \mathcal{V}_s$ such that

$$\begin{cases} \tilde{v} = v \text{ on } J_0 \cup \cdots \cup J_{l-1} \cup J_{l+1} \cup \cdots \cup J_\kappa, \\ \tilde{v}_t = M_t^T \left(M_t M_t^T\right)^{-1} \left(h_t - N_t(z[\tilde{v}]_t, \tilde{v}, z[\tilde{v}])\right) + w_t \text{ for a.a. } t \in J_l, \end{cases} \tag{A.14}$$

Indeed, one can define a mapping from $\mathcal{V}_s$ to $\mathcal{V}_s$, using the right-hand side of (A.14). Then it can be shown, as in the proof of lemma 1, that this mapping is a contraction for a well-suited norm, using lemmas 3, 4 and 40. The existence and uniqueness follow. Moreover, a version of the contraction mapping theorem with parameter (see e.g. Théorème 21-5 in [10]) shows that $\tilde{v}$ depends continuously on (v, h, w).

(iii) Let us prove a): let $(\bar{h}, v) \in L^s(J_l; \mathbb{R}^{|I_l|}) \times \mathcal{V}_s$ and let $w := 0$. Let $\tilde{v} \in \mathcal{V}_s$ be the unique solution of (A.14) for $(v, \bar{h}, w)$. Then $\tilde{v}$ is a solution of (4.17) by *(i)*.

(iv) Let us prove b): let $(\bar{h}, \bar{v}) \in L^s(J_l; \mathbb{R}^{|I_l|}) \times \mathcal{V}_s$ as in the statement and let $\bar{w}$ be given by *(i)*. Then $\bar{v}$ is the unique solution of (A.14) for $(\bar{v}, \bar{h}, \bar{w})$.

Let $(h^k, v^k) \in L^\infty(J_l; \mathbb{R}^{|I_l|}) \times \mathcal{U}$, $k \in \mathbb{N}$, be such that $(h^k, v^k) \xrightarrow{L^s \times L^s} (\bar{h}, \bar{v})$ and let $w^k \in L^\infty(J_l; \mathbb{R}^m)$, $k \in \mathbb{N}$, be the truncation of $\bar{w}$. It is obvious from definition 41 that

$$M_t w_t^k = 0 \text{ for a.a. } t \in J_l.$$

Let $\tilde{v}^k \in \mathcal{U}$ be the unique solution of (A.14) for (v^k, h^k, w^k), $k \in \mathbb{N}$. Then by uniqueness and continuity in *(ii)*,

$$\tilde{v}^k \xrightarrow{L^s} \bar{v}. \tag{A.15}$$

And $\tilde{v}^k$ is a solution of (4.19) by *(i)*. □

A.3 Approximations in $W^{q,2}$

We will prove in this section lemmas 24 and 31. First we give the statement and the proof of a general result:

Lemma 42. *Let $\hat{x} \in W^{q,2}([0,1])$. For $j = 0, \ldots, q-1$, we denote*

$$\begin{cases} \hat{\alpha}_j := \hat{x}^{(j)}(0), \\ \hat{\beta}_j := \hat{x}^{(j)}(1), \end{cases} \tag{A.16}$$

and we consider $\alpha_j^k, \beta_j^k \in \mathbb{R}^q$, $k \in \mathbb{N}$, such that $(\alpha_j^k, \beta_j^k) \longrightarrow (\hat{\alpha}_j, \hat{\beta}_j)$. Then there exists $x^k \in W^{q,\infty}([0,1])$, $k \in \mathbb{N}$, such that $x^k \xrightarrow{W^{q,2}} \hat{x}$ and, for $j = 0, \ldots, q-1$,

$$\begin{cases} (x^k)^{(j)}(0) = \alpha_j^k, \\ (x^k)^{(j)}(1) = \beta_j^k. \end{cases} \tag{A.17}$$

Proof. Given $u \in L^2([0,1])$, we define $x_u \in W^{q,2}([0,1])$ by

$$x_u(t) := \int_0^t \int_0^{s_1} \cdots \int_0^{s_{q-1}} u(s_q) \mathrm{d}s_q \mathrm{d}s_{q-1} \cdots \mathrm{d}s_1, \ t \in [0,1].$$

Then $x_u^{(q)} = u$ and, for $j = 0, \ldots, q-1$,

$$x_u^{(j)}(1) = \gamma_j \iff \langle a_j, u \rangle_{L^2} = \gamma_j$$

where $a_j \in C([0,1])$ is defined by

$$a_j(t) := \frac{(1-t)^{q-1-j}}{(q-1-j)!}, \ t \in [0,1].$$

Indeed, a straightforward induction shows that

$$x_u^{(j)}(1) = \int_0^t \int_0^{s_{j+1}} \cdots \int_0^{s_{q-1}} u(s_q) \mathrm{d}s_q \mathrm{d}s_{q-1} \cdots \mathrm{d}s_{j+1}.$$

Then integrations by parts give the expression of the a_j. Note that the a_j $(j = 0, \dots, q-1)$ are linearly independent in $L^2([0,1])$. Then

$$A\colon \quad \begin{array}{ccc} \mathbb{R}^q & \longrightarrow & L^2([0,1]) \\ \begin{pmatrix} \lambda_0 \\ \vdots \\ \lambda_{q-1} \end{pmatrix} & \longmapsto & \displaystyle\sum_{j=0}^{q-1} \lambda_j a_j \end{array}$$

is such that A^*A is invertible (A^* is here the adjoint operator). And

$$x_u^{(j)}(1) = \gamma_j, \ j = 0, \dots, q-1 \iff A^*u = (\gamma_0, \dots, \gamma_{q-1})^T. \tag{A.18}$$

Going back to the lemma, let $\hat{u} := \hat{x}^{(q)} \in L^2([0,1])$. Observe that

$$\hat{x}(t) = \sum_{l=0}^{q-1} \frac{\hat{\alpha}_l}{l!} t^l + x_{\hat{u}}(t), \ t \in [0,1],$$

and that $A^*\hat{u} = (\hat{\gamma}_0, \dots, \hat{\gamma}_{q-1})^T$ where

$$\hat{\gamma}_j := \hat{\beta}_j - \sum_{l=j}^{q-1} \frac{\hat{\alpha}_l}{(l-j)!}, \ j = 0, \dots, q-1.$$

Then we consider, for $k \in \mathbb{N}$, the truncation (definition 41) $\hat{u}^k \in L^\infty([0,1])$ of $\hat{u}$, and

$$\gamma_j^k := \beta_j^k - \sum_{l=j}^{q-1} \frac{\alpha_l^k}{(l-j)!}, \ j = 0, \dots, q-1, \tag{A.19}$$

$$\gamma^k := (\gamma_0^k, \dots, \gamma_{q-1}^k)^T,$$

$$u^k := \hat{u}^k + A(A^*A)^{-1}\left(\gamma^k - A^*\hat{u}^k\right),$$

$$x^k(t) := \sum_{l=0}^{q-1} \frac{\alpha_l^k}{l!} t^l + x_{u^k}(t), \ t \in [0,1]. \tag{A.20}$$

It is clear that $u^k \in L^\infty([0,1])$ (by definition of A); then $x^k \in W^{q,\infty}([0,T])$. Since $A^*u^k = \gamma^k$ and in view of (A.18), (A.19) and (A.20), (A.17) is satisfied. Finally, $\gamma_j^k \longrightarrow \hat{\gamma}_j$ $(j = 0, \dots, q-1)$; then $\gamma^k \longrightarrow A^*\hat{u}$ and $u^k \longrightarrow \hat{u}$.

□

We can also prove the following:

Lemma 43. *Let $\hat{x} \in W^{q,2}([0,1])$ be such that $\hat{x}^{(j)}(0) = 0$ for $j = 0, \dots, q-1$. Then there exists $x^\delta \in W^{q,\infty}([0,1])$ for $\delta > 0$ such that $x^\delta \xrightarrow[\delta \to 0]{W^{q,2}} \hat{x}$ and*

$$x^\delta = 0 \ \textit{on} \ [0, \delta]. \tag{A.21}$$

Proof. We consider $u^\delta \in L^\infty([0,1])$, $\delta > 0$, such that $u^\delta = 0$ on $[0,\delta]$ and $u^\delta \xrightarrow[\delta \to 0]{L^2} \hat{u} := \hat{x}^{(q)}$. Then we define $x^\delta := x_{u^\delta}$ (see the previous proof). □

Now the proof of lemma 31 is straightforward.

Proof of lemma 31. We observe that $\bar{b}_i = 0$ on $\mathcal{I}_i$ implies that $\bar{b}_i^{(j)} = 0$ at the end points of $\mathcal{I}_i$ for $j = 0, \dots, q_i - 1$ (note that with the definition (3.14), if one component of $\mathcal{I}_i$ is a singleton, then $q_i = 1$). Then the conclusion follows with lemma 43 applied on each component of $\mathcal{I}_i^\varepsilon \setminus \mathcal{I}_i$. □

Finally, we use lemma 42 to prove lemma 24.

Proof of lemma 24. In the sequel we omit z_0 in the notations. We define a *connection in* $W^{q,\infty}$ between ψ_1 at t_1 and ψ_2 at t_2 as any $\psi \in W^{q,\infty}([t_1, t_2])$ such that

$$\begin{cases} \psi^{(j)}(t_1) = \psi_1^{(j)}(t_1), \\ \psi^{(j)}(t_2) = \psi_2^{(j)}(t_2), \end{cases} \quad j = 0, \dots, q-1.$$

a) We define $\tilde{b}_i$ on $[0, t_0]$ by $\tilde{b}_i := g_i'(\bar{y})z[v]$, $i = 1, \dots, r$. We need to explain how we define $\tilde{b}_i$ on $(t_0, T]$, using $\bar{b}_i$ and connections, to have $\tilde{b}_i \in W^{q_i,s}([0,T])$ and $\tilde{b}_i = \bar{b}_i$ on each component of $\mathcal{I}_i^\varepsilon \cap (t_0, T]$. The construction is slightly different whether $t_0 \in \mathcal{I}_i^\varepsilon$ or not, i.e. whether $i \in I_{t_0}^\varepsilon$ or not. Note that by definition of ε_0 and of t_0, I_t^ε is constant for t in a neighbourhood of t_0. We now distinguish the 2 cases just mentioned:

1. $i \in I_{t_0}^\varepsilon$: We denote by $[t_1, t_2]$ the connected component of $\mathcal{I}_i^\varepsilon$ such that $t_0 \in (t_1, t_2)$. We derive from (4.12) that $\tilde{b}_i = \bar{b}_i$ on $[t_1, t_0]$. Then we define $\tilde{b}_i := \bar{b}_i$ on $(t_0, t_2]$.

   If $\mathcal{I}_i^\varepsilon$ has another component in $(t_2, T]$, we denote the first one by $[t_1', t_2']$. Let ψ be a connection in $W^{q_i,\infty}$ between $\bar{b}_i$ at t_2 to $\bar{b}_i$ at t_1'. We define $\tilde{b}_i := \psi$ on (t_2, t_1'), $\tilde{b}_i := \bar{b}_i$ on $[t_1', t_2']$, and so forth on $(t_2', T]$.

   If $\mathcal{I}_i^\varepsilon$ has no more component, we define $\tilde{b}_i$ on what is left as a connection in $W^{q_i,\infty}$ between $\bar{b}_i$ and $g_i'(\bar{y})z[v]$ at T.

2. $i \notin I_{t_0}^\varepsilon$: If $\mathcal{I}_i^\varepsilon$ has a component in $[t_0, T]$, we denote the first one by $[t_1, t_2]$. Note that $t_1 - t_0 \geq \varepsilon_0 - \varepsilon > 0$. We consider a connection in $W^{q_i,\infty}$ between $\tilde{b}_i$ at t_0 and $\bar{b}_i$ at t_1 and we continue as in 1.

   If $\mathcal{I}_i^\varepsilon$ has no component in $[t_0, T]$, we do as in 1.

b) For all $k \in \mathbb{N}$, we apply a) to (b^k, v^k) and we get $\tilde{b}^k$. We just need to explain how we can get, for $i = 1, \dots, r$,

$$\tilde{b}_i^k \xrightarrow[k\to\infty]{W^{q_i,2}} g_i'(\bar{y})z[\bar{v}].$$

By construction we have

$$\begin{array}{ll} \text{on } [0, t_0], & \tilde{b}_i^k = g_i'(\bar{y})z[v^k] \longrightarrow g_i'(\bar{y})z[\bar{v}], \\ \text{on } \mathcal{I}_i^\varepsilon, & \tilde{b}_i^k = b_i^k \longrightarrow \bar{b}_i = g_i'(\bar{y})z[\bar{v}]. \end{array}$$

Then it is enough to show that every connection which appears when we apply a) to (b^k, v^k), for example $\psi_i^k \in W^{q_i,\infty}([t_1, t_2])$, can be chosen in such a way that

$$\psi_i^k \longrightarrow g_i'(\bar{y})z[\bar{v}] \text{ on } [t_1, t_2].$$

This is possible by lemma 42. □

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