A staggered discontinuous Galerkin method for wave propagation in media with dielectrics and meta-materials

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Eric T. Chung* Patrick Ciarlet, Jr.†

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Abstract

Some electromagnetic materials exhibit, in a given frequency range, effective dielectric permittivity and/or magnetic permeability which are negative. In the literature, they are called negative index materials, left-handed materials or meta-materials. We propose in this paper a numerical method to solve a wave transmission between a classical dielectric material and a meta-material. The method we investigate can be considered as an alternative method compared to the method presented by the second author and co-workers. In particular, we shall use the abstract framework they developed to prove well-posedness of the exact problem. We recast this problem to fit later discretization by the staggered discontinuous Galerkin method developed by the first author and co-worker, a method which relies on introducing an auxiliary unknown. Convergence of the numerical method is proven, with the help of explicit inf-sup operators, and numerical examples are provided to show the efficiency of the method.

Keywords: wave diffraction problem, interface problem, negative index materials, left-handed materials, meta-materials, inf-sup theory, T-coercivity, staggered discontinuous Galerkin finite elements, convergence and stability.

1 Introduction

Consider a bounded domain Ω of $\mathbb{R}^d$, with $d = 1, 2, 3$. The model problem we study is a scalar electromagnetic wave equation in the time-frequency domain, e.g.

$$\begin{aligned} &\text{find } u \in H^1(\Omega) \text{ such that} \\ &\operatorname{div}\left(\mu^{-1}\nabla u\right) + \omega^2 \varepsilon u = f \text{ in } \Omega \\ &u = 0 \text{ on } \partial\Omega. \end{aligned} \tag{1}$$

*Department of Mathematics, The Chinese University of Hong Kong, Hong Kong SAR. Email: tschung@math.cuhk.edu.hk.

†POEMS Laboratory, CNRS-INRIA-ENSTA UMR 7231, ENSTA ParisTech, 32, Boulevard Victor, 75739 Paris Cedex 15, France. Email: patrick.ciarlet@ensta-paristech.fr.

Above, f is a volume source function of $L^2(\Omega)$, $\omega \geq 0$ is the given pulsation, and ε, μ are respectively the electric permittivity and the magnetic permeability.

Remark 1.1 *For instance, assume that we study the Transverse Magnetic, or TM, mode in Ω, a subset of $\mathbb{R}^2$. Classically, the right-hand side f in (1) is proportional to the current density, and the solution u is the scalar potential of the magnetic field. See [10], §5, for an alternate approach.*

An equivalent variational formulation is obtained simply via integration by parts:

$$\begin{array}{l} \text{find } u \in H_0^1(\Omega) \text{ such that} \\ \left(\mu^{-1}\nabla u, \nabla v\right)_{\boldsymbol{L}^2(\Omega)} - \omega^2(\varepsilon u, v)_{L^2(\Omega)} = -(f,v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega). \end{array} \tag{2}$$

By introducing an additional unknown, namely $\boldsymbol{U} = \mu^{-1}\nabla u$, we can recast equivalently this problem, and obtain a suitable framework for Discontinuous Galerkin discretization, the so-called *two-unknown problem*:

$$\begin{array}{l} \text{find } (u, \boldsymbol{U}) \in H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega) \text{ such that} \\ (\mu \boldsymbol{U}, \boldsymbol{V})_{\boldsymbol{L}^2(\Omega)} - (\nabla u, \boldsymbol{V})_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \boldsymbol{V} \in \boldsymbol{L}^2(\Omega), \\ (\boldsymbol{U}, \nabla v)_{\boldsymbol{L}^2(\Omega)} - \omega^2(\varepsilon u, v)_{L^2(\Omega)} = -(f,v)_{L^2(\Omega)} \quad \forall v \in H_0^1(\Omega). \end{array} \tag{3}$$

In the same spirit, (1) or (2) is called the *one-unknown problem*. By construction, we find that $\operatorname{div} \boldsymbol{U} + \omega^2 \varepsilon u = f$, so $\operatorname{div} \boldsymbol{U}$ automatically belongs to $L^2(\Omega)$.

If there is a dielectric in the domain Ω, one has $0 < \varepsilon_{min} < \varepsilon < \varepsilon_{max}$ and $0 < \mu_{min} < \mu < \mu_{max}$ a.e. in Ω, so the model problem (1) fits into the well-known Fredholm, or coercive + compact, framework. Indeed, the form $(u, v) \mapsto (\mu^{-1}\nabla u, \nabla v)_{\boldsymbol{L}^2(\Omega)}$ is coercive over $H_0^1(\Omega) \times H_0^1(\Omega)$, whereas the form $(u, v) \mapsto (\varepsilon u, v)_{L^2(\Omega)}$ is a compact perturbation.
Then, a number of materials can be modeled at a given frequency (or within a given frequency range) by considering negative real values for their dielectric permittivity and/or magnetic permeability: these are the so-called meta-materials. Interestingly, if the domain Ω is made entirely of a meta-material, the problem (1) still fits into the Fredholm framework, because the form $(u, v) \mapsto \text{sign}(\mu)(\mu^{-1}\nabla u, \nabla v)_{\boldsymbol{L}^2(\Omega)}$ remains coercive.
On the other hand, in a setting which includes an interface between a dielectric and a meta-material, the situation can be much more complex. In this case, ε and/or μ can exhibit a sign-shift. Note however that if only ε has a sign-shift, then there is no difficulty. The difficulty arises if μ has a sign-shift, because in this case the form $(u, v) \mapsto (\mu^{-1}\nabla u, \nabla v)_{\boldsymbol{L}^2(\Omega)}$ is indefinite, so it is certainly not coercive.

Our aim is to consider a domain made of a dielectric and a meta-material, separated by an interface across which the magnetic permeability μ exhibits a sign-shift, and to solve the two-unknown problem in this case, both from theoretical and numerical points of view. In Section 2, we introduce the abstract framework, and we recall how T-coercivity, i.e. the use of explicit inf-sup operators (see [2, 1]), can be used to solve indefinite problems. We prove in the following section that the two-unknown problem is well-posed, under suitable assumptions. Then, we introduce the staggered discontinuous Galerkin finite element discretization of [6, 7] in Section 4. In particular, this method gives some local and global conservation properties in the discrete level that mimic the conservation properties arising from the continuous problem [8]. In the next section, we prove that it converges in a classic manner for the class of indefinite problems under scrutiny. Finally in Section 6 we report some numerical experiments.

2 The theory of T-coercivity

We propose below a well-known reformulation of the classical inf-sup theory [3, 9], using *explicit* operators to achieve the inf-sup condition for the exact and discrete problems. This operator is sometimes called an inf-sup operator. This approach will be used in the forthcoming sections to prove the well-posedness, and then the convergence of the numerical approximation, of the interface problem with sign-shifting permeability. We choose the vocabulary T-coercivity, in the spirit of [2, 1].

2.1 Abstract theory

Consider a Hilbert space V, with scalar product $(\cdot,\cdot)_V$ and norm $\|\cdot\|_V$. To a continuous bilinear form b defined on $V \times V$, one associates a unique continuous and linear operator $\mathtt{B}$ ($\mathtt{B} \in \mathcal{L}(V)$): $\forall u, v \in V,\ b(u,v) = (\mathtt{B}u, v)_V$.
Given $\ell \in V'$, we focus on the variational problem:

$$\text{find } u \in V \text{ such that} \quad b(u,v) = \ell(v) \quad \forall v \in V. \tag{4}$$

Below, we recall the definition of T-coercivity of the form b and its consequence (cf. Definition 2.1 and Theorem 2.1 in [2]).

Definition 2.1 (T-coercivity) Let $\mathbb{T}$ be a continuous, bijective, linear operator on V. A bilinear form b is $\mathbb{T}$-coercive on $V \times V$ if

$$\exists \gamma > 0,\ \forall v \in V, \quad |b(v, \mathbb{T}v)| \geq \gamma \|v\|_V^2 .$$

Proposition 2.2 *Assume that the T-coercivity assumption is fulfilled. Then, the variational problem (4) is well-posed:* $\mathtt{B}^{-1}$ *exists and* $\mathtt{B}^{-1} \in \mathcal{L}(V)$.

The notion of T-coercivity can be applied to a problem involving a more general continuous bilinear form a, defined on $V \times V$. In this case, the problem to be solved writes:

$$\text{find } u \in V \text{ such that} \quad a(u,v) = \ell(v) \quad \forall v \in V. \tag{5}$$

Above, the form a can be split as $a = b + c$, where forms b and c are both continuous and bilinear on $V \times V$. Let us assume that

(H1) There exists $\mathbb{T} \in \mathcal{L}(V)$, bijective, such that b is $\mathbb{T}$-coercive on $V \times V$;

(H2) the operator associated with the bilinear form c is compact.

Remark 2.3 *For the one-unknown problem, one introduces respectively*

$$V_1 = H_0^1(\Omega),\ b_1(u,v) = (\mu^{-1}\nabla u, \nabla v)_{\boldsymbol{L}^2(\Omega)} \text{ and } c_1(u,v) = -\omega^2(\varepsilon u, v)_{L^2(\Omega)}.$$

Proposition 2.4 *Assume that the conditions (H1) and (H2) are fulfilled. Then, the variational problem (5) is well-posed if, and only if, the uniqueness principle of the solution to (5) holds, i.e. $\ell = 0 \Longrightarrow u = 0$.*

In our case, we shall use a variant of this result, namely we relax assumption (H1) to:

(H1') The mapping $\mathtt{B}^{-1}$ exists and belongs to $\mathcal{L}(V)$.

It is straightforward to check that the statement of the previous proposition holds with (H1') replacing (H1).

2.2 Convergence theory

Let us recall some additional results of [2], §2, in the case of a *conforming* discrete version of the problem (5), which writes

$$\text{find } u^h \in V^h \text{ such that} \quad a(u^h, v^h) = \ell(v^h) \quad \forall v^h \in V^h, \tag{6}$$

where $(V^h)_h$ is a family of finite dimensional vector subspaces of V. We assume the usual approximability property below

(H3) For all $v \in V$, one has $\lim\limits_{h\to 0} \inf\limits_{v^h \in V^h} \|v - v^h\|_V = 0$.

The idea is to prove the uniform stability of the form a over $(V^h)_h$:

$$\exists\sigma > 0,\ \exists h_0 > 0,\ \forall h \in]0, h_0[,\ \forall v^h \in V^h, \quad \sup_{w^h \in V^h} \frac{|a(v^h, w^h)|}{\|w^h\|_V} \geq \sigma \|v^h\|_V\,. \tag{7}$$

In [2] (Theorem 2.2), the result below is proved.

Proposition 2.5 *Assume that hypotheses (H1) and (H2) hold, together with the uniqueness principle so that problem (5) is well-posed.*
Assume that the approximability property (H3) holds.
Assume further that: $\exists\ \delta > 0$, $\gamma > 0$, such that $\forall h$, $\exists \mathbb{T}^h \in \mathcal{L}(V^h)$, satisfying

(a) $\displaystyle\sup_{v^h \in V^h} \frac{\|\mathbb{T}^h v^h\|_V}{\|v^h\|_V} \leq \delta$,

(b) the form b is $\mathbb{T}^h$-coercive over $V^h \times V^h$ with a coercivity constant equal to γ.

Then, the bilinear form a is uniformly stable.

As a consequence of (7), the standard error estimate is recovered with the help of the Strang lemma [12]:

$$\exists \mathcal{C} > 0\,, \exists h_0 > 0\,, \forall h \in]0, h_0] \quad \|u - u^h\|_V \leq \mathcal{C} \inf_{v^h \in V^h} \|u - v^h\|_V\,. \tag{8}$$

3 Well-posedness of the two-unknown problem

Let us begin by some notations and functional spaces. Let Ω be an open bounded subset of $\mathbb{R}^d$, $d = 1, 2, 3$. It is assumed that this domain can be split in two sub-domains Ω_1 and Ω_2 with Lipschitz boundaries: $\overline{\Omega} = \overline{\Omega_1} \cup \overline{\Omega_2}$, $\Omega_1 \cap \Omega_2 = \emptyset$. Moreover, if we let $\Sigma = \partial\Omega_1 \cap \partial\Omega_2$ be the interface, we define $\Gamma_\ell = \partial\Omega_\ell \setminus \Sigma$ for $\ell = 1, 2$. Finally, we introduce

$$H^1_{0,\Gamma_\ell}(\Omega_\ell) = \{v \in H^1(\Omega_\ell) \mid v_{|\Gamma_\ell} = 0\}, \text{ for } \ell = 1, 2.$$

Throughout this paper we will consider that the electromagnetic parameters verify

$$\mu, \varepsilon \in L^\infty(\Omega), \quad \mu^{-1}, \varepsilon^{-1} \in L^\infty(\Omega).$$

To fix ideas, we assume that $\mu_{|\Omega_1} > 0$ a. e., and $\mu_{|\Omega_2} < 0$ a. e..
Hereafter we adopt the notation, for all quantities v defined on Ω, $v_\ell = v_{|\Omega_\ell}$, for $\ell = 1, 2$. Let us now introduce the ratios

$$\frac{\inf_{\Omega_1} \mu_1}{\sup_{\Omega_2} |\mu_2|} \quad \text{and} \quad \frac{\inf_{\Omega_2} |\mu_2|}{\sup_{\Omega_1} \mu_1}. \tag{9}$$

In the case where μ is piecewise constant (equal to the constant μ_ℓ over Ω_ℓ, for $\ell = 1, 2$), they are respectively equal to $1/|\kappa_\mu|$ and $|\kappa_\mu|$, where

$$\kappa_\mu = \frac{\mu_2}{\mu_1} \tag{10}$$

defines the *contrast* of the magnetic permeabilities.
It turns out that we can not prove T-coercivity directly, i.e. exhibit some *ad hoc* operator $\mathbb{T}$, for the two-unknown problem (3). Instead, we verify its well-posedness (cf. (H1')), using the T-coercivity results [2, 1] for the one-unknown problem (2). For this latter problem, consider a continuous, linear operator $R : H^1_{0,\Gamma_1}(\Omega_1) \to H^1_{0,\Gamma_2}(\Omega_2)$, such that one has the *compatibility condition* $(Rv)_{|\Sigma} = v_{|\Sigma}$ for all $v \in H^1_{0,\Gamma_1}(\Omega_1)$, and let the explicit operator be defined by

$$Tu = \begin{cases} u_1 & \text{in } \Omega_1 \\ -u_2 + 2Ru_1 & \text{in } \Omega_2 \end{cases}. \tag{11}$$

Due to the compatibility condition at the interface, the operator T belongs to $\mathcal{L}(H^1_0(\Omega))$, and moreover $T^2 = \mathbb{I}$.

Remark 3.1 *The roles of Ω_1 and Ω_2 can be reversed. Indeed, considering a continuous, linear operator $R' : H^1_{0,\Gamma_2}(\Omega_2) \to H^1_{0,\Gamma_1}(\Omega_1)$, such that one has the compatibility condition $(R'v)_{|\Sigma} = v_{|\Sigma}$ for all $v \in H^1_{0,\Gamma_2}(\Omega_2)$, one can define $T \in \mathcal{L}(H^1_0(\Omega))$ by*

$$Tu = \begin{cases} u_1 - 2R'u_2 & \textit{in } \Omega_1 \\ -u_2 & \textit{in } \Omega_2 \end{cases}.$$

For the one-unknown problem (with form b_1), one can prove T-coercivity using such an operator, under *suitable conditions* [2, 1] on the ratios (9), or on the contrast κ_μ in the piecewise-constant case (see the end of this section for a precise statement). Below, well-posedness of the two-unknown problem is shown to hold under *identical* conditions. To that aim, we introduce $V_2 = H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)$ and

$$b_2((u,\boldsymbol{U}),(v,\boldsymbol{V})) = (\mu\boldsymbol{U},\boldsymbol{V})_{\boldsymbol{L}^2(\Omega)} - (\nabla u,\boldsymbol{V})_{\boldsymbol{L}^2(\Omega)} + (\boldsymbol{U},\nabla v)_{\boldsymbol{L}^2(\Omega)}, \qquad (12)$$

a bilinear form defined on $V_2 \times V_2$. Let $\mathtt{B}_2$ be the associated linear operator of $\mathcal{L}(V_2)$.

Theorem 3.2 *Assume that the T-coercivity is true for the form b_1 of the one-unknown problem. Then, (H1') holds for $\mathtt{B}_2$: ${\mathtt{B}_2}^{-1}$ exists and ${\mathtt{B}_2}^{-1} \in \mathcal{L}(V_2)$.*

Proof. To prove (H1') for $\mathtt{B}_2$, we need to establish that, given any $(f,\boldsymbol{G}) \in V_2'$ (by definition, $V_2' = H^{-1}(\Omega) \times \boldsymbol{L}^2(\Omega)$), there exists one, and only one, solution to

$$\begin{array}{l} \text{find } (u,\boldsymbol{U}) \in V_2 \text{ such that} \\ b_2((u,\boldsymbol{U}),(v,\boldsymbol{V})) = -\langle f,v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} + (\boldsymbol{G},\boldsymbol{V})_{\boldsymbol{L}^2(\Omega)} \quad \forall (v,\boldsymbol{V}) \in V_2. \end{array} \qquad (13)$$

(We can then use the open mapping theorem to conclude, V_2 being a Banach space.)

We assume the conditions ensuring T-coercivity for the one-unknown problem are met, using some *ad hoc* bijective operator T of $\mathcal{L}(H_0^1(\Omega))$. On the other hand, for the two-unknown problem (with form b_2), we introduce the operator $\mathbb{T}$ of $\mathcal{L}(V_2)$, defined by $\mathbb{T}((u,\boldsymbol{U})) = (Tu,\boldsymbol{T}\boldsymbol{U})$, where the action of $\boldsymbol{T} \in \mathcal{L}(\boldsymbol{L}^2(\Omega))$ is simply

$$\boldsymbol{T}\boldsymbol{U} = \begin{cases} \boldsymbol{U}_1 & \text{in } \Omega_1 \\ -\boldsymbol{U}_2 & \text{in } \Omega_2 \end{cases}.$$

Now, we are in a position to prove that (H1') holds for $\mathtt{B}_2$ under the *same* suitable conditions. We note that, by definition, $\boldsymbol{T}^2 = \mathbb{I}$ in $\mathcal{L}(L^2(\Omega))$, so $\mathbb{T}$ is a bijection: in (13), we can thus replace the test-fields $(v,\boldsymbol{V})$ by $\mathbb{T}((v,\boldsymbol{V}))$. This writes

$$\begin{array}{l} \text{find } (u,\boldsymbol{U}) \in V_2 \text{ such that} \\ b_2((u,\boldsymbol{U}),\mathbb{T}((v,\boldsymbol{V}))) = -\langle f,Tv\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \\ \qquad\qquad +(\boldsymbol{G},\boldsymbol{T}\boldsymbol{V})_{\boldsymbol{L}^2(\Omega)} \quad \forall (v,\boldsymbol{V}) \in V_2. \end{array} \qquad (14)$$

Let us prove the existence of a solution to (14): to that aim, we provide a constructive proof.
First, consider that $v = 0$. Then, we have that $(u,\boldsymbol{U})$ is governed by

$$(\mu\boldsymbol{U},\boldsymbol{T}\boldsymbol{V})_{\boldsymbol{L}^2(\Omega)} = (\boldsymbol{G}+\nabla u,\boldsymbol{T}\boldsymbol{V})_{\boldsymbol{L}^2(\Omega)} \ \forall \boldsymbol{V} \in \boldsymbol{L}^2(\Omega).$$

Now, $(\boldsymbol{U},\boldsymbol{V}) \mapsto (\mu\boldsymbol{U},\boldsymbol{T}\boldsymbol{V})_{\boldsymbol{L}^2(\Omega)}$ is a bilinear, continuous and coercive form over $\boldsymbol{L}^2(\Omega) \times \boldsymbol{L}^2(\Omega)$, and $\boldsymbol{V} \mapsto (\boldsymbol{G}+\nabla u,\boldsymbol{T}\boldsymbol{V})_{\boldsymbol{L}^2(\Omega)}$ is a linear and continuous form over $\boldsymbol{L}^2(\Omega)$. According to Lax-Milgram theorem, there exists one, and only one, solution $\boldsymbol{U} \in \boldsymbol{L}^2(\Omega)$ to the above variational formulation, set in $\boldsymbol{L}^2(\Omega)$. Also, $\mu\boldsymbol{U} = \boldsymbol{G} + \nabla u$

in $\boldsymbol{L}^2(\Omega)$.
Second, consider that $\boldsymbol{V} = \mathbf{0}$ in (14). We have that $(u, \boldsymbol{U})$ is governed by

$$(\boldsymbol{U}, \nabla(Tv))_{\boldsymbol{L}^2(\Omega)} = -\langle f, Tv\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} \ \forall v \in H^1_0(\Omega).$$

Replacing $\boldsymbol{U}$ by $\boldsymbol{U} = \mu^{-1}(\boldsymbol{G} + \nabla u)$, we find that

$$b_1(u, Tv) = -\langle f, Tv\rangle_{H^{-1}(\Omega), H^1_0(\Omega)} - (\mu^{-1}\boldsymbol{G}, \nabla(Tv))_{\boldsymbol{L}^2(\Omega)} \ \forall v \in H^1_0(\Omega).$$

By assumption (the suitable conditions are met), b_1 is T-coercive, so this variational formulation is well-posed in $H^1_0(\Omega)$: it has one, and only one solution, u, and also $\operatorname{div}(\mu^{-1}\nabla u) = f - \operatorname{div}(\mu^{-1}\boldsymbol{G})$ in $H^{-1}(\Omega)$.
Last, taking $u \in H^1_0(\Omega)$ characterized by $\operatorname{div}(\mu^{-1}\nabla u) = f - \operatorname{div}(\mu^{-1}\boldsymbol{G})$ in $H^{-1}(\Omega)$ (which is possible according to the T-coercivity of b_1), and then defining $\boldsymbol{U} = \mu^{-1}(\boldsymbol{G}+ \nabla u)$ that belongs to $\boldsymbol{L}^2(\Omega)$, it is straightforward to check that $(u, \boldsymbol{U})$ solves (14).
There remains to prove the uniqueness of a solution to (14).
For that, let $(u, \boldsymbol{U})$ be governed by (14) with zero right-hand side. Retracing our steps, we find as previously that $\mu\boldsymbol{U} = \nabla u$ in $\boldsymbol{L}^2(\Omega)$, and then that $u \in H^1_0(\Omega)$ is characterized by $b_1(u, Tv) = 0$ for all $v \in H^1_0(\Omega)$. Since the suitable conditions are met, we have that $u = 0$, and it follows that $(u, \boldsymbol{U}) = (0, \mathbf{0})$.

□

Finally we define the bilinear form on $V_2 \times V_2$

$$a_2((u, \boldsymbol{U}), (v, \boldsymbol{V})) = b_2((u, \boldsymbol{U}), (v, \boldsymbol{V})) - \omega^2(\varepsilon u, v)_{L^2(\Omega)}. \tag{15}$$

We remark that (3) can be recast as, for a given $f \in L^2(\Omega)$,

$$\begin{array}{l} \text{find } (u, \boldsymbol{U}) \in V_2 \text{ such that} \\ a_2((u, \boldsymbol{U}), (v, \boldsymbol{V})) = -(f, v)_{L^2(\Omega)} \quad \forall (v, \boldsymbol{V}) \in V_2. \end{array} \tag{16}$$

Using the abstract proposition 2.4 with (H1'), we conclude as below on the well-posedness of the two-unknown problem (3). Indeed, (H1') has been proven in Theorem 3.2.

Corollary 3.3 ***(Fredholm framework)*** *Assume that the T-coercivity is true for the form b_1 of the one-unknown problem. Then the two-unknown problem (3) is well-posed if, and only if, the uniqueness principle of the solution to (3) holds, i.e. $f = 0 \Longrightarrow (u, \boldsymbol{U}) = 0$.*

We assume from now on that the uniqueness principle of the solution to (3) holds.

To conclude this section, let us recall briefly the *suitable conditions* that allow one to prove the T-coercivity of the bilinear form b_1. We follow here [2, 1] and references therein. Basically, they write in the general case

$$\frac{\inf_{\Omega_1} \mu_1}{\sup_{\Omega_2} |\mu_2|} > \check{I}_\Sigma \quad \text{or} \quad \frac{\inf_{\Omega_2} |\mu_2|}{\sup_{\Omega_1} \mu_1} > \hat{I}_\Sigma\,, \tag{17}$$

with $\hat{I}_\Sigma, \check{I}_\Sigma \geq 1$. In addition, these numbers $\hat{I}_\Sigma$, $\check{I}_\Sigma$ depend critically on the geometry of the interface. For instance, if Ω_1 and Ω_2 can be mapped from one to the other with the help of a reflection symmetry, then $\hat{I}_\Sigma = \check{I}_\Sigma = 1$. If Σ is only piecewise smooth, then $\hat{I}_\Sigma > 1$ or $\check{I}_\Sigma > 1$. Finally, the conditions (17) can be refined, to include only *local* suprema near the interface Σ (see [1] for details).
In the piecewise-constant case, the conditions write equivalently

$$\kappa_\mu \in]-\infty, -\hat{I}_\Sigma[\cup]-\frac{1}{\check{I}_\Sigma}, 0[\,. \tag{18}$$

Remark 3.4 *In [1], sufficient conditions are also proven for* $\mathtt{B}_1$ *to be Fredholm of index* 0*. In this case, it can happen that Ker* $\mathtt{B}_1 \neq \{0\}$.

4 Discontinuous Galerkin discretization

Following Chung and Engquist [6, 7], we first define the initial triangulation $\mathcal{T}_u$. Suppose the domain Ω is triangulated by a set of tetrahedra in 3D (or triangles in 2D, segments in 1D). We use the notation $\mathcal{F}_u$ to denote the set of all faces in this triangulation and use the notation $\mathcal{F}_u^0$ to denote the subset of all interior faces – that is faces that are not embedded in $\partial\Omega$ – in $\mathcal{F}_u$. For each tetrahedron, we take an interior point ν and call this tetrahedron $\mathcal{S}(\nu)$. Using the point ν, we can further subdivide each tetrahedron into 4 sub-tetrahedra by connecting the point ν to the 4 vertices of the tetrahedron. We denote by $\mathcal{T}$ the triangulation made up of all sub-tetrahedra. We use the notation $\mathcal{F}_p$ to denote all new faces obtained by the subdivision of tetrahedra, and we let $\mathcal{F} = \mathcal{F}_u \cup \mathcal{F}_p$, respectively $\mathcal{F}^0 = \mathcal{F}_u^0 \cup \mathcal{F}_p$. For each face $\kappa \in \mathcal{F}_u$, we let $\mathcal{R}(\kappa)$ be the union of the two sub-tetrahedra sharing the face κ. If κ is a boundary face, we let $\mathcal{R}(\kappa)$ be the only tetrahedron having the face κ. For an illustration in 2D, see Figure 1.

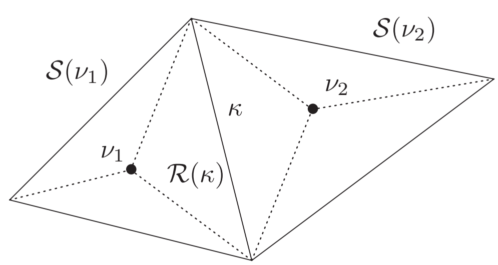

Figure 1: Triangulation in 2D.

We will also define a unit normal vector $\boldsymbol{n}_\kappa$ on each face κ in $\mathcal{F}$ by the following way. If $\kappa \in \mathcal{F} \backslash \mathcal{F}^0$, then we define $\boldsymbol{n}_\kappa$ as the unit normal vector of κ pointing outside of Ω. If $\kappa \in \mathcal{F}^0$ is an interior face, then we fix $\boldsymbol{n}_\kappa$ as one of the two possible unit normal vectors on κ. When it is clear which face we are considering, we will use $\boldsymbol{n}$ instead of $\boldsymbol{n}_\kappa$ to simplify the notations.

Now, we will discuss the finite element spaces. Let $k \geq 0$ be a non-negative integer. Let $\tau \in \mathcal{T}$. We define $P^k(\tau)$ as the space of polynomials of degree less than or equal to k on τ. Then we introduce the following discrete space for scalar fields.

4.1 Locally $H^1(\Omega)$-conforming finite element space for scalar fields

$$\mathcal{S}_h = \{v \mid v_{|\tau} \in P^k(\tau),\ \forall \tau \in \mathcal{T};\ v \text{ is continuous on } \kappa \in \mathcal{F}_u^0;\ v_{|\partial\Omega} = 0\}. \tag{19}$$

In the space $\mathcal{S}_h$ we define the following norms

$$\|v\|_X^2 = \int_\Omega v^2\, dx + \sum_{\kappa \in \mathcal{F}_u^0} h_\kappa \int_\kappa v^2\, d\sigma, \tag{20}$$

$$\|v\|_Z^2 = \int_{\Omega^h} |\nabla v|^2\, dx + \sum_{\kappa \in \mathcal{F}_p} h_\kappa^{-1} \int_\kappa [v]^2\, d\sigma \tag{21}$$

where we remark that the integral of ∇v in (21) is defined elementwise:

$$\int_{\Omega^h} |\nabla v|^2\, dx = \sum_{\tau \in \mathcal{T}} \int_\tau |\nabla (v_{|\tau})|^2\, dx.$$

Here we recall that, by definition, $v \in \mathcal{S}_h$ is always continuous on each face κ in the set $\mathcal{F}_u^0$, whereas it can be discontinuous on each face κ in the set $\mathcal{F}_p$. We say $\|v\|_X$ is the discrete L^2-norm of v and $\|v\|_Z$ is the discrete H^1-norm of v. In the above definition, the jump $[v]$ is defined in the following way. For each $\kappa \in \mathcal{F}_p$, there exist two (sub-)tetrahedra τ_1 and τ_2 such that κ is a common face of them. Moreover, each τ_i, $i = 1, 2$, has a face κ_i that belongs to $\mathcal{F}_u$. Thus, $\kappa \subset \partial\mathcal{R}(\kappa_i)$ for $i = 1, 2$. Then for such $\kappa \in \mathcal{F}_p$, we write $\boldsymbol{m}_i$ as the outward unit normal vector of $\partial\mathcal{R}(\kappa_i)$ for $i = 1, 2$, and define

$$\delta_\kappa^{(i)} = \begin{cases} 1 & \text{if } \boldsymbol{m}_i = \boldsymbol{n} \text{ on } \kappa \\ -1 & \text{if } \boldsymbol{m}_i = -\boldsymbol{n} \text{ on } \kappa \end{cases}$$

where $\boldsymbol{n}$ is the unit normal vector of the face κ. Then the jump $[v]$ on the face κ is defined as

$$[v] = \delta_\kappa^{(1)} v_1 + \delta_\kappa^{(2)} v_2$$

where $v_i = v_{|\tau_i}$.
Note that one can prove, by the argument used in the proof of Theorem 3.1 of [7], that there exists a constant $\alpha > 0$, independent of h, such that

$$\|v\|_{L^2(\Omega)}^2 \leq \|v\|_X^2 \leq \alpha \|v\|_{L^2(\Omega)}^2 \quad \forall v \in \mathcal{S}_h.$$

4.2 Locally $\boldsymbol{H}(\mathrm{div}\,;\Omega)$-conforming finite element space for vector fields

Now, we introduce the following discrete space for vector fields.

$$\mathcal{V}_h = \{\boldsymbol{V} \mid \boldsymbol{V}_{|\tau} \in P^k(\tau)^3,\ \forall \tau \in \mathcal{T};\ \boldsymbol{V} \cdot \boldsymbol{n} \text{ is continuous on } \kappa \in \mathcal{F}_p\}. \tag{22}$$

In the space $\mathcal{V}_h$, we define the following norms

$$\|\boldsymbol{V}\|_{\boldsymbol{X}'}^2 = \int_\Omega |\boldsymbol{V}|^2 \, dx + \sum_{\kappa \in \mathcal{F}_p} h_\kappa \int_\kappa (\boldsymbol{V} \cdot \boldsymbol{n})^2 \, d\sigma, \tag{23}$$

$$\|\boldsymbol{V}\|_{\boldsymbol{Z}'}^2 = \int_{\Omega^h} (\operatorname{div} \boldsymbol{V})^2 \, dx + \sum_{\kappa \in \mathcal{F}_u^0} h_\kappa^{-1} \int_\kappa [\boldsymbol{V} \cdot \boldsymbol{n}]^2 \, d\sigma \tag{24}$$

where we remark again that the integral of $\operatorname{div} \boldsymbol{V}$ in (24) is defined elementwise. Here we recall that, by definition, $\boldsymbol{V} \in \mathcal{V}_h$ has continuous normal component on each face $\kappa \in \mathcal{F}_p$. We say $\|\boldsymbol{V}\|_{\boldsymbol{X}'}$ is the discrete $\boldsymbol{L}^2$-norm of $\boldsymbol{V}$ and $\|\boldsymbol{V}\|_{\boldsymbol{Z}'}$ is the discrete $\boldsymbol{H}(\operatorname{div}; \Omega)$-norm of $\boldsymbol{V}$. In the above definition, the jump $[\boldsymbol{V} \cdot \boldsymbol{n}]$ is defined in the following way. Let $\kappa \subset \mathcal{F}_u^0$. Then there are exactly two tetrahedra τ_1 and τ_2 such that κ is a common face of them. Let ν_i be an interior node of τ_i. Then we have $\kappa \in \partial\mathcal{S}(\nu_i)$ for $i = 1, 2$. Let $\boldsymbol{m}_i$ be the outward unit normal vector of $\partial\mathcal{S}(\nu_i)$. We define

$$\delta_\kappa^{(i)} = \begin{cases} 1 & \text{if } \boldsymbol{m}_i = \boldsymbol{n} \text{ on } \kappa \\ -1 & \text{if } \boldsymbol{m}_i = -\boldsymbol{n} \text{ on } \kappa \end{cases}$$

where $\boldsymbol{n}$ is the unit normal vector of the face κ. Then the jump $[\boldsymbol{V} \cdot \boldsymbol{n}]$ on the face κ is defined as

$$[\boldsymbol{V} \cdot \boldsymbol{n}] = \delta_\kappa^{(1)} \boldsymbol{V}_1 \cdot \boldsymbol{n} + \delta_\kappa^{(2)} \boldsymbol{V}_2 \cdot \boldsymbol{n},$$

where $\boldsymbol{V}_i = \boldsymbol{V}_{|\tau_i}$.

One can prove, by the argument used in the proof of Theorem 3.2 of [7], that there exists a constant $\beta > 0$, independent of h, such that

$$\|\boldsymbol{V}\|_{\boldsymbol{L}^2(\Omega)}^2 \leq \|\boldsymbol{V}\|_{\boldsymbol{X}'}^2 \leq \beta \, \|\boldsymbol{V}\|_{\boldsymbol{L}^2(\Omega)}^2 \quad \forall \boldsymbol{V} \in \mathcal{V}_h. \tag{25}$$

We define

$$B_h(\boldsymbol{V}, v) = \int_{\Omega^h} \boldsymbol{V} \cdot \nabla v \, dx - \sum_{\kappa \in \mathcal{F}_p} \int_\kappa \boldsymbol{V} \cdot \boldsymbol{n} \, [v] \, d\sigma, \ \boldsymbol{V} \in \mathcal{V}_h, v \in \mathcal{S}_h \tag{26}$$

$$B_h^*(v, \boldsymbol{V}) = -\int_{\Omega^h} v \operatorname{div} \boldsymbol{V} \, dx + \sum_{\kappa \in \mathcal{F}_u^0} \int_\kappa v \, [\boldsymbol{V} \cdot \boldsymbol{n}] \, d\sigma, \ v \in \mathcal{S}_h, \boldsymbol{V} \in \mathcal{V}_h. \tag{27}$$

Remark 4.1 *A natural question to ask is: can we use those forms with respectively $v = u \in H_0^1(\Omega)$ and $\boldsymbol{V} \in \mathcal{V}_h$ in (27), and $\boldsymbol{V} = \boldsymbol{U} \in \boldsymbol{H}(\operatorname{div}; \Omega)$ and $v \in \mathcal{S}_h$ in (26)? On the one hand, given $v \in H_0^1(\Omega)$ and $\boldsymbol{V} \in \mathcal{V}_h$, $B_h^*(v, \boldsymbol{V})$ can be defined by (27): indeed, over faces $\kappa \in \mathcal{F}_u^0$, both v and $[\boldsymbol{V} \cdot \boldsymbol{n}]$ belong to $L^2(\kappa)$.*
On the other hand, given $v \in \mathcal{S}_h$ and $\boldsymbol{V} \in \boldsymbol{H}(\operatorname{div}; \Omega)$, then $B_h(\boldsymbol{V}, v)$ cannot be defined as in (26). As a matter of fact, integrals over faces $\kappa \in \mathcal{F}_p$ must be understood as duality brackets, but one has only $[v] \in H^{\frac{1}{2}}(\kappa)$, whereas there is no guarantee that $(\boldsymbol{V} \cdot \boldsymbol{n})$ belongs to its dual space $(H^{\frac{1}{2}}(\kappa))'$. Nevertheless, we remark that one can consider the alternate definition below:

$$B_h(\boldsymbol{V}, v) = -\int_\Omega v \operatorname{div} \boldsymbol{V} \, dx, \quad \boldsymbol{V} \in \boldsymbol{H}(\operatorname{div}; \Omega), v \in \mathcal{S}_h. \tag{28}$$

Indeed, if $\boldsymbol{V}$ belongs to $\mathcal{V}_h \cap \boldsymbol{H}(\mathrm{div}\,;\Omega)$, we can integrate by parts, element by element, to find:

$$\begin{aligned}
-\int_\Omega v \,\mathrm{div}\, \boldsymbol{V}\, dx &= -\sum_{\tau\in\mathcal{T}} \int_\tau v\,\mathrm{div}\,\boldsymbol{V}\, dx \\
&= \sum_{\tau\in\mathcal{T}} \left\{ \int_\tau \boldsymbol{V}\cdot\nabla v\, dx - \int_{\partial\tau} v\,(\boldsymbol{V}\cdot\boldsymbol{n}_{|\partial\tau})\, d\sigma \right\} \\
&= \int_{\Omega^h} \boldsymbol{V}\cdot\nabla v\, dx - \sum_{\kappa\in\mathcal{F}_p} \int_\kappa [v]\,(\boldsymbol{V}\cdot\boldsymbol{n})\, d\sigma.
\end{aligned}$$

To go from the first to the second line, we used on the one hand the fact that $v \in \mathcal{S}_h$ being continuous across faces of $\mathcal{F}_u^0$, there is no contribution on those faces. Then, to compute the contribution on the remaining faces (i.e. those of $\mathcal{F}_p$), we used the coupled definitions of the unit normal vectors and jumps on those faces (see §4.1). As a result, we recover the original definition of $B_h(\boldsymbol{V}, v)$, that is (26): hence, the two definitions are consistent.

According to Lemma 2.4 of [7], we have

$$B_h(\boldsymbol{V}, v) = B_h^*(v, \boldsymbol{V}), \quad \forall (v, \boldsymbol{V}) \in \mathcal{S}_h \times \mathcal{V}_h. \tag{29}$$

Moreover, the following holds

$$B_h(\boldsymbol{V}, v) \le \|v\|_Z \,\|\boldsymbol{V}\|_{\boldsymbol{X}'}, \quad \forall (v, \boldsymbol{V}) \in \mathcal{S}_h \times \mathcal{V}_h. \tag{30}$$

We say that the discrete fields $(v, \boldsymbol{V}) \in \mathcal{S}_h \times \mathcal{V}_h$ are *aligned* if they satisfy

$$(\mu\boldsymbol{V}, \boldsymbol{W})_{\boldsymbol{L}^2(\Omega)} - B_h^*(v, \boldsymbol{W}) = 0 \quad \forall\, \boldsymbol{W} \in \mathcal{V}_h. \tag{31}$$

Accordingly, let us introduce the subspace of aligned fields

$$\mathtt{A}_h = \{(v, \boldsymbol{V}) \in \mathcal{S}_h \times \mathcal{V}_h \mid (v, \boldsymbol{V}) \text{ satisfy } (31)\}.$$

The discrete variational formulation, or numerical method, is

$$\begin{aligned}
&\text{find } (u_h, \boldsymbol{U}_h) \in \mathcal{S}_h \times \mathcal{V}_h \text{ such that} \\
&(\mu\boldsymbol{U}_h, \boldsymbol{V})_{\boldsymbol{L}^2(\Omega)} - B_h^*(u_h, \boldsymbol{V}) = 0, \quad \forall\, \boldsymbol{V} \in \mathcal{V}_h \\
&B_h(\boldsymbol{U}_h, v) - \omega^2(\varepsilon\, u_h, v)_{L^2(\Omega)} = -(f, v)_{L^2(\Omega)}, \quad \forall\, v \in \mathcal{S}_h.
\end{aligned} \tag{32}$$

In particular, the discrete solutions $(u_h, \boldsymbol{U}_h)$ are aligned: $(u_h, \boldsymbol{U}_h) \in \mathtt{A}_h$.
For our subsequent analysis, we finally define

$$\begin{aligned}
&b_h\Big((u, \boldsymbol{U}),\ (v, \boldsymbol{V})\Big) = B_h(\boldsymbol{U}, v) + (\mu\boldsymbol{U}, \boldsymbol{V})_{\boldsymbol{L}^2(\Omega)} - B_h^*(u, \boldsymbol{V}), \\
&a_h\Big((u, \boldsymbol{U}),\ (v, \boldsymbol{V})\Big) = b_h\Big((u, \boldsymbol{U}),\ (v, \boldsymbol{V})\Big) - \omega^2(\varepsilon\, u, v)_{L^2(\Omega)}, \\
&\ell_h(v) = -(f, v)_{L^2(\Omega)}.
\end{aligned}$$

5 Convergence theory for the two-unknown problem

Here, we choose a conforming triangulation $\mathcal{T}_u$, in the sense that the interface Σ is a union of faces: in other words, $\mathring{\tau} \cap \Sigma = \emptyset$, for all tetrahedra $\tau \in \mathcal{T}_u$. Obviously, this is possible as soon as the interface Σ is *piecewise plane*. In this manner, one can split $\mathcal{T}$ (resp. $\mathcal{F}_p$, etc.) as $\mathcal{T} = \mathcal{T}^{(1)} \cup \mathcal{T}^{(2)}$ (resp. $\mathcal{F}_p = \mathcal{F}_p^{(1)} \cup \mathcal{F}_p^{(2)}$, etc.), with $\mathcal{T}^{(\ell)}$ made up of tetrahedra (resp. faces, etc.) embedded in Ω_ℓ, for $\ell = 1, 2$. It follows that one can consider the discrete spaces over Ω_1 and Ω_2 respectively. For $\ell = 1, 2$:

$$\mathcal{S}_h^{(\ell)} = \{v \mid v_{|\tau} \in P^k(\tau),\ \forall\tau \in \mathcal{T}^{(\ell)};\ v \text{ is continuous on } \kappa \in \mathcal{F}_u^{0,(\ell)};\ v_{|\Gamma_\ell} = 0\},$$
$$\mathcal{V}_h^{(\ell)} = \{\boldsymbol{V} \mid \boldsymbol{V}_{|\tau} \in P^k(\tau)^3,\ \forall\tau \in \mathcal{T}^{(\ell)};\ \boldsymbol{V}\cdot\boldsymbol{n} \text{ is continuous on } \kappa \in \mathcal{F}_p^{(\ell)}\}.$$

One can define the norms $\|\cdot\|_{X_\ell}$, $\|\cdot\|_{Z_\ell}$, $\|\cdot\|_{\boldsymbol{X}'_\ell}$, $\|\cdot\|_{\boldsymbol{Z}'_\ell}$, for $\ell = 1, 2$.
Finally, we note that given any $\boldsymbol{V}_\ell$ in $\mathcal{V}_h^{(\ell)}$, the discrete field defined by

$$\boldsymbol{V}_\ell^{ext} = \begin{cases} 0 & \text{in } \Omega\setminus\overline{\Omega_\ell} \\ \boldsymbol{V}_\ell & \text{in } \Omega_\ell \end{cases},$$

automatically belong to $\mathcal{V}_h$.

To simplify the proofs[1], we assume that μ is *piecewise constant*, namely $\mu_\ell = \mu_{|\Omega_\ell}$ is constant, for $\ell = 1, 2$. In particular, the relevant quantities to ensure well-posedness are the absolute value of either the contrast κ_μ or its inverse $1/\kappa_\mu$, namely $|\mu_2|/\mu_1$ or $\mu_1/|\mu_2|$.

5.1 Inf-sup conditions and measures for aligned fields

From Theorem 3.2 of [7], we know that there is a uniform constant $K > 0$ such that the global inf-sup condition below holds:

$$\inf_{v\in\mathcal{S}_h}\ \sup_{\boldsymbol{V}\in\mathcal{V}_h} \frac{B_h(\boldsymbol{V}, v)}{\|\boldsymbol{V}\|_{\boldsymbol{X}'}\ \|v\|_Z} \geq K. \tag{33}$$

Furthermore, by using a similar proof, one can prove the following localized inf-sup condition on Ω_ℓ, $\ell = 1, 2$, with $K_\ell > 0$ independent of h:

$$\inf_{v\in\mathcal{S}_h^{(\ell)}}\ \sup_{\boldsymbol{V}\in\mathcal{V}_h^{(\ell)}} \frac{B_h(\boldsymbol{V}, v)}{\|\boldsymbol{V}\|_{\boldsymbol{X}'_\ell}\ \|v\|_{Z_\ell}} \geq K_\ell. \tag{34}$$

Consider next *aligned* fields $(u, \boldsymbol{U}) \in \mathtt{A}_h$: we infer the global measure

$$\begin{aligned} \|\mu\boldsymbol{U}\|_{\boldsymbol{L}^2(\Omega)} &\geq \sup_{\boldsymbol{V}\in\mathcal{V}_h} \frac{(\mu\boldsymbol{U}, \boldsymbol{V})_{\boldsymbol{L}^2(\Omega)}}{\|\boldsymbol{V}\|_{\boldsymbol{L}^2(\Omega)}} \\ &\geq \sup_{\boldsymbol{V}\in\mathcal{V}_h} \frac{(\mu\boldsymbol{U}, \boldsymbol{V})_{\boldsymbol{L}^2(\Omega)}}{\|\boldsymbol{V}\|_{\boldsymbol{X}'}} = \sup_{\boldsymbol{V}\in\mathcal{V}_h} \frac{B_h(\boldsymbol{V}, u)}{\|\boldsymbol{V}\|_{\boldsymbol{X}'}} \geq K\|u\|_Z. \end{aligned} \tag{35}$$

[1]To remove this last assumption, the proof of Proposition 5.1 after (42) has to be modified, in the spirit of the results of [1].

In addition, we can find local measures (for $\ell = 1, 2$). Indeed, as the extension of discrete fields of $\mathcal{V}_h^{(\ell)}$ by 0 automatically belongs to $\mathcal{V}_h$, we have

$$\begin{aligned}
\|\mu \boldsymbol{U}_\ell\|_{\boldsymbol{L}^2(\Omega_\ell)} &\geq \sup_{\boldsymbol{V}_\ell \in \mathcal{V}_h^{(\ell)}} \frac{(\mu \boldsymbol{U}_\ell, \boldsymbol{V}_\ell)_{\boldsymbol{L}^2(\Omega_\ell)}}{\|\boldsymbol{V}_\ell\|_{\boldsymbol{L}^2(\Omega_\ell)}} = \sup_{\boldsymbol{V}_\ell \in \mathcal{V}_h^{(\ell)}} \frac{(\mu \boldsymbol{U}, \boldsymbol{V}_\ell^{ext})_{\boldsymbol{L}^2(\Omega)}}{\|\boldsymbol{V}_\ell^{ext}\|_{\boldsymbol{L}^2(\Omega)}} \\
&\overset{(31)}{=} \sup_{\boldsymbol{V}_\ell \in \mathcal{V}_h^{(\ell)}} \frac{B_h^*(u, \boldsymbol{V}_\ell^{ext})_{\boldsymbol{L}^2(\Omega)}}{\|\boldsymbol{V}_\ell^{ext}\|_{\boldsymbol{L}^2(\Omega)}} \overset{(29)}{=} \sup_{\boldsymbol{V}_\ell \in \mathcal{V}_h^{(\ell)}} \frac{B_h(\boldsymbol{V}_\ell^{ext}, u)_{\boldsymbol{L}^2(\Omega)}}{\|\boldsymbol{V}_\ell^{ext}\|_{\boldsymbol{L}^2(\Omega)}}.
\end{aligned}$$

Going back to the definition (26) of the form B_h, we find

$$\begin{aligned}
B_h(\boldsymbol{V}_\ell^{ext}, u) &= \int_{\Omega^h} \boldsymbol{V}_\ell^{ext} \cdot \nabla u \, dx - \sum_{\kappa \in \mathcal{F}_p} \int_\kappa \boldsymbol{V}_\ell^{ext} \cdot \boldsymbol{n} \, [u] \, d\sigma \\
&= \int_{\Omega_\ell^h} \boldsymbol{V}_\ell \cdot \nabla u_\ell \, dx - \sum_{\kappa \in \mathcal{F}_p^{(\ell)}} \int_\kappa \boldsymbol{V}_\ell \cdot \boldsymbol{n} \, [u_\ell] \, d\sigma = B_h(\boldsymbol{V}_\ell, u_\ell).
\end{aligned}$$

But u_ℓ belongs to $\mathcal{S}_h^{(\ell)}$. Using (34), we conclude that, for *aligned* fields and $\ell = 1, 2$,

$$\|\mu \boldsymbol{U}_\ell\|_{\boldsymbol{L}^2(\Omega_\ell)} \geq K_\ell \|u_\ell\|_{Z_\ell}. \tag{36}$$

5.2 Uniform discrete T-coercivity for aligned fields

We have already defined the exact operator $\mathbb{T}$ over $H_0^1(\Omega) \times \boldsymbol{L}^2(\Omega)$ that ensures well-posedness, provided the absolute value of κ_μ or $1/\kappa_\mu$ is large enough. Let us now introduce the discrete operator $\mathbb{T}_h$ over $\mathcal{S}_h \times \mathcal{V}_h$. Given $(u, \boldsymbol{U}) \in \mathcal{S}_h \times \mathcal{V}_h$, let $\mathbb{T}_h(u, \boldsymbol{U}) = (\tilde{u}, \tilde{\boldsymbol{U}})$ be defined by

$$\tilde{u} = \begin{cases} u_1 & \text{in } \Omega_1 \\ -u_2 + 2R_h u_1 & \text{in } \Omega_2 \end{cases} \quad \text{and} \quad \tilde{\boldsymbol{U}} = \boldsymbol{T}\boldsymbol{U} = \begin{cases} \boldsymbol{U}_1 & \text{in } \Omega_1 \\ -\boldsymbol{U}_2 & \text{in } \Omega_2 \end{cases}, \tag{37}$$

where R_h is a discrete operator from $\mathcal{S}_h^{(1)}$ to $\mathcal{S}_h^{(2)}$, such that one has the *compatibility condition* $(R_h u_1)_{|\Sigma} = (u_1)_{|\Sigma}$ for all $u_1 \in \mathcal{S}_h^{(1)}$. We introduce

$$\|R_h\| = \sup_{u_1 \in \mathcal{S}_h^{(1)}} \frac{\|R_h u_1\|_{Z_2}}{\|u_1\|_{Z_1}}.$$

The roles of Ω_1 and Ω_2 can be reversed, meaning that one can define $(\tilde{u}, \tilde{\boldsymbol{U}})$ by

$$\tilde{u} = \begin{cases} u_1 - 2R'_h u_2 & \text{in } \Omega_1 \\ -u_2 & \text{in } \Omega_2 \end{cases} \quad \text{and} \quad \tilde{\boldsymbol{U}} = \boldsymbol{T}\boldsymbol{U},$$

where R'_h is a discrete operator from $\mathcal{S}_h^{(2)}$ to $\mathcal{S}_h^{(1)}$, with the *compatibility condition* $(R'_h u_2)_{|\Sigma} = (u_2)_{|\Sigma}$ for all $u_2 \in \mathcal{S}_h^{(2)}$ ($\|R'_h\| = \sup_{u_2 \in \mathcal{S}_h^{(2)}} \|R'_h u_2\|_{Z_1}/\|u_2\|_{Z_2}$).

Let us define the norm on $(\mathcal{S}_h + H_0^1(\Omega)) \times \boldsymbol{L}^2(\Omega)$ by

$$\|(u, \boldsymbol{U})\|_h = \left(\||\mu|^{\frac{1}{2}} \boldsymbol{U}\|_{\boldsymbol{L}^2(\Omega)}^2 + \|u\|_Z^2 \right)^{\frac{1}{2}}, \quad \forall\, u \in \mathcal{S}_h + H_0^1(\Omega),\ \boldsymbol{U} \in \mathcal{V}_h. \tag{38}$$

Proposition 5.1 *Suppose that the discrete operators* $(R_h)_h$ *and* $(R'_h)_h$ *are such that*

$$\exists h_\star > 0, \quad \max_{h\in]0,h_\star[} \|R_h\|^2 < \frac{K_1}{\beta}\,\frac{|\mu_2|}{\mu_1} \quad \textit{or} \quad \max_{h\in]0,h_\star[} \|R'_h\|^2 < \frac{K_2}{\beta}\,\frac{\mu_1}{|\mu_2|}\,. \tag{39}$$

Then, one has the uniform discrete T-coercivity of $(b_h)_h$ *for* aligned fields*:*

$$\exists\gamma > 0,\ \forall h \in]0, h_\star[,\ \forall (u, \boldsymbol{U}) \in \mathtt{A}_h, \quad b_h\Big((u,\boldsymbol{U}), \mathbb{T}_h(u,\boldsymbol{U})\Big) \geq \gamma \|(u,\boldsymbol{U})\|_h^2\,. \tag{40}$$

Proof. To fix ideas, we consider that the condition (39) holds for the operators $(R_h)_h$. In (37), we note that $\tilde{u}$, $\tilde{\boldsymbol{U}}$ can be split as $\tilde{u} = u + u'$ and $\tilde{\boldsymbol{U}} = \boldsymbol{U} + \boldsymbol{U}'$ where

$$u' = \begin{cases} 0 & \text{in } \ \Omega_1 \\ -2u_2 + 2R_h u_1 & \text{in } \ \Omega_2 \end{cases} \quad \text{and} \quad \boldsymbol{U}' = \begin{cases} 0 & \text{in } \ \Omega_1 \\ -2\boldsymbol{U}_2 & \text{in } \ \Omega_2 \end{cases}. \tag{41}$$

Then we have by construction

$$\begin{aligned} & b_h\Big((u,\boldsymbol{U}), \mathbb{T}_h(u,\boldsymbol{U})\Big) \\ &= B_h(\boldsymbol{U}, u) + (\mu\boldsymbol{U}, \tilde{\boldsymbol{U}})_{\boldsymbol{L}^2(\Omega)} - B_h^*(u, \boldsymbol{U}) + B_h(\boldsymbol{U}, u') - B_h^*(u, \boldsymbol{U}') \\ &= (|\mu|\boldsymbol{U}, \boldsymbol{U})_{\boldsymbol{L}^2(\Omega)} + B_h(\boldsymbol{U}, u') - B_h^*(u, \boldsymbol{U}') \end{aligned}$$

On the other hand, due to the conforming assumption on the triangulation $\mathcal{T}_u$, it follows that $\boldsymbol{U}'$ belongs to $\mathcal{V}_h$. Indeed, according to (22), the matching of the normal component is enforced on faces κ of $\mathcal{F}_p$ only, but each of those faces is embedded either in Ω_1 or Ω_2, so no matching condition is required on the interface Σ. Therefore, $B_h^*(u, \boldsymbol{U}') = B_h(\boldsymbol{U}', u)$, and we have

$$B_h(\boldsymbol{U}, u') - B_h^*(u, \boldsymbol{U}') = B_h(\boldsymbol{U}, u') - B_h(\boldsymbol{U}', u).$$

By the definition of u' and $\boldsymbol{U}'$, we can further write

$$\begin{aligned} & B_h(\boldsymbol{U}, u') - B_h(\boldsymbol{U}', u) \\ &= \int_{\Omega^h} \boldsymbol{U}\cdot\nabla u'\,dx - \sum_{\kappa\in\mathcal{F}_p}\int_\kappa \boldsymbol{U}\cdot\boldsymbol{n}\,[u']\,d\sigma - \int_{\Omega^h} \boldsymbol{U}'\cdot\nabla u\,dx + \sum_{\kappa\in\mathcal{F}_p}\int_\kappa \boldsymbol{U}'\cdot\boldsymbol{n}\,[u]\,d\sigma \\ &= 2\int_{\Omega_2^h} \boldsymbol{U}_2\cdot\nabla(R_h u_1)\,dx - 2\sum_{\kappa\in\mathcal{F}_p^{(2)}}\int_\kappa \boldsymbol{U}_2\cdot\boldsymbol{n}\,[R_h u_1]\,d\sigma. \end{aligned}$$

Combining the previous results, we get

$$\begin{aligned} b_h\Big((u,\boldsymbol{U}), \mathbb{T}_h(u,\boldsymbol{U})\Big) &= (|\mu|\boldsymbol{U}, \boldsymbol{U})_{\boldsymbol{L}^2(\Omega)} \\ & \quad +2\int_{\Omega_2^h} \boldsymbol{U}_2\cdot\nabla(R_h u_1)\,dx - 2\sum_{\kappa\in\mathcal{F}_p^{(2)}}\int_\kappa \boldsymbol{U}_2\cdot\boldsymbol{n}\,[R_h u_1]\,d\sigma. \end{aligned} \tag{42}$$

By the Cauchy-Schwarz inequality and the definitions of $\boldsymbol{X}'$ and Z norms, we find

$$
\begin{aligned}
& -2\int_{\Omega_2^h} \boldsymbol{U}_2\cdot\nabla(R_h u_1)\,dx + 2\sum_{\kappa\in\mathcal{F}_p^{(2)}}\int_\kappa \boldsymbol{U}_2\cdot\boldsymbol{n}\,[R_h u_1]\,d\sigma \\
\le\ & \frac{\eta}{|\mu_2|}\|R_h u_1\|_{Z_2}^2 + \frac{|\mu_2|}{\eta}\|\boldsymbol{U}_2\|_{\boldsymbol{X}_2'}^2 \\
\le\ & \frac{\eta}{|\mu_2|}\|R_h\|^2\|u_1\|_{Z_1}^2 + \frac{|\mu_2|}{\eta}\|\boldsymbol{U}_2\|_{\boldsymbol{X}_2'}^2, \qquad \forall\,\eta>0.
\end{aligned}
$$

Since the norms $\|\cdot\|_{\boldsymbol{X}'}$ and $\|\cdot\|_{\boldsymbol{L}^2(\Omega)}$ are equivalent (cf. (25)), we have that $\|\boldsymbol{U}_2^{ext}\|_{\boldsymbol{X}'}^2 \le \beta\|\boldsymbol{U}_2^{ext}\|_{\boldsymbol{L}^2(\Omega)}^2$, i.e. $\|\boldsymbol{U}_2\|_{\boldsymbol{X}_2'}^2 \le \beta\|\boldsymbol{U}_2\|_{\boldsymbol{L}^2(\Omega_2)}^2$, and it follows that

$$
\begin{aligned}
& -2\int_{\Omega_2^h} \boldsymbol{U}_2\cdot\nabla(R_h u_1)\,dx + 2\sum_{\kappa\in\mathcal{F}_p^{(2)}}\int_\kappa \boldsymbol{U}_2\cdot\boldsymbol{n}\,[R_h u_1]\,d\sigma \\
\le\ & \frac{\eta}{|\mu_2|}\|R_h\|^2\|u_1\|_{Z_1}^2 + \beta\frac{|\mu_2|}{\eta}\|\boldsymbol{U}_2\|_{\boldsymbol{L}^2(\Omega_2)}^2, \qquad \forall\,\eta>0.
\end{aligned}
$$

Thus, (42) yields

$$
b_h\Big((u,\boldsymbol{U}),\mathbb{T}_h(u,\boldsymbol{U})\Big) \ge (|\mu|\boldsymbol{U},\boldsymbol{U})_{\boldsymbol{L}^2(\Omega)} - \frac{\eta}{|\mu_2|}\|R_h\|^2\|u_1\|_{Z_1}^2 - \beta\frac{|\mu_2|}{\eta}\|\boldsymbol{U}_2\|_{\boldsymbol{L}^2(\Omega_2)}^2.
$$

To obtain uniform T-coercivity, we assume from now on that the discrete fields $(\boldsymbol{U},u)$ are *aligned*: $(u,\boldsymbol{U})\in\mathtt{A}_h$. Using the local measure (36), we have

$$
\begin{aligned}
b_h\Big((u,\boldsymbol{U}),\mathbb{T}_h(u,\boldsymbol{U})\Big) \ \ge\ & (|\mu|\boldsymbol{U},\boldsymbol{U})_{\boldsymbol{L}^2(\Omega)} - \frac{\eta\mu_1}{K_1|\mu_2|}\|R_h\|^2(\mu_1\boldsymbol{U}_1,\boldsymbol{U}_1)_{\boldsymbol{L}^2(\Omega_1)} \\
& -\beta\frac{|\mu_2|}{\eta}\|\boldsymbol{U}_2\|_{\boldsymbol{L}^2(\Omega_2)}^2 \\
\ge\ & \left(1-\frac{\eta\mu_1}{K_1|\mu_2|}\|R_h\|^2\right)(\mu_1\boldsymbol{U}_1,\boldsymbol{U}_1)_{\boldsymbol{L}^2(\Omega_1)} \\
& +\left(1-\frac{\beta}{\eta}\right)(|\mu_2|\boldsymbol{U}_2,\boldsymbol{U}_2)_{\boldsymbol{L}^2(\Omega_2)}. \qquad (43)
\end{aligned}
$$

According to (39), for $h\in]0,h_\star[$, one can choose η such that

$$
\beta<\eta<\frac{K_1|\mu_2|}{\max_h\|R_h\|^2\mu_1}, \text{ i.e. } \min_h\left(1-\frac{\eta\mu_1}{K_1|\mu_2|}\|R_h\|^2\right)>0 \text{ and } \left(1-\frac{\beta}{\eta}\right)>0.
$$

Consider again (43) with this choice of the parameter η. Using finally the global measure (35), we derive the uniform discrete T-coercivity of $(b_h)_h$ for *aligned* fields (40).

□

Remark 5.2 *The result of the previous proposition holds under condition (39) which is independent of the pulsation ω.*

Going back to the definition of the discrete operators $(\mathbb{T}_h)_h$, another straightforward consequence of (39) is that these operators are uniformly bounded for h "small" enough, i.e.

$$\exists\delta > 0,\ \forall h \in]0, h_\star[,\ \sup_{(u,\boldsymbol{U})\in\mathtt{A}_h} \frac{\|\mathbb{T}_h(u,\boldsymbol{U})\|_h}{\|(u,\boldsymbol{U})\|_h} \leq \delta. \tag{44}$$

5.3 Stability for aligned fields

Below, we consider separately the cases $\omega = 0$ and $\omega \neq 0$, which can be solved by two very different approaches. Our aim is to prove the uniform stability of the forms $(a_h)_h$ for *aligned* fields:

$$\exists\sigma > 0,\ \exists h_0 > 0,\ \forall h \in]0, h_0[,\ \forall \mathtt{v}_h \in \mathtt{A}_h,\ \sup_{\mathtt{w}_h\in\mathcal{S}_h\times\mathcal{V}_h} \frac{|a_h(\mathtt{v}_h,\mathtt{w}_h)|}{\|\mathtt{w}_h\|_h} \geq \sigma\|\mathtt{v}_h\|_h\,, \tag{45}$$

under the condition on the contrast (39), so one has necessarily $h_0 < h_\star$. Indeed, it is natural to assume this condition on the contrast, as (40) and (44) are true when this condition is met.

Case $\omega = 0$ In this case, we need to prove

$$\exists\sigma > 0,\ \exists h_0 \in]0, h_\star[,\ \forall h \in]0, h_0[,\ \forall \mathtt{v}_h \in \mathtt{A}_h,\ \sup_{\mathtt{w}_h\in\mathcal{S}_h\times\mathcal{V}_h} \frac{|b_h(\mathtt{v}_h,\mathtt{w}_h)|}{\|\mathtt{w}_h\|_h} \geq \sigma\|\mathtt{v}_h\|_h\,.$$

Let us proceed by contradiction. Namely, we assume that

$$\exists(\mu_q)_{q\in\mathbb{N}},\ \lim_{q\to\infty}\mu_q = 0,\ \exists(h_q)_{q\in\mathbb{N}},\ \lim_{q\to\infty} h_q = 0,$$
$$\forall q\in\mathbb{N},\ \exists\mathtt{v}_{h_q}\in\mathtt{A}_{h_q},\ \sup_{\mathtt{w}_{h_q}\in\mathcal{S}_{h_q}\times\mathcal{V}_{h_q}} \frac{|b_{h_q}(\mathtt{v}_{h_q},\mathtt{w}_{h_q})|}{\|\mathtt{w}_{h_q}\|_{h_q}} \leq \mu_q\,\|\mathtt{v}_{h_q}\|_{h_q}\,.$$

Without loss of generality, we normalize $\mathtt{v}_{h_q}\in\mathtt{A}_{h_q}$ above ($\|\mathtt{v}_{h_q}\|_{h_q} = 1$), for all q. Now, using the uniform discrete T-coercivity of $(b_h)_h$ for aligned fields (40) and the uniform boundedness (44), we have, for all q,

$$\gamma \leq b_{h_q}(\mathtt{v}_{h_q}, \mathbb{T}_{h_q}\mathtt{v}_{h_q}) \leq \mu_q\,\|\mathbb{T}_{h_q}\mathtt{v}_{h_q}\|_{h_q} \leq \delta\,\mu_q.$$

But $\lim_q \mu_q = 0$, which leads to a contradiction, so (45) holds when $\omega = 0$.

Case $\omega \neq 0$ Let us proceed again by contradiction. Namely, we assume that

$$\exists(\mu_q)_{q\in\mathbb{N}},\ \lim_{q\to\infty}\mu_q = 0,\ \exists(h_q)_{q\in\mathbb{N}},\ \lim_{q\to\infty} h_q = 0,$$
$$\forall q\in\mathbb{N},\ \exists\mathtt{v}_{h_q}\in\mathtt{A}_{h_q},\ \|\mathtt{v}_{h_q}\|_{h_q} = 1,\ \sup_{\mathtt{w}_{h_q}\in\mathcal{S}_{h_q}\times\mathcal{V}_{h_q}} \frac{|a_{h_q}(\mathtt{v}_{h_q},\mathtt{w}_{h_q})|}{\|\mathtt{w}_{h_q}\|_{h_q}} \leq \mu_q.$$

Let us write $\mathtt{v}_{h_q} = (v_{h_q}, \boldsymbol{V}_{h_q})$. Then we have $\|v_{h_q}\|_Z \leq 1$ and $\||\mu|^{\frac{1}{2}}\boldsymbol{V}_{h_q}\|_{\boldsymbol{L}^2(\Omega)} \leq 1$, for all q. According to general properties of Discontinuous Galerkin discrete spaces and norms (see Theorem 5.2 and Lemma 8 of [4]), we infer that one can extract a subsequence from $(v_{h_q})_q$ that converges strongly in $L^2(\Omega)$. Namely, if we still denote this subsequence by $(v_{h_q})_q$, there exists $u^* \in L^2(\Omega)$ such that

$$\lim_q \|v_{h_q} - u^*\|_{L^2(\Omega)} = 0\,. \tag{46}$$

Let us assume provisionally that $u^* = 0$, that is $\lim_q \|v_{h_q}\|_{L^2(\Omega)} = 0$. This result will be proved below, see Lemma 5.3. Then, using the uniform discrete T-coercivity of $(b_h)_h$ for aligned fields (40), the uniform boundedness (44) and our assumption on the lack of stability of the forms $(a_h)_h$ (made at the start of the paragraph), we find now, for all q,

$$\begin{aligned}
\gamma &\leq a_{h_q}(\mathrm{v}_{h_q}, \mathbb{T}_{h_q}\mathrm{v}_{h_q}) + \omega^2(\varepsilon v_{h_q}, \tilde{v}_{h_q})_{L^2(\Omega)} \\
&\leq \delta\,\mu_q + \omega^2(\varepsilon v_{h_q}, \tilde{v}_{h_q})_{L^2(\Omega)} \leq \delta\,\mu_q + \omega^2\|\varepsilon\|_{L^\infty(\Omega)}\|v_{h_q}\|_{L^2(\Omega)}\,\|\tilde{v}_{h_q}\|_{L^2(\Omega)}.
\end{aligned}$$

Above, $\tilde{v}_{h_q}$ is defined as in (37). Using the discrete version of Poincaré's inequality[2] in $\mathcal{S}_{h_q}$, we get

$$\gamma \leq \delta\,\mu_q + C_P\,\omega^2\|\varepsilon\|_{L^\infty(\Omega)}\,\delta\,\|v_{h_q}\|_{L^2(\Omega)}.$$

But we have both $\lim_q \mu_q = 0$ and $\lim_q \|v_{h_q}\|_{L^2(\Omega)} = 0$, which leads to a contradiction. So, we conclude that we have the uniform stability of the forms $(a_h)_h$ for aligned fields, that is (45), when $\omega \neq 0$.

Lemma 5.3 *Under the condition on the contrast (18), one has $u^* = 0$ in (46).*

Proof. We remark that the sequence $(\boldsymbol{V}_{h_q})_q$ is bounded in $\boldsymbol{L}^2(\Omega)$, so one can extract a subsequence – still denoted by $(\boldsymbol{V}_{h_q})_q$ – and introduce $\boldsymbol{U}^* \in \boldsymbol{L}^2(\Omega)$ such that $(\boldsymbol{V}_{h_q})_q$ converges *weakly* to $\boldsymbol{U}^*$ in $\boldsymbol{L}^2(\Omega)$:

$$\boldsymbol{V}_{h_q} \rightharpoonup \boldsymbol{U}^* \text{ weakly in } \boldsymbol{L}^2(\Omega)\,. \tag{47}$$

Up to the extraction of another subsequence, we keep the same set of indices q in (46) and (47). From this point on, our aim is to prove that $(u^*, \boldsymbol{U}^*)$ solves the two-unknown problem (3), with $f = 0$. For that, we need to prove that u^* belongs to $H^1_0(\Omega)$. First, we check that ∇u^* belongs to $\boldsymbol{L}^2(\Omega)$, using differentiation in the sense of distributions. So, given $\boldsymbol{Z} \in \mathcal{D}(\Omega)^d$, let us compute $\langle \nabla u^*, \boldsymbol{Z}\rangle$:

$$\begin{aligned}
\langle \nabla u^*, \boldsymbol{Z}\rangle &= -\langle u^*, \operatorname{div}\boldsymbol{Z}\rangle = -\int_\Omega u^* \operatorname{div}\boldsymbol{Z}\,dx \\
&= -\lim_q \int_\Omega v_{h_q} \operatorname{div}\boldsymbol{Z}\,dx = \lim_q B_{h_q}(\boldsymbol{Z}, v_{h_q})\,.
\end{aligned}$$

For the last equality, we refer to Remark 4.1.
According to (3.15) and (3.22) of [7], given $\boldsymbol{Z} \in \boldsymbol{H}^{k+1}(\Omega)$, for all q, there exists $\boldsymbol{Z}_{h_q} \in \mathcal{V}_{h_q}$ such that

$$\begin{aligned}
&B_{h_q}(\boldsymbol{Z}_{h_q} - \boldsymbol{Z}, w) = 0, \quad \forall\, w \in \mathcal{S}_{h_q}\,; \\
&\|\boldsymbol{Z}_{h_q} - \boldsymbol{Z}\|_{\boldsymbol{L}^2(\Omega)} \leq C {h_q}^{k+1}\,|\boldsymbol{Z}|_{\boldsymbol{H}^{k+1}(\Omega)}.
\end{aligned}$$

[2]The discrete Poincaré inequality writes: $\|w\|_{L^2(\Omega)} \leq C_P\|w\|_Z$ for all $w \in \mathcal{S}_h$, with C_P independent of h (see Corollary 4.3 of [4]).

Above, C is independent of $\boldsymbol{Z}$ and h_q.
As $(v_{h_q}, \boldsymbol{V}_{h_q}) \in \mathtt{A}_{h_q}$, we can write successively:

$$-\int_\Omega v_{h_q} \operatorname{div} \boldsymbol{Z}\, dx = B_{h_q}(\boldsymbol{Z}_{h_q}, v_{h_q}) \overset{(29)}{=} B^*_{h_q}(v_{h_q}, \boldsymbol{Z}_{h_q}) \overset{(31)}{=} \int_\Omega \mu \boldsymbol{V}_{h_q} \cdot \boldsymbol{Z}_{h_q}\, dx.$$

Since $(\mu \boldsymbol{Z}_{h_q})_q$ converges strongly to $\mu \boldsymbol{Z}$ in $\boldsymbol{L}^2(\Omega)$ and $(\boldsymbol{V}_{h_q})_q$ converges *weakly* to $\boldsymbol{U}^*$ in $\boldsymbol{L}^2(\Omega)$, we conclude that

$$\langle \nabla u^*, \boldsymbol{Z} \rangle = \int_\Omega \mu \boldsymbol{U}^* \cdot \boldsymbol{Z}\, dx, \quad \forall \boldsymbol{Z} \in \mathcal{D}(\Omega)^d.$$

In other words, $u^* \in H^1(\Omega)$ and moreover $\nabla u^* = \mu \boldsymbol{U}^*$.
Second, one has $u^* \in H^1_0(\Omega)$ if, and only if, there holds

$$\int_\Omega (u^* \operatorname{div} \boldsymbol{Z} + \nabla u^* \cdot \boldsymbol{Z})\; dx = 0, \quad \forall \boldsymbol{Z} \in \mathcal{C}^\infty(\bar{\Omega})^d.$$

This time, we find

$$\begin{aligned}
\int_\Omega u^* \operatorname{div} \boldsymbol{Z}\, dx &= \lim_q \int_\Omega v_{h_q} \operatorname{div} \boldsymbol{Z}\, dx = -\lim_q B_{h_q}(\boldsymbol{Z}, v_{h_q}) \\
&= -\lim_q B_{h_q}(\boldsymbol{Z}_{h_q}, v_{h_q}) = -\lim_q (\mu \boldsymbol{V}_{h_q}, \boldsymbol{Z}_{h_q})_{\boldsymbol{L}^2(\Omega)} \\
&= -\int_\Omega \mu \boldsymbol{U}^* \cdot \boldsymbol{Z}\, dx = -\int_\Omega \nabla u^* \cdot \boldsymbol{Z}\, dx,
\end{aligned}$$

which proves that $u^* \in H^1_0(\Omega)$.
Third, let us check that $(u^*, \boldsymbol{U}^*) \in H^1_0(\Omega) \times \boldsymbol{L}^2(\Omega)$ solves the original two-unknown problem (3), with $f = 0$. As $\nabla u^* = \mu \boldsymbol{U}^*$, we obviously have that

$$(\mu \boldsymbol{U}^*, \boldsymbol{Z})_{\boldsymbol{L}^2(\Omega)} - (\nabla u^*, \boldsymbol{Z})_{\boldsymbol{L}^2(\Omega)} = 0 \quad \forall \boldsymbol{Z} \in \boldsymbol{L}^2(\Omega).$$

Consider next $z \in \mathcal{D}(\Omega)$:

$$(\boldsymbol{U}^*, \nabla z)_{\boldsymbol{L}^2(\Omega)} - \omega^2 (\varepsilon u^*, z)_{L^2(\Omega)} = \lim_q \{ (\boldsymbol{V}_{h_q}, \nabla z)_{\boldsymbol{L}^2(\Omega)} - \omega^2 (\varepsilon v_{h_q}, z)_{L^2(\Omega)} \}.$$

Again, let us integrate the first term by parts, element by element:

$$\begin{aligned}
(\boldsymbol{V}_{h_q}, \nabla z)_{\boldsymbol{L}^2(\Omega)} &= \sum_{\tau \in \mathcal{T}} \int_\tau \boldsymbol{V}_{h_q} \cdot \nabla z\, dx \\
&= \sum_{\tau \in \mathcal{T}} \left\{ -\int_\tau z \operatorname{div} \boldsymbol{V}_{h_q}\, dx + \int_{\partial\tau} z (\boldsymbol{V}_{h_q} \cdot \boldsymbol{n}_{|\partial\tau})\, d\sigma \right\} \\
&= -\int_{\Omega^{h_q}} z \operatorname{div} \boldsymbol{V}_{h_q}\, dx + \sum_{\kappa \in \mathcal{F}^0_u} \int_\kappa z [\boldsymbol{V}_{h_q} \cdot \boldsymbol{n}]\, d\sigma \\
&= B^*_{h_q}(z, \boldsymbol{V}_{h_q}).
\end{aligned} \tag{48}$$

Above, we used the fact that $\boldsymbol{V}_{h_q} \cdot \boldsymbol{n}$ is continuous across faces of $\mathcal{F}_p$. Also, to compute the contribution on the remaining faces (i.e. those of $\mathcal{F}_u^0$), we used the definition of the jumps of the normal component on those faces (see §4.2).
According to (3.13) and (3.19) of [7], given $z \in H^{k+1}(\Omega)$, for all q, there exists $z_{h_q} \in \mathcal{S}_{h_q}$ such that

$$B^*_{h_q}(z_{h_q} - z, \boldsymbol{W}) = 0, \quad \forall\, \boldsymbol{W} \in \mathcal{V}_{h_q}\,;$$
$$\|z_{h_q} - z\|_{L^2(\Omega)} \leq C{h_q}^{k+1}\,|z|_{H^{k+1}(\Omega)}, \quad \|z_{h_q} - z\|_Z \leq C{h_q}^{k}\,|z|_{H^{k+1}(\Omega)}.$$

Above, C is independent of z and h_q.
Therefore, we reach

$$\begin{aligned}
(\boldsymbol{V}_{h_q}, \nabla z)_{\boldsymbol{L}^2(\Omega)} &- \omega^2(\varepsilon v_{h_q}, z)_{L^2(\Omega)} \\
&= B^*_{h_q}(z_{h_q}, \boldsymbol{V}_{h_q}) - \omega^2(\varepsilon v_{h_q}, z)_{L^2(\Omega)} \\
&\overset{(29)}{=} B_{h_q}(\boldsymbol{V}_{h_q}, z_{h_q}) - \omega^2(\varepsilon v_{h_q}, z)_{L^2(\Omega)} \\
&= B_{h_q}(\boldsymbol{V}_{h_q}, z_{h_q}) - \omega^2(\varepsilon v_{h_q}, z_{h_q})_{L^2(\Omega)} + \omega^2(\varepsilon v_{h_q}, z_{h_q} - z)_{L^2(\Omega)} \\
&= a_{h_q}(\mathtt{v}_{h_q}, (z_{h_q}, \boldsymbol{0})) + \omega^2(\varepsilon v_{h_q}, z_{h_q} - z)_{L^2(\Omega)}.
\end{aligned}$$

Let us consider each term of the right-hand side separately, when q goes to infinity:

$$|a_{h_q}(\mathtt{v}_{h_q}, (z_{h_q}, \boldsymbol{0}))| \leq \mu_q \|(z_{h_q}, \boldsymbol{0})\|_{h_q} = \mu_q \|z_{h_q}\|_Z \to 0.$$

For the other term:

$$|(\varepsilon v_{h_q}, z_{h_q} - z)_{L^2(\Omega)}| \leq \|\varepsilon\|_{L^\infty(\Omega)} \|v_{h_q}\|_{L^2(\Omega)}\, \|z_{h_q} - z\|_{L^2(\Omega)} \to 0.$$

We thus conclude that

$$(\boldsymbol{U}^*, \nabla z)_{\boldsymbol{L}^2(\Omega)} - \omega^2(\varepsilon u^*, z)_{L^2(\Omega)} = 0, \quad \forall z \in \mathcal{D}(\Omega).$$

By density, this is also true for all $z \in H^1_0(\Omega)$. In other words, $(u^*, \boldsymbol{U}^*)$ solves (3), with $f = 0$. As a consequence, under the condition on the contrast (18), we find that $(u^*, \boldsymbol{U}^*) = (0, \boldsymbol{0})$.

□

Remark 5.4 *Since we proceed by contradiction, no value of the stability parameter can be exhibited (cf. (45)). In particular, the sensivity of σ to the pulsation ω is not provided. To our knowledge, no such result can be found in the literature, including research works that rely on the use of the standard, conforming finite element method for the interface problem we consider [2, 11, 5]. A possible explanation is that, for a setting that includes an interface between a dielectric and a metamaterial, little is known on the spectral behavior of the (exact) operator.*

5.4 Error estimates

We use the notation $\|(v,\boldsymbol{V})\|_{0,\mu} = \left(\||\mu|^{\frac{1}{2}}\boldsymbol{V}\|^2_{\boldsymbol{L}^2(\Omega)} + \|v\|_{L^2(\Omega)}\right)^{\frac{1}{2}}$ to represent the weighted $L^2(\Omega)$ norm on $(\mathcal{S}_h + H^1_0(\Omega)) \times \boldsymbol{L}^2(\Omega)$.

We recall that $(u,\boldsymbol{U})$ (resp. $(u_h,\boldsymbol{U}_h)$) denotes the solution to the exact two-unknown problem (3) (resp. discrete two-unknown problem (32)). Let v be an arbitrary element in $\mathcal{S}_h$. Then we define[3] $\boldsymbol{V} \in \mathcal{V}_h$ by

$$(\mu\boldsymbol{V},\boldsymbol{W})_{\boldsymbol{L}^2(\Omega)} - B_h^*(v,\boldsymbol{W}) = 0 \quad \forall\, \boldsymbol{W} \in \mathcal{V}_h. \tag{49}$$

Thus, $(\boldsymbol{V},v)$ satisfy (31): they are *aligned* fields. Let us now use the uniform stability of the forms $(a_h)_h$, i.e. condition (45), to establish error estimates. Accordingly, we have

$$\begin{aligned}
&\|(v-u_h,\boldsymbol{V}-\boldsymbol{U}_h)\|_h \\
&\le \frac{1}{\sigma}\sup_{w\in\mathcal{S}_h,\boldsymbol{W}\in\mathcal{V}_h} \frac{a_h\Big((v-u_h,\boldsymbol{V}-\boldsymbol{U}_h),\ (w,\boldsymbol{W})\Big)}{\|(w,\boldsymbol{W})\|_h} \\
&\le \frac{1}{\sigma}\sup_{w\in\mathcal{S}_h,\boldsymbol{W}\in\mathcal{V}_h} \frac{a_h\Big((u-u_h,\boldsymbol{U}-\boldsymbol{U}_h),\ (w,\boldsymbol{W})\Big)}{\|(w,\boldsymbol{W})\|_h} \\
&\qquad\qquad + \frac{1}{\sigma}\sup_{w\in\mathcal{S}_h,\boldsymbol{W}\in\mathcal{V}_h} \frac{a_h\Big((v-u,\boldsymbol{V}-\boldsymbol{U}),\ (w,\boldsymbol{W})\Big)}{\|(w,\boldsymbol{W})\|_h} \\
&= \frac{1}{\sigma}\sup_{w\in\mathcal{S}_h,\boldsymbol{W}\in\mathcal{V}_h} \frac{a_h\Big((u,\boldsymbol{U}),\ (w,\boldsymbol{W})\Big) + (f,w)_{L^2(\Omega)}}{\|(w,\boldsymbol{W})\|_h} \\
&\qquad\qquad + \frac{1}{\sigma}\sup_{w\in\mathcal{S}_h,\boldsymbol{W}\in\mathcal{V}_h} \frac{a_h\Big((v-u,\boldsymbol{V}-\boldsymbol{U}),\ (w,\boldsymbol{W})\Big)}{\|(w,\boldsymbol{W})\|_h}.
\end{aligned} \tag{50}$$

The first term on the right hand side of (50) represents the consistency error while the second term on the right hand side of (50) represents the approximation error.

Approximation error By the definition of a_h, we have

$$\begin{aligned}
a_h\Big((v-u,\boldsymbol{V}-\boldsymbol{U}),\ (w,\boldsymbol{W})\Big) &= B_h(\boldsymbol{V}-\boldsymbol{U},w) + (\mu(\boldsymbol{V}-\boldsymbol{U}),\boldsymbol{W})_{\boldsymbol{L}^2(\Omega)} \\
&\quad - B_h^*(v-u,\boldsymbol{W}) - \omega^2(\varepsilon\,(v-u),w)_{L^2(\Omega)}.
\end{aligned} \tag{51}$$

[3] We recall that $(\boldsymbol{V},\boldsymbol{W}) \mapsto (\mu\boldsymbol{V},\boldsymbol{T}\boldsymbol{W})_{\boldsymbol{L}^2(\Omega)} = (|\mu|\boldsymbol{V},\boldsymbol{W})_{\boldsymbol{L}^2(\Omega)}$ is a bilinear, continuous and coercive form over $\boldsymbol{L}^2(\Omega)\times\boldsymbol{L}^2(\Omega)$, where $\boldsymbol{T}$ is the operator used to prove Theorem 3.2. In addition, $\boldsymbol{T}$ belongs to $\mathcal{L}(\mathcal{V}_h)$ and it is bijective because there are no continuity requirements across the interface for elements of the space $\mathcal{V}_h$.

According to (3.15) and (3.13) of [7], there exist elements $\pi_h u \in \mathcal{S}_h$ and $\Pi_h \boldsymbol{U} \in \mathcal{V}_h$ such that

$$\begin{aligned} B_h(\Pi_h \boldsymbol{U} - \boldsymbol{U}, w) &= 0, \quad \forall\, w \in \mathcal{S}_h, \\ B_h^*(\pi_h u - u, \boldsymbol{W}) &= 0, \quad \forall\, \boldsymbol{W} \in \mathcal{V}_h. \end{aligned}$$

Now we choose $v = \pi_h u$ and note that the corresponding $\boldsymbol{V}$ is defined via (49).

Then, for all $\boldsymbol{W} \in \mathcal{V}_h$, we have

$$(\mu \boldsymbol{V}, \boldsymbol{W})_{\boldsymbol{L}^2(\Omega)} = B_h^*(\pi_h u, \boldsymbol{W}) = B_h^*(u, \boldsymbol{W}) = (\mu \boldsymbol{U}, \boldsymbol{W})_{\boldsymbol{L}^2(\Omega)}.$$

Thus, $\boldsymbol{V}$ is merely the $\boldsymbol{L}^2$-projection of $\boldsymbol{U}$ with respect to the weighted inner product $(\mu\cdot,\cdot)_{\boldsymbol{L}^2(\Omega)}$. Therefore, for $v = \pi_h u$, (51) becomes

$$a_h\Big((\pi_h u - u, \boldsymbol{V} - \boldsymbol{U}),\ (w, \boldsymbol{W})\Big) = B_h(\boldsymbol{V} - \boldsymbol{U}, w) - \omega^2(\varepsilon\,(\pi_h u - u), w)_{L^2(\Omega)}.$$

Using the definition of $\Pi_h \boldsymbol{U}$,

$$a_h\Big((\pi_h u - u, \boldsymbol{V} - \boldsymbol{U}),\ (w, \boldsymbol{W})\Big) = B_h(\boldsymbol{V} - \Pi_h \boldsymbol{U}, w) - \omega^2(\varepsilon\,(\pi_h u - u), w)_{L^2(\Omega)}.$$

By the inequality (30) and the equivalence of norms $\|\cdot\|_{\boldsymbol{X}'}$ and $\|\cdot\|_{\boldsymbol{L}^2(\Omega)}$,

$$\begin{aligned} & a_h\Big((\pi_h u - u, \boldsymbol{V} - \boldsymbol{U}),\ (w, \boldsymbol{W})\Big) \\ & \leq\ \|\boldsymbol{V} - \Pi_h \boldsymbol{U}\|_{\boldsymbol{X}'}\, \|w\|_Z + \omega^2 \max(\varepsilon_1, |\varepsilon_2|)\, \|\pi_h u - u\|_{L^2(\Omega)}\, \|w\|_{L^2(\Omega)} \\ & \leq \frac{\beta}{\min(\mu_1, |\mu_2|)^{\frac{1}{2}}}\, \||\mu|^{\frac{1}{2}}(\boldsymbol{V} - \Pi_h \boldsymbol{U})\|_{\boldsymbol{L}^2(\Omega)}\, \|w\|_Z \\ & \qquad\qquad\qquad + \omega^2 \max(\varepsilon_1, |\varepsilon_2|)\, \|\pi_h u - u\|_{L^2(\Omega)}\, \|w\|_{L^2(\Omega)}. \end{aligned}$$

With the help of the discrete version of Poincaré's inequality in $\mathcal{S}_h$, we obtain

$$a_h\Big((\pi_h u - u, \boldsymbol{V} - \boldsymbol{U}),\ (w, \boldsymbol{W})\Big) \leq C\, \Big\|(\pi_h u - u, \boldsymbol{V} - \Pi_h \boldsymbol{U})\Big\|_{0,\mu}\, \Big\|(w, \boldsymbol{W})\Big\|_h,$$

where $C = C(\omega^2, C_P, \beta, \varepsilon_1, \varepsilon_2, \mu_1, \mu_2)$. Hence

$$\sup_{w \in \mathcal{S}_h, \boldsymbol{W} \in \mathcal{V}_h} \frac{a_h\Big((\pi_h u - u, \boldsymbol{V} - \boldsymbol{U}),\ (w, \boldsymbol{W})\Big)}{\|(w, \boldsymbol{W})\|_h} \leq C \Big\|(\pi_h u - u, \boldsymbol{V} - \Pi_h \boldsymbol{U})\Big\|_{0,\mu}.$$

We observe first that by the triangle inequality

$$\||\mu|^{\frac{1}{2}}\,(\boldsymbol{V} - \Pi_h \boldsymbol{U})\|_{\boldsymbol{L}^2(\Omega)} \leq \||\mu|^{\frac{1}{2}}\,(\boldsymbol{V} - \boldsymbol{U})\|_{\boldsymbol{L}^2(\Omega)} + \||\mu|^{\frac{1}{2}}\,(\boldsymbol{U} - \Pi_h \boldsymbol{U})\|_{\boldsymbol{L}^2(\Omega)},$$

and then since $\boldsymbol{V}$ is the $\boldsymbol{L}^2$-projection of $\boldsymbol{U}$ with respect to the weighted inner product $(\mu\cdot,\cdot)_{\boldsymbol{L}^2(\Omega)}$, the following holds (see footnote [3]):

$$\begin{aligned}
\||\mu|^{\frac{1}{2}}(\boldsymbol{V}-\boldsymbol{U})\|^2_{\boldsymbol{L}^2(\Omega)} &= (\mu(\boldsymbol{V}-\boldsymbol{U}),\boldsymbol{T}(\boldsymbol{V}-\boldsymbol{U}))_{\boldsymbol{L}^2(\Omega)}\\
&= (\mu(\boldsymbol{V}-\boldsymbol{U}),\boldsymbol{T}(\boldsymbol{V}-\boldsymbol{U})+\boldsymbol{T}(\Pi_h\boldsymbol{U}-\boldsymbol{V}))_{\boldsymbol{L}^2(\Omega)}\\
&= (|\mu|(\boldsymbol{V}-\boldsymbol{U}),(\Pi_h\boldsymbol{U}-\boldsymbol{U}))_{\boldsymbol{L}^2(\Omega)}\\
&\le \||\mu|^{\frac{1}{2}}(\boldsymbol{V}-\boldsymbol{U})\|_{\boldsymbol{L}^2(\Omega)}\,\||\mu|^{\frac{1}{2}}(\Pi_h\boldsymbol{U}-\boldsymbol{U})\|_{\boldsymbol{L}^2(\Omega)}\\
\text{so } \||\mu|^{\frac{1}{2}}(\boldsymbol{V}-\boldsymbol{U})\|_{\boldsymbol{L}^2(\Omega)} &\le \||\mu|^{\frac{1}{2}}(\boldsymbol{U}-\Pi_h\boldsymbol{U})\|_{\boldsymbol{L}^2(\Omega)}\,. \qquad (52)
\end{aligned}$$

With that, we can obtain error estimates. According to Theorem 3.4 and Theorem 3.5 of [7], we have respectively

$$\begin{aligned}
\|\pi_h u-u\|_{L^2(\Omega)} &\le C\,h^{\min(k+1,s+1)}\,|u|_{H^{s+1}(\Omega)} && \text{if } u\in H^{s+1}(\Omega)\,,\\
\|\Pi_h\boldsymbol{U}-\boldsymbol{U}\|_{\boldsymbol{L}^2(\Omega)} &\le C\,h^{\min(k+1,S+1)}\,|\boldsymbol{U}|_{\boldsymbol{H}^{S+1}(\Omega)} && \text{if } \boldsymbol{U}\in \boldsymbol{H}^{S+1}(\Omega)\,,
\end{aligned}$$

where k is the maximal degree of the polynomials that define the discrete fields, and C is independent of $u,\boldsymbol{U}$ and h. It is possible to obtain more precise results.
First, we can obtain similar estimates, under the weaker assumptions that u and $\boldsymbol{U}$ be *piecewise smooth*[4], namely

$$u_\ell\in H^{s+1}(\Omega_\ell),\ \ell=1,2 \quad ; \quad \boldsymbol{U}_\ell\in \boldsymbol{H}^{S+1}(\Omega_\ell),\ \ell=1,2. \qquad (53)$$

Within this setting, using the identity $\boldsymbol{U}=\mu^{-1}\nabla u$, we have automatically $S=s-1$, as soon as μ is piecewise smooth (which is the case as it is piecewise constant). Second, the results can also be extended[4] to *non-integer values* of s (and S). Hence, we find that

$$\|\pi_h u-u\|_{L^2(\Omega)}\le C\,h^{\min(k+1,s+1)}\ ,\ \|\Pi_h\boldsymbol{U}-\boldsymbol{U}\|_{\boldsymbol{L}^2(\Omega)}\le C\,h^{\min(k+1,s)}\ , \qquad (54)$$

where $s>0$ defines the piecewise smoothness of u (cf. (53)), and C is independent of h.
Thus, we conclude that for the term representing the approximation error, we have

$$\sup_{w\in\mathcal{S}_h,\boldsymbol{W}\in\mathcal{V}_h}\frac{a_h\Big((\pi_h u-u,\boldsymbol{V}-\boldsymbol{U}),\ (w,\boldsymbol{W})\Big)}{\|(w,\boldsymbol{W})\|_h}\le C\,h^{\min(k+1,s)}.$$

Consistency error By the definition of a_h, we have

$$\begin{aligned}
&a_h\Big((u,\boldsymbol{U}),\ (w,\boldsymbol{W})\Big)+(f,w)_{L^2(\Omega)}\\
&=B_h(\boldsymbol{U},w)+(\mu\,\boldsymbol{U},\boldsymbol{W})_{\boldsymbol{L}^2(\Omega)}-B_h^*(u,\boldsymbol{W})-\omega^2(\varepsilon\,u,w)_{L^2(\Omega)}+(f,w)_{L^2(\Omega)}.
\end{aligned}$$

[4]Indeed, the proofs of the above results are obtained with the help of the standard theory of polynomial preserving interpolation operators such as π_h and Π_h (see pp. 3836-3837 of [7]).

Integrating by parts, element by element (cf. (48)), we find that $B_h^*(u,\boldsymbol{W}) = (\boldsymbol{W}, \nabla u)_{\boldsymbol{L}^2(\Omega)}$. Using the definition (28) for $B_h(\boldsymbol{U}, w)$, we have

$$\begin{aligned} & a_h\Big((u,\boldsymbol{U}),\ (w,\boldsymbol{W})\Big) + (f,w)_{L^2(\Omega)} \\ = & \ -(\operatorname{div}\boldsymbol{U} + \omega^2 \varepsilon\, u - f, w)_{L^2(\Omega)} + (\mu\, \boldsymbol{U} - \nabla u, \boldsymbol{W})_{\boldsymbol{L}^2(\Omega)}. \end{aligned}$$

Therefore, as $\mu \boldsymbol{U} = \nabla u$ in $\boldsymbol{L}^2(\Omega)$ and $\operatorname{div}\boldsymbol{U} + \omega^2\,\varepsilon\, u - f = 0$ in $L^2(\Omega)$, we conclude that the consistency term is zero.

Error estimate We obtain finally the following estimates.

Theorem 5.5 *Assume the condition on the contrast (39) holds. Let $s > 0$ define the piecewise smoothness of u as in (53) and let k be the maximal degree of the polynomials that define the discrete fields. Then one has*

$$\||\mu|^{\frac{1}{2}}\,(\boldsymbol{U} - \boldsymbol{U}_h)\|_{\boldsymbol{L}^2(\Omega)} \le C\, h^{\min(k+1,s)}\,, \tag{55}$$

$$\|u - u_h\|_Z \le C\, h^{\min(k,s)}\,, \tag{56}$$

$$\|u - u_h\|_{L^2(\Omega)} \le C\, h^{\min(k+1,s)}\,. \tag{57}$$

Proof. Starting from (50) and combining all the previous results, we know that

$$\|(\pi_h u - u_h, \boldsymbol{V} - \boldsymbol{U}_h)\|_h \le C\, h^{\min(k+1,s)},$$

where $\boldsymbol{V}$ is the $\boldsymbol{L}^2$-projection of $\boldsymbol{U}$ with respect to the weighted inner product $(\mu\cdot,\cdot)_{\boldsymbol{L}^2(\Omega)}$ (see (49)).
Then, using (52) and (54), we find

$$\||\mu|^{\frac{1}{2}}\,(\boldsymbol{U}-\boldsymbol{U}_h)\|_{\boldsymbol{L}^2(\Omega)} \le \||\mu|^{\frac{1}{2}}\,(\boldsymbol{U}-\boldsymbol{V})\|_{\boldsymbol{L}^2(\Omega)} + \||\mu|^{\frac{1}{2}}\,(\boldsymbol{V}-\boldsymbol{U}_h)\|_{\boldsymbol{L}^2(\Omega)} \le C\, h^{\min(k+1,s)}.$$

Next, we recall from Theorem 3.4 of [7] that $\|u - \pi_h u\|_Z \le C\, h^{\min(k,s)}$, so we get

$$\|u - u_h\|_Z \le \|u - \pi_h u\|_Z + \|\pi_h u - u_h\|_Z \le C\, h^{\min(k,s)}.$$

Moreover, by the discrete Poincaré inequality on the space $\mathcal{S}_h$,

$$\|\pi_h u - u_h\|_{L^2(\Omega)} \le C_P \|\pi_h u - u_h\|_Z \le C\, h^{\min(k+1,s)}.$$

Using again Theorem 3.4 of [7] to reach $\|u - \pi_h u\|_{L^2(\Omega)} \le C\, h^{\min(k+1,s+1)}$, we conclude that

$$\|u - u_h\|_{L^2(\Omega)} \le \|u - \pi_h u\|_{L^2(\Omega)} + \|\pi_h u - u_h\|_{L^2(\Omega)} \le C\, h^{\min(k+1,s)}.$$

$\square$

6 Numerical experiments

In this section, numerical examples will be provided. We take $\Omega = [0,5] \times [0,2]$. The data in (1) are defined as follows:

$$f(x,y) = \begin{cases} \sin(\frac{\pi}{2}y), & \text{if } x < 1 \\ 0, & \text{otherwise} \end{cases}, \qquad \epsilon = \mu = \begin{cases} 1, & \text{if } x < 1 \text{ or } x > 3 \\ -3, & \text{otherwise} \end{cases}.$$

The exact solution for (1) with data defined above can be easily found by the method of separation of variables. For all numerical results shown below, piecewise linear approximation is used ($k = 1$).

6.1 The case $\omega = 0$

In Figure 2, results are shown for the scalar unknowns u and u_h. In the left and the middle figures, we have shown the exact and the numerical solutions on the whole domain respectively. On the right figure, we compare the numerical and the exact solutions at $y = 0.98$. We use blue curve with circles to represent the numerical solution and red curve to represent the exact solution.

In Figure 3 and Figure 4, results are shown for the vector unknowns $\boldsymbol{U}$ and $\boldsymbol{U}_h$, with Figure 3 showing the first components U_1 and $(U_h)_1$ and Figure 4 showing the second components U_2 and $(U_h)_2$. In the left and the middle figures, we have shown the exact and the numerical solutions on the whole domain respectively. On the right figure, we compare the numerical and the exact solutions at $y = 0.98$. We use blue curve with circles to represent the numerical solution and red curve to represent the exact solution.

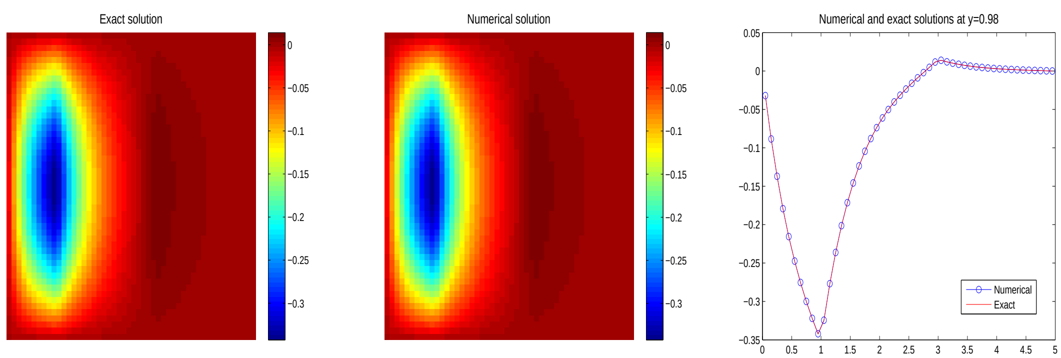

Figure 2: Case $\omega = 0$. Left: Exact solution u. Middle: Numerical solution u_h. Right: Comparison of numerical and exact solutions at $y = 0.98$.

In Table 1, L^2-norm errors are shown for various mesh sizes. We see that the DG method we propose achieves the expected second order accuracy. In addition, we compare the accuracy of the DG method and that of the conforming finite element

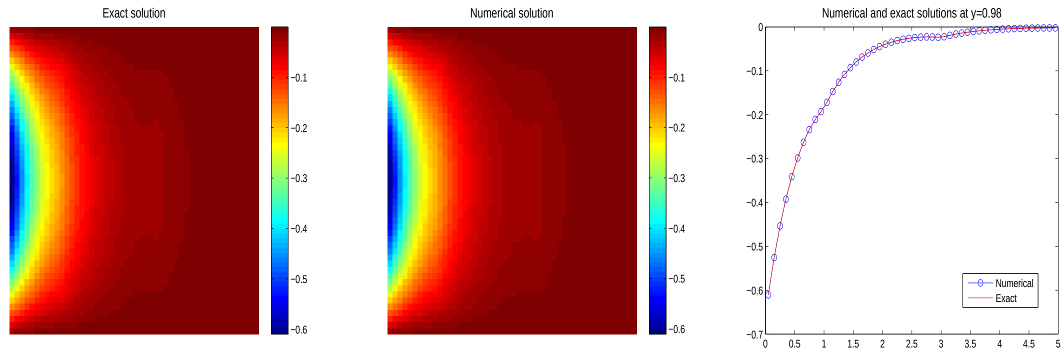

Figure 3: Case $\omega = 0$. Left: Exact solution U_1. Middle: Numerical solution $(U_h)_1$. Right: Comparison of numerical and exact solutions at $y = 0.98$.

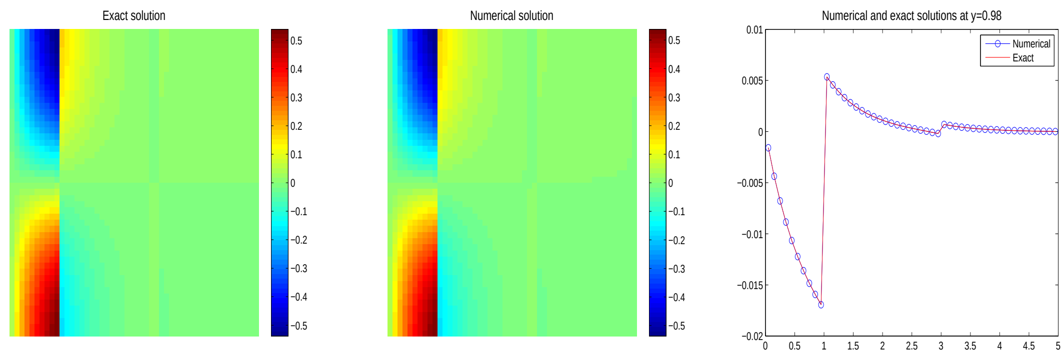

Figure 4: Case $\omega = 0$. Left: Exact solution U_2. Middle: Numerical solution $(U_h)_2$. Right: Comparison of numerical and exact solutions at $y = 0.98$.

method (FEM) [2, 11, 5]. The conforming FEM is defined on the finer triangulation $\mathcal{T}$ and the corresponding L^2-norm errors for various mesh sizes are shown again in Table 1. We observe the second order accuracy of both the DG method and the conforming FEM. Furthermore, we see that the error of the DG method is approximately 3.5 times smaller than that of the conforming FEM. For the sake of completeness, we mention that the above errors are computed by using the quadrature rule with quadrature points located on the mid-points of the edges.

6.2 The case $\omega \neq 0$

In this subsection, we present an example with $\omega = 1.6$.

In Figure 5, results are shown for the scalar unknowns. In Figure 6 and Figure 7, numerical results are shown for the vector unknowns, with Figure 6 showing the first

h	Our method	Order	Conforming FEM	Order
0.1768	6.9685e-004	–	2.5235e-003	–
0.0884	1.7429e-004	1.99936	6.3346e-004	1.99412
0.0442	4.3577e-005	1.99986	1.5853e-004	1.99851
0.0221	1.0894e-005	1.99997	3.9643e-005	1.99963
0.0110	2.7236e-006	1.99999	9.9113e-006	1.99991

Table 1: Case $\omega = 0$. L^2-norm errors with the DG method and the conforming FEM.

components and Figure 7 showing the second components. In Table 2, L^2-norm errors with the DG method are shown for various mesh sizes. We see that the DG method achieves the expected second order accuracy. In addition, the L^2-norm errors with the conforming FEM are shown. We observe the second order accuracy of the conforming FEM. In this instance, we see that the error of the DG method is approximately 34 times smaller than that of the conforming FEM.

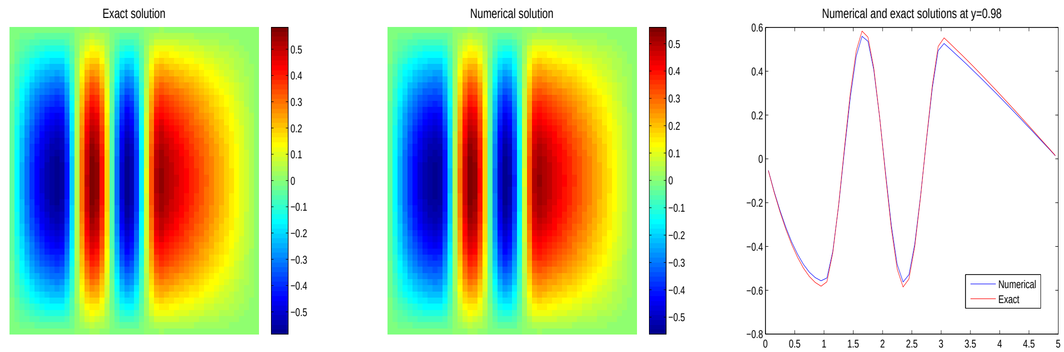

Figure 5: Case $\omega = 1.6$. Left: Exact solution u. Middle: Numerical solution u_h. Right: Comparison of numerical and exact solutions at $y = 0.98$.

h	Our method	Order	Conforming FEM	Order
0.1768	6.1975e-003	–	2.7622e-001	–
0.0884	1.5507e-003	1.99878	5.7594e-002	2.26184
0.0442	3.8775e-004	1.99972	1.3586e-002	2.08379
0.0221	9.6941e-005	1.99995	3.3548e-003	2.01785
0.0110	2.4236e-005	1.99996	8.3616e-004	2.00435

Table 2: Case $\omega = 1.6$. L^2-norm errors with the DG method and the conforming FEM.

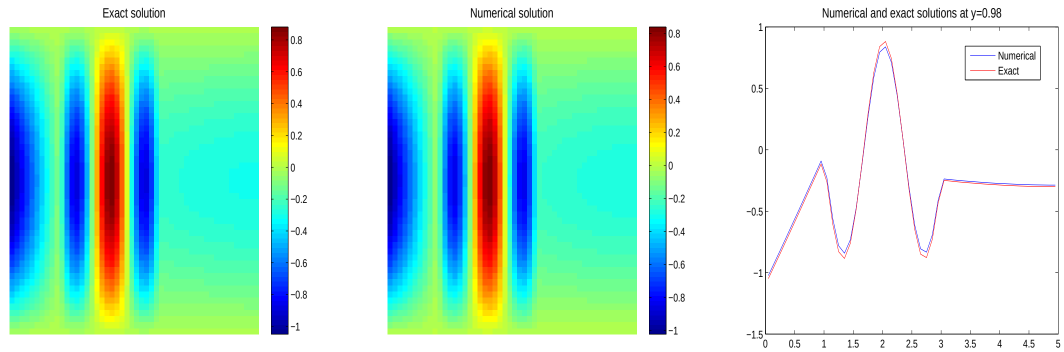

Figure 6: Case $\omega = 1.6$. Left: Exact solution U_1. Middle: Numerical solution $(U_1)_h$. Right: Comparison of numerical and exact solutions at $y = 0.98$.

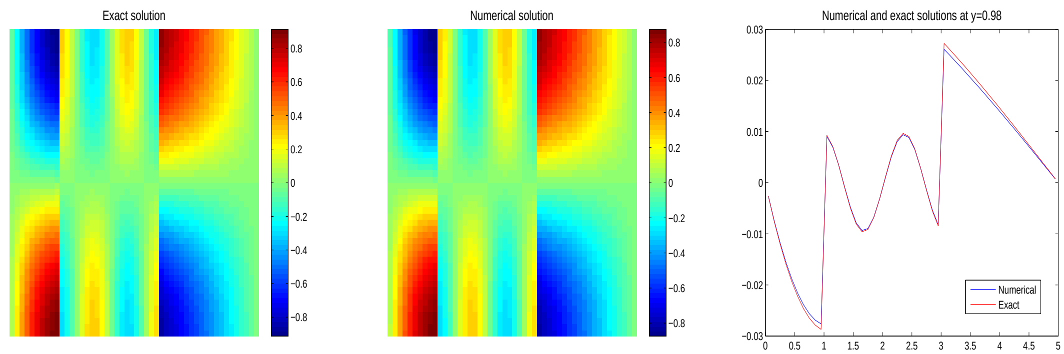

Figure 7: Case $\omega = 1.6$. Left: Exact solution U_2. Middle: Numerical solution $(U_2)_h$. Right: Comparison of numerical and exact solutions at $y = 0.98$.

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