Optimal encoding of triangular and quadrangular meshes with fixed topology

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Optimal encoding of triangular and quadrangular meshes with fixed topology

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Abstract

Extending a bijection recently introduced by Poulalhon and Schaeffer [15] for triangulations of the sphere we design an efficient algorithm for encoding (topological) triangulations and bipartite quadrangulations on an orientable surface of *fixed* topology τ (given by the genus g and number of boundaries b). To our knowledge, our encoding procedure is the first to be asymptotically optimal (in the information theory sense) with respect to two natural parameters, the number n of inner vertices and the number k of boundary vertices.

1 Introduction

The origin of our work is a nice bijection due to Poulalhon and Schaeffer [15] between planar triangulations (without loops and multiple edges) and a special class of plane trees, providing a combinatorial proof of the counting formula found by Tutte [18] and yielding efficient procedures for random sampling and encoding. The construction in [15] associates bijectively to a triangulation (endowed with a certain "canonical" orientation of its edges) a tree having the special property that each node is incident to exactly two leaves (also referred to as *stems*). In this work we extend the construction to triangulations (and quadrangulations by similar principles) on a surface of arbitrary topology, given by the genus and number of boundaries. Compared to the planar case with no boundary, the bijective correspondence is lost but the encoding is still asymptotically optimal in the information theory sense (and with topology fixed):

Theorem 1 *Given an orientable surface S of fixed topology $\tau = (g, b)$, it is possible to encode any triangulation (bipartite quadrangulation) on S having n inner vertices and k boundary vertices (resp. $2k$ boundary vertices), such that the length $\ell(n, k)$ of the encoding word satisfies, as $n + k \longrightarrow \infty$:*

- $\ell(n, k) \sim \log_2 |\mathcal{T}^{(\tau)}_{n,k}|$ *for triangulations,*
- $\ell(n, k) \sim \log_2 |\mathcal{Q}^{(\tau)}_{n,k}|$ *for quadrangulations*

where $\mathcal{T}^{(\tau)}_{n,k}$ (resp. $\mathcal{Q}^{(\tau)}_{n,k}$) denotes the set of triangulations (resp. bipartite quadrangulations) on S with n inner vertices and k boundary vertices (resp. $2k$ boundary vertices). Moreover, the encoding phase requires $O(n + k)$ time if $g = 0$ and $O((n + k) \log(n + k))$ time if $g > 0$, while decoding takes $O(n + k)$ time.

Actually the result above is still valid when $b = o(\frac{n+k}{\log(n+k)})$. For larger values of b we cannot prove the tightness of our bounds, since no enumeration formula is known for counting simple triangulations or bipartite quadrangulations with multiple boundaries.

Related works on graph counting and coding. Our encoding procedure extends the bijection introduced by Poulalhon and Schaeffer [15] for planar triangulations with no boundary (case $g = 0$, $b = 0$). It achieves asymptotically $\log_2(\frac{256}{27}) \approx 3.2451$ bits per vertex, which is (asymptotically) optimal since it matches the information theory lower bound. The case $g = 0$, $b = 1$ is also combinatorially tractable; there is an exact counting formula due to Brown [5] for the number of triangulations with n inner vertices and k boundary vertices, and there are two different bijective constructions in [15] (Section 5) and [2], the second construction being amenable to an optimal encoding scheme according to n and k. When there are more boundaries, no counting formula nor bijective constructions is known. (The triangulated maps counted in [11] have multiple boundaries, but loops and multiple edges are allowed.) Our construction yields an injection from planar triangulations with $b > 0$ boundaries to a certain family of plane trees with boundaries that can be encoded optimally. Let us now review other types of encoding schemes; the topological approach of the popular *Edgebreaker* encoder [16] (requiring $3.67n$ bits for the planar case without boundaries) has been extended to the case of boundaries [13], but the compression ratio is higher and far from the optimal when the overall size k of the boundaries is not negligible. The compact encoding [7] requires $2.175m + o(m)$ bits for triangulations with b boundaries and m triangles: this is optimal with respect to m (one-parameter optimality), but only when $k = \frac{13}{17}m + o(m)$, whereas our algorithm yields optimality with respect to (m, k) in full generality.

2 Maps, orientations and canonical spanning trees

Maps. A *map* (also called cellular embedding) is a graph G embedded on a closed orientable surface S (of a certain genus g) such that all components of $S\backslash G$ are topological disks; each component being called a

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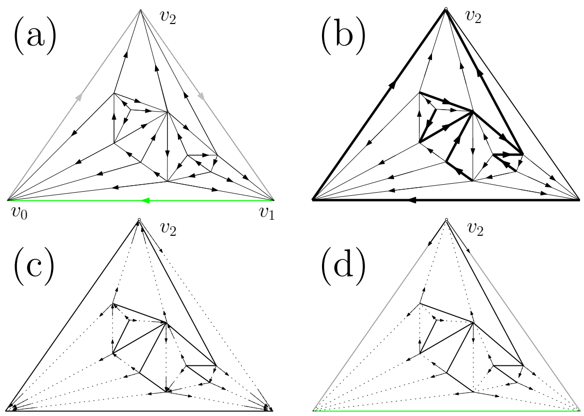

Figure 1: A planar rooted triangulation M endowed with its minimal 3-orientation (a), and the corresponding spanning tree T, rooted at v_2 (b).

face of the map. A *map with b boundaries* is a map with b marked faces, called *boundaries*, which are pairwise vertex disjoint and without self intersections. Each boundary-face can be considered as a "hole", so a map of genus g with b boundaries can as well be considered as a cellular embedding on the surface of genus g with b boundaries. A *planar map* is a map of genus 0 (no boundary if not specified); a *planar map with boundaries* is a map of genus 0 with $b > 0$ boundaries. A *plane tree with $b \geq 0$ boundaries* is a planar map with b boundaries and a unique non-boundary face (shortly called a plane tree if there is no boundary). A map (possibly with boundaries) is *rooted* if it has a marked corner, called the root, incident to a non-boundary face. The root-face (root-vertex) is the face (vertex, resp.) incident to the root. A triangulation (quadrangulation) is a loopless simple map possibly with boundaries where all non-boundary faces have degree 3 (degree 4, resp.).

Orientations and canonical spanning trees. An *orientation* O of a rooted planar map M (possibly with boundaries), is the choice of a direction for its edges. An orientation is *minimal* if there is no counter-clockwise *circuit* (a directed cycle of edges), and is *accessible* if from every vertex one can reach the root-vertex by an oriented path. To such an orientation O is associated the so-called *canonical spanning tree* [1] for O [1], which is the unique spanning tree T of M satisfying:

1. the edges of T are oriented toward the root-vertex,
2. every edge $e \in M \backslash T$ has on its right the interior of the unique cycle of $e + T$.

The canonical spanning tree can be computed in linear time (according to the number of edges) by a traversal algorithm [15, 1]. The *fully decorated spanning tree* F for O is obtained by cutting each edge $e \in M \backslash T$ in its middle, leaving an *outgoing stem* (incident to the origin of e) and an *ingoing stem* (incident to the end of e), see Figure 1(c). Property 2 ensures that O can be recovered from F, since the edges of $M \backslash T$ correspond to the matchings of the cyclic parenthesis word formed by the stems in clockwise order around the unique face of F (outgoing stems being seen as opening parentheses and ingoing stems as closing parentheses).

Canonical spanning trees are of great help to encode a planar map M; indeed if one can endow M with a minimal accessible orientation, encoding M reduces to encoding the associated fully decorated spanning tree, a much easier task since trees are amenable to encoding by (contour) words [1, 2, 15]. We will thoroughly exploit this strategy for triangulations and quadrangulations.

3 Encoding planar triangulations

Planar triangulations with no boundary We recall the procedure in [15] to encode a planar triangulation with no boundary. Let M be a triangulation with $n + 2$ vertices. Fix an outer face $\{v_0, v_1, v_2\}$ for M, and endow M with its unique minimal 3-*orientation* O, where a 3-orientation is an orientation where each of the 3 outer vertices has outdegree 1 and each inner vertex has outdegree 3. The existence of a 3-orientation follows from work by Schnyder [17], which guarantees that any 3-orientation is accessible with respect to every outer vertex. Existence and uniqueness of the minimal 3-orientation of M follows from [9], and a linear time algorithm is given in [4]. Let F be the fully decorated spanning tree for O, see Fig. 1(c). Since all faces are of degree 3, there is no loss of information in deleting ingoing stems (because there is a unique way to place the ingoing stems in such a way that the map obtained by matching outgoing with ingoing stems is a triangulation). One can also delete the branch (v_1, v_0), (v_0, v_2) without loss of information. The obtained tree with only outgoing stems is called the *reduced decorated spanning tree* R of M (see Fig. 1(d)); R belongs to the set $\mathcal{P}_n$ of rooted plane trees with 2 stems at each of the n nodes (the extremity of a stem being not considered as a node); as shown in [15], trees in $\mathcal{P}_n$ can be encoded by binary words of length $\ell_n \sim \log_2 \binom{4n}{n} \sim \log_2(|\mathcal{T}_{n+2}|)$, where $\mathcal{T}_n$ is the set of planar triangulations with n nodes and no boundary, so the encoding is asymptotically optimal.

Planar triangulations with boundaries Here starts our contribution, which is to keep an optimal encoding scheme in case of boundaries. Let M be a plane triangulation with $b > 0$ boundaries, $n + 2$ non boundary vertices, and k boundary vertices. Assume without loss of generality that an outer (non-boundary) face $\{v_0, v_1, v_2\}$ for M is fixed that does not touch any of

[1]The term "canonical spanning trees" denotes sometimes different things in prior works: namely, one of the spanning trees in the Schnyder tree decomposition [17] of a planar triangulation.

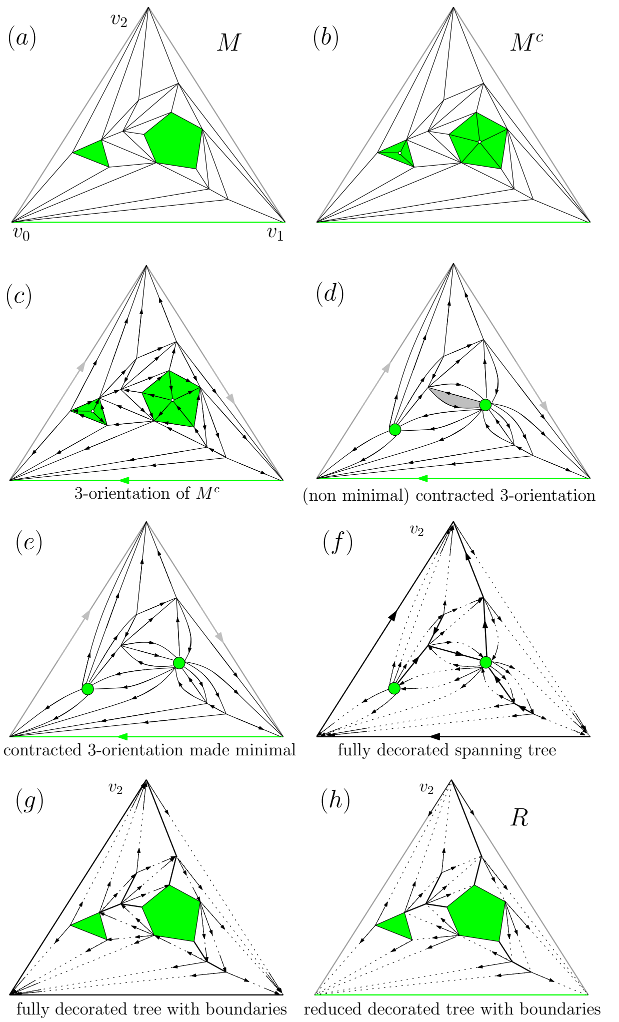

Figure 2: Correspondence between a triangulation M with 2 boundaries (a), and a decorated plane tree with 2 boundaries, spanning all vertices of $M \setminus \{v_0, v_1\}$ (h).

the boundaries (if no such face exists, create it inside an arbitrary non-boundary face, this adds only 3 vertices and will have no effect on the length of the coding word asymptotically). Define the *completed triangulation* for M as the planar triangulation M^c obtained by adding a star in each boundary face (see Fig. 2(b)); M^c is a triangulation with $n+k+b$ vertices and no boundary. Endow M^c with a 3-orientation. Then contract each of the b stars $S_1, \ldots, S_b$ into a single so-called *special* vertex $s_1, \ldots, s_b$ (each contraction deletes the edges of the star and the edges on the contour of the corresponding boundary face). The orientation inherited by the 3-orientation of M^c is such that every non special vertex has outdegree 3, while every special vertex s_i has outdegree $k_i + 3$, with k_i the size of the corresponding boundary. The contracted orientation is still accessible with respect to vertices $\{v_0, v_1, v_2\}$, but it may not be minimal, even if coming from the minimal 3-orientation of M^c; indeed a path connecting two vertices on the same boundary might become a ccw circuit after contraction (as shown in Fig. 2(d)). However a procedure discussed in [3] allows us to make —in linear time— the contracted orientation minimal (by successively reversing ccw circuits) while keeping the same outdegree at each vertex. Moreover the obtained minimal orientation is still accessible, because returning circuits does not affect accessibility. So we can now consider the fully decorated spanning tree for the orientation, see Fig. 2(f). Then we uncontract each special vertex back into the original boundary face and obtain a fully decorated plane tree with boundaries. As in Section 3, without loss of information we can delete ingoing stems and the branch (v_0, v_1), (v_0, v_2), to obtain the so-called *reduced decorated plane tree with boundaries* R for M. The tree R belongs to the family $\mathcal{P}_{n,k}^{(b)}$ of plane trees with b boundaries, n non-boundary vertices, k boundary vertices, and decorated with stems as follows: 1) each non-boundary vertex carries two stems, 2) for $1, \ldots, b$, the i-th boundary, of size called k_i, carries overall $k_i + 2$ stems [2]. We obtain:

Lemma 2 *For fixed $b > 0$, any tree in $\mathcal{P}_{n,k}^{(b)}$ can be encoded in a number $\ell(n,k)$ of bits that satisfies*

$$\ell(n,k) \sim 2k + \log_2 \binom{4n+2k}{n} \quad \text{as } n+k \to \infty.$$

In addition, the encoding is asymptotically optimal with respect to n and k, as the number $a_{n,k}^{(b)}$ of plane triangulations with b boundaries, n non-boundary vertices, and k boundary vertices satisfies

$$\log_2 \left(a_{n,k}^{(b)}\right) \sim 2k + \log_2 \binom{4n+2k}{n} \quad \text{as } n+k \to \infty,$$

The encoding of the tree is done by a contour word similarly as in [15]. Concerning the second statement, it is already known for $b = 1$ thanks to Brown's counting formula [5], from which one can derive a lower bound for general $b > 0$. Finally, $\ell(n,k)$ gives an upper bound since the encoding procedure is injective.

4 Encoding planar bipartite quadrangulations

Bipartite quadrangulations (in genus 0 bipartiteness is equivalent to the property that all boundaries are of even sizes) can be treated in a completely similar way as triangulations. For planar quadrangulations with no boundary the bijection (presented in [10]) relies on the unique minimal *2-orientation of a simple quadrangulation*, where a 2-orientation has inner vertices of outdegree 2 and outer vertices have outdegree 1. Existence

[2] R satisfies this property since, in the contracted tree, the total outdegree $k_i + 3$ of the special vertex s_i consists of one outgoing edge to the father of s_i plus $k_i + 2$ outgoing stems

of such an orientation has been shown by De Fraysseix et al. [8]. Similarly as for triangulations, one can consider the fully decorated spanning tree obtained from the minimal 2-orientation, and then reduce it (deleting ingoing stems and the leftmost branch, of length 3) into a so-called *reduced decorating spanning tree*, a plane tree where each node carries one stem (indeed the two outgoing edges at a vertex become the edge going to the father plus one stem in the tree). The trees from this family are then readily encoded in an asymptotically optimal way, see [10]. For a bipartite planar quadrangulation Q with $b > 0$ boundaries (with black and white vertices) the treatment is similar as for triangulations.

Lemma 3 *Let $\mathcal{Q}_{n,k}^{(b)}$ be the set of bipartite planar quadrangulations with b boundaries, n non-boundary vertices, and $2k$ boundary vertices. For fixed $b > 0$, any $Q \in \mathcal{Q}_{n,k}^{(b)}$ can be encoded with $\ell(n,k)$ bits, where*

$$\ell(n,k) \sim k \cdot \log_2(\tfrac{27}{4}) + \log_2 \binom{3n+3k}{n} \text{ as } n+k \to \infty,$$

In addition $\ell(n,k) \sim \log_2(|\mathcal{Q}_{n,k}^{(b)}|)$ as $n+k \to \infty$, so the encoding is asymptotically optimal w.r.t. n and k.

5 Encoding in higher genus

For dealing with the higher genus case, it suffices to make some simple observations (we discuss triangulations only, the discussion for quadrangulations is similar). First, as discussed in [14] (Lemma 4.1), for any graph G on a surface S of genus $g > 0$ with n vertices, there exists a non-contractible cycle C on S such that C crosses G at vertices only, and $|G \cap C| \leq \sqrt{2n}$; C is in fact the cycle with smallest number of intersections and can be computed in time $O(n \log(n))$ for fixed genus g [12]. For a triangulation M on S with b boundaries (boundaries seen as faces), C can be deformed in each triangular face f to pass by one edge around f (but we do not deform C inside the boundary faces). After this, cut S along C; this yields a triangulation M' of genus $g-1$ with two *special* boundary-faces f_1, f_2 bounded each by C (indeed, cutting splits C into two copies), with otherwise at most $2b$ boundaries (because each of the b boundaries of M might be crossed by C, thus becoming two boundaries after cutting). We add a star into f_1 and f_2, so the boundaries are only the ones arising from the boundaries of M (the locations of the two special stars have be stored to recover M from M', which costs only $O(\log(n))$ in memory). If M has n non-boundary vertices and k boundary vertices, M' will have $n' \leq n + 2 + |C|$ non-boundary vertices and $k' \leq k + 2b + |C|$ boundary vertices, with $|C| \leq \sqrt{2(n+k)}$. By induction on g, M' can be encoded asymptotically optimally, i.e., with a word of length $\ell(n', k') \sim 2k' + \log_2 \binom{4n'+2k'}{n'}$. Since $\ell(n', k') \sim \ell(n, k)$ when $n + k \to \infty$ and when $n' + k' = n + k + O(\sqrt{n+k})$, and since only memory $O(\log(n))$ is necessary to recover M from M', the encoding in genus g is also asymptotically optimal.

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