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Pierre Bremaud, Jean-Marc Lasgouttes. Stationary IPA Estimates for Non-Smooth G/G/1/∞ Functionals via Palm Inversion and Level-Crossing Analysis.. Discrete Event Dynamic Systems, Springer Verlag, 1993, 3 (4), pp.347-374. <10.1007/BF01439159>. <hal-00717364>

HAL Id: hal-00717364
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Submitted on 13 Jul 2012

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Stationary IPA Estimates for Non-Smooth $G/G/1/\infty$ Functionals via Palm Inversion and Level-Crossing Analysis. *

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April 1992; revised January 1993, September 1993

Abstract

We give stationary estimates for the derivative of the expectation of a non-smooth function of bounded variation f of the workload in a $G/G/1/\infty$ queue, with respect to a parameter influencing the distribution of the input process. For this, we use an idea of Konstantopoulos and Zazanis [15] based on the Palm inversion formula, however avoiding a limiting argument by performing the level-crossing analysis thereof globally, via Fubini's theorem. This method of proof allows to treat the case where the workload distribution has a mass at discontinuities of f and where the formula of [15] has to be modified. The case where the parameter is the speed of service or/and the time scale factor of the input process is also treated using the same approach.

1 Introduction.

Consider a stationary $G/G/1/\infty$ queue in which customers arrive according to a stationary process $\{T_n\}_{n\in\mathbb{Z}}$. The customer n asks for a service time $\sigma_n(\theta)$, where θ is a real parameter in the compact interval Θ and $\{\sigma_n(\theta)\}_{n\in\mathbb{Z}}$ is an i.i.d sequence. Let $\{\tau_n\}_{n\in\mathbb{Z}}$ denote the inter-arrival times process satisfying $\tau_n = T_{n+1} - T_n$. Assume that the queue is stationary and let $W_\theta(t)$ be the remaining work in the system at time t—see Figure 1— given by Lindley's equation

*This article has been presented at the 31^{st} IEEE CDC, Dec. 16-18 1992, Tucson, Arizona, USA.

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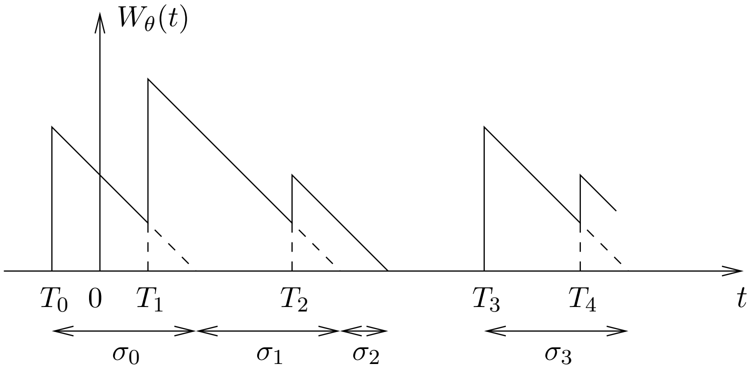


Figure 1: workload of a G/G/1 queue.

$$W_\theta(t) = \Big(W_\theta(T_n-) + \sigma_n(\theta) - (t - T_n)\Big)^+, \ t \in [T_n, T_{n+1}), \tag{1}$$

with the notation $x^+ \stackrel{\text{def}}{=} \max(x, 0)$. Given a real function f, consider the functional $J(\theta)$ defined as

$$J(\theta) \stackrel{\text{def}}{=} \mathbb{E} f(W_\theta(0)).$$

We want to estimate, if it exists, the derivative of J with respect to θ. To this end, we use Infinitesimal Perturbation Analysis (IPA), a method first introduced by Ho and Cao [13] and further developed by Cao [6], Suri and Zazanis [19] and recently Konstantopoulos and Zazanis [15]. Glasserman [8] and Ho and Cao [14] summarize and review most previous results on IPA. Alternative methods have been used to estimate derivatives, namely Smooth Perturbation Analysis (SPA, see Suri and Zazanis [20], Gong and Ho [12], Glasserman and Gong [9], Fu and Hu [7]), Likelihood Ratio Method (LRM, see e.g. Reiman and Weiss [17] or Glynn [10]) and Rare Perturbation Analysis (RPA, see Brémaud and Vázquez-Abad [5] and Brémaud [2]).

In this article, we aim to prove that, under appropriate conditions

$$\begin{aligned} &\lim_{h\to 0} \frac{1}{h}\Big[\mathbb{E} f(W_{\theta+h}(0)) - \mathbb{E} f(W_\theta(0))\Big] \\ &\quad = \ \mathbb{E} \lim_{h\to 0} \frac{1}{h}\Big[f(W_{\theta+h}(0)) - f(W_\theta(0))\Big] = \mathbb{E} \frac{\partial}{\partial\theta} f(W_\theta(0)) \end{aligned} \tag{2}$$

and we give a formula replacing (2) when f is not differentiable but is of bounded variation. This formula was obtained by Konstantopoulos and Zazanis [15] under stronger assumptions on the service times distributions. However, due to the difficulty of passing to the limit in their approximation procedure, their formula does not give any insight on the equality of the left-hand and right-hand derivatives ; this information is crucial for practical use of the derivative estimator. Our method of proof avoids the passage to the limit and therefore allows for better control of the computations. Moreover, it can be extended in many ways to handle different situations.

The article is organized as follow: in Section 2, we give a construction of the G/G/1 queue and we derive some basic properties. The main result of the article is given in Section 3 and the same method is applied to second-order derivatives in Section 4; Section 5 shows how our method can be extended to other parameters, respectively the speed of the server and the rate of arrival in the system. Section 6 discusses the implementation of the estimates and a short review of Palm probabilities can be found in the appendix.

2 Construction of the G/G/1 queue.

In a formula like (2), the probability space does not depend on θ. To obtain this independence, we use the inversion representation (see Suri [18]) to generate service times: let $\{\xi_n\}_{n\in\mathbb{Z}}$ be a sequence of random variables uniformly distributed on $[0,1]$. Let $F(\cdot,\theta)$ be the common distribution function of service times; we can define its inverse function

$$G(\xi,\theta) = \sup(x \geq 0 \ : \ F(x,\theta) \leq \xi).$$

Then $\sigma_n(\theta) \stackrel{\text{def}}{=} G(\xi_n,\theta)$ is distributed according to $F(\cdot,\theta)$. This means that, if we choose as basic stationary random sequences $\{\tau_n\}_{n\in\mathbb{Z}}$ and $\{\xi_n\}_{n\in\mathbb{Z}}$, we define the queue on a probability space independent from θ. We note λ the intensity of the input process and P^0 the associated Palm probability—see Appendix for notations and details. In order to apply IPA, the following assumption on service times is needed:

Assumption A1 *The distribution of service times verifies the following conditions:*

(*i*) $\theta \mapsto G(\xi,\theta)$ *is differentiable and Lipschitz, that is*

$$|G(\xi,\theta_1) - G(\xi,\theta_2)| \leq K^{\sigma}(\xi)|\theta_1 - \theta_2|, \ \forall\, \theta_1,\theta_2 \in \Theta;$$

(*ii*) $\lambda\mathbb{E}^0\sigma_0^* < 1$, *with the notation* $\sigma_n^* \stackrel{\text{def}}{=} \sup_{\theta\in\Theta}\sigma_n(\theta) = \sup_{\theta\in\Theta} G(\xi_n,\theta)$.

Condition **A1**-(*i*) ensures that we have enough smoothness with respect to θ in the distribution of the service times. However, in a number of cases, ξ_n will not be directly known, in particular when observing a real experiment; this difficulty can be overcome with the following classical proposition (Suri [18]; for this formulation see Glasserman [8]):

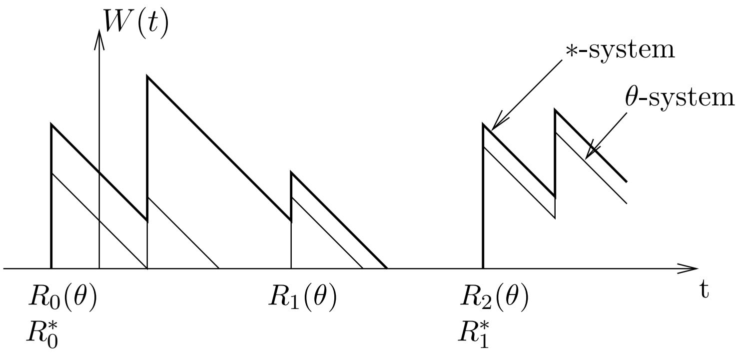


Figure 2: the domination property.

Proposition *Suppose that (i) $F(\cdot,\theta)$ has a density $\partial_x F(\cdot,\theta)$ which is strictly positive on an open interval I_θ and zero elsewhere; and (ii) F is continuously differentiable on $I_\theta \times \Theta$. Then*

$$\sigma'(\theta) = -\frac{\partial_\theta F(\sigma(\theta),\theta)}{\partial_x F(\sigma(\theta),\theta)}.$$

In the above formula, the prime denotes the derivative with respect to θ. A case of particular interest is when θ is a scale parameter of the service times, that is when $\sigma(\theta) = \theta\eta$ for some random variable η. Then we have directly

$$\sigma'(\theta) = \eta = \frac{\sigma(\theta)}{\theta}.$$

In particular, we do not need to know the real distribution of service times unless we actually want to simulate them. Note that **A1**-(i) is similar to assumption (i) of Section 1 in Konstantopoulos and Zazanis [15]; it is the classical assumption on smoothed distributions needed for IPA.

Using Assumption **A1**-(ii), we derive a bound on the size of the busy periods of the system for all possible values of θ. We shall note that we don't know *a priori* whether σ_n^* is P^0-a.s. finite or not; however, this condition is weaker than assumption (ii) and (iii) of [15]. With that in mind, let $\{R_k^*\}_{k\in\mathbb{Z}}$ be the regeneration times at which the arriving customers of the $*$-system find the queue empty—the $*$-system is the queue with service times $\{\sigma_n^*\}_{n\in\mathbb{Z}}$, whereas the θ-system uses service times $\{\sigma_n(\theta)\}_{n\in\mathbb{Z}}$. We can build the θ-system from the busy period process $\{R_k^*\}_{k\in\mathbb{Z}}$ but with the service times given by $\{\sigma_n(\theta)\}_{n\in\mathbb{Z}}$, so that the following domination property holds for the respective stationary workload of the queues—see Figure 2:

$$W_\theta(t) \le W^*(t),\ \forall\,\theta\in\Theta,\ \forall\,t\in\mathbb{R}. \tag{3}$$

With the above construction, we get $\{R_k^*\}_{k\in\mathbb{Z}} \subseteq \{R_k(\theta)\}_{k\in\mathbb{Z}}$, where $\{R_k(\theta)\}_{k\in\mathbb{Z}}$—or simply $\{R_k\}_{k\in\mathbb{Z}}$—denotes the beginning of busy period process for the θ-system. Moreover, we have the boundary property

$$R_-^*(t) \le R_-(\theta)(t) \le t < R_+(\theta)(t) \le R_+^*(t). \tag{4}$$

3 An IPA estimator for general non-decreasing functions.

In this section, we show that IPA applies with any non-decreasing *càdlàg* function f. But since f is not required to be continuous, we cannot apply (2) as such. First of all, we need to introduce an assumption similar to assumptions **A1**, **A2** and **A3′** of Konstantopoulos and Zazanis [15]:

Assumption A2 *The following inequalities hold:*

(*i*) $\mathbb{E}^0[K^\sigma(\xi_0)]^4 < \infty$;

(*ii*) $\mathbb{E}^0[A([R_0^*, R_1^*))]^4 < \infty$;

(*iii*) $\mathbb{E}^0[f(W^*(0))]^2 < \infty$.

Theorem 1 *Let μ_f be the measure on $\mathbb{R}$ associated with f. Assume* **A1** *and* **A2** *hold. Then J admits a right derivative with respect to θ given by*

$$\begin{aligned} J_r'(\theta) &= \lambda \mathbb{E}^0 W_\theta'(0)\Big[f(W_\theta(0)) - f(W_\theta(T_1-)) \\ &\qquad - \mathbb{1}_{\{W_\theta'(0)<0\}}\big[\mu_f(\{W_\theta(0)\}) - \mu_f(\{W_\theta(T_1-)\})\big]\Big], \end{aligned} \tag{5}$$

and its left derivative is

$$\begin{aligned} J_l'(\theta) &= \lambda \mathbb{E}^0 W_\theta'(0)\Big[f(W_\theta(0)) - f(W_\theta(T_1-)) \\ &\qquad - \mathbb{1}_{\{W_\theta'(0)>0\}}\big[\mu_f(\{W_\theta(0)\}) - \mu_f(\{W_\theta(T_1-)\})\big]\Big]. \end{aligned} \tag{6}$$

Example 2 With $f(w) = \mathbb{1}_{\{w \ge x\}}$, Theorem 1 yields

$$\begin{aligned} \frac{\partial_r}{\partial\theta}\mathrm{P}(W_\theta(0) > x) &= \lambda \mathbb{E}^0 W_\theta'(0)\Big[\mathbb{1}_{(W_\theta(T_1-), W_\theta(0)]}(x) \\ &\qquad - \mathbb{1}_{\{W_\theta'(0)<0\}}\big[\mathbb{1}_{\{W_\theta(0)=x\}} - \mathbb{1}_{\{W_\theta(T_1-)=x\}}\big]\Big] \\ \frac{\partial_l}{\partial\theta}\mathrm{P}(W_\theta(0) > x) &= \lambda \mathbb{E}^0 W_\theta'(0)\Big[\mathbb{1}_{(W_\theta(T_1-), W_\theta(0)]}(x) \\ &\qquad - \mathbb{1}_{\{W_\theta'(0)>0\}}\big[\mathbb{1}_{\{W_\theta(0)=x\}} - \mathbb{1}_{\{W_\theta(T_1-)=x\}}\big]\Big]. \end{aligned}$$

Theorem 1 shows that $J(\theta)$ admits right and left derivatives even when f is not continuous. But in a number of cases, we can get the equality of these two derivatives:

Corollary 3 *Assume* **A1** *and* **A2** *hold. If f is continuous or if $W_\theta(0)$ and $W_\theta(T_1-)$ admit densities with respect to* P^0 *then $J(\theta)$ is differentiable and*

$$J'(\theta) = \lambda \mathbb{E}^0 W'_\theta(0)[f(W_\theta(0)) - f(W_\theta(T_1-))]. \tag{7}$$

Proof If f is continuous, then $w \mapsto \mu_f(\{w\}) \equiv 0$. If $W_\theta(0)$ admits a P^0-density, say $\gamma^0(w)$, we can use the fact that $\mu_f(\{\cdot\}) = 0$ almost everywhere for the Lebesgue measure:

$$\begin{aligned} |\mathbb{E}^0 \mathbb{1}_{\{W'_\theta(0)<0\}} \mu_f(\{W_\theta(0)\})| &\leq \mathbb{E}^0 \mu_f(\{W_\theta(0)\}) \\ &= \int_0^\infty \mu_f(\{w\}) \gamma^0(w)\, dw = 0. \end{aligned}$$

In either case, the result is proved. ■

Remark In the case where f admits a derivative f', we can use the inversion formula (22) of Appendix and write (7) as

$$J'(\theta) = \lambda \mathbb{E}^0 [\int_0^{T_1} W'_\theta(t) f'(W_\theta(t)) dt] = \mathbb{E}[W'_\theta(0) f'(W_\theta(0))],$$

thus obtaining the expected IPA estimate (2). In this computation, we used the fact that $W'_\theta(t)$ is constant between arrivals during busy periods, and zero during idle periods. A comparison between the two estimates is made in Section 6.

Before starting the proof of the theorem, let us mention that our derivation is different from Konstantopoulos and Zazanis [15] in two respects: first we do not require an approximation procedure and we treat directly a non decreasing function f. This is made possible by the simple crucial observation that

$$f(y) - f(x) = \int_{(x,y]} \mu_f(dz) \quad \text{for all } x \leq y,$$

which allows us to have a better view of the residual terms in the level crossing analysis that follows. The result can be applied to any function of bounded variation if assumption **A2** is verified by both the increasing and decreasing parts of the function. Secondly, we do not need switch back and forth between the Palm probabilities with respect to the arrival process and with respect to the regeneration points as in [15]. However, we retain the fundamental idea of [15] by starting with its expression in terms of the Palm probability P^0.

Proof of Theorem 1 Assume that $f(0) = 0$, so that f is non-negative. The Palm inversion formula (22) gives

$$\begin{aligned}
\mathbb{E}f(W_\theta(0)) &= \lambda \mathbb{E}^0 \int_0^{T_1} f(W_\theta(t))\,dt \\
&= \lambda \mathbb{E}^0 \int_0^{T_1} \int_{\mathbb{R}_+} \mathbb{1}_{\{W_\theta(t)>x\}}\, \mu_f(dx)\,dt \\
&= \lambda \mathbb{E}^0 \int_{\mathbb{R}_+} \int_0^{T_1} \mathbb{1}_{\{W_\theta(t)>x\}}\, dt\, \mu_f(dx)
\end{aligned}$$

and therefore

$$\begin{aligned}
&\frac{1}{h}\mathbb{E}[f(W_{\theta+h}(0)) - f(W_\theta(0))] \\
&\quad = \frac{\lambda}{h}\mathbb{E}^0 \int_{\mathbb{R}_+} \int_0^{T_1} [\mathbb{1}_{\{W_{\theta+h}(t)>x\}} - \mathbb{1}_{\{W_\theta(t)>x\}}]\,dt\,\mu_f(dx).
\end{aligned}$$

In order to simplify the notations, let:

$$\begin{aligned}
\varphi(x,t) &\stackrel{\text{def}}{=} \mathbb{1}_{\{W_{\theta+h}(t)>x\}} - \mathbb{1}_{\{W_\theta(t)>x\}} \\
\Phi(\theta,h) &\stackrel{\text{def}}{=} \int_{\mathbb{R}_+} \int_0^{T_1} \varphi(x,t)\,dt\,\mu_f(dx).
\end{aligned}$$

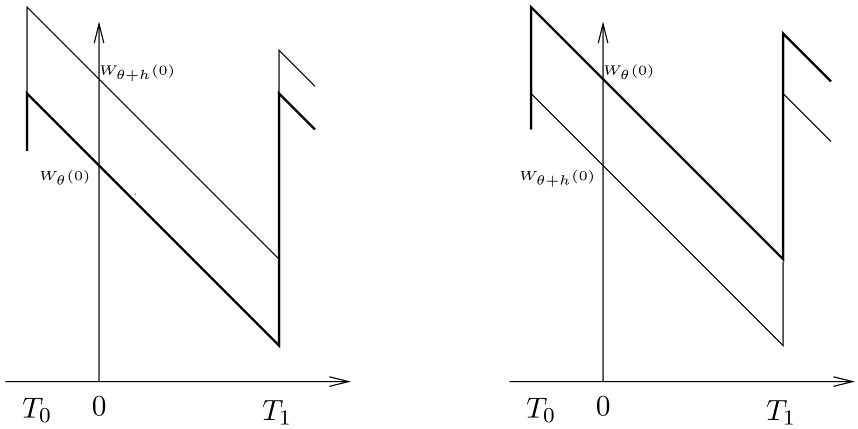
The first step of our proof is to compute $\lim_{h\to 0} \Phi(\theta,h)/h$. We will have to integrate a function taking its values in $\{-1,0,1\}$ with respect to $dt\,\mu_f(dx)$. Define also for any $t \in [0,T_1)$:

$$\Delta W_{\theta,h} \stackrel{\text{def}}{=} W_{\theta+h}(t) - W_\theta(t).$$

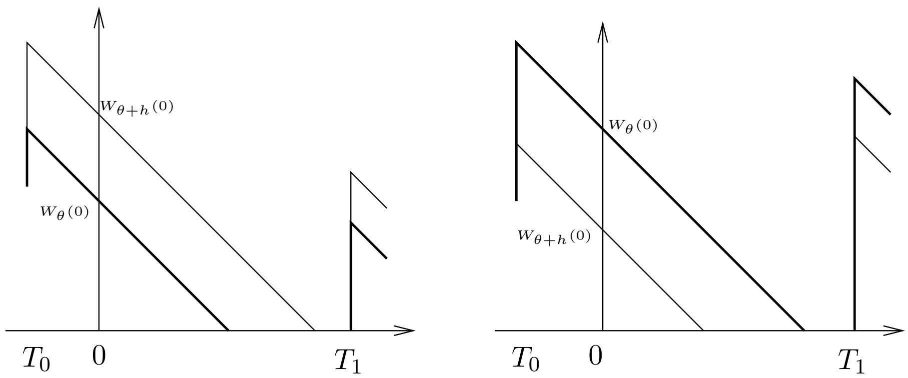
Assume first that $h > 0$. As shown in Figure 3, we must consider different cases depending on the relative position of $W_\theta(0)$ and $W_{\theta+h}(0)$. We have to add cases 3 and $3'$, where $W'_\theta(0) = 0$, preventing us to guess their relative positions. In fact, all the terms of the formula can be found in the first two cases and we will leave the other ones to the reader's attention.

Case 1: for h small enough, $W_{\theta+h}(0) > W_\theta(0)$ and $\varphi = 1$. The way to compute $\Phi(\theta,h)$ can be best understood with the help of Figure 4. Φ is equal to the area with a dashed border plus the dotted triangle on the left, minus the right one. Here the borders included in the areas are in bold; since all functions are *càdlàg*, these borders are the top and right ones.

$$\begin{aligned}
\frac{1}{h}\Phi(\theta,h) &= \Big[f(W_\theta(0)) - f(W_\theta(T_1-))\Big]\frac{\Delta W_{\theta,h}}{h} \\
&\quad + \frac{1}{h}\int_0^{\Delta W_{\theta,h}} \mu_f((W_\theta(0), W_\theta(0)+y])\,dy \\
&\quad - \frac{1}{h}\int_0^{\Delta W_{\theta,h}} \mu_f((W_\theta(T_1-), W_\theta(T_1-)+y])\,dy. \qquad (8)
\end{aligned}$$



Case 1: $W'(0) > 0$ and $W(T_1-) > 0$ Case 2: $W'(0) < 0$ and $W(T_1-) > 0$



Case 1′: $W'(0) > 0$ and $W(T_1-) = 0$ Case 2′: $W'(0) < 0$ and $W(T_1-) = 0$

Figure 3: Four different cases for the computation of Φ.

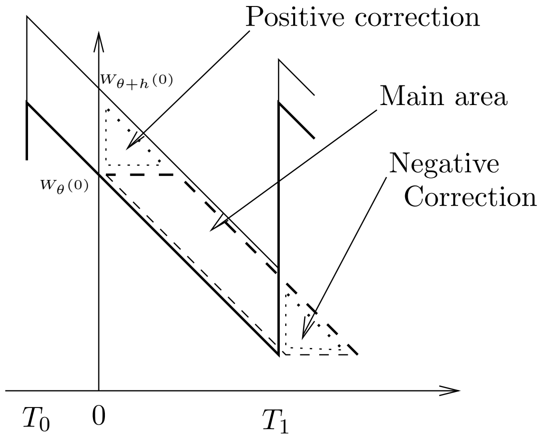


Figure 4: computation of Φ in case 1.

The first term converges to $[f(W_\theta(0)) - f(W_\theta(T_1-))]W'_\theta(0)$. Moreover,

$$\mu_f((W_\theta(0), W_\theta(0) + y]) = \mu_f((W_\theta(0), W_\theta(0) + y)) + \mu_f(\{W_\theta(0) + y\})$$

and since $\mu_f(\{W_\theta(0) + y\}) = 0$ dy-a.e., the second term of r.h.s. of equation (8) reads

$$\frac{1}{h}\int_0^{\Delta W_{\theta,h}} \mu_f((W_\theta(0), W_\theta(0) + y))\, dy,$$

which is less or equal than

$$(W'_\theta(0) + o(1))\, \mu_f((W_\theta(0), W_{\theta+h}(0))).$$

Since $W_\theta(0)$ is continuous in the neighborhood of θ, this goes to zero with h. The third term converges to 0 for the same reasons. So we have in this case:

$$\lim_{h\to 0+} \frac{1}{h}\Phi(\theta, h) = [f(W_\theta(0)) - f(W_\theta(T_1-))]W'_\theta(0).$$

Case 2: here $W_{\theta+h}(0) < W_\theta(0)$ and $\varphi = -1$. Due to the order of $W_{\theta+h}(0)$ and $W_\theta(0)$, we find a formula different from equation (8)—see Figure 5:

$$\begin{aligned}\frac{1}{h}\Phi(\theta, h) \;&=\; -\Bigg\{\Big[f(W_\theta(0)) - f(W_\theta(T_1-))\Big]\frac{-\Delta W_{\theta,h}}{h} \\ &\quad - \frac{1}{h}\int_0^{-\Delta W_{\theta,h}} \mu_f((W_\theta(0) - y, W_\theta(0)])\, dy \\ &\quad + \frac{1}{h}\int_0^{-\Delta W_{\theta,h}} \mu_f((W_\theta(T_1-) - y, W_\theta(T_1-)])\, dy\Bigg\}. \qquad (9)\end{aligned}$$

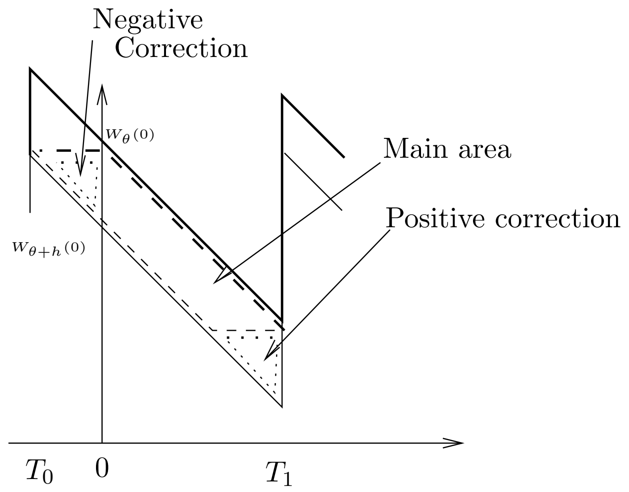


Figure 5: computation of Φ in case 2.

The first term is the same as in case 1, but the second is equal to

$$\frac{1}{h}\int_0^{-\Delta W_{\theta,h}} \mu_f((W_\theta(0)-y, W_\theta(0)))\,dy \;-\; \mu_f(\{W_\theta(0)\})\frac{\Delta W_{\theta,h}}{h}$$

and its limit is $\mu_f(\{W_\theta(0)\})W'_\theta(0)$. The last term of $\Phi(\theta,h)/h$ is computed in a similar way. Finally:

$$\begin{aligned}\lim_{h\to 0+}\frac{1}{h}\Phi(\theta,h) &= \Big[f(W_\theta(0)) - f(W_\theta(T_1-)) \\ &\qquad - \mu_f(\{W_\theta(0)\}) + \mu_f(\{W_\theta(T_1-)\})\Big]W'_\theta(0).\end{aligned}$$

We can summarize the above cases in the following formula:

$$\begin{aligned}\lim_{h\to 0+}\frac{1}{h}\Phi(\theta,h) &= W'_\theta(0)\Big[f(W_\theta(0)) - f(W_\theta(T_1-)) \\ &\qquad - \mathbb{1}_{\{W'_\theta(0)<0\}}[\mu_f(\{W_\theta(0)\}) - \mu_f(\{W_\theta(T_1-)\})]\Big].\end{aligned}$$

The next step is to find a bound for $\Phi(\theta,h)/h$ which has a finite mean with respect to P^0. The formulas for each case give:

$$\begin{aligned}\Big|\frac{1}{h}\Phi(\theta,h)\Big| &\le \Big(f(W_\theta(0)) - f(W_\theta(T_1-))\Big)\Big|\frac{\Delta W_{\theta,h}}{h}\Big| \\ &\quad + |f(W_{\theta+h}(0)) - f(W_\theta(0))|\cdot\Big|\frac{\Delta W_{\theta,h}}{h}\Big| \\ &\quad + |f(W_{\theta+h}(T_1-)) - f(W_\theta(T_1-))|\cdot\Big|\frac{\Delta W_{\theta,h}}{h}\Big| \\ &\le 3f(W^*(0))K_\theta^W(0).\end{aligned}$$

The last inequality takes advantage of the fact that f is non-decreasing and of the domination property (3). $K^W_\theta(t)$ is a Lipschitz coefficient for $W(t)$ w.r.t. θ. Finally,

$$\Big|\frac{1}{h}\Phi(\theta,h)\Big| \leq 3f(W^*(0))K^W_\theta(0).$$

The latter expression is independent from h. Moreover, it has a finite mean under P^0: from Cauchy-Schwartz inequality,

$$\mathbb{E}^0\Big[f(W^*(0))K^W_\theta(0)\Big] \leq \sqrt{\mathbb{E}^0[f(W^*(0))]^2}\sqrt{\mathbb{E}^0[K^W_\theta(0)]^2}.$$

The first mean is finite from assumption **A2**-(iii). To prove that the second one is also finite, we must first give an expression of $K^W_\theta(0)$:

$$\begin{aligned}
\Big|\frac{W_{\theta+h}(0)-W_\theta(0)}{h}\Big| &\leq \Big|\frac{W_{\theta+h}(T_{-1})-W_\theta(T_{-1})}{h}\Big| + \Big|\frac{\sigma_0(\theta+h)-\sigma_0(\theta)}{h}\Big| \\
&\leq \sum_{n\in\mathbb{Z}}\Big|\frac{\sigma_n(\theta+h)-\sigma_n(\theta)}{h}\Big|\mathbb{1}_{[R^*_-(T_0),0)}(T_n) \\
&\leq \sum_{n\in\mathbb{Z}} K^\sigma(\xi_n)\mathbb{1}_{[R^*_-(0),R^*_+(0))}(T_n) \\
&\stackrel{\text{def}}{=} K^W_\theta(0).
\end{aligned}$$

The first inequality comes from equation (1) and inequality $|a^+ - b^+| \leq |a-b|$; then we use the boundary property (4) and last the Lipschitz property **A1**-(i). To prove that $\mathbb{E}^0[K^W_\theta(0)]^2$ is finite, we can use the inequality $(x_1+\cdots+x_n)^p \leq n^{p-1}(x_1^p+\cdots+x_n^p)$ and

$$\begin{aligned}
&\mathbb{E}^0\Big[\sum_{n\in\mathbb{Z}} K^\sigma(\xi_n)\mathbb{1}_{[R^*_-(0),R^*_+(0))}(T_n)\Big]^2 \\
&\quad\leq \mathbb{E}^0\Big[\sum_{n\in\mathbb{Z}} A([R^*_0,R^*_1))[K^\sigma(\xi_n)]^2\mathbb{1}_{[R^*_-(0),R^*_+(0))}(T_n)\Big] \\
&\quad\leq \mathbb{E}^0[A([R^*_0,R^*_1))]^2[K^\sigma(\xi_0)]^2 \\
&\quad\leq \sqrt{\mathbb{E}^0[A([R^*_0,R^*_1))]^4}\sqrt{\mathbb{E}^0[K^\sigma(\xi_0)]^4},
\end{aligned}$$

which is finite from **A2**-(i) and **A2**-(ii). Here, the second inequality uses Lemma 9.

Summing up our results, we can apply the Dominated Convergence Theorem:

$$\begin{aligned}
J'_r(\theta) &\stackrel{\text{def}}{=} \lim_{h\to 0+}\mathbb{E}[f(W_{\theta+h}(0))-f(W_\theta(0))] \\
&= \lim_{h\to 0+}\lambda\mathbb{E}^0\frac{1}{h}\Phi(\theta,h) \\
&= \lambda\mathbb{E}^0\lim_{h\to 0+}\frac{1}{h}\Phi(\theta,h)
\end{aligned}$$

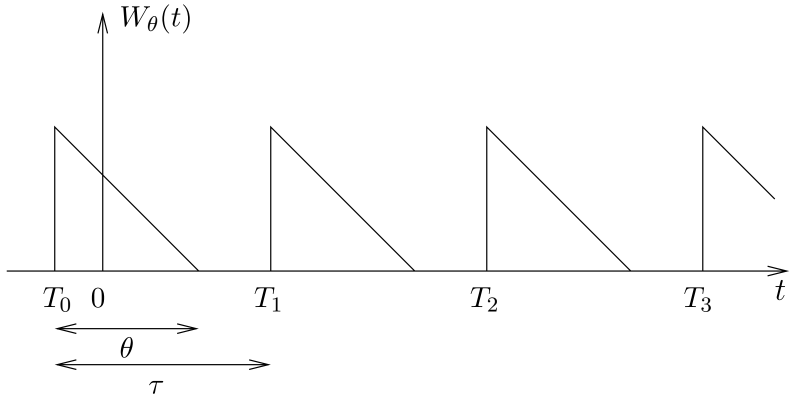


Figure 6: workload of the D/D/1 queue.

This gives equation (5). The case of $h < 0$ is handled in the same way and gives equation (6)—loosely speaking, the above cases used the sign of $W_{\theta+h}(0) - W_\theta(0)$; this sign is inverted if $h < 0$. This concludes the proof of the theorem. ■

Remark Assumption **A2** ensures that $\mathbb{E}^0[f(W^*(0))K_\theta^W(0)] < \infty$. If we know that f is bounded, for example, the only assumptions we need are

(i) $\mathbb{E}^0[K^\sigma(\xi_0)]^2 < \infty$;

(ii) $\mathbb{E}^0[A([R_0^*, R_1^*))]^2 < \infty$.

This reduced set of assumptions can for instance be used in Example 2.

It is important to point out that Corollary 3 cannot always be applied. We show such a case in next example :

Example 4 Consider a D/D/1 queue, that is with deterministic inter-arrival time τ and service time $\theta < \tau$. In order to have a stationary queue, T_0 must be uniformly spread in $[-\tau, 0]$. As we can see in Figure 6, we have

$$W(T_n) = \theta,\ W(T_n-) = 0,\ W'_\theta(0) = 1\ \mathrm{P}^0\text{-a.s.}$$

For $x > 0$, take $f(w) = \mathbb{1}_{\{w \geq x\}}$ as in Example 2. Then if $\theta \leq x$, $J(\theta) = 0$, else

$$J(\theta) = \int_{-\tau}^{0} \mathbb{1}_{\{\theta+t \geq x\}} \frac{dt}{\tau} = \frac{\theta - x}{\tau}.$$

Finally $J(\theta) \equiv \mathrm{P}(W_\theta(0) \geq x) = \left(\frac{\theta-x}{\tau}\right)^+$, which is not differentiable at point $\theta = x$. Besides,

$$J'_r(\theta) \quad = \quad \lambda\mathbb{E}^0[\mathbb{1}_{\{\theta \geq x\}} - \mathbb{1}_{\{0 \geq x\}}]$$

$$
\begin{aligned}
&= \frac{1}{\tau}\mathbb{1}_{\{\theta\geq x\}} \\
J_l'(\theta) &= \lambda \mathbb{E}^0[\mathbb{1}_{\{\theta\geq x\}} - \mathbb{1}_{\{0\geq x\}} + \mathbb{1}_{\{0=x\}} - \mathbb{1}_{\{\theta=x\}}] \\
&= \frac{1}{\tau}\mathbb{1}_{\{\theta> x\}}
\end{aligned}
$$

4 Second order derivative.

The method used in Section 3 can be used for higher-order derivatives. We need assumptions on the properties of our system and some new moment conditions:

Assumption A3 *G and f verify the following:*

(i) *$\theta \mapsto G(\xi,\theta)$ is twice differentiable and there exists a function $\xi \mapsto K^{\sigma'}(\xi)$ such that*

$$
|G(\xi,\theta+2h) - 2G(\xi,\theta+h) + G(\xi,\theta)| \leq h^2 K^{\sigma'}(\xi);
$$

(ii) *$w \mapsto f(w)$ is non-decreasing and differentiable.*

Assumption A4 *The following inequalities hold:*

(i) $\mathbb{E}^0[K^{\sigma}(\xi_0)]^8 < \infty$;

(ii) $\mathbb{E}^0[K^{\sigma'}(\xi_0)]^4 < \infty$;

(iii) $\mathbb{E}^0[A([R_0^*, R_1^*))]^8 < \infty$;

(iv) $\mathbb{E}^0[f(W^*(0))]^2 < \infty$;

(v) $\mathbb{E}^0[\sup_\theta f'(W_\theta(0))]^2 < \infty$;

(vi) $\mathbb{E}^0[\sup_\theta f'(W_\theta(T_1-))]^2 < \infty$.

The main result of this section is:

Theorem 5 *Assume* **A1**, **A3** *and* **A4** *hold; then J admits a right second derivative with respect to θ given by*

$$
\begin{aligned}
J_r''(\theta) = \lambda \mathbb{E}^0\Big[& W_\theta''(0)\Big[f(W_\theta(0)) - f(W_\theta(T_1-))\Big] \\
& + [W'(0)]^2\Big[f'(W_\theta(0)) - f'(W_\theta(T_1-)) \\
& - \mathbb{1}_{\{W_\theta'(0)<0\}}[\mu_{f'}(\{W_\theta(0)\}) - \mu_{f'}(\{W_\theta(T_1-)\})]\Big]\Big], \quad (10)
\end{aligned}
$$

and its left second derivative is

$$
\begin{aligned}
J_l''(\theta) \;\; = \;\; & \lambda \mathbb{E}^0 \Big[W_\theta''(0) \Big[f(W_\theta(0)) - f(W_\theta(T_1-)) \Big] \\
& + [W'(0)]^2 \Big[f'(W_\theta(0)) - f'(W_\theta(T_1-)) \\
& - \mathbb{1}_{\{W_\theta'(0)>0\}} [\mu_{f'}(\{W_\theta(0)\}) - \mu_{f'}(\{W_\theta(T_1-)\})] \Big] \Big]. \quad (11)
\end{aligned}
$$

Corollary 6 *Assume* **A1**, **A3** *and* **A4** *hold; if* f' *is continuous or if* $W_\theta(0)$ *and* $W_\theta(T_1-)$ *admit densities w.r.t.* P^0 *then* $J(\theta)$ *is differentiable twice and*

$$
\begin{aligned}
J''(\theta) \;\; = \;\; & \lambda \mathbb{E}^0 \Big[W_\theta''(0) \Big[f(W_\theta(0)) - f(W_\theta(T_1-)) \Big] \\
& + [W'(0)]^2 \Big[f'(W_\theta(0)) - f'(W_\theta(T_1-)) \Big] \Big].
\end{aligned}
$$

Proof of Theorem 5 As this proof is very similar to that of Theorem 1, we will omit the parts of it which are not new. We want to compute the limit as $h \to 0$ of

$$
\frac{1}{h^2} \mathbb{E} \Big[f(W_{\theta+2h}(0)) - 2f(W_{\theta+h}(0)) + f(W_\theta(0)) \Big] = \frac{\lambda}{h^2} \mathbb{E}^0 \Phi_2(\theta, h),
$$

with

$$
\begin{aligned}
\Phi_2(\theta, h) \;\; \stackrel{\text{def}}{=} \;\; & \int_{\mathbb{R}_+} \int_0^{T_1} \Big[[\mathbb{1}_{\{W_{\theta+2h}(t)>x\}} - \mathbb{1}_{\{W_{\theta+h}(t)>x\}}] \\
& - [\mathbb{1}_{\{W_{\theta+h}(t)>x\}} - \mathbb{1}_{\{W_\theta(t)>x\}}] \Big] dt \, \mu_f(dx).
\end{aligned}
$$

We will once more distinguish two important cases among all possible ones, depending on the sign of $W_\theta'(0)$. Suppose first that $h > 0$.

Case 1: $W_\theta'(0) > 0$; for h small enough, $W_\theta(0) < W_{\theta+h}(0) < W_{\theta+2h}(0)$— see Figure 7. We have here to subtract the areas of two bands which are of the same sort as in Theorem 1:

$$
\begin{aligned}
\Phi_2(\theta, h) \;\; = \;\; & \Delta W_{\theta+h,h} \Big[f(W_{\theta+h}(0)) - f(W_{\theta+h}(T_1-)) \Big] \\
& - \Delta W_{\theta,h} \Big[f(W_\theta(0)) - f(W_\theta(T_1-)) \Big] \\
& + A_{\theta,h}(0) - A_{\theta,h}(T_1-)
\end{aligned}
$$

where

$$
\begin{aligned}
A_{\theta,h}(t) \;\; = \;\; & \int_0^{\Delta W_{\theta+h,h}} \mu_f((W_{\theta+h}(t), W_{\theta+h}(t) + y]) \, dy \\
& - \int_0^{\Delta W_{\theta,h}} \mu_f((W_\theta(t), W_\theta(t) + y]) \, dy.
\end{aligned}
$$

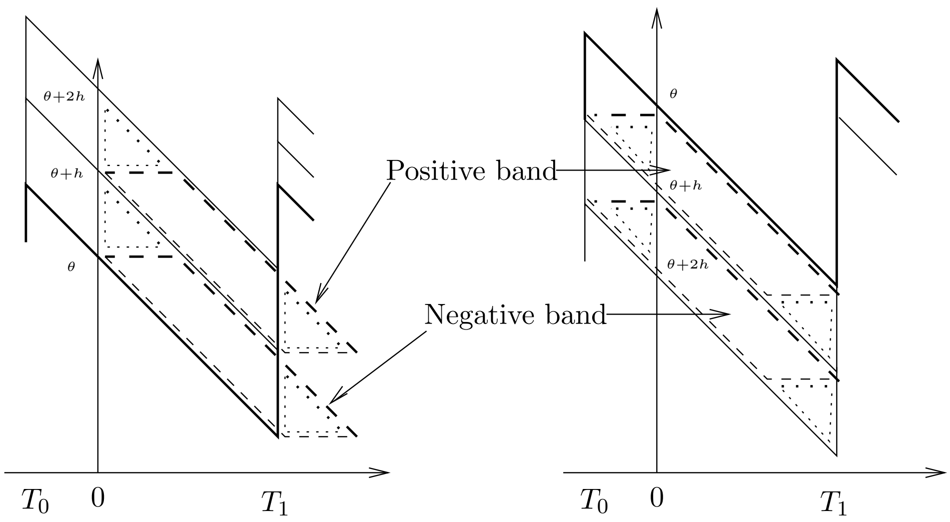


Figure 7: computation of Φ_2 in cases 1 and 2

The main term is equal to

$$\Delta W_{\theta+h,h}\Big[f(W_{\theta+h}(0)) - f(W_{\theta+h}(T_1-)) - f(W_\theta(0)) + f(W_\theta(T_1-))\Big]$$
$$+ \Delta^2 W_{\theta,h}\Big[f(W_\theta(0)) - f(W_\theta(T_1-))\Big].$$

Moreover,

$$\begin{aligned} A_{\theta,h}(0) &= \int_0^{\Delta W_{\theta,h}} \Big[f(W_{\theta+h}(0)+y) - f(W_{\theta+h}(0)) \\ &\qquad - f(W_\theta(0)+y) + f(W_\theta(0))\Big]dy + o(h^2) \\ &= \int_0^{\Delta W_{\theta,h}} hW'_\theta(0)\mu_{f'}((W_\theta(0), W_\theta(0)+y])dy + o(h^2) \end{aligned}$$

As in Theorem 1-case 1, $\lim_{h\to 0} A_{\theta,h}(0)/h^2 = 0$; the limit is the same for $A_{\theta,h}(T_1-)$. Consequently,

$$\begin{aligned} \lim_{h\to 0+} \frac{1}{h^2}\Phi_2(\theta,h) &= [W'_\theta(0)]^2\Big[f'(W_\theta(0)) - f'(W_\theta(T_1-))\Big] \\ &\quad + W''_\theta(0)\Big[f(W_\theta(0)) - f(W_\theta(T_1-))\Big]. \end{aligned}$$

Case 2: $W'_\theta(0) < 0$; for h small enough, $W_\theta(0) > W_{\theta+h}(0) > W_{\theta+2h}(0)$ and

$$\begin{aligned} \Phi_2(\theta,h) &= -1\cdot\Big\{-\Delta W_{\theta+h,h}\Big[f(W_{\theta+h}(0)) - f(W_{\theta+h}(T_1-))\Big] \\ &\quad - \Delta W_{\theta,h}\Big[f(W_\theta(0)) - f(W_\theta(T_1-))\Big] \\ &\quad - B_{\theta,h}(0) + B_{\theta,h}(T_1-)\Big\} \end{aligned}$$

where

$$\begin{aligned}B_{\theta,h}(t) &\stackrel{\text{def}}{=} \int_0^{-\Delta W_{\theta+h,h}} \mu_f((W_{\theta+h}(t)-y, W_{\theta+h}(t)])\, dy \\ &\quad - \int_0^{-\Delta W_{\theta,h}} \mu_f((W_\theta(t)-y, W_\theta(t)])\, dy.\end{aligned}$$

While the main part has the same limit as in case 1, we have

$$\begin{aligned}B_{\theta,h}(0) &= \int_0^{-\Delta W_{\theta,h}} hW'_\theta(0)\mu_{f'}((W_\theta(0)-y, W_\theta(0)])dy + o(h^2) \\ &= -h^2[W'_\theta(0)]^2\mu_{f'}(\{W_\theta(0)\}) + o(h^2).\end{aligned}$$

Finally, in case 2,

$$\begin{aligned}\lim_{h\to 0+} \frac{1}{h^2}\Phi_2(\theta,h) &= [W'_\theta(0)]^2\Big[f'(W_\theta(0)) - f'(W_\theta(T_1-))\Big] \\ &\quad + W''_\theta(0)\Big[f(W_\theta(0)) - f(W_\theta(T_1-))\Big] \\ &\quad - [W'_\theta(0)]^2\Big[\mu_{f'}(\{W_\theta(0)\}) - \mu_{f'}(\{W_\theta(T_1-)\})\Big].\end{aligned}$$

Besides,

$$\begin{aligned}\Big|\frac{1}{h^2}\Phi_2(\theta,h)\Big| &\le 3\Big[\sup_\theta f'(W_\theta(0)) + \sup_\theta f'(W_\theta(T_1-))\Big]\Big[K_\theta^W(0)\Big]^2 \\ &\quad + f(W^*(0))K_\theta^{W'}(0),\end{aligned}$$

where $K_\theta^W(0)$ is the same as in Theorem 1 and

$$K_\theta^{W'}(0) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} K^{\sigma'}(\xi_n)\mathbb{1}_{[R_-^*(0),R_+^*(0))}(T_n).$$

As in theorem 1, we use Lemma 9, Cauchy-Schwartz inequality and assumption **A4** to prove that $|\Phi_2(\theta,h)/h^2|$ has a finite mean under P^0. Using the Dominated Convergence Theorem, we find expressions (10) and (11) for the second derivatives of J. ■

5 Other parameters of the queue.

Let us consider a setting slightly different from the original one: we still deal with a G/G/1 queue, but now working at speed ν. Lindley's equation for the workload of the queue reads:

$$W_\nu(t) = \Big(W_\nu(T_n-) + \sigma_n - \nu(t-T_n)\Big)^+,\ t\in[T_n, T_{n+1}).$$

We address the same problem as in Section 3 in this new setting. Our method can apply in this case in the same way as for variable service times; we will try to keep the notations as close as possible to those of Section 2 to point out the similitudes, replacing θ with ν when necessary.

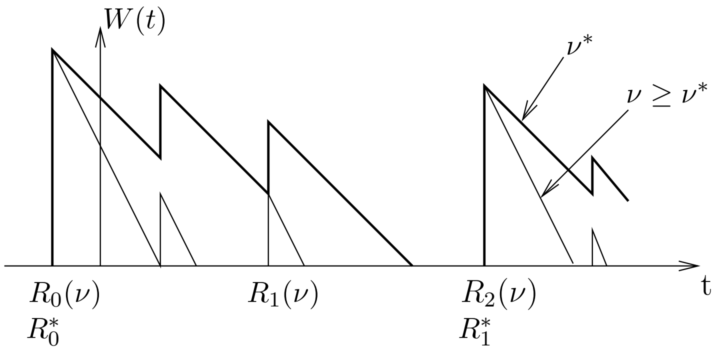


Figure 8: the domination property for the speed

Remark As an anonymous reviewer pointed out, if we define $\bar{W}_\nu(t) = \nu W_\nu(t)$ we have the relation:

$$\bar{W}_\nu(t) = \left(\bar{W}_\nu(T_n-) + \frac{\sigma_n}{\nu} - (t - T_n)\right)^+, \ t \in [T_n, T_{n+1}).$$

This means that the queue with workload $\bar{W}_\nu$ fits in the framework of Sections 2 and 3. Nevertheless, what we want to estimate is $(\partial/\partial\nu)\mathbb{E}f(\nu\bar{W}_\nu(0))$, which does not follow directly from Theorem 1. The extra computations needed would cancel the gain of using Theorem 1. Note also that this result will be useful in the second part of this section to deal with parameters of the arrival process.

Assume that $\nu \geq \nu^* > 0$; then as in Section 2, we can construct all the queues for different values of ν so that for all $\nu \geq \nu^*$ and $t \in \mathbb{R}$, we have the relation (see Figure 8)

$$W_\nu(t) \leq W_{\nu^*}(t) \tag{12}$$

$$R^*_-(t) \leq R_-(\nu)(t) \leq t < R_+(\nu)(t) \leq R^*_+(t), \tag{13}$$

The assumption on moments we need is much like **A2**:

Assumption A5 *The following moments are finite:*

(i) $\mathbb{E}^0[\tau_0]^4 < \infty$;

(ii) $\mathbb{E}^0[A([R_0^*, R_1^*))]^4 < \infty$;

(iii) $\mathbb{E}^0[f(W_{\nu^*}(0))]^2 < \infty$.

The first real difference with the results of Section 3 is that the expressions for the derivative use a primitive of f, whereas only f appeared in Theorem 1.

Theorem 7 *Let F be a primitive of f; if* **A5** *holds, then J has a right-hand derivative equal to:*

$$\begin{aligned} J'_r(\nu) &= \frac{\lambda}{\nu^2}\mathbb{E}^0\Big\{\nu W'_\nu(0)\Big[f(W_\nu(0)) - f(W_\nu(T_1-))\Big] \\ &\quad + F(W_\nu(0)) - F(W_\nu(T_1-)) \\ &\quad - [W_\nu(0) - W_\nu(T_1-)]f(W_\nu(T_1-))\Big\} \end{aligned} \tag{14}$$

and its left-hand derivative is

$$\begin{aligned} J'_l(\nu) &= \frac{\lambda}{\nu^2}\mathbb{E}^0\Big\{\nu W'_\nu(0)\Big[f(W_\nu(0)) - f(W_\nu(T_1-))\Big] \\ &\quad + F(W_\nu(0)) - F(W_\nu(T_1-)) \\ &\quad - [W_\nu(0) - W_\nu(T_1-)]f(W_\nu(T_1-)) \\ &\quad + \nu W'_\nu(0)[\mu_f(\{W_\nu(0)\}) - \mu_f(\{W_\nu(T_1-)\})] \\ &\quad - [W_\nu(0) - W_\nu(T_1-)]\mu_f(\{W_\nu(T_1-)\})\Big\}. \end{aligned}$$

If f is continuous or if both $W_\nu(0)$ and $W_\nu(T_1-)$ admit densities w.r.t. P^0*, then J is differentiable and its derivative is equal to J'_r.*

Remark The expressions in Theorem 7 seem really complicated when compared to those obtained in Theorem 1; in fact, in the case where f is differentiable, the inversion formula applied to (14) gives the classic IPA formula

$$J'(\nu) = \mathbb{E}W'_\nu(0)f(W_\nu(0)).$$

The complexity of (14) comes from the fact that $W'_\nu(t)$ is not constant on $[T_0, T_1)$.

Proof of Theorem 7 We once more proceed as in the proof of Theorem 1—more details can be found in [4]. Define

$$\Phi(\nu, h) \stackrel{\text{def}}{=} \int_{\mathbb{R}_+}\int_0^{T_1}[\mathbb{1}_{\{W_{\nu+h}(t)>x\}} - \mathbb{1}_{\{W_\nu(t)>x\}}]\,dt\,\mu_f(dx)$$

and remark that

$$\frac{1}{h}\mathbb{E}[f(W_{\nu+h}(0)) - f(W_\nu(0))] = \frac{\lambda}{h}\mathbb{E}^0\Phi(\nu, h).$$

We will consider only right-hand derivatives; left-hand derivatives are obtained with the same method. Figure 9 shows how Φ can be computed: the main area is equal to the area of the trapezium on the right. As W_ν is linear in ν, we have

$$\Delta W_{\nu,h}(T_1-) - \Delta W_{\nu,h}(0) = hT'_1,$$

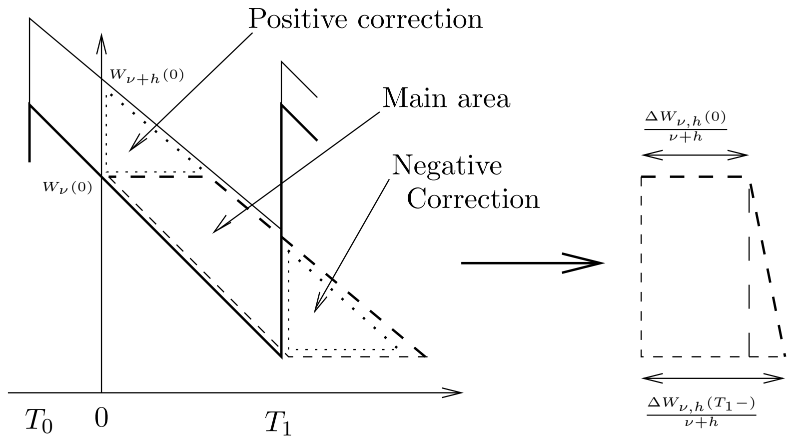


Figure 9: computation of Φ for $h > 0$.

where

$$T_1' \stackrel{\text{def}}{=} \min\Big[\frac{W_\nu(0)}{\nu}, T_1\Big] = \frac{W_\nu(0) - W_\nu(T_1-)}{\nu}.$$

The area of the trapezium of Figure 9 is equal to

$$\begin{aligned} \mathcal{A} &= \frac{\Delta W_{\nu,h}(0)}{\nu+h}\Big[f(W_\nu(0)) - f(W_\nu(T_1-))\Big] \\ &\quad + \int_0^{\frac{hT_1'}{\nu+h}} \mu_f\Big(\Big(W_\nu(T_1-), W_\nu(0) - \frac{\nu(\nu+h)}{h}y\Big]\Big)\, dy \\ &= \frac{h}{\nu+h}\Big\{\frac{\Delta W_{\nu,h}(0)}{h}\Big[f(W_\nu(0)) - f(W_\nu(T_1-))\Big] \\ &\quad + \frac{1}{\nu}\Big[F(W_\nu(0)) - F(W_\nu(T_1-))\Big] \\ &\quad - \frac{W_\nu(0) - W_\nu(T_1-)}{\nu} f(W_\nu(T_1-))\Big\}. \end{aligned}$$

The additional terms read:

$$\begin{aligned} &\int_0^{\frac{\Delta W_{\nu,h}(0)}{\nu+h}} \mu_f((W_\nu(0), W_\nu(0) + (\nu+h)y])\, dy \\ &- \int_0^{\frac{\Delta W_{\nu,h}(T_1-)}{\nu+h}} \mu_f((W_\nu(T_1-), W_\nu(T_1-) + (\nu+h)y])\, dy. \end{aligned}$$

As we have shown in the proof of Theorem 1, this kind of expression is an $o(h)$ and

$$\begin{aligned} \lim_{h\to 0+} \Phi(\nu, h) &= \frac{W_\nu'(0)}{\nu}\Big[f(W_\nu(0)) - f(W_\nu(T_1-))\Big] \\ &\quad + \frac{1}{\nu^2}\Big[F(W_\nu(0)) - F(W_\nu(T_1-))\Big] \end{aligned}$$

$$- \frac{W_\nu(0) - W_\nu(T_1-)}{\nu} f(W_\nu(T_1-)).$$

Moreover, as in Theorem 1, we have

$$\left|\frac{1}{h}\Phi(\nu,h)\right| \leq \frac{1}{\nu}[3K_\nu^W(0) + 2\tau_0] f(W_{\nu^*}(0)),$$

where $K_\nu^W(0)$ is a Lipschitz coefficient for $W_\nu(0)$ w.r.t. ν, which can be expressed as in Theorem 1 as

$$K_\nu^W(0) \stackrel{\text{def}}{=} \sum_{n\in\mathbb{Z}} \tau_n \mathbb{1}_{[R_-^*(0), R_+^*(0))}(T_n).$$

One can easily check that assumption **A5** suffices to prove that $|\Phi/h|$ is bounded by an integrable variable. Consequently, we can apply the Dominated Convergence Theorem and find the expected result. ∎

The method used so far does not apply to the case where the parameter of interest is a parameter of the inter-arrival times; in this case, the Palm measure associated to the arrival process depends on the parameter and the method fails. We show how a change of time scale can be used in some cases. We consider a G/G/1 queue with inter-arrival times $\{\tau_n(\alpha)\}_{n\in\mathbb{Z}}$, $\alpha \geq \alpha^* > 0$ and we will restrict our attention to the following case:

Assumption A6 *α is a scale parameter for $\tau_n(\alpha)$, that is $\tau_n(\alpha) = \alpha\eta_n$.*

Lindley's equation takes the form

$$W_\alpha(t) = \Big(W_\alpha(T_n(\alpha)-) + \sigma_n - (t - T_n(\alpha))\Big)^+, \ t \in [T_n(\alpha), T_{n+1}(\alpha)) \quad (15)$$

Now define a G/G/1 queue *with speed* α which inter-arrival times, service times and arrival process are given by:

$$\begin{aligned} \tilde{\tau}_n &\stackrel{\text{def}}{=} \frac{\tau_n(\alpha)}{\alpha} = \eta_n \\ \tilde{\sigma}_n &\stackrel{\text{def}}{=} \sigma_n \\ \widetilde{T}_n &\stackrel{\text{def}}{=} \frac{T_n(\alpha)}{\alpha}. \end{aligned}$$

These processes are stationary with respect to the measurable flow $\tilde{\theta}_t \stackrel{\text{def}}{=} \theta_{\alpha t}$ and the queue they define is stable whenever the original one is; this queue will be referred to as the "auxiliary system". Throughout this section, we will use the same notations as for the main system, but with a tilde. Lindley's equation for the auxiliary system reads:

$$\widetilde{W}_\alpha(t) = \Big(\widetilde{W}_\alpha(\widetilde{T}_n-) + \tilde{\sigma}_n - \alpha(t - \widetilde{T}_n)\Big)^+, \ t \in [\widetilde{T}_n, \widetilde{T}_{n+1}). \quad (16)$$

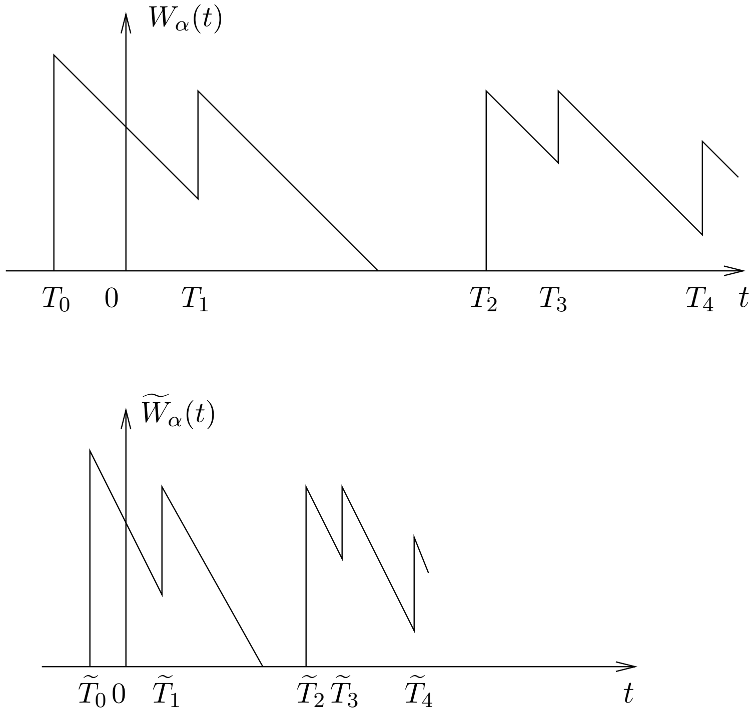


Figure 10: Change of time scale for $\alpha = 2$.

Comparing equations (15) and (16) and noting that the process $W_\alpha(\alpha t)$ is stationary with respect to the flow $\tilde{\theta}_t$, uniqueness in Loynes' Stability Theorem—see Baccelli and Brémaud [1]— yields

$$W_\alpha(t) = \widetilde{W}_\alpha(t/\alpha).$$

The effect of the change of time scale can be seen on Figure 10. Moreover,

$$\begin{aligned} \tilde{\lambda} &= \mathbb{E}\widetilde{A}((0,1]) \\ &= \mathbb{E}A((0,\alpha]) \;=\; \alpha\lambda(\alpha) \\ \widetilde{W}'_\alpha(0) &= W'_\alpha(0). \end{aligned}$$

In the computation of $\tilde{\lambda}$, we use the fact that the auxiliary system is defined on the same probability space than the main one. It has its own Palm measure associated to $\{\widetilde{T}_n\}_{n\in\mathbb{Z}}$, say $\widetilde{\mathrm{P}}^0$. The way to switch between probability measures P^0_α and $\widetilde{\mathrm{P}}^0$ will be shown in the proof of Theorem 8. Before proceeding, we need a set of **A5**-like conditions:

Assumption A7 *The following conditions hold:*

(*i*) $\mathbb{E}^0_\alpha[\tau_0]^4 < \infty$;

(*ii*) $\mathbb{E}^0_{\alpha^*}[A([R^*_0, R^*_1))]^4 < \infty$;

(*iii*) $\mathbb{E}^0_{\alpha^*}[f(W_{\alpha^*}(0))]^2 < \infty$.

Using this model, we find the following result:

Theorem 8 *Assume* **A6** *and* **A7** *hold; then*

$$\begin{aligned} J'_r(\alpha) &= \frac{\lambda}{\alpha}\mathbb{E}^0_\alpha\Big\{\alpha W'_\alpha(0)\Big[f(W_\alpha(0)) - f(W_\alpha(T_1-))\Big] \\ &\quad + \Big[F(W_\alpha(0)) - F(W_\alpha(T_1-))\Big] \\ &\quad - [W_\alpha(0) - W_\alpha(T_1-)]f(W_\alpha(T_1-))\Big\} \qquad (17) \\ J'_l(\alpha) &= \frac{\lambda}{\alpha}\mathbb{E}^0_\alpha\Big\{\alpha W'_\alpha(0)\Big[f(W_\alpha(0)) - f(W_\alpha(T_1-))\Big] \\ &\quad + \Big[F(W_\alpha(0)) - F(W_\alpha(T_1-))\Big] \\ &\quad - [W_\alpha(0) - W_\alpha(T_1-)]f(W_\alpha(T_1-)) \\ &\quad - \alpha W'_\alpha(0)\Big[\mu_f(\{W_\alpha(0)\}) - \mu_f(\{W_\alpha(T_1-)\})\Big] \\ &\quad + [W_\alpha(0) - W_\alpha(T_1-)]\mu_f(\{W_\alpha(T_1-)\})\Big\}. \qquad (18) \end{aligned}$$

If f is continuous or if $W_\alpha(0)$ and $W_\alpha(T_1(\alpha)-)$ admit densities with respect to P^0_α then J is differentiable w.r.t. α and its derivative is equal to J'_r.

Proof We have

$$J(\alpha) = \mathbb{E}f(W_\alpha(0)) = \mathbb{E}f(\widetilde{W}_\alpha(0))$$

where $\widetilde{W}_\alpha(0)$ is the workload of the auxiliary queue with speed α. We aim to apply Theorem 7 to this queue and then adapt the result to the main queue. The three conditions of **A5** correspond to the three ones of **A7**: for condition **A5**-(i), note that

$$\begin{aligned} \widetilde{\mathbb{E}}^0[\tilde{\tau}_0]^4 &= \frac{1}{\tilde{\lambda}}\mathbb{E}\sum_{n\in\mathbb{Z}}[\tilde{\tau}_n]^4 \mathbb{1}_{\{\widetilde{T}_n\in(0,1]\}} \\ &= \frac{1}{\alpha\lambda(\alpha)}\mathbb{E}\sum_{n\in\mathbb{Z}}\Big[\frac{\tau_n(\alpha)}{\alpha}\Big]^4 \mathbb{1}_{\{T_n(\alpha)\in(0,\alpha]\}} \\ &= \frac{1}{\alpha^4}\mathbb{E}^0_\alpha[\tau_0]^4 < \infty \end{aligned}$$

and for **A5**-(ii),

$$\widetilde{\mathbb{E}}^0[\widetilde{A}([\widetilde{R}^*_0, \widetilde{R}^*_1))]^4 = \frac{1}{\tilde{\lambda}}\mathbb{E}\sum_{n\in\mathbb{Z}}[\widetilde{A}([\widetilde{R}^*_-(\widetilde{T}_n), \widetilde{R}^*_+(\widetilde{T}_n)))]^4 \mathbb{1}_{\{\widetilde{T}_n\in(0,1]\}}$$

$$= \frac{1}{\alpha\lambda(\alpha)} \mathbb{E} \sum_{n\in\mathbb{Z}} [A([R^*_-(T_n), R^*_+(T_n)))]^4 \mathbb{1}_{\{T_n(\alpha^*)\in(0,\alpha^*]\}}$$
$$= \mathbb{E}^0_{\alpha^*}[A([R^*_0, R^*_1))]^4 < \infty.$$

Finally, for **A5**-(iii),

$$\widetilde{\mathbb{E}}^0[f(\widetilde{W}_{\alpha^*}(0))]^2 = \mathbb{E}^0_{\alpha^*}[f(W_{\alpha^*}(0))]^2 < \infty.$$

So we apply Theorem 7 and find for the right-hand derivative:

$$\begin{aligned} J'_r(\alpha) &= \frac{\tilde{\lambda}}{\alpha^2} \widetilde{\mathbb{E}}^0 \Big\{ \alpha \widetilde{W}'_\alpha(0) \Big[f(\widetilde{W}_\alpha(0)) - f(\widetilde{W}_\alpha(\widetilde{T}_1-)) \Big] \\ &\quad + \Big[F(\widetilde{W}_\alpha(0)) - F(\widetilde{W}_\alpha(\widetilde{T}_1-)) \Big] \\ &\quad - [\widetilde{W}_\alpha(0) - \widetilde{W}_\alpha(\widetilde{T}_1-)] f(\widetilde{W}_\alpha(\widetilde{T}_1-)) \Big\}. \end{aligned}$$

This gives Equation (17); the left-hand derivative is derived similarly. ■

6 Implementation of the method.

The formulas given in preceding sections will be interesting only if they provide estimates which are (i) easy to compute and (ii) strongly consistent, which means that they converge a.s. to their expected values. In this section we show how the estimate given by Corollary 3 can be used in simulation when the system is ergodic. In this case, ergodic theorem (26) applied to equation (7) reads:

$$\begin{aligned} J'(\theta) &= \lim_{n\to\infty} \frac{\lambda}{n} \sum_{k=0}^{n-1} W'_\theta(T_k)[f(W_\theta(T_k)) - f(W_\theta(T_{k+1}-))] \\ &= \lim_{n\to\infty} \frac{\lambda}{n} \sum_{k=0}^{n-1} W'_\theta(T_k)[f(W_\theta(T_k)) - f(W_\theta(T_k-))]. \end{aligned} \tag{19}$$

The different ingredients of this formula are easy to evaluate once the simulation of the queue is set up: $W_\theta(T_k)$ and $W_\theta(T_k-)$ are known when customer k joins the queue; to get $W'_\theta(T_k)$, we use equation (1), keeping in mind that both $W(t)$ and $W'(t)$ are *càdlàg* processes and find:

$$W'_\theta(T_k) = \begin{cases} \sigma'_k(\theta) & \text{if customer } k \text{ finds the system empty} \\ W'_\theta(T_{k-1}) + \sigma'_k(\theta) & \text{else.} \end{cases}$$

As shown in Section 2, σ'_k can most of the time be expressed as a function of σ_k and θ, say $\sigma'_k = D(\sigma_k, \theta)$. So if we define $w_k = W_\theta(T_k-)$ and $d_k = W'_\theta(T_k)$, we have

$$\begin{aligned} w_k &= (w_{k-1} + \sigma_{k-1} - \tau_{k-1})^+ \\ d_k &= d_{k-1}\mathbb{1}\{w_k > 0\} + D(\sigma_k, \theta), \end{aligned}$$

and equation (19) shows that

$$\phi_n \stackrel{\text{def}}{=} \frac{\lambda}{n} \sum_{k=0}^{n-1} d_k [f(w_k + \sigma_k) - f(w_k)]$$

is a strongly consistent estimate of $J'(\theta)$. Since our Palm estimate does not require differentiability for f, one will want to check whether it is as accurate as the classic IPA estimate: if the system is ergodic, the ergodic theorem (25) of the appendix applied to equation (2) gives

$$\begin{aligned} J'(\theta) &= \lim_{t\to\infty} \frac{1}{t} \int_0^t W'_\theta(s) f'(W_\theta(s))\, ds \\ &= \lim_{n\to\infty} \frac{1}{T_n} \sum_{k=0}^{n-1} \int_{T_k}^{T_{k+1}} W'_\theta(s) f'(W_\theta(s))\, ds \\ &= \lim_{n\to\infty} \frac{1}{T_n} \sum_{k=0}^{n-1} W'_\theta(T_k)[f(W_\theta(T_k)) - f(W_\theta(T_{k+1}-))], \end{aligned} \tag{20}$$

where all the limits are valid P^0-a.s. or P-a.s. indifferently. In the third equality, we used the fact that $W'_\theta(t)$ is zero during idle periods. Comparing equations (19) and (20), we see that the estimates based on the same amount of data give very close expressions; in fact, they are even equal when λ needs to be estimated. For comparisons between time-average and customer-average estimates, see for instance Glynn and Whitt [11]. The implementation of an estimate of the second derivative of J would be done exactly in the same way, except that the formulas involved are slightly more complicated.

Appendix: a short introduction to Palm theory.

In this appendix we give without proof some basic results on Palm theory; interested readers can refer to Baccelli and Brémaud [1] for a more complete presentation of the subject. The stationary framework is the following: given a probability space $(\Omega, \mathcal{F}, \mathrm{P})$, let θ_t, $t \in \mathbb{R}$ be a measurable flow $(\Omega, \mathcal{F}) \mapsto (\Omega, \mathcal{F})$, i.e.:

- $(t, \omega) \mapsto \theta_t \omega$ is measurable w.r.t. $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}$,
- θ_t is bijective for all $t \in \mathbb{R}$,
- $\theta_t \circ \theta_s = \theta_{t+s}$ for all $t, s \in \mathbb{R}$. In particular, θ_0 =identity and $\theta_t^{-1} = \theta_{-t}$.

Note that there is nothing common between the flow θ_t and the parameter θ of the queue; these are the traditional notations in sensitivity analysis and Palm theory. We assume that $\mathrm{P} \circ \theta_t = \mathrm{P}$. Let $\{T_n\}_{n\in\mathbb{Z}}$ and $\{U_n\}_{n\in\mathbb{Z}}$

be two simple point process and let A and D be the associated counting measures, that is, for all borelian set $C \in \mathbb{R}$,

$$A(C) = \sum_{n \in \mathbb{Z}} \mathbb{1}_C(T_n), \;\; D(C) = \sum_{n \in \mathbb{Z}} \mathbb{1}_C(U_n)$$

and assume that for each $n \in \mathbb{Z}$, $U_n - T_n \stackrel{\text{def}}{=} W_n > 0$. We take the convention $T_0 \leq 0 < T_1$ and note:

$$\begin{aligned} T_-(t) &= \sup(T_n : T_n \leq t), \\ T_+(t) &= \inf(T_n \; : \; T_n > t). \end{aligned}$$

A and D are viewed as arrival and departure processes and we note $X(t)$ a queueing process associated with them. Let $B(t)$ be a non decreasing *càdlàg*—i.e. right continuous with left limits—real valued process and $Z(t)$ a non-negative real-valued stochastic process. We assume that these processes are compatible with the flow θ_t, that is

$$\begin{aligned} A(\omega, C+t) &= A(\theta_t \omega, C) \\ Z(\omega, t) &= Z(\theta_t \omega, 0). \end{aligned}$$

Similar equalities hold for D and X; if we define $\lambda_A = \mathbb{E}[A((0,1])]$, then there exists a probability measure called the Palm probability of the stationary process $(A, \theta_t, \mathrm{P})$ verifying the Swiss Army Formula (Brémaud [3]):

$$\lambda_A \mathbb{E}_A^0 [\int_{(0,W_0]} Z(s) dB(s)] = \frac{1}{t} \mathbb{E}[\int_{(0,t]} X(s-) Z(s) dB(s)] \tag{21}$$

The Swiss Army Formula is not the definition of the Palm measure, but we will see that it contains this definition and the classic formulas of Palm theory. We shall add that under P_A^0, $T_0 = 0$ a.s. We derive now some useful formulas from (21). The first one is the inversion formula: Take $U_n = T_{n+1}$ and $B(s) = s$; then $X(t) = 1$ and

$$\mathbb{E}[Z(0)] = \lambda_A \mathbb{E}_A^0 [\int_0^{T_1} Z(s) ds]. \tag{22}$$

The second formula is Neveu's exchange formula (Neveu [16]): we take D as for the inversion formula and remark that if $B \equiv A$, (21) reads

$$\lambda_A \mathbb{E}_A^0 [Z(0)] = \frac{1}{t} \mathbb{E}[\int_{(0,t]} Z(s) dA(s)].$$

This is Mecke's definition of Palm probability. Now, if B is a point process, we use the above equation and (21) to derive the exchange formula:

$$\lambda_A \mathbb{E}_A^0 [\int_0^{T_1} Z(s) dB(s)] = \lambda_B \mathbb{E}_B^0 [Z(0)]. \tag{23}$$

We will now prove a simple lemma which replaces Wald's identity for stationary systems:

Lemma 9 *Let $\{R_n\}_{n\in\mathbb{Z}}$ be a stationary stochastic point process with associated measure B. The following holds:*

$$\mathbb{E}_A^0\Big[\int_{[R_0,R_1)} Z(s)dA(s)\Big] = \mathbb{E}_A^0[A([R_0,R_1))\, Z(0)]. \tag{24}$$

Proof If we note Y the random variable inside the expectation of the l.h.s. of equation (24), then

$$Y\circ\theta_{T_i} = \int_{[R_-(T_i),R_+(T_i))} Z(s)dA(s) = \sum_{n\in\mathbb{Z}} Z(T_n)\mathbb{1}_{[R_-(T_i),R_+(T_i))}(T_n).$$

Since $R_\pm(T_i) = R_\pm(0)$ if $T_i \in [R_-(0),R_+(0))$,

$$\begin{aligned}
\sum_{T_i\in[R_-(0),R_+(0))} Y\circ\theta_{T_i} &= \sum_{i\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} Z(T_n)\mathbb{1}_{[R_-(0),R_+(0))}(T_n)\mathbb{1}_{[R_-(0),R_+(0))}(T_i)\\
&= A([R_0,R_1))\sum_{n\in\mathbb{Z}} Z(T_n)\,\mathbb{1}_{[R_0,R_1)}(T_n).
\end{aligned}$$

If we now apply Neveu's exchange formula (23) between P_A^0 and P_B^0, we obtain:

$$\begin{aligned}
\lambda_A\mathbb{E}_A^0 Y &= \lambda_B\mathbb{E}_B^0 \sum_{T_i\in[R_-(0),R_+(0))} Y\circ\theta_{T_i}\\
&= \lambda_B\mathbb{E}_B^0\Big[\sum_{n\in\mathbb{Z}} A([R_0,R_1))\,Z(T_n)\,\mathbb{1}_{[R_0,R_1)}(T_n)\Big]\\
&= \lambda_A\mathbb{E}_A^0[A([R_0,R_1))\,Z(0)],
\end{aligned}$$

which is exactly equality (24). ■

Formula (24) can be seen as an extension of Wald's identity that can be used for stationary sequences instead of i.i.d. variables and applies to any stationary process. It is not as convenient as Wald's identity is, but is valid in a wider setting.

Remark Lemma 9 is also a corollary of the extended $H=\lambda G$ formula (6.2) of Brémaud [3].

Palm probabilities can also be given an interpretation which relates them to simulation. When (P,θ_t) is ergodic, the ergodic theorems for P and P^0 read:

$$\begin{aligned}
\mathbb{E}[Y] &= \lim_{t\to\infty}\frac{1}{t}\int_0^t Y\circ\theta_s ds\\
\mathbb{E}_A^0[Y] &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} Y\circ\theta_{T_k}
\end{aligned}$$

which imply that:

$$\mathbb{E}[Z(0)] = \lim_{t\to\infty} \frac{1}{t}\int_0^t Z(s)ds \tag{25}$$

$$\mathbb{E}^0_A[Z(0)] = \lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} Z(T_k). \tag{26}$$

These equalities are valid P-a.s. and P^0_A-a.s. This shows that $\mathbb{E}[Z(0)]$ is the *time-average* of the process $Z(t)$, whether $\mathbb{E}^0_A[Z(0)]$ is its *customer-average*.

References

[1] Baccelli, F., and Brémaud, P. *Palm Probabilities and Stationary Queues*, vol. 41 of *Lecture Notes in Statistics*. Springer-Verlag, 1987.

[2] Brémaud, P. Maximal coupling and rare perturbation analysis. *Queueing Systems, Theory and Applications 11*, 4 (1992), 307–333.

[3] Brémaud, P. A Swiss Army formula of Palm calculus. *Journal of Applied Probability 30* (1993), 40–51.

[4] Brémaud, P., and Lasgouttes, J.-M. Stationary IPA estimates for non-smooth functions of the GI/G/1/∞ workload. Rapport de Recherche 1677, INRIA, Rocquencourt BP 105 – 78153 Le Chesnay – France, May 1992.

[5] Brémaud, P., and Vázquez-Abad, F. J. On the pathwise computation of derivatives with respect to the rate of a point process: the phantom RPA method. *Queueing Systems, Theory and Applications 10*, 3 (1992), 249–270.

[6] Cao, X.-R. Convergence of parameter sensitivity estimates in a stochastic experiment. *IEEE Transactions on Automatic Control 30*, 9 (1985), 834–843.

[7] Fu, M. C., and Hu, J.-Q. Extensions and generalizations of smoothed perturbation analysis in a generalized semi-Markov process framework. *IEEE Transactions on Automatic Control 37*, 10 (1992), 1483–1500.

[8] Glasserman, P. *Gradient Estimation via Perturbation Analysis.* Kluwer Academic Publishers, Norwell, Massachusetts, 1991.

[9] Glasserman, P., and Gong, W.-B. Smoothed perturbation analysis for a class of discrete event systems. *IEEE Transactions on Automatic Control 35*, 11 (1991), 1218–1230.

[10] Glynn, P. W. Likelihood ratio gradient estimation for stochastic systems. *Communications of the ACM 33*, 10 (1990), 76–84.

[11] Glynn, P. W., and Whitt, W. Indirect estimation via $L = \lambda W$. *Operations Research 37*, 1 (1989), 82–103.

[12] Gong, W.-B., and Ho, Y.-C. Smoothed (conditional) perturbation analysis of discrete event dynamical systems. *IEEE Transactions on Automatic Control 32*, 10 (1987), 858–866.

[13] Ho, Y.-C., and Cao, X.-R. Optimization and perturbation analysis of queueing networks. *Journal of Optimization Theory and Applications 40* (1983), 559–582.

[14] Ho, Y.-C., and Cao, X.-R. *Perturbation Analysis of Discrete Event Dynamic Systems.* Kluwer Academic Publishers, 1991.

[15] Konstantopoulos, P., and Zazanis, M. Sensitivity analysis for stationary and ergodic queues. *Advances in Applied Probability 24*, 3 (1992), 738–750.

[16] Neveu, J. Sur les mesures de Palm de deux processus ponctuels stationnaires. *Zeitschrift für Wahrscheinlichkeitstheorie und verw. Geb. 34* (1976), 199–203.

[17] Reiman, M. I., and Weiss, A. Sensitivity analysis for simulations via likelihood ratios. *Operations Research 37*, 5 (1989), 830–844.

[18] Suri, R. Implementation of sensitivity calculations on a Monte Carlo experiment. *Journal of Optimization Theory and Applications 40*, 4 (1983), 625–630.

[19] Suri, R., and Zazanis, M. Perturbation analysis gives strongly consistent sensitivity estimates for the M/G/1 queue. *Management Science 34*, 1 (1988), 39–64.

[20] Zazanis, M. A., and Suri, R. Estimating first and second derivatives of response time for $G/G/1$ queues from a single sample path. Technical report, Division of Applied Sciences, Harvard University, Cambridge, MA, 1985.