A State-Dependent Polling Model with Markovian Routing

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Abstract

A state-dependent 1-limited polling model with N queues is analyzed. The routing strategy generalizes the classical Markovian polling model, in the sense that two routing matrices are involved, the choice being made according to the state of the last visited queue. The stationary distribution of the position of the server is given. Ergodicity conditions are obtained by means of an associated dynamical system. Under rotational symmetry assumptions, average queue length and mean waiting times are computed.

1 Introduction

Consider a taxicab in a city in which there are N stations at which clients arrive and wait for the vehicle. When their turn has come to be served, they ask for a transit to a destination (one of the other stations) where they leave the system. Whenever the taxicab finds a station empty, it goes somewhere else to look for a client. The choice of the destinations by a client or by the empty taxicab are made via the two distinct routing matrices P and $\widetilde{P}$.

This system can also be seen as a polling model with N queues at which customers arrive and one server which visits the queues according to the following rules: if the server finds a client at the current queue, it serves this client and chooses a new queue according to the routing matrix P; otherwise it selects the next queue according to $\widetilde{P}$. This polling scheme is an extension of the classical Markovian polling model, with routing probabilities depending on the state of the last visited queue. As a taxicab only takes one client at a time, the service strategy is the 1-*limited* strategy, which is known to be more difficult to analyze than either the *gated* or *exhaustive* strategies—in which the server respectively serves the clients present on its arrival or serves clients until the queue is empty.

Since the bibliography on polling models is plethoric, we refer the reader to the references given in Takagi [16] and Levy and Sidi [10]. Among the studies related to our work, we can cite Kleinrock and Levy [9], who compute waiting times in random polling models in which the destination of the server is chosen independently of its provenance, and Boxma and Weststrate [2] who give a pseudo-conservation law for a model with Markovian routing. Ferguson [6] and Bradlow and Byrd [3] study approximatively a model where the switching time

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is station-dependent. Srinivasan [15] analyzes a polling system with the same routing policy as in our model, and with different service policies at each queue. However, the waiting times are not computed in the case where the routing is state dependent. Another important issue concerns state-independent ergodicity conditions. In this case, necessary and sufficient conditions were obtained by Fricker and Jaïbi [7, 8] for deterministic and Markovian routing with various service disciplines. Borovkov and Schassberger [1] and Fayolle *et al.* [5] give necessary and sufficient conditions for a system with Markovian routing and 1-limited service. Finally, Schassberger [13] gives a necessary condition for the ergodicity of a polling model with 1-limited service and a routing that depends on the whole state of the system.

Here, we give a new method to get the stationary distribution of the position of the server and, in the case of a fully symmetrical system, a way to compute the mean waiting time of a customer. This extends known results on 1-limited polling models, especially for the symmetrical case. However, we do not provide a pseudo-conservation law as for example Boxma and Weststrate [2], since the computations are not as simple as in the state-independent routing case. Our method of proof allows to make only minimal assumptions on the arrival process and applies to either discrete or continuous time models. Moreover, in the symmetrical case the results allow to compare various polling strategies.

The paper is organized as follows: in Section 2, we present the model and give a functional equation describing its evolution. In Section 3, stationary probabilities of the position of the server are obtained and Section 4 is devoted to the general problem of ergodicity. Section 5 presents formulas to compute the first moment of the queue length at polling instants. These results are used in Section 6 to calculate the waiting time of an arbitrary customer. Applications to known models are also given.

2 Description of the model

The system consists of N stations at which clients arrive according to a stationary process. Let $\mathcal{S} \stackrel{\text{def}}{=} \{1, \dots, N\}$ be the set of stations. Assume that the server arrives after $n-1$ moves at station $i \in \mathcal{S}$ where a client is waiting. Then the server loads this client and goes to station $j \in \mathcal{S}$ with probability $p_{i,j}$, such that $p_{i,1} + \cdots + p_{i,N} = 1$. Conversely, if station i is empty, the server polls station j with probability $\tilde{p}_{i,j}$. The service policy is known as *1-limited* policy, since at most 1 customer is served each time a station is visited. The two transition matrices will be denoted respectively by

$$P = (p_{i,j})_{i,j \in \mathcal{S}}, \quad \widetilde{P} = (\tilde{p}_{i,j})_{i,j \in \mathcal{S}}.$$

The number of new customers arriving to station $q \in \mathcal{S}$ between the polling of stations i and j when there have been a service (resp. no service) at station i is $B_{i,j;q}(n)$ (resp. $\widetilde{B}_{i,j;q}(n)$). The vectors $B_{i,j}(n) = (B_{i,j;1}(n), \dots, B_{i,j;N}(n))$ (resp. $\widetilde{B}_{i,j}(n)$) are i.i.d. for different n, but their components may be dependents and not identically distributed. In the classical polling terminology, $B_{i,j}(n)$ is the number of arrivals during *both* the service and the switchover and $\widetilde{B}_{i,j}(n)$ is the number of arrivals during a switchover. However, our model covers a wider class of applications. In particular, the service time may depend on both i and

j. In addition, the switchover time distribution may depend on the state of the last visited queue (empty or not). Remark also that, unlike in Srinivasan [15], we can also have $\mathbb{E}B_{i,j}(n) < \mathbb{E}\widetilde{B}_{i,j}(n)$, which means that serving a customer can actually take less time than just moving. This would amount to *negative* service times in the usual polling formulation, which are not easy to handle!

It is worth noting that our setting is valid for both discrete-time and continuous-time models with or without batch arrivals. We do not need a separate analysis for each case, until the computation of waiting times, where we will have to give precise definitions of the vectors $B_{i,j}(n)$ and $\widetilde{B}_{i,j}(n)$. We do not describe yet the exact arrival process and the time taken to travel from one station to another, since we only need the distribution of the number of clients arriving during such a move. However, due to the assumption of independence between successive arrivals, the model applies mainly to the following situations:

- discrete-time evolution, when a batch of customers arrives at each station at the beginning of each time slot; batches are i.i.d. with respect to the time slots but they can be correlated between stations;
- continuous time evolution, when the arrival process is a compound Poisson process: customers arrive at the system in batches at the epochs of a Poisson process and the batches have the same properties as in the previous case.

Let $X_i(n)$ be the number of clients waiting at station i at polling instant n and $S(n)$ be the corresponding position of the server. If we define $X(n) \stackrel{\text{def}}{=} (X_1(n), \ldots, X_N(n))$, then $\mathcal{L} \stackrel{\text{def}}{=} (S, X)$ forms a Markov chain. Throughout this paper, we will assume that this chain is irreducible. For any vector $\vec{z} = (z_1, \ldots, z_N) \in \mathcal{D}^N$ (where $\mathcal{D}$ denotes the unit disc in the complex plane), we note $\vec{z}^B = z_1^{B_1} \cdot z_2^{B_2} \cdots z_N^{B_N}$ and define the following generating functions:

$$
\begin{aligned}
a_{i,j}(\vec{z}) &\stackrel{\text{def}}{=} \mathbb{E}[\vec{z}^{B_{i,j}(n)}],\\
\tilde{a}_{i,j}(\vec{z}) &\stackrel{\text{def}}{=} \mathbb{E}[\vec{z}^{\widetilde{B}_{i,j}(n)}],\\
F_i(\vec{z}; n) &\stackrel{\text{def}}{=} \mathbb{E}[\vec{z}^{X(n)} \mathbb{1}_{\{S(n)=i\}}],\\
\widetilde{F}_i(\vec{z}; n) &\stackrel{\text{def}}{=} \mathbb{E}[\vec{z}^{X(n)} \mathbb{1}_{\{S(n)=i, X_i(n)=0\}}],\\
&= F_i(z_1, \ldots, z_{i-1}, 0, z_{i+1}, \ldots, z_N; n),\\
F_i(\vec{z}) &\stackrel{\text{def}}{=} \lim_{n\to\infty} F_i(\vec{z}; n),\\
\widetilde{F}_i(\vec{z}) &\stackrel{\text{def}}{=} \lim_{n\to\infty} \widetilde{F}_i(\vec{z}; n),
\end{aligned}
$$

where $\mathbb{1}_{\mathcal{E}}$ is as usual the indicator function of the set $\mathcal{E}$. The following result holds:

Theorem 1 *Let $A(\vec{z})$, $\widetilde{A}(\vec{z})$ and $\Delta(\vec{z})$ be matrices defined as follows: for all $i, j \in \mathcal{S}$, the elements of row i and column j are given by*

$$
\begin{aligned}
[A(\vec{z})]_{i,j} &= p_{j,i} a_{j,i}(\vec{z}),\\
[\widetilde{A}(\vec{z})]_{i,j} &= \tilde{p}_{j,i} \tilde{a}_{j,i}(\vec{z}),\\
[\Delta(\vec{z})]_{i,j} &= \frac{1}{z_i} \mathbb{1}_{\{i=j\}}.
\end{aligned}
$$

Then the vectors $F(\vec{z}) = (F_1(\vec{z}), \dots, F_N(\vec{z}))$ and $\widetilde{F}(\vec{z}) = (\widetilde{F}_1(\vec{z}), \dots, \widetilde{F}_N(\vec{z}))$ are related by the functional equation

$$[I - A\Delta(\vec{z})]F(\vec{z}) = [\widetilde{A}(\vec{z}) - A\Delta(\vec{z})]\widetilde{F}(\vec{z}), \tag{2.1}$$

where $A\Delta(\vec{z})$ stands for $A(\vec{z})\Delta(\vec{z})$.

Proof We have, for all $i, j \in \mathcal{S}$ and $n > 0$,

$$\begin{aligned}
&\mathbb{E}[\vec{z}^{X(n+1)} \mathbb{1}_{\{S(n)=i, S(n+1)=j\}}] \\
&\quad = \mathbb{E}[\vec{z}^{X(n)+B_{i,j}(n)-\vec{e}_i} \mathbb{1}_{\{S(n)=i,S(n+1)=j,X_i(n)>0\}}] \\
&\qquad + \mathbb{E}[\vec{z}^{X(n)+\widetilde{B}_{i,j}(n)} \mathbb{1}_{\{S(n)=i,S(n+1)=j,X_i(n)=0\}}] \\
&\quad = p_{i,j}a_{i,j}(\vec{z})\mathbb{E}[\vec{z}^{X(n)-\vec{e}_i} \mathbb{1}_{\{S(n)=i,X_i(n)>0\}}] \\
&\qquad + \tilde{p}_{i,j}\tilde{a}_{i,j}(\vec{z})\mathbb{E}[\vec{z}^{X(n)} \mathbb{1}_{\{S(n)=i,X_i(n)=0\}}] \\
&\quad = p_{i,j}a_{i,j}(\vec{z})\frac{F_i(\vec{z};n) - \widetilde{F}_i(\vec{z};n)}{z_i} + \tilde{p}_{i,j}\tilde{a}_{i,j}(\vec{z})\widetilde{F}_i(\vec{z};n).
\end{aligned}$$

The second equality above uses the independence of the routing and the arrivals with respect to the past. Summing over all possible i, we get

$$F_j(\vec{z}; n+1) = \sum_{i=1}^{N} p_{i,j}a_{i,j}(\vec{z})\frac{F_i(\vec{z};n) - \widetilde{F}_i(\vec{z};n)}{z_i} + \sum_{i=1}^{N} \tilde{p}_{i,j}\tilde{a}_{i,j}(\vec{z})\widetilde{F}_i(\vec{z};n),$$

so that, letting $n \to \infty$,

$$F_j(\vec{z}) - \sum_{i=1}^{N} \frac{p_{i,j}a_{i,j}(\vec{z})}{z_i}F_i(\vec{z}) = \sum_{i=1}^{N} \tilde{p}_{i,j}\tilde{a}_{i,j}(\vec{z})\widetilde{F}_i(\vec{z}) - \sum_{i=1}^{N} \frac{p_{i,j}a_{i,j}(\vec{z})}{z_i}\widetilde{F}_i(\vec{z}),$$

which is equivalent to (2.1). ■

Defining, when it exists,

$$D(\vec{z}) \stackrel{\text{def}}{=} [I - A\Delta(\vec{z})]^{-1}[\widetilde{A}(\vec{z}) - A\Delta(\vec{z})], \tag{2.2}$$

one sees that (2.1) can be rewritten as

$$F(\vec{z}) = D(\vec{z})\widetilde{F}(\vec{z}). \tag{2.3}$$

This functional equation contains all the information sufficient to characterize $F(\vec{z})$ and $\widetilde{F}(\vec{z})$. Although its solution for $N \geq 3$ is still an open question, partial results can be derived as shown in the following sections.

3 The stationary distribution of the position of the server

What renders the polling model presented in the previous section difficult to analyze is, among other things, the fact that the movements of the server depend on the state of the visited stations. In particular, $\{S(n)\}_{n\geq 0}$ is not a Markov process, as it would be when $p_{i,j} = \tilde{p}_{i,j}$ for all i and j. Some computations are needed to get the stationary probabilities of the position of the server, that is

$$\begin{aligned} F_i(\vec{e}) &= P(S = i) \\ \widetilde{F}_i(\vec{e}) &= P(S = i, X_i = 0). \end{aligned}$$

It appears that these stationary probabilities depend not only on the transition probabilities, but also on the following mean values:

$$\alpha_{i;q} \stackrel{\text{def}}{=} \sum_{j=1}^{N} p_{i,j} \mathbb{E} B_{i,j;q}(n), \tag{3.1}$$

$$\tilde{\alpha}_{i;q} \stackrel{\text{def}}{=} \sum_{j=1}^{N} \tilde{p}_{i,j} \mathbb{E} \widetilde{B}_{i,j;q}(n). \tag{3.2}$$

Here $\alpha_{i;q}$ (resp. $\tilde{\alpha}_{i;q}$) is the mean number of new clients at station q between the arrival of the server at station i which was non-empty (resp. empty) and its arrival at the next (arbitrary) polled station. Since the arrival process is stationary, it is possible to define for all $q \in \mathcal{S}$ the mean number λ_q of arrivals at station q per unit of time and write, with obvious definitions for τ_i and $\tilde{\tau}_i$,

$$\alpha_{i;q} = \lambda_q \tau_i, \tag{3.3}$$

$$\tilde{\alpha}_{i;q} = \lambda_q \tilde{\tau}_i. \tag{3.4}$$

Throughout this paper we will sometimes write F (resp. $\widetilde{F}$) instead of $F(\vec{e})$ (resp. $\widetilde{F}(\vec{e})$) in order to shorten the notation. Moreover, ${}^*F$ denotes the transpose of the vector F.

Theorem 2 *Define the matrices $\mathcal{A} \stackrel{\text{def}}{=} (\alpha_{i;q})$ and $\widetilde{\mathcal{A}} \stackrel{\text{def}}{=} (\tilde{\alpha}_{i;q})$. When the Markov chain $\mathcal{L}$ is ergodic, F and $\widetilde{F}$ satisfy the following linear system of equations:*

$${}^*F\,\vec{e} = 1, \tag{3.5}$$

$${}^*F\,[I - P] = {}^*\widetilde{F}\,[\widetilde{P} - P], \tag{3.6}$$

$${}^*F\,[I - \mathcal{A}] = {}^*\widetilde{F}\,[I - \mathcal{A} + \widetilde{\mathcal{A}}]. \tag{3.7}$$

Define

$$\hat{\rho} \stackrel{\text{def}}{=} \sum_{i=1}^{N} \lambda_i (\tau_i - \tilde{\tau}_i).$$

When $\hat{\rho} \neq 1$, Equations (3.6) and (3.7) can be rewritten as

$$F_j = \sum_{i \in \mathcal{S}} \tilde{p}_{i,j} F_i + \bar{\tau} \sum_{i \in \mathcal{S}} \lambda_i (p_{i,j} - \tilde{p}_{i,j}), \quad \text{for all } j \in \mathcal{S}, \tag{3.8}$$

$$\bar{\tau} = \frac{1}{1 - \hat{\rho}} \sum_{j \in \mathcal{S}} F_j \tilde{\tau}_j, \tag{3.9}$$

and $\widetilde{F}$ is given by

$$\widetilde{F}_j = F_j - \lambda_j \bar{\tau}. \tag{3.10}$$

Moreover the mean time between two consecutive visits to queue j is

$$\mathbb{E} T_{j,j} = \frac{\bar{\tau}}{F_j} \tag{3.11}$$

The quantity $\bar{\tau}$ defined in (3.9) can be seen as the mean time between two polling instants. Note that, with the above definitions, $\hat{\rho}$ is in general different from the classical ρ defined in queueing theory. However, they coincide if $P = \widetilde{P}$ and the time between two polling instants can be decomposed into a state-independent switchover time and a service time which only depends on the station where the customer is.

Proof of Theorem 2 Although this theorem could be proved analytically using the functional equation (2.1), we give here a simple probabilistic interpretation of (3.6) and (3.7). Indeed, when the system is ergodic, (3.6) follows directly from

$$P(S = j) = \sum_{i=1}^{N} \Big[P(S = i, X_i > 0) p_{i,j} + P(S = i, X_i = 0) \tilde{p}_{i,j} \Big].$$

Moreover, writing the equality of the outgoing and ingoing flows at station q gives

$$\begin{aligned} P(S = q, X_q > 0) \quad &= \quad \sum_{i=1}^{N} \sum_{j=1}^{N} \Big[P(S = i, X_i > 0) p_{i,j} \mathbb{E} B_{i,j;q}(n) \\ &\qquad + P(S = i, X_i = 0) \tilde{p}_{i,j} \mathbb{E} \widetilde{B}_{i,j;q} \Big], \end{aligned}$$

which is equivalent to (3.7) if we take into account (3.1) and (3.2). It is straightforward to see, using (3.3) and (3.4), that, when $\hat{\rho} \neq 1$,

$$[I - \mathcal{A}][I - \mathcal{A} + \widetilde{\mathcal{A}}]^{-1} = I - \frac{\widetilde{\mathcal{A}}}{1 - \hat{\rho}}.$$

With this relation and $\bar{\tau}$ as in (3.9), Equations (3.8) and (3.10) follow easily from (3.6) and (3.7). Let $N_{j,j;j}$ be the number of arrivals to a queue j between two consecutive visits of the server. The equality of flows reads

$$P(X_j > 0 \mid S = j) = \mathbb{E} N_{j,j;j}.$$

Since the last equation can be rewritten as

$$\frac{F_j - \widetilde{F}_j}{F_j} = \lambda_j \mathbb{E} T_{j,j} \ ,$$

we obtain (3.11) and the proof of the theorem is concluded. ■

The set of equations (3.5), (3.8), (3.9) and (3.10) provides a convenient way to compute F and $\widetilde{F}$. The following proposition gives simple conditions under which its rank is full.

Proposition 3 *The linear system given by (3.5), (3.8) and (3.9) has a unique solution when $\widetilde{P}$ has exactly one essential class. When $\widetilde{P}$ has $K > 1$ essential classes $\mathcal{E}_1, \ldots, \mathcal{E}_K$, the Markov chain $\mathcal{L}$ is never ergodic unless*

$$\sum_{j \in \mathcal{E}_m} \sum_{i \in \mathcal{S}} \lambda_i (p_{i,j} - \tilde{p}_{i,j}) = 0, \ \ \textit{for all } m \leq K. \tag{3.12}$$

Proof Define $\psi_i \stackrel{\text{def}}{=} F_i/\bar{\tau}$. Then the system (3.8)–(3.9) reads

$$\psi_j - \sum_{i\in\mathcal{S}} \tilde{p}_{i,j}\psi_i = \sum_{i\in\mathcal{S}} \lambda_i(p_{i,j} - \tilde{p}_{i,j}), \quad \text{for all } j \in \mathcal{S}, \tag{3.13}$$

$$\sum_{i\in\mathcal{S}} \psi_i\tilde{\tau}_i = 1 - \hat{\rho}. \tag{3.14}$$

The system (3.13) have rank $N-1$ unless the stochastic matrix $\widetilde{P}$ has several essential classes. Then, for each $m \leq K$, $i \in \mathcal{E}_m$ and $j \notin \mathcal{E}_m$, $\tilde{p}_{i,j} = 0$. In this case, (3.13) has solutions only when (3.12) holds. ■

The conditions (3.12) represent in some sense "zero drift" relationships which, as usual, imply more involved derivations. However, when the arrivals form independent compound Poisson processes at each queue, the system is never ergodic! The proof is of analytic nature. It requires Taylor expansions of second order in equation (2.1), similar to those extensively used in Section 5. In general, when the batches are correlated, it is difficult to conclude and we suspect that all situations might occur (ergodicity, null recurrence or transience).

It is worth noting the special role played by $\widetilde{P}$. In particular, when the arrivals are Poisson, the Markov chain is *never* ergodic if $\widetilde{P}$ admits several essential classes (obviously, P has then to be chosen to ensure the irreducibility of $\mathcal{L}$). In our opinion, this result is not intuitive and we cannot be explained just by waving hands.

4 Conditions for ergodicity

The purpose of this section is to classify the process $\mathcal{L}$, viewed as a random walk on $\mathcal{S} \times \mathbb{Z}_+^N$, in terms of ergodicity and non-ergodicity.

4.1 Necessary condition

We give a necessary condition for ergodicity which is a simple consequence of the results of Theorem 2. When the system is ergodic, we have $\widetilde{F}_i > 0$ for all $i \in \mathcal{S}$ and Equation (3.10) implies the following.

Theorem 4 *If the Markov chain $\mathcal{L}$ is ergodic, then*

$$\hat{\rho} < 1, \tag{4.1}$$

$$\lambda_i\bar{\tau} < F_i, \quad \text{for all } i \in \mathcal{S}. \tag{4.2}$$

These conditions extend in the state-dependent case the results obtained in [1]. When they hold, we prove another useful inequality. Indeed, instantiating (4.2) in (3.9) yields

$$\bar{\tau} > \frac{\sum_{j\in\mathcal{S}} \bar{\tau}\lambda_j\tilde{\tau}_j}{1-\hat{\rho}},$$

or, equivalently,

$$1 - \sum_{j\in\mathcal{S}} \lambda_j\tau_j > 0. \tag{4.3}$$

4.2 Sufficient condition

Let us emphasize that this part could be skipped by readers not strongly interested (if any!) in ergodicity. The mathematical understanding requires the reading of Appendix A.1, which summarizes deep works of [12], [4] and [5]. The approach relies on the study of a dynamical system associated to $\mathcal{L}$. For the sake of readability, we recall hereafter two main notions.

- *Induced chain* $\mathcal{L}^\wedge$. It is a Markov chain corresponding to a polling system in which the queues belonging to the *face* $\wedge \subset \mathcal{S}$ are kept saturated. From a purely notational point of view, the original system would correspond to the case $\wedge = \emptyset$. The behaviours of $\mathcal{L}$ and $\mathcal{L}^\wedge$ are not *directly* connected to each other.

- *Second vector field* $\vec{v}^\wedge$. One can imagine that the random walk starts from a point which is close to $\wedge$, but sufficiently far from all other faces $\wedge'$, with $\wedge \not\subset \wedge'$. After some time—sufficiently long, but less than the minimal distance from $\wedge'$—, the stationary regime in the induced chain will be installed. In this regime, one can ask about the mean along $\wedge$: it is defined exactly by $\vec{v}^\wedge$.

As in Theorem 2, the stationary position of the server for any ergodic induced chain can be obtained as the solution of a linear system consisting of $N+1$ equations. Indeed, for all $t \in \mathcal{S}$, $j \notin \wedge$, we have

$$\begin{aligned}
\pi^\wedge(t) &= \sum_{s\in\wedge} \pi^\wedge(s) p_{s,t} + \sum_{s\notin\wedge} \pi^\wedge(s) \tilde{p}_{s,t} \\
&\quad + \sum_{s\notin\wedge} \pi^\wedge(s, x_s > 0)(p_{s,t} - \tilde{p}_{s,t}), \qquad (4.4)\\
\pi^\wedge(j, x_j > 0) &= \sum_{s\in\wedge} \pi^\wedge(s) \lambda_j \tau_s + \sum_{s\notin\wedge} \pi^\wedge(s) \lambda_j \tilde{\tau}_s \\
&\quad + \sum_{s\notin\wedge} \pi^\wedge(s, x_s > 0)\lambda_j(\tau_s - \tilde{\tau}_s), \qquad (4.5)\\
\sum_{s\in\mathcal{S}} \pi^\wedge(s) &= 1. \qquad (4.6)
\end{aligned}$$

Let us introduce the quantities

$$\begin{aligned}
\hat{\rho}^\wedge &\stackrel{\text{def}}{=} \sum_{s\notin\wedge} \lambda_s(\tau_s - \tilde{\tau}_s),\\
\bar{\tau}^\wedge &\stackrel{\text{def}}{=} \frac{1}{1-\hat{\rho}^\wedge}\left[\sum_{s\in\wedge} \pi^\wedge(s)\tau_s + \sum_{s\notin\wedge} \pi^\wedge(s)\tilde{\tau}_s\right].
\end{aligned}$$

Note that when $\wedge$ is ergodic, $\pi^\wedge(j, x_j > 0) = \lambda_j \bar{\tau}^\wedge$, for all $j \notin \wedge$, and $\bar{\tau}^\wedge$ can be interpreted as the mean time between two polling instants of the induced chain. Then the system defined by (4.4) and (4.5) can be replaced by the forthcoming $N+1$ equations: for all $t \in \mathcal{S}$,

$$\begin{aligned}
\pi^\wedge(t) &= \sum_{s\in\wedge} \pi^\wedge(s) p_{s,t} + \sum_{s\notin\wedge} \pi^\wedge(s) \tilde{p}_{s,t} + \bar{\tau}^\wedge \sum_{s\notin\wedge} \lambda_s(p_{s,t} - \tilde{p}_{s,t}), \qquad (4.7)\\
\bar{\tau}^\wedge &= \frac{1}{1-\hat{\rho}^\wedge}\left[\sum_{s\in\wedge} \pi^\wedge(s)\tau_s + \sum_{s\notin\wedge} \pi^\wedge(s)\tilde{\tau}_s\right]. \qquad (4.8)
\end{aligned}$$

When $\wedge = \emptyset$, the system (4.6), (4.7) and (4.8) coincides with (3.5), (3.8) and (3.9). In addition, under (4.3), we have

$$\hat{\rho}^{\wedge} < 1, \text{ for all } \wedge\text{'s}. \tag{4.9}$$

An easy flow computation shows that the components of the drift vector $\vec{M}(s, \vec{x})$ can be expressed as

$$M_j(s, \vec{x}) \;=\; \lambda_j \tau_s \mathbb{1}_{\{x_s > 0\}} + \lambda_j \tilde{\tau}_s \mathbb{1}_{\{x_s = 0\}} - \mathbb{1}_{\{x_s > 0, s = j\}}.$$

Then the computation of the second vector field becomes easy. For $j \in \wedge$,

$$\begin{aligned}
v_j^{\wedge} &= \sum_{\substack{x \in C^{\wedge} \\ s \in \mathcal{S}}} \pi^{\wedge}(s, \vec{x}) M_j(s, \vec{x}) \\
&= \sum_{\substack{x \in C^{\wedge} \\ s \notin \wedge}} \pi^{\wedge}(s, \vec{x}) M_j(s, \vec{x}) + \sum_{\substack{x \in C^{\wedge} \\ s \in \wedge}} \pi^{\wedge}(s, \vec{x}) M_j(s, \vec{x}) \\
&= \lambda_j \sum_{s \notin \wedge} \Big[\pi^{\wedge}(s, x_s > 0) \tau_s + \pi^{\wedge}(s, x_s = 0) \tilde{\tau}_s \Big] \\
&\quad + \lambda_j \sum_{s \in \wedge} \pi^{\wedge}(s) \tau_s - \pi^{\wedge}(j) \\
&= \lambda_j \bar{\tau}^{\wedge} - \pi^{\wedge}(j).
\end{aligned}$$

In order to apply Theorem 11, it will be convenient to introduce

$$\begin{aligned}
f_i(\vec{x}) &\stackrel{\text{def}}{=} \Big[{}^{*}\vec{x}\, [I - \mathcal{A} + \widetilde{\mathcal{A}}]^{-1} \Big]_i \\
&= x_i + \frac{\lambda_i \sum_{j=1}^{N} x_j (\tau_j - \tilde{\tau}_j)}{1 - \hat{\rho}}.
\end{aligned} \tag{4.10}$$

This function is not as outlandish as it could seem at first sight. It is indeed directly related to flow conservation equations (see [5]). Let $\wedge$ be any ergodic face. For $i \in \wedge$, we get after a straightforward computation

$$\begin{aligned}
f_i(\vec{v}^{\wedge}) &= v_i^{\wedge} + \frac{\lambda_i}{1 - \hat{\rho}} \sum_{j=1}^{N} (\tau_j - \tilde{\tau}_j) v_j^{\wedge} \\
&= \frac{\lambda_i \sum_{j=1}^{N} \pi^{\wedge}(j) \tilde{\tau}_j}{1 - \hat{\rho}} - \pi^{\wedge}(i).
\end{aligned}$$

For $i \notin \wedge$, and from the very definition of $f_i(\vec{x})$, we have

$$f_i(\vec{v}^{\wedge}) - \sum_{j=1}^{N} f_j(\vec{v}^{\wedge}) \lambda_i (\tau_j - \tilde{\tau}_j) = v_i^{\wedge} = 0,$$

or

$$f_i(\vec{v}^{\wedge}) = \frac{\lambda_i \sum_{j \in \wedge} f_j(\vec{v}^{\wedge})(\tau_j - \tilde{\tau}_j)}{1 - \hat{\rho}^{\wedge}}.$$

Theorem 5 *Assume that (4.1)–(4.2) hold and that, for any ergodic face* $\wedge$*,*

$$f_i(\vec{v}^\wedge) \equiv \frac{\lambda_i \sum_{j=1}^N \pi^\wedge(j)\tilde{\tau}_j}{1-\hat{\rho}} - \pi^\wedge(i) < 0, \quad \textit{for all } i \in \wedge. \tag{4.11}$$

Then the random walk $\mathcal{L}$ *is ergodic. In particular, when* $P = \widetilde{P}$*, the conditions (4.1) and (4.2) are necessary and sufficient for the random walk* $\mathcal{L}$ *to be ergodic.*

Proof We shall apply Theorem 11 (quoted in Appendix A.1), which in itself contains a principle of gluing Lyapounov functions together. Most of the time, these functions are piecewise linear. Since the $f_i(\vec{v}^\wedge)$ enjoy "nice" properties, it is not unnatural to search for a linear combination like

$$f(\vec{x}) \stackrel{\text{def}}{=} \sum_{i=1}^N u_i f_i(\vec{x}),$$

where $\vec{u} = (u_1, \ldots, u_N)$ is a positive vector to be properly determined. Then, for any ergodic $\wedge$,

$$f(\vec{v}^\wedge) = \sum_{i\in\wedge}\Big[u_i + \frac{\sum_{j\notin\wedge}\lambda_j u_j}{1-\hat{\rho}^\wedge}(\tau_i - \tilde{\tau}_i)\Big] f_i(\vec{v}^\wedge).$$

The basic constraint for $\vec{u}$ is to ensure the positivity of f. Using (4.3) and (4.9), it appears that a suitable choice is $u_i = \max(\tilde{\tau}_i - \tau_i, \varepsilon)$, for some ε, positive and sufficiently small. Then one can directly check that $f(\vec{x}) > 0$, for any $\vec{x} \in \mathbb{R}_+^N$, and $f(\vec{v}^\wedge) < 0$, for any ergodic face $\wedge$. ■

Although the conditions of Theorem 5 are sufficient for ergodicity, they might well be implied by conditions (4.1)–(4.2) only. We did not check this fact, since it is difficult to compare algebraically the solutions of system (4.7)–(4.8), for different $\wedge$'s. Let us simply formulate the conjecture that (4.1)–(4.2) are also sufficient for ergodicity when $P \neq \widetilde{P}$.

Theorem 6 *Assume that* $P = \widetilde{P}$ *and that one of the following hold:*

(i) *there exists* $j \in \mathcal{S}$*, such that* $\lambda_j \bar{\tau} - F_j > 0$*;*

(ii) $\hat{\rho} > 1$*.*

Then the random walk $\mathcal{L}$ *is transient.*

Proof We assume that the queues are numbered according to the following order:

$$\frac{\lambda_1}{F_1} \leq \frac{\lambda_2}{F_2} \leq \ldots \leq \frac{\lambda_N}{F_N}. \tag{4.12}$$

Consider all faces $\wedge_i \stackrel{\text{def}}{=} \{i, \ldots, N\}$, for $i = 1, \ldots, N$. The idea is to show the transience of $\mathcal{L}$ by visiting the ordered set $\wedge_1, \wedge_2, \ldots \wedge_N$. In the usual terminology of dynamical systems, this amounts to finding a set of trajectories going to infinity with positive probability. To this end, the following algebraic relationship will be useful: for any $\wedge$ and $k \notin \wedge$, we have

$$(\lambda_k \bar{\tau}^{\wedge+\{k\}} - F_k)(1 - \hat{\rho}^{\wedge+\{k\}}) = (\lambda_k \bar{\tau}^\wedge - F_k)(1 - \hat{\rho}^\wedge). \tag{4.13}$$

By definition, the face $\wedge_1 \equiv \mathcal{S}$ is ergodic (see Appendix A.1). Assume now $\wedge_i$ is ergodic, for some fixed $i \in \mathcal{S}$. If $v_i^{\wedge_i} < 0$, then one can prove the inequality

$$1 - \sum_{s=1}^{i} \lambda_s \tau_s \leq \frac{1}{\bar{\tau}^{\wedge_i}} \sum_{s=i}^{N} F_s \tau_s,$$

which in turn yields $1 - \hat{\rho}^{\wedge_i} > 0$. By using (4.13), (4.12) and Theorem 5, it follows that the face $\wedge_{i+1}$ is ergodic. Conversely, if $v_i^{\wedge_i} > 0$, then $\vec{v}^{\wedge_i} > \vec{0}$ and the random walk $\mathcal{L}$ is transient (see [4]).

Thus, by induction, we have shown that either $\mathcal{L}$ is transient or the face $\wedge_N \equiv \{N\}$ is ergodic. In the latter case, the assumptions of the theorem, together with (4.13) and (4.12), yield $v_N^{\wedge_N} \equiv \lambda_N \bar{\tau}^{\wedge_N} - F_N > 0$, so that $\mathcal{L}$ is transient. ■

4.3 Remarks and problems

(*i*) In general, the second vector field requires to compute invariant measures of Markov chains in dimensions strictly smaller than N (the induced chains). This is rarely possible for $N > 3$, but there are some miracles, the most noticeable ones concerning Jackson networks and some polling systems [5].

(*ii*) No stochastic monotonicity argument seems to work, when the $\tau_k - \tilde{\tau}_k$'s are not all of the same sign, even for $P = \widetilde{P}$. This monotonicity exists and is crucial in [1], [7] and [8].

(*iii*) The analysis of the vector field $\vec{v}^{\wedge}$ is interesting in itself and will be the subject of a future work. Let us just quote one negative property: when $\wedge$ is ergodic and $\vec{v}_k^{\wedge} < 0$, then $\wedge_1 = \wedge \setminus \{k\}$ can be non-ergodic, contrary to what happens in Jackson networks (see [5]). In the case $P = \widetilde{P}$, we conjecture nonetheless that the dynamical system formed from the velocity vectors $\vec{v}^{\wedge}$ is strongly acyclic (in the sense that the same face is not visited twice).

5 The first moment of the queue length

This section shows that, when the system enjoys some symmetry properties, it becomes possible to derive the stationary mean queue length $\mathbb{E}[X_m \mid S = m]$ seen by the server at polling instants. To this end, ergodicity of $\mathcal{L}$ will be assumed, as well as the existence of second moments of all random variables of interest.

Assumption A$_1$ *The system has a* rotational *symmetry*

(*i*) *for all* $i, j \in \mathcal{S}$, $p_{i,j} = p_{j-i}$, $\tilde{p}_{i,j} = \tilde{p}_{j-i}$, $a_{i,j}(\vec{z}) = a_{j-i}(\vec{z})$ *and* $\tilde{a}_{i,j}(\vec{z}) = \tilde{a}_{j-i}(\vec{z})$;

(*ii*) *for all* $1 \leq s, d \leq N$,

$$\begin{aligned} a_d(z_1, \ldots, z_N) &= a_d(z_{1+s}, \ldots, z_{N+s}), \\ \tilde{a}_d(z_1, \ldots, z_N) &= \tilde{a}_d(z_{1+s}, \ldots, z_{N+s}). \end{aligned}$$

Note that this assumption is less restrictive than the ones appearing in the literature, even for state-independent situations. It allows in particular cyclic and random polling strategies. From A_1, we directly get, for all $1 \le i, s \le N$,

$$
\begin{aligned}
F_i(\vec{z}) &= F_{i+s}(z_{1+s}, \dots, z_{N+s}), &(5.1)\\
\widetilde{F}_i(\vec{z}) &= \widetilde{F}_{i+s}(z_{1+s}, \dots, z_{N+s}). &(5.2)
\end{aligned}
$$

Consequently, $P(S = i) = F_i(\vec{e}) = 1/N$. In addition, $A(\vec{z})$ and $\widetilde{A}(\vec{z})$ are *circulant* matrices—see Appendix A.2 for the results and notation which will be used from now on. It implies in particular that, for $k \in \mathcal{S}$, $\vec{v}_k$ is an eigenvector of $A(\vec{z})$ and $\widetilde{A}(\vec{z})$ with respective eigenvalues

$$
\begin{aligned}
\mu_k(\vec{z}) &\stackrel{\text{def}}{=} \sum_{d=1}^{N} p_d a_d(\vec{z}) \omega_k^d ,\\
\tilde{\mu}_k(\vec{z}) &\stackrel{\text{def}}{=} \sum_{d=1}^{N} \tilde{p}_d \tilde{a}_d(\vec{z}) \omega_k^d .
\end{aligned}
$$

Note that $\mu_N(\vec{z})$ (resp. $\tilde{\mu}_N(\vec{z})$) is the generating function of the number of arrivals during the transportation of one client (resp. a move without client). Moreover, $\mu_k \stackrel{\text{def}}{=} \mu_k(\vec{e})$ and $\tilde{\mu}_k \stackrel{\text{def}}{=} \tilde{\mu}_k(\vec{e})$ are eigenvalues of P and $\widetilde{P}$. For *a priori* technical reasons we shall also need

Assumption $\mathbf{A}_2$ *For any $k < N$, $\tilde{\mu}_k \neq 1$.*

These relations are not surprising in view of Proposition 3. Since the system is symmetrical, the mean number of customers arriving during a travel of the taxicab $\alpha_{i;q}$ and $\tilde{\alpha}_{i;q}$ as defined by (3.1)–(3.2) does not depend on i and q; we note them α and $\tilde{\alpha}$ and remark that

$$
\begin{aligned}
\alpha &= \frac{\partial \mu_N}{\partial z_m}(\vec{e}),\\
\tilde{\alpha} &= \frac{\partial \tilde{\mu}_N}{\partial z_m}(\vec{e}).
\end{aligned}
$$

The second derivatives of $\mu_N(\vec{z})$ and $\tilde{\mu}_N(\vec{z})$ are defined as

$$
\begin{aligned}
\alpha^{(2)}_{q,r} &\stackrel{\text{def}}{=} \frac{\partial^2 \mu_N(\vec{z})}{\partial z_q \partial z_r}(\vec{e}), &(5.3)\\
\tilde{\alpha}^{(2)}_{q,r} &\stackrel{\text{def}}{=} \frac{\partial^2 \tilde{\mu}_N(\vec{z})}{\partial z_q \partial z_r}(\vec{e}) &(5.4)
\end{aligned}
$$

Note that these values depend only of the unsigned distance between q and r and that we can write $\alpha^{(2)}_{q,q+d} = \alpha^{(2)}_{q+d,q} \stackrel{\text{def}}{=} \alpha^{(2)}_d$, $\tilde{\alpha}^{(2)}_{q,q+d} = \tilde{\alpha}^{(2)}_{q+d,q} \stackrel{\text{def}}{=} \tilde{\alpha}^{(2)}_d$. It will be also convenient to introduce the first and second moments of the number of arrivals between two polling instants, that is

$$
\begin{aligned}
\bar{\alpha} &\stackrel{\text{def}}{=} P(X_m > 0 \mid S = m)\alpha + \mathrm{P}(X_m = 0 \mid S = m)\tilde{\alpha}, &(5.5)\\
\bar{\alpha}^{(2)}_d &\stackrel{\text{def}}{=} P(X_m > 0 \mid S = m)\alpha^{(2)}_d + \mathrm{P}(X_m = 0 \mid S = m)\tilde{\alpha}^{(2)}_d . &(5.6)
\end{aligned}
$$

Using the symmetry of the system and according to the properties of generating functions, we have for any $m \in \mathcal{S}$

$$\mathbb{E}[X_m \mid S = m] = N \frac{\partial F_m}{\partial z_m}(\vec{e}). \tag{5.7}$$

By symmetry, this expression is independent from m. In order to derive $\mathbb{E}[X_m \mid S = m]$, we use the fact that the matrices $I - A\Delta(\vec{z})$ and $\widetilde{A}(\vec{z}) - A\Delta(\vec{z})$ are not invertible when $\vec{z} = \vec{e}$. This implies that the matrix $D(\vec{z})$ as defined in (2.2) is not continuous in a neighborhood of $\vec{z} = \vec{e}$ and provides a mean to compute the derivatives of $F(\vec{z})$ and $\widetilde{F}(\vec{z})$ for $\vec{z} = \vec{e}$. However in our setting, even the existence of $D(\vec{z})$ in a neighborhood of $\vec{e}$ is difficult to prove and the computations become really involved. The approach that we present here avoids the theoretical problems and simplifies the computations. In the next lemma, we show how it is possible to choose $\vec{z}$ such that the left member of Equation (2.1) can be rendered of "small" order w.r.t. the right one.

Lemma 7 *Let $t \to \vec{z}_t$ be a function from $\mathbb{R}_- \to \mathcal{D}^N$ such that in a neighborhood of $t = 0$, we can write $\vec{z}_t = \vec{e} + t\dot{\vec{z}} + \frac{1}{2}t^2\ddot{\vec{z}} + \vec{o}(t^2)$, where $\dot{\vec{z}} = \dot{z}_1\vec{e}_1 + \cdots + \dot{z}_N\vec{e}_N = \dot{\zeta}_1\vec{v}_1 + \cdots + \dot{\zeta}_N\vec{v}_N \in \mathbb{C}^N$ and $\ddot{\vec{z}} = (\ddot{z}_1, \ldots, \ddot{z}_N) \in \mathbb{C}^N$. Assume that $\dot{z}_1 + \cdots + \dot{z}_N = 0$. Then there exists for $t > 0$ a vector $\vec{u}_t$ such that, for $1 \le k < N$,*

$$\begin{aligned} {}^*\vec{u}_t\,[I - A\Delta(\vec{z}_t)]\vec{v}_k &= o(t), & (5.8)\\ {}^*\vec{u}_t\,[\widetilde{A}(\vec{z}_t) - A\Delta(\vec{z}_t)]\vec{v}_k &= t\frac{1-\tilde{\mu}_k}{1-\mu_k}\dot{\zeta}_{N-k} + o(t), & (5.9)\end{aligned}$$

and

$$\begin{aligned} {}^*\vec{u}_t\,[I - A\Delta(\vec{z}_t)]\vec{e} &= -t^2\sum_{l=1}^{N-1}\dot{\zeta}_l\dot{\zeta}_{N-l}\left[\frac{1}{1-\mu_l} + \frac{N}{2}\sum_{d=1}^{N}\alpha_d^{(2)}\omega_l^d\right] \\ &\quad + t^2\frac{1-N\alpha}{2N}\sum_{q=1}^{N}\ddot{z}_q + o(t^2), & (5.10)\\ {}^*\vec{u}_t\,[\widetilde{A}(\vec{z}_t) - A\Delta(\vec{z}_t)]\vec{e} &= -t^2\sum_{l=1}^{N-1}\dot{\zeta}_l\dot{\zeta}_{N-l}\left[\frac{1}{1-\mu_l} + \frac{N}{2}\sum_{d=1}^{N}(\alpha_d^{(2)} - \tilde{\alpha}_d^{(2)})\omega_l^d\right] \\ &\quad + t^2\frac{1-N\alpha+N\tilde{\alpha}}{2N}\sum_{q=1}^{N}\ddot{z}_q + o(t^2). & (5.11)\end{aligned}$$

Proof See Appendix A.3. ∎

For the sake of simplicity, the computations have been carried out in the most natural way which apparently requires $\mu_k \neq 1$ for $k < N$. In fact, the reader can convince himself that all derivations could be achieved by rendering the right member of (2.1) small w.r.t. the left member. This would yield the same result without any restriction on μ_k, at the expense of a somewhat longer proof.

One problem in polling models with 1-limited service strategy is that the rank of the systems of equations that we can write is $\lfloor n/2 \rfloor$, which means that $\partial F/\partial z_m(\vec{e})$ cannot be computed. Fortunately, under the following assumption, we are able to compute the most important value, that is $\partial F_m/\partial z_m(\vec{e})$.

Assumption $\mathbf{A}_3$ *The routing matrices are such that*

$$[I-P][I-{}^*\widetilde{P}] = [I-{}^*P][I-\widetilde{P}].$$

It is not easy to describe all models satisfying A_3. The simplest one is the classical Markov polling obtained when $P=\widetilde{P}$, for which the results of Theorem 8 thereafter are greatly simplified. Another admissible model is when the routing probabilities depend only on the absolute distance between stations, that is when $P={}^*P$ and $\widetilde{P}={}^*\widetilde{P}$.

Theorem 8 *Assume that A_1, A_2 and A_3 hold. Then, for any station m, the stationary probability that the server finds station m empty is given by*

$$P(X_m=0 \mid S=m) = \frac{1-N\alpha}{1-N\alpha+N\tilde{\alpha}}\,. \tag{5.12}$$

Moreover the mean number of customers found by the server when it arrives at station m is obtained by means of the following formula

$$\begin{aligned}
\mathbb{E}[X_m \mid S=m] &= N\bar{\alpha} + \frac{N\tilde{\alpha}\bar{\alpha}}{1-N\alpha}\sum_{l=1}^{N-1}\frac{1}{1-\tilde{\mu}_l} \\
&\quad + \frac{1-N\alpha+N\tilde{\alpha}}{1-N\alpha}\frac{N}{2}\bar{\alpha}_N^{(2)} + \frac{N\alpha-N\tilde{\alpha}}{1-N\alpha}\frac{1}{2}\sum_{d=1}^{N}\bar{\alpha}_d^{(2)} \\
&\quad - \frac{N\tilde{\alpha}}{1-N\alpha}\sum_{l=1}^{N-1}\frac{\mu_l-\tilde{\mu}_l}{1-\tilde{\mu}_l}\frac{1}{2}\sum_{d=1}^{N}\bar{\alpha}_d^{(2)}\omega_l^d\,.
\end{aligned} \tag{5.13}$$

Finally, the mean number of customers present at arbitrary station m at a polling instant can be expressed as

$$\begin{aligned}
\mathbb{E}X_m &= \bar{\alpha} + \frac{\tilde{\alpha}}{1-N\alpha}\sum_{l=1}^{N-1}\frac{1}{1-\tilde{\mu}_l} \\
&\quad + \frac{1-N\alpha+N\tilde{\alpha}}{1-N\alpha}\frac{N}{2}\bar{\alpha}_N^{(2)} + \frac{N\alpha-N\tilde{\alpha}}{1-N\alpha}\frac{1}{2}\sum_{d=1}^{N}\bar{\alpha}_d^{(2)} \\
&\quad - \frac{1-N\alpha+N\tilde{\alpha}}{1-N\alpha}\sum_{l=1}^{N-1}\frac{\mu_l-\tilde{\mu}_l}{1-\tilde{\mu}_l}\frac{1}{2}\sum_{d=1}^{N}\bar{\alpha}_d^{(2)}\omega_l^d\,.
\end{aligned} \tag{5.14}$$

Proof See Appendix A.3. ∎

6 The mean waiting time of a customer

Theorem 8 is the key to compute of the waiting time of an arbitrary customer. In this section, we focus on the continuous-time case with compound Poisson arrivals and further assume that the arrivals at each station are independent. While Theorem 8 is valid in discrete time or with correlated arrivals, we do not give any formula in these cases, since the notation would become too involved.

The notation is as follows: customers arrive in i.i.d. batches of mean length b and second moment $b^{(2)}$ at the instants of a Poisson stream with intensity $\hat{\lambda}$

at each station. The intensity of the arrival process at each queue is $\lambda = \hat{\lambda}b$. The time between two polling instants when the first queue is not empty (resp. empty) has mean and second moment τ and $\tau^{(2)}$ (resp. $\tilde{\tau}$ and $\tilde{\tau}^{(2)}$). We must keep in mind that in general τ depends on P like $\alpha_{i;q}$ in Equation (3.1): if τ_d is the mean time between two consecutive polling instants at stations i and $i+d$, then we have $\tau = p_1\tau_1 + \cdots + p_N\tau_N$. However, in most symmetrical polling models that have been previously analyzed, the switchover times are equal and τ does not depend on P. We will say in this case that the stations are *equidistant*, which is obviously not the case in our original taxicab problem. The same holds for $\tau^{(2)}$, $\tilde{\tau}$ and $\tilde{\tau}^{(2)}$.

Theorem 9 *Assume that A_1, A_2 and A_3 hold. Then the mean stationary waiting time of a customer is given by the relation*

$$\begin{aligned} \mathbb{E}[W] &= \frac{\tilde{\tau}}{1-N\lambda\tau}\sum_{l=1}^{N-1}\left[\frac{1}{1-\tilde{\mu}_l}\right] + \frac{N\lambda\tau^{(2)}}{2(1-N\lambda\tau)} + \frac{\tilde{\tau}^{(2)}}{2\tilde{\tau}} \\ &\quad + \frac{b^{(2)}-b}{2b}\left[\frac{\tau+(N-1)\tilde{\tau}}{1-N\lambda\tau} - \frac{\tilde{\tau}}{1-N\lambda\tau}\sum_{l=1}^{N-1}\frac{\mu_l-\tilde{\mu}_l}{1-\tilde{\mu}_l}\right]. \end{aligned} \tag{6.1}$$

Proof See Appendix A.3. ■

One surprising consequence of this formula is that when the arrival process is Poisson and the stations are *equidistant*, the mean waiting time does not depend on the routing P. In other words, it does not depend on where the server goes after serving a customer.

Beside giving the mean waiting time for a customer in our taxicab model, Equation (6.1) contains in fact many known formulas for waiting times corresponding to various service and polling strategies. In the next subsections, we give some of these applications.

A notation closer to the classical queuing theory notation is necessary to make the link with known results. Let w (resp. $\tilde{w}$) be the mean walking time—or switchover time—from a non-empty (resp. empty) station and let σ be the mean service time required by the customers. We denote by $w^{(2)}$, $\tilde{w}^{(2)}$ and $\sigma^{(2)}$ the associated second moments.

6.1 A state-independent Markovian polling model

The first possible application of our model is the classical state independent polling model with 1-limited service strategy. In this model, we have $P = \widetilde{P}$, $\tau = w + \sigma$, $\tilde{\tau} = w$ and (6.1) becomes

$$\begin{aligned} \mathbb{E}[W] &= \frac{w}{1-N\lambda(w+\sigma)}\sum_{l=1}^{N-1}\frac{1}{1-\mu_l} + \frac{N\lambda(w^{(2)}+2w\sigma+\sigma^{(2)})}{2(1-N\lambda(w+\sigma))} + \frac{w^{(2)}}{2w} \\ &\quad - \frac{b^{(2)}-b}{2b}\frac{Nw+\sigma}{1-N\lambda(w+\sigma)}. \end{aligned}$$

This formula is valid for all symmetric polling models with Markovian polling. It is interesting to note that, in the case of equidistant stations, there is only one term of the formula which depends on the routing. To compute this term, we remark that, if $\mathcal{P}(x)$ is the characteristic polynomial of P,

$$\sum_{l=1}^{N-1} \frac{1}{1-\mu_l} = \sum_{l=1}^{N-1} \frac{d}{dx} \log(x-\mu_l)\Big|_{x=1} = \frac{d}{dx} \log\Big[\frac{\mathcal{P}(x)}{x-1}\Big]\Big|_{x=1}.$$

The case of the cyclic polling is obtained by taking $p_1 = 1$ and $p_d = 0$ for $d \neq 1$. Then the eigenvalues of P are $\omega_1, \dots, \omega_N$ and $\mathcal{P}(x) = x^N - 1$. So in this case, we have

$$\sum_{l=1}^{N-1} \frac{1}{1-\mu_l} = \frac{N-1}{2}.$$

Another classical polling strategy is the random polling with $p_d = 1/N$ for all d. In this case, we have $\mu_l = 0$ for $l < N$ and

$$\sum_{l=1}^{N-1} \frac{1}{1-\mu_l} = N-1.$$

The comparison between formulas for cyclic polling and random polling shows that the mean waiting time is smaller in the cyclic case. In fact this property can be generalized to all Markovian polling models.

Lemma 10 *Among all Markovian polling strategies for symmetric* 1*-limited polling models with* equidistant *queues, the cyclic polling strategy minimizes the waiting time of the customers.*

Proof This result is very easy to prove from Equation (6.1). We have

$$\begin{aligned}\sum_{l=1}^{N-1} \frac{1}{1-\mu_l} &= \sum_{l=1}^{N-1} \Re\Big(\frac{1}{1-\mu_l}\Big) \\ &= \sum_{l=1}^{N-1} \frac{1-\Re(\mu_l)}{2(1-\Re(\mu_l)) + |\mu_l|^2 - 1},\end{aligned}$$

where $\Re(z)$ is the real part of z. Since $|\mu_l| \leq 1$ and $\Re(\mu_l) \leq 1$, we find that

$$\sum_{l=1}^{N-1} \frac{1}{1-\mu_l} \geq \frac{N-1}{2}.$$

The bound is attained if and only if for all l, $|\mu_l| = 1$. Since μ_l is the center of gravity of $\omega_l^1, \dots, \omega_l^N$ with weights $p_1, \dots, p_N$, this is only possible when for some $s \in \mathcal{S}$ we have $p_s = 1$ and $p_d = 0$ for $d \neq s$. When s is not a divider of N (to ensure that the routing matrix is irreducible), this is equivalent to cyclic polling. ■

This property also holds for discrete time systems and systems with correlated arrivals. The computation would be more difficult in the case where the stations are not equidistant.

6.2 The exhaustive and Bernoulli service strategies

An interesting application of our model is to show that some service strategies can be obtained by a proper choice of polling strategy. In this subsection, we show how we can apply our model to exhaustive and Bernoulli service strategies.

Consider the case where the stations are *equidistant*, except that the switchover time from a station to itself after a service is zero. Then, if we denote $p_N = 1-\pi$, we have $\tau = \pi w + \sigma$, $\tau^{(2)} = \pi w^{(2)} + 2\pi w\sigma + \sigma^{(2)}$, $\tilde{\tau} = w$ and $\tilde{\tau}^{(2)} = w^{(2)}$. If we assume for the sake of simplicity that the arrival process is Poisson, (6.1) yields

$$\mathbb{E}[W] = \frac{w}{1 - N\lambda(\pi w + \sigma)} \sum_{l=1}^{N-1} \frac{1}{1-\tilde{\mu}_l} + \frac{N\lambda(\pi w^{(2)} + 2\pi w\sigma + \sigma^{(2)})}{2(1 - N\lambda(\pi w + \sigma))} + \frac{w^{(2)}}{2w}.$$

The value of $\mathbb{E}[W]$ depends on P only through the value of p_N. One known model that is described by this equation is the Bernoulli strategy of Servi [14]: when the server has served a customer, it quits the queue with probability π and continues to serve it with probability $1-\pi$. In the original model, the server polls the queues in cyclic order. Here, we have a Markovian Bernoulli polling model if we take $P = (1-\pi)I + \pi\widetilde{P}$. It is easy to check that this choice of P satisfies A_3. Moreover, $\mathbb{E}[W]$ is an increasing function of π that is minimal when $\pi = 0$. This case corresponds to an exhaustive service strategy with Markovian routing $\widetilde{P}$: the server polls the same queue until it is empty and then moves to another queue using the routing matrix $\widetilde{P}$. In this case, Equation (6.1) simply becomes

$$\mathbb{E}[W] \;=\; \frac{w}{1 - N\lambda\sigma} \sum_{l=1}^{N-1} \frac{1}{1-\tilde{\mu}_l} + \frac{N\lambda\sigma^{(2)}}{2(1-N\lambda\sigma)} + \frac{w^{(2)}}{2w}.$$

Using Lemma 10, we find that the above expression is minimal when $\widetilde{P}$ describes a cyclic polling scheme. Further comparisons between different polling strategies can be found in Levy, Sidi and Boxma [11].

Appendix

A.1 Second vector field

This appendix contains some basic definitions and results from the theory of dynamical systems. These definitions have been adapted for the Markov chain $\mathcal{L}$ defined in Section 2. We refer the reader to [12] and [4] for a more complete treatment of the subject.

Faces. For any $\wedge \subset \{1,\ldots,N\}$, define $B^\wedge \subset \mathbb{R}_+^N$ as

$$B^\wedge \stackrel{\text{def}}{=} \{(x_1,\ldots,x_n) : x_i > 0 \Leftrightarrow i \in \wedge\}.$$

$B^\wedge$ is the *face* of $\mathbb{R}_+^N$ associated to $\wedge$. Whenever not ambiguous, the face $B^\wedge$ will be identified with $\wedge$.

Induced chains. For any $\wedge \neq \mathcal{S}$ we choose an arbitrary point $\vec{a} \in B^\wedge \cap \mathbb{Z}_+^N$ and draw a plane $C^\wedge$ of dimension $N - |\wedge|$, perpendicular to $B^\wedge$ and containing $\vec{a}$. We define the *induced Markov chain* $\mathcal{L}^\wedge$, with state space $\mathcal{S} \times (C^\wedge \cap \mathbb{Z}_+^N)$ (by an obvious abuse in the notation, we shall write most of the time $\mathcal{S} \times C^{\wedge}$) and transition probabilities

$$
{}_{\wedge}p_{(s,\vec{x})(t,\vec{y})} = p_{(s,\vec{x})(t,\vec{y})} + \sum_{\vec{z} \neq \vec{y}} p_{(s,\vec{x})(t,\vec{z})}, \ \forall \vec{x}, \vec{y} \in C^{\wedge}, \ s, t \in \mathcal{S},
$$

where the summation is performed over all $\vec{z} \in \mathbb{Z}_{+}^{N}$, such that the straight line connecting $\vec{z}$ and $\vec{y}$ is perpendicular to $C^{\wedge}$. It is important to note that this construction does not depend on $\vec{a}$.

Assumption $\mathbf{A}_4$ *For any $\wedge$ the chain $\mathcal{L}^{\wedge}$ is irreducible and aperiodic. $\wedge$ is called* ergodic *(resp.* non-ergodic, transient*) according as $\mathcal{L}^{\wedge}$ is ergodic (resp. non-ergodic, transient).*

For an ergodic $\mathcal{L}^{\wedge}$, let $\pi^{\wedge}(s, \vec{x}), \ (s, \vec{x}) \in \mathcal{S} \times C^{\wedge}$ be its stationary transition probabilities. Moreover, let $\pi^{\wedge}(s)$ (resp. $\pi^{\wedge}(s, x_s > 0)$) be the stationary probability that the server is at station s (resp. and finds it nonempty).

First vector field. For any $(s, \vec{x}) \in \mathcal{S} \times \mathbb{Z}_{+}^{N}$, the *first vector field* is simply the mean drift of the random walk $\mathcal{L}$ at point $(s, \vec{x})$:

$$
\vec{M}(s, \vec{x}) \stackrel{\text{def}}{=} \sum_{r \in \mathcal{S}, \ \vec{y} \in \mathbb{Z}_{+}^{N}} (\vec{y} - \vec{x}) P\Big[X(n+1) = \vec{y}, S(n+1) = r \,\Big|\, X(n) = \vec{x}, S(n) = s\Big].
$$

Let $\vec{v}^{\wedge} \stackrel{\text{def}}{=} (v_1^{\wedge}, \ldots, v_N^{\wedge})$ such that

$$
\begin{aligned}
v_i^{\wedge} &= 0, \ i \notin \wedge, \\
v_i^{\wedge} &= \sum_{(s,\vec{x}) \in \mathcal{S} \times C^{\wedge}} \pi^{\wedge}(s, \vec{x}) \vec{M}_i(s, \vec{x}), \ i \in \wedge.
\end{aligned}
$$

Intuitively, one can imagine that the random walk starts from a point which is close to $\wedge$, but sufficiently far from all other faces $B^{\wedge'}$, with $\wedge \not\subset \wedge'$. After some time (sufficiently long, but less than the minimal distance from the above mentioned $B^{\wedge'}$), the stationary regime in the induced chain will be installed. In this regime, one can ask about the mean drift along $\wedge$: it is defined exactly by $\vec{v}^{\wedge}$. For $\wedge = \mathcal{S}$, we call $\wedge$ *ergodic*, by definition, and put

$$
\vec{v}^{\{1,\ldots,N\}} \equiv \sum_{s \in \mathcal{S}} \pi^{\{1,\ldots,N\}}(s) \vec{M}(s, \vec{x}), \ \vec{x} \in B^{\{1,\ldots,N\}}.
$$

From now on, when speaking about the *components* of $\vec{v}^{\wedge}$, we mean the components $v_i^{\wedge}$ with $i \in \wedge$.

Assumption $\mathbf{A}_5$ $v_i^{\wedge} \neq 0$*, for each $i \in \wedge$.*

Ingoing, outgoing and neutral faces. Let us fix $\wedge, \wedge_1$, so that $\wedge \supset \wedge_1, \wedge \neq \wedge_1$, that is to say $\overline{B}^{\wedge} \supset B^{\wedge_1}$ ($\overline{B}^{\wedge}$ is the closure of $B^{\wedge}$). Let $B^{\wedge}$ be ergodic. Thus $\vec{v}^{\wedge}$ is well defined. There are three possibilities for the direction of $\vec{v}^{\wedge}$ w.r.t. $B^{\wedge_1}$. We say that $B^{\wedge}$ is an *ingoing* (resp. *outgoing*) *face* for $B^{\wedge_1}$, if all the coordinates $v_i^{\wedge}$ for $i \in \wedge \setminus \wedge_1$ are negative (resp. positive). Otherwise we say that $B^{\wedge}$ is *neutral*. As an example we give simple sufficient criteria for a face to be ergodic.

The second vector field. To any point $\vec{x} \in \mathbb{R}_+^N$, we assign a vector $v(\vec{x})$ and call this function the *second vector field*. It can be multivalued on some non-ergodic faces. We put, for ergodic faces $B^\wedge$,

$$\vec{v}(\vec{x}) \equiv \vec{v}^\wedge, \ \vec{x} \in B^\wedge.$$

If $B^{\wedge_1}$ is non-ergodic, then at any point $\vec{x} \in B^{\wedge_1}$, $\vec{v}(\vec{x})$ takes all values $\vec{v}^\wedge$ for which $B^\wedge$ is an outgoing face with respect to $B^{\wedge_1}$. In other words, for $\vec{x}$ belonging to non-ergodic faces, with $\|\vec{x}\|$ sufficiently large,

$$\vec{x} + \vec{v}(\vec{x}) \ \in \mathbb{R}_+^N ,$$

for any value $\vec{v}(\vec{x})$. If there is no such vector, we put $\vec{v}(\vec{x}) = 0$, for $\vec{x} \in B^{\wedge_1}$. Points $\vec{x} \in \mathbf{R}_+^N$, where $\vec{v}(\vec{x})$ is more than one-valued, are called branch points.

There are few interesting examples, for which only the first vector field suffices to obtain ergodicity conditions for the random walk of interest, but it is nevertheless the case for Jackson networks. In general, the second vector field must be introduced as shown in the following theorem, taken from [12] (and extended in [5] to the case of upward unbounded jumps).

Theorem 11 *Assume that there exists a nonnegative function* $f : \mathbb{R}_+^N \to \mathbb{R}_+$ *such that*

(i) *for some constant* $c > 0$

$$f(\vec{x}) - f(\vec{y}) \le c\|\vec{x} - \vec{y}\|;$$

(ii) *there exists* $\delta > 0$, $p > 0$ *such that for all* $x \in B^\wedge$

$$f(\vec{x} + v(\vec{x})) - f(\vec{x}) \le -\delta.$$

Then the Markov chain $\mathcal{L}$ *is ergodic.*

A.2 Some simple results about circulant matrices

In this appendix, we recall some well-known properties of the circulant matrices which are used throughout this paper. These properties are given without proof, since they can easily be verified at hand. Throughout the paper, we take the convention that every subscript less than 1 or greater than N should be shifted into the correct range.

Let $(\vec{e}_1, \ldots, \vec{e}_N)$ denote the canonical basis of $\mathbb{C}^N$ and let $\vec{e} = (1, \ldots, 1)$. Moreover, we define $\omega_k \stackrel{\text{def}}{=} \exp(2\hat{\imath}\pi k/N)$, with $\hat{\imath}^2 = -1$.

Definition *A circulant matrix* M *of size* N *is a matrix whose coefficients* $m_{i,j}$ *verify the relation:*

$$m_{i+k,j+k} = m_{i,j}, \text{ for all } 1 \le i, j, k \le N.$$

The following Lemma summarizes some key properties of these matrices:

Lemma 12 *Let M be a circulant matrix of the form:*

$$M = \begin{pmatrix} m_N & m_{N-1} & \cdots & m_1 \\ m_1 & m_N & \cdots & m_2 \\ \vdots & & \ddots & \vdots \\ m_{N-1} & m_{N-2} & \cdots & m_N \end{pmatrix}$$

Then for each $1 \leq k \leq N$, the vector $\vec{v}_k \stackrel{\text{def}}{=} \sum_{i=1}^{N} \omega_k^{-i} \vec{e}_i$ is an eigenvector of the matrix M with eigenvalue $\sum_{i=1}^{N} \omega_k^i m_i$. Moreover, $(\vec{v}_1, \ldots, \vec{v}_N)$ is an orthogonal basis of $\mathbb{C}^N$ in which $\vec{e}_1, \ldots, \vec{e}_N$ can be expressed as

$$\vec{e}_i = \frac{1}{N} \sum_{k=1}^{N} \omega_k^i \vec{v}_k.$$

One important feature of circulant matrices is that they share the same basis of eigenvectors. Note that, with the notation given before Lemma 12, we have $\vec{v}_N = \vec{e}$. Circulant matrices enjoy other properties that are not used here: for example, the product of two circulant matrices is a circulant matrix and this product commutes.

A.3 Proofs of the results of Sections 5 and 6

The derivations in this section are essentially analytic.

Proof of Lemma 7 This proof uses specific properties of circulant matrices to perform a fine analysis of the behavior of $A(\vec{z})$, $\widetilde{A}(\vec{z})$ and $\Delta(\vec{z})$ in the neighborhood of $\vec{z} = \vec{e}$. As pointed out later in the proof, we study these matrices only for $\vec{z} \in \mathcal{D}^N$, thus avoiding any analytical continuation. The basic relation used thereafter is a simple consequence of the definitions of $A(\vec{z})$ and $\Delta(\vec{z})$:

$$A\Delta(\vec{z})\vec{v}_k = \mu_k(\vec{z})\vec{v}_k + \sum_{q=1}^{N} (\frac{1}{z_q} - 1)\omega_k^{-q} A_{\cdot q}(\vec{z}), \tag{A.1}$$

where $A_{\cdot q}(\vec{z})$ stands for the q-th column of $A(\vec{z})$ and can be written as

$$\begin{aligned} A_{\cdot q}(\vec{z}) &= \sum_{i=1}^{N} p_{i-q} a_{i-q}(\vec{z}) \vec{e}_i \\ &= \sum_{i=1}^{N} p_{i-q} a_{i-q}(\vec{z}) \frac{1}{N} \sum_{l=1}^{N} \omega_l^i \vec{v}_l \\ &= \sum_{l=1}^{N} \omega_l^q \mu_l(\vec{z}) \vec{v}_l. \end{aligned}$$

So, if we define

$$\varepsilon_l(\vec{z}) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{q=1}^{N} (\frac{1}{z_q} - 1)\omega_l^q,$$

(A.1) can be rewritten as

$$[I - A\Delta(\vec{z})]\vec{v}_k = (1 - \mu_k(\vec{z}))\vec{v}_k - \sum_{l=1}^{N} \varepsilon_{l-k}(\vec{z})\mu_l(\vec{z})\,\vec{v}_l \tag{A.2}$$

$$[\widetilde{A}(\vec{z}) - A\Delta(\vec{z})]\vec{v}_k = (\tilde{\mu}_k(\vec{z}) - \mu_k(\vec{z}))\vec{v}_k - \sum_{l=1}^{N} \varepsilon_{l-k}(\vec{z})\mu_l(\vec{z})\,\vec{v}_l \tag{A.3}$$

Since ${}^*\vec{e}\,[I - A\Delta(\vec{e})] = {}^*\vec{e}\,[\widetilde{A}(\vec{e}) - A\Delta(\vec{e})] = \vec{0}$, we define, for any set of arbitrary complex numbers $(c_1, \ldots, c_{N-1})$, the vector $\vec{u}_t$ as follows:

$$\vec{u}_t \stackrel{\text{def}}{=} \frac{1}{N}\vec{e} + \frac{t}{N}\sum_{l=1}^{N-1} c_l \vec{v}_l.$$

With this definition, we have

$$\begin{aligned} {}^*\vec{u}_t\,[I - A\Delta(\vec{z})]\vec{e} &= \frac{{}^*\vec{e}}{N}[I - A\Delta(\vec{z})]\vec{e} + \frac{t}{N}\sum_{l=1}^{N-1} c_l {}^*\vec{v}_l\,[I - A\Delta(\vec{z})]\vec{e} \\ &= 1 - \mu_N(\vec{z}) - \varepsilon_0(\vec{z})\mu_N(\vec{z}) - t\sum_{l=1}^{N-1} c_l\varepsilon_l(\vec{z})\mu_l(\vec{z}). \end{aligned} \tag{A.4}$$

When t is small, we have the relations:

$$\begin{aligned} \mu_N(\vec{z}_t) &= 1 + t\alpha\sum_{q=1}^{N}\dot{z}_q + \frac{t^2}{2}\sum_{q,r=1}^{N}\alpha^{(2)}_{q,r}\dot{z}_q\dot{z}_r \\ &\quad + \frac{t^2\alpha}{2}\sum_{q=1}^{N}\ddot{z}_q + o(t^2) \end{aligned} \tag{A.5}$$

$$\begin{aligned} \tilde{\mu}_N(\vec{z}_t) &= 1 + t\tilde{\alpha}\sum_{q=1}^{N}\dot{z}_q + \frac{t^2}{2}\sum_{q,r=1}^{N}\tilde{\alpha}^{(2)}_{q,r}\dot{z}_q\dot{z}_r \\ &\quad + \frac{t^2\tilde{\alpha}}{2}\sum_{q=1}^{N}\ddot{z}_q + o(t^2) \end{aligned} \tag{A.6}$$

$$\begin{aligned} \varepsilon_l(\vec{z}_t) &= -\frac{t}{N}\sum_{q=1}^{N}\dot{z}_q\omega_l^q + \frac{t^2}{N}\sum_{q=1}^{N}\dot{z}_q^2\omega_l^q \\ &\quad - \frac{t^2}{2N}\sum_{q=1}^{N}\ddot{z}_q\omega_l^q + o(t^2). \end{aligned} \tag{A.7}$$

The formulas given in Appendix A.2 allow to relate $\dot{z}_1, \ldots, \dot{z}_N$ to $\dot{\zeta}_1, \ldots, \dot{\zeta}_N$:

$$\dot{\zeta}_k = \frac{1}{N}\sum_{q=1}^{N}\dot{z}_q\omega_k^q, \qquad \dot{z}_q = \sum_{l=1}^{N}\dot{\zeta}_q\omega_l^{-q}, \qquad \sum_{q=1}^{N}\dot{z}_q\dot{z}_{q+d} = N\sum_{l=1}^{N}\dot{\zeta}_l\dot{\zeta}_{N-l}\omega_l^d.$$

Using the remark after the definition of $\alpha^{(2)}_{q,r}$ and $\tilde{\alpha}^{(2)}_{q,r}$, we find that:

$$\sum_{q,r=1}^{N}\alpha^{(2)}_{q,r}\dot{z}_q\dot{z}_r = \sum_{d=1}^{N}\alpha^{(2)}_d\sum_{q=1}^{N}\dot{z}_q\dot{z}_{q+d} = N\sum_{l=1}^{N}\dot{\zeta}_l\dot{\zeta}_{N-l}\sum_{d=1}^{N}\alpha^{(2)}_d\omega_l^d.$$

Applying (A.5) and (A.7) to (A.4), we see that the first-order term in the expression (A.4) is $(\alpha - 1/N)t[\dot{z}_1 + \cdots + \dot{z}_N]$. So a necessary condition to have a formula like (5.10) is $\dot{z}_1 + \cdots + \dot{z}_N = 0$. Using this relation, we have

$$\begin{aligned} {}^*\vec{u}_t\,[I - A\Delta(\vec{z}_t)]\vec{e} &= t^2 \sum_{l=1}^{N-1} c_l \dot{\zeta}_l \mu_l - t^2 \sum_{l=1}^{N} \dot{\zeta}_l \dot{\zeta}_{N-l} \\ &\quad - t^2 \frac{N}{2} \sum_{l=1}^{N} \dot{\zeta}_l \dot{\zeta}_{N-l} \sum_{d=1}^{N} \alpha_d^{(2)} \omega_l^d \\ &\quad + t^2 \frac{1 - N\alpha}{2N} \sum_{q=1}^{N} \ddot{z}_q + o(t^2). \end{aligned} \tag{A.8}$$

We have to check that it is possible to have $\vec{z}_t \in \mathcal{D}^N$ and $\dot{z}_1 + \cdots + \dot{z}_N = 0$. One easy way to satisfy these constraints is to ensure that all coordinates of $\vec{z}_t$ arrive to 1 tangentially to the unit circle as t goes to 0. This is the case when, for any q, $\dot{z}_q$ is an imaginary number and $\ddot{z}_q < 0$. Moreover, for $k < N$,

$$\begin{aligned} {}^*\vec{u}_t\,[I - A\Delta(\vec{z}_t)]\vec{v}_k &= -\varepsilon_{N-k}(\vec{z}_t)\mu_N(\vec{z}_t) + tc_k(1 - \mu_k(\vec{z}_t)) \\ &\quad - t \sum_{l=1}^{N-1} c_l \varepsilon_{l-k}(\vec{z}_t)\mu_l(\vec{z}_t) \\ &= t\dot{\zeta}_{N-k} + tc_k(1 - \mu_k) + o(t). \end{aligned} \tag{A.9}$$

This shows that we get equations (5.8) and (5.10) from (A.8) and (A.9) if we take

$$c_k = -\frac{\dot{\zeta}_{N-k}}{1 - \mu_k}.$$

With this choice of $c_1, \ldots, c_{N-1}$, we obtain equations (5.11) and (5.9) in exactly the same way. ■

Proof of Theorem 8 The idea of this proof is to apply the results of Lemma 7 to the equation

$$ {}^*\vec{u}_t\,[I - A\Delta(\vec{z}_t)]F(\vec{z}_t) = {}^*\vec{u}_t\,[\widetilde{A}(\vec{z}_t) - A\Delta(\vec{z}_t)]\widetilde{F}(\vec{z}_t). \tag{A.10}$$

Define $\vec{z}_t$ as in Lemma 7. Then

$$F(\vec{z}_t) = F(\vec{e}) + t \sum_{r=1}^{N} \dot{z}_r \frac{\partial F}{\partial z_r}(\vec{e}) + \vec{o}(t).$$

Moreover, Equation (5.1) implies that $\partial F_i/\partial z_r(\vec{e}) = \partial F_{i-r+m}/\partial z_m(\vec{e})$ for any fixed $m \in \mathcal{S}$ and

$$\begin{aligned} \frac{\partial F}{\partial z_r}(\vec{e}) &= \sum_{i=1}^{N} \frac{\partial F_i}{\partial z_r}(\vec{e})\vec{e}_i \\ &= \sum_{i=1}^{N} \Big[\frac{\partial F_{i-r+m}}{\partial z_m}(\vec{e}) \frac{1}{N} \sum_{k=1}^{N} \omega_k^i \vec{v}_k \Big] \end{aligned}$$

$$= \sum_{k=1}^{N} \omega_k^{r-m} \Big[\frac{1}{N} \sum_{i=1}^{N} \frac{\partial F_{i-r+m}}{\partial z_m}(\vec{e}) \omega_k^{i-r+m} \Big] \vec{v}_k$$

$$= \sum_{k=1}^{N} \omega_k^{r-m} \frac{\partial \varphi_k}{\partial z_m}(\vec{e}) \vec{v}_k,$$

where $(\varphi_1(\vec{z}), \ldots, \varphi_N(\vec{z}))$, defined as

$$\varphi_k(\vec{z}) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{i=1}^{N} F_i(\vec{z}) \omega_k^i,$$

are the coordinates of $F(\vec{z})$ in the basis $(\vec{v}_1, \ldots, \vec{v}_N)$. Finally,

$$F(\vec{z}_t) = F(\vec{e}) + tN \sum_{k=1}^{N} \dot{\zeta}_k \omega_k^{-m} \frac{\partial \varphi_k}{\partial z_m}(\vec{e}) \vec{v}_k + \vec{o}(t). \tag{A.11}$$

With a similar definition for $\tilde{\varphi}(\vec{z})$,

$$\widetilde{F}(\vec{z}_t) = \widetilde{F}(\vec{e}) + tN \sum_{k=1}^{N} \dot{\zeta}_k \omega_k^{-m} \frac{\partial \tilde{\varphi}_k}{\partial z_m}(\vec{e}) \vec{v}_k + \vec{o}(t). \tag{A.12}$$

We now apply Lemma 7 and Equations (A.11) and (A.12) to Equation (A.10) and use the fact that, by symmetry, $F(\vec{e}) = F_1(\vec{e})\vec{e}$ and $\widetilde{F}(\vec{e}) = \widetilde{F}_1(\vec{e})\vec{e}$

$$t^2 F_1(\vec{e}) \Bigg\{ - \sum_{l=1}^{N-1} \dot{\zeta}_l \dot{\zeta}_{N-l} \Bigg[\frac{1}{1-\mu_l} + \frac{N}{2} \sum_{d=1}^{N} \alpha_d^{(2)} \omega_l^d \Bigg] + \frac{1-N\alpha}{2N} \sum_{q=1}^{N} \ddot{z}_q \Bigg\}$$

$$= t^2 \widetilde{F}_1(\vec{e}) \Bigg\{ - \sum_{l=1}^{N-1} \dot{\zeta}_l \dot{\zeta}_{N-l} \Bigg[\frac{1}{1-\mu_l} + \frac{N}{2} \sum_{d=1}^{N} (\alpha_d^{(2)} - \tilde{\alpha}_d^{(2)}) \omega_k^d \Bigg]$$

$$+ \frac{1-N\alpha+N\tilde{\alpha}}{2N} \sum_{q=1}^{N} \ddot{z}_q \Bigg\}$$

$$+ t^2 N \sum_{k=1}^{N-1} \dot{\zeta}_k \dot{\zeta}_{N-k} \omega_k^{-m} \frac{1-\tilde{\mu}_k}{1-\mu_k} \frac{\partial \widetilde{\varphi}_k}{\partial z_m}(\vec{e}) + o(t^2).$$

As $\vec{\ddot{z}}$ can be chosen freely, we can derive a first equality from this equation, namely

$$F_1(\vec{e}) \frac{1-N\alpha}{2N} = \widetilde{F}_1(\vec{e}) \frac{1-N\alpha+N\tilde{\alpha}}{2N}.$$

Since $P(X_m = 0 \mid S = m) = \widetilde{F}_1(\vec{e})/F_1(\vec{e})$, this yields Equation (5.12). Taking in account (5.6), we find

$$\sum_{k=1}^{N-1} \dot{\zeta}_k \dot{\zeta}_{N-k} \omega_k^{-m} \frac{1-\tilde{\mu}_k}{1-\mu_k} N \frac{\partial \widetilde{\varphi}_k}{\partial z_m}(\vec{e})$$

$$= - \sum_{l=1}^{N-1} \dot{\zeta}_l \dot{\zeta}_{N-l} \Bigg[\frac{F_1(\vec{e}) - \widetilde{F}_1(\vec{e})}{1-\mu_l} + \frac{1}{2} \sum_{d=1}^{N} \bar{\alpha}_d^{(2)} \omega_l^d \Bigg]. \tag{A.13}$$

This equation contains in fact a system of linear equations that can be built by choosing $\vec{\dot{z}}$. The problem is that in general the rank of this system is only $\lfloor n/2 \rfloor$. However, since μ_k (resp. μ_{N-k}, $\tilde{\mu}_k$, $\tilde{\mu}_{N-k}$) is the eigenvalue of ${}^*P$ (resp. P, ${}^*\widetilde{P}$, $\widetilde{P}$) associated to the eigenvector $\vec{v}_k$, A$_3$ implies that, for $1 \le k < N$,

$$\frac{1-\mu_k}{1-\tilde{\mu}_k} = \frac{1-\mu_{N-k}}{1-\tilde{\mu}_{N-k}} \in \mathbb{R}.$$

As noted in the proof of Lemma 7, $\dot{z}_1, \ldots, \dot{z}_N$ are imaginary numbers and $\dot{\zeta}_k \dot{\zeta}_{N-k}$ is a negative real number for any $k < N$. Hence we can choose in (A.13)

$$\dot{\zeta}_k \dot{\zeta}_{N-k} = -\min\Big[\frac{1-\mu_k}{1-\tilde{\mu}_k}, 0\Big].$$

The combination of the resulting equation with a similar equation containing only the terms where $(1-\mu_k)/(1-\tilde{\mu}_k) < 0$ yields

$$\begin{aligned}
&\sum_{k=1}^{N-1} \omega_k^{-m} N \frac{\partial \widetilde{\varphi}_k}{\partial z_m}(\vec{e}) \\
&\qquad = -\sum_{l=1}^{N-1} \frac{1-\mu_k}{1-\tilde{\mu}_k}\left[\frac{F_1(\vec{e}) - \widetilde{F}_1(\vec{e})}{1-\mu_l} + \frac{1}{2}\sum_{d=1}^{N} \bar{\alpha}_d^{(2)} \omega_l^d\right]. \qquad \text{(A.14)}
\end{aligned}$$

We know that, by definition, $\widetilde{F}_m(\vec{z})$ does not depend on z_m. Hence,

$$\frac{\partial \widetilde{F}_m}{\partial z_m}(\vec{e}) = \sum_{k=1}^{N} \omega_k^{-m} \frac{\partial \widetilde{\varphi}_k}{\partial z_m}(\vec{e}) = 0,$$

and, using Equation (A.14)

$$N \frac{\partial \tilde{\varphi}_N}{\partial z_m}(\vec{e}) = \sum_{l=1}^{N-1}\left[\frac{F_1(\vec{e}) - \widetilde{F}_1(\vec{e})}{1-\tilde{\mu}_l} + \frac{1-\mu_l}{1-\tilde{\mu}_l}\frac{1}{2}\sum_{d=1}^{N} \bar{\alpha}_d^{(2)} \omega_l^d\right]. \qquad \text{(A.15)}$$

The last step of the demonstration is to get $\partial F_m / \partial z_m(\vec{e})$ from these results. This can be done as in Lemma 7, but with $\vec{u}_t = \vec{e}/N$ and $\vec{z}_t$ chosen differently. Using Equations (A.2), (A.3) and (A.5)–(A.7) with the same notation as in Lemma 7, we have for $k < N$

$$\begin{aligned}
\frac{{}^*\vec{e}}{N}[I - A\Delta(\vec{z}_t)]\vec{e} &= 1 - \mu_N(\vec{z}_t) - \varepsilon_0(\vec{z}_t)\mu_N(\vec{z}_t) \\
&= t(1-N\alpha)\dot{\zeta}_N - t^2 \sum_{l=1}^{N} \dot{\zeta}_l \dot{\zeta}_{N-l} + t^2 N\alpha \dot{\zeta}_N^2 \\
&\quad - t^2 \frac{N}{2} \sum_{l=1}^{N} \dot{\zeta}_l \dot{\zeta}_{N-l} \sum_{d=1}^{N} \alpha_d^{(2)} \omega_l^d + o(t^2) \\
\frac{{}^*\vec{e}}{N}[I - A\Delta(\vec{z}_t)]\vec{v}_k &= -\varepsilon_{N-k}(\vec{z}_t)\mu_N(\vec{z}_t) \;=\; t\dot{\zeta}_{N-k} + o(t) \\
\frac{{}^*\vec{e}}{N}[\widetilde{A}(\vec{z}_t) - A\Delta(\vec{z}_t)]\vec{e} &= \tilde{\mu}_N(\vec{z}_t) - \mu_N(\vec{z}_t) - \varepsilon_0(\vec{z}_t)\mu_N(\vec{z}_t)
\end{aligned}$$

$$
\begin{aligned}
&= t(1 - N\alpha + N\tilde{\alpha})\dot{\zeta}_N - t^2 \sum_{l=1}^{N} \dot{\zeta}_l \dot{\zeta}_{N-l} + t^2 N\alpha\dot{\zeta}_N^2 \\
&\quad + t^2 \frac{N}{2} \sum_{l=1}^{N} \dot{\zeta}_l \dot{\zeta}_{N-l} \sum_{d=1}^{N} [\tilde{\alpha}_d^{(2)} - \alpha_d^{(2)}]\omega_l^d + o(t^2)
\end{aligned}
$$

$$
\frac{{}^*\vec{e}}{N}[\widetilde{A}(\vec{z}_t) - A\Delta(\vec{z}_t)]\vec{v}_k \;=\; -\varepsilon_{N-k}(\vec{z}_t)\mu_N(\vec{z}_t) \;=\; t\dot{\zeta}_{N-k} + o(t)
$$

Combining these equations with (A.11) and (A.12), we find

$$
\begin{aligned}
&-t^2 F_1(\vec{e})\left\{ \sum_{l=1}^{N} \dot{\zeta}_l \dot{\zeta}_{N-l}\left[1 + \frac{N}{2}\sum_{d=1}^{N} \alpha_d^{(2)}\omega_l^d\right] - N\alpha\dot{\zeta}_N^2 \right\} \\
&\quad + t^2 N \sum_{k=1}^{N-1} \dot{\zeta}_k \dot{\zeta}_{N-k} \omega_k^{-m} \frac{\partial \varphi_k}{\partial z_m}(\vec{e}) + t^2 N(1 - N\alpha)\dot{\zeta}_N^2 \frac{\partial \varphi_N}{\partial z_m}(\vec{e}) \\
&\qquad = -t^2 \widetilde{F}_1(\vec{e})\left\{ \sum_{l=1}^{N} \dot{\zeta}_l \dot{\zeta}_{N-l}\left[1 + \frac{N}{2}\sum_{d=1}^{N} [\alpha_d^{(2)} - \tilde{\alpha}_d^{(2)}]\omega_l^d\right] - N\alpha\dot{\zeta}_N^2 \right\} \\
&\qquad\quad + t^2 N \sum_{k=1}^{N-1} \dot{\zeta}_k \dot{\zeta}_{N-k} \omega_k^{-m} \frac{\partial \tilde{\varphi}_k}{\partial z_m}(\vec{e}) \\
&\qquad\quad + t^2 N(1 - N\alpha + N\tilde{\alpha})\dot{\zeta}_N^2 \frac{\partial \tilde{\varphi}_N}{\partial z_m}(\vec{e}) + o(t^2).
\end{aligned} \tag{A.16}
$$

This equation in turn gives for $(1 - N\alpha)\dot{\zeta}_N^2 = 1$ and $\dot{\zeta}_l \dot{\zeta}_{N-l} = 1$ if $1 \leq l < N$

$$
\begin{aligned}
N \sum_{k=1}^{N} \omega_k^{-m} \frac{\partial \varphi_k}{\partial z_m}(\vec{e}) &= \frac{N\tilde{\alpha}}{1 - N\alpha} N \frac{\partial \tilde{\varphi}_N}{\partial z_m}(\vec{e}) + N(F_1(\vec{e}) - \widetilde{F}_1(\vec{e})) \\
&\quad + \frac{1}{2} \sum_{l=1}^{N-1} \sum_{d=1}^{N} \bar{\alpha}_d^{(2)} \omega_l^d + \frac{1}{1 - N\alpha} \frac{1}{2} \sum_{d=1}^{N} \bar{\alpha}_d^{(2)}
\end{aligned}
$$

This expression, together with (A.15) and (5.7), yields Equation (5.13). The derivation of (5.14) is done in a similar way: we use the fact that

$$
\mathbb{E}X_m \;=\; \sum_{i=1}^{N} \frac{\partial F_i}{\partial z_m}(\vec{e}) \;=\; N \frac{\partial \varphi_N}{\partial z_m}(\vec{e}).
$$

We use Equation (A.16) with $(1 - N\alpha)\dot{\zeta}_N^2 = 1$ and $\dot{\zeta}_k \dot{\zeta}_{N-k} = 0$ for $k < N$ and find

$$
N \frac{\partial \varphi_N}{\partial z_m}(\vec{e}) = \frac{1 - N\alpha + N\tilde{\alpha}}{1 - N\alpha} N \frac{\partial \tilde{\varphi}_N}{\partial z_m}(\vec{e}) + F_1(\vec{e}) - \widetilde{F}_1(\vec{e}) + \frac{1}{1 - N\alpha} \frac{1}{2} \sum_{d=1}^{N} \bar{\alpha}_d^{(2)}.
$$

This gives Equation (5.14) and concludes the proof of the theorem. ■

Proof of Theorem 9 We see easily that the mean numbers of clients arriving between polling times are respectively $\alpha = \lambda\tau$ and $\tilde{\alpha} = \lambda\tilde{\tau}$. Moreover, using

Equations (5.3) and (5.4) and the properties of generating functions we find that, for $d < N$

$$\begin{aligned}
\alpha_d^{(2)} &= \alpha^{(2)} = \lambda^2 \tau^{(2)} \\
\alpha_N^{(2)} &= \alpha^{(2)} + \hat{\lambda}(b^{(2)} - b)\tau \\
\tilde{\alpha}_d^{(2)} &= \tilde{\alpha}^{(2)} = \lambda^2 \tilde{\tau}^{(2)} \\
\tilde{\alpha}_N^{(2)} &= \tilde{\alpha}^{(2)} + \hat{\lambda}(b^{(2)} - b)\tilde{\tau}.
\end{aligned}$$

The computation of waiting times uses the following classical argument: a non empty queue visited by the server can be decomposed into

- the head of line customer;
- the clients who arrived after him during his waiting time;
- clients who arrived in the same batch as the first client, but are not yet served; since, by renewal arguments, the mean size of this batch is $b^{(2)}/b$, the mean number of remaining clients is $(b^{(2)} - b)/2b$.

This can be written as

$$\mathbb{E}[X_m \mid S = m, X_m > 0] = 1 + \lambda \mathbb{E}[W] + \frac{b^{(2)} - b}{2b}$$

and, using the fact that $\sum_{d=1}^{N} \alpha_d^{(2)} \omega_l^d = \alpha_N^{(2)} - \alpha^{(2)}$,

$$\begin{aligned}
\lambda b \mathbb{E}[W] &= \frac{\mathbb{E}[X_m \mid S = m]}{1 - P(X_m = 0 \mid S = m)} - 1 - \frac{b^{(2)} - b}{2b} \\
&= \frac{\tilde{\alpha}}{1 - N\alpha} \sum_{l=1}^{N-1} \frac{1}{1 - \mu_l} + \frac{1}{1 - N\alpha} \frac{N}{2} \alpha^{(2)} + \frac{1}{N\tilde{\alpha}} \frac{N}{2} \tilde{\alpha}^{(2)} \\
&\quad + \frac{b^{(2)} - b}{2b} \left[\frac{\alpha + (N-1)\tilde{\alpha}}{1 - N\alpha} - \frac{N\tilde{\alpha}}{1 - N\alpha} \frac{1}{N} \sum_{l=1}^{N-1} \frac{\mu_l - \tilde{\mu}_l}{1 - \tilde{\mu}_l} \right] \\
&= \frac{\lambda \tilde{\tau}}{1 - N\lambda\tau} \sum_{l=1}^{N-1} \frac{1}{1 - \mu_l} + \frac{N\lambda^2 \tau^{(2)}}{2(1 - N\lambda\tau)} + \frac{\lambda \tilde{\tau}^{(2)}}{2\tilde{\tau}} \\
&\quad + \frac{b^{(2)} - b}{2b} \left[\frac{\lambda\tau + (N-1)\lambda\tilde{\tau}}{1 - N\lambda\tau} - \frac{\lambda\tilde{\tau}}{1 - N\lambda\tau} \sum_{l=1}^{N-1} \frac{\mu_l - \tilde{\mu}_l}{1 - \tilde{\mu}_l} \right].
\end{aligned}$$

This gives Equation (6.1). ∎

References

[1] A. A. Borovkov and R. Schassberger, *Ergodicity of a polling network*, Stochastic Processes and their Applications, 50 (1994), pp. 253–262.

[2] O. J. Boxma and J. A. Weststrate, *Waiting time in polling systems with Markovian server routing*, in Messung, Modellierung und Bewertung von Rechensysteme, Berlin, 1989, Proc. Conference Braunschweig, Springer-Verlag, pp. 89–104.

[3] H. S. Bradlow and H. F. Byrd, *Mean waiting time evaluation of packet switches for centrally controlled PBX's*, Performance Evaluation, 7 (1987), pp. 309–327.

[4] G. Fayolle, V. A. Malyshev, and M. V. Menshikov, *Topics in the Constructive Theory of Countable Markov Chains*, Cambridge University Press, 1995.

[5] G. Fayolle and A. A. Zamyatin, *Controlled random walks in Z_+^N and their applications to queueing networks*, Preprint, (1993). To appear.

[6] M. J. Ferguson, *Mean waiting time for a token ring with station dependent overheads*, in Local Area & Multiple Access Networks, R. L. Pickholtz, ed., Computer Science Press, Rockville, Maryland, 1986, ch. 3, pp. 43–67.

[7] C. Fricker and M. R. Jaïbi, *Monotonicity and stability of periodic polling models*, Queueing Systems, Theory and Applications, 15 (1994), pp. 211–238.

[8] ——, *Stability of a polling model with a Markovian scheme*, Rapport de Recherche 2278, INRIA, Rocquencourt BP 105 – 78153 Le Chesnay – France, May 1994.

[9] L. Kleinrock and H. Levy, *The analysis of random polling systems*, Operations Research, 36 (1988), pp. 716–732.

[10] H. Levy and M. Sidi, *Polling systems: Applications, modeling and optimization*, IEEE Transactions on Communications, 38 (1990), pp. 1750–1760.

[11] H. Levy, M. Sidi, and O. J. Boxma, *Dominance relations in polling systems*, Queueing Systems, Theory and Applications, 6 (1990), pp. 155–172.

[12] V. A. Malyshev and M. V. Menshikov, *Ergodicity, continuity and analyticity of countable Markov chains*, Trans. Moscow. Math. Soc., 39 (1979), pp. 3–48.

[13] R. Schassberger, *Stability of polling networks with state-dependent server routing*, Probab. Engrg. Inform. Sci., 9 (1995), pp. 539–550.

[14] L. D. Servi, *Average delay approximation of M/G/1 cyclic service queues with Bernoulli schedules*, IEEE Journal on Selected Areas in Communications, SAC-4 (1986), pp. 813–822. Correction in Vol. Sac-5, No. 3, p. 547, 1987.

[15] M. M. Srinivasan, *Nondeterministic polling systems*, Management Science, 37 (1991), pp. 667–681.

[16] H. Takagi, *Queueing analysis of polling models: an update*, in Stochastic Analysis of Computer and Communication Systems, H. Takagi, ed., North Holland, Amsterdam, 1990, pp. 267–318.