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Abstract

As a complement to a recently proposed quadratic programming (QP)-based method for linear regression estimation from quantized or binary data, this report presents a complete input condition ensuring the uniqueness of result of the QP-based method.

1 Introduction

Quantized data are typically produced by the process of analog-to-digital conversion and has been widely studied in signal encoding and digital representation. System identification based on quantized data has been investigated in quite a few articles [7][10][18][13] [14][17][1][2][6][5][8][3][4]. Binary data are in principle special cases of quantized data; however, as the information of original data is greatly lost during this particular quantization procedure, most of the existing identification methods for quantized data cannot be directly applied to binary measurements. Hence, some special methods have been designed [11][12][15][16][6] [3] for this case.

Recently a quadratic programming (QP)-based method for quantized system identification has been proposed in [9] with the the following advantages:

- The QP-based method has mild requirements on identification experiments.
- The QP-based method is equally applicable to both general quantized data and to binary measurements without any modification.

After a brief recall of the QP-based method in Section 2, a condition ensuring the uniqueness of its result will be presented in the following sections.

2 The QP-based method

Consider the linear regression

$$z(t) = \phi^T(t)\theta + e(t) \tag{1}$$

with the regression vector $\phi(t) \in \mathbb{R}^n$, the regression coefficient vector $\theta \in \mathbb{R}^n$, an additive white noise $e(t)$, and the discrete sampling index $t = 1, 2, 3, \ldots, N$.

The output $z(t) \in \mathbb{R}$ is not directly accessible; instead, its quantized counterpart $y(t)$ is available, which is related to $z(t)$ by a known quantificaton function $q(\cdot)$, i.e.,

$$y(t) = q\left(z(t)\right) = q\left(\phi^T(t)\theta + e(t)\right).$$

A simple example of the quantification rule $q(\cdot)$ is to round the real value $z(t)$ to the nearest integer value.

Given the quantized output $y(t)$, the input $u(t)$ and the quantification rule q, the objective is to estimate the unknown parameter vector θ.

As the quantification rule $q(\cdot)$ is a nonlinear stairwise function and its derivative is zero except at some singular points, it is impossible to apply gradient-based optimization methods to minimize the errors between quantified measurement and the nonlinear model output.

The QP-based method proposed in [9] is as follows. As the quantification rule q is assumed available, for any observed quantized output $y(t)$ (which takes discrete values), it is known that

$$\alpha\left(y(t)\right) \le z(t) < \beta\left(y(t)\right), \tag{2}$$

where α and β are known nonlinear functions depending on q. For example, if the quantification function q simply rounds the real value $z(t)$ to the nearest integer, then,

$$y(t) - 0.5 \le z(t) < y(t) + 0.5.$$

If the system output before quantification $z(t)$ was directly accessible, the parameters in θ could be estimated by minimizing the least squares criterion:

$$\hat{\theta} = \arg\min_{\theta} \frac{1}{N} \sum_{t=1}^{N} \left(z(t) - \phi^T(t)\theta\right)^2. \tag{3}$$

Such a simple method cannot be applied, because $z(t)$ is not directly accessible. However, some information about $z(t)$ is available through the quantized output $y(t)$, and this information is expressed by the inequalities (2). By combining the least squares criterion (3) and the inequalities (2), the parameter vector θ can be estimated through the constrained optimization problem:

$$\left(\hat{z}(t), \hat{\theta}\right) = \arg\min_{z(t),\theta} \frac{1}{N} \sum_{t=1}^{N} \left(z(t) - \phi^T(t)\theta\right)^2, \tag{4}$$
$$\text{s.t. } \alpha\left(y(t)\right) \le z(t) < \beta\left(y(t)\right), \ \forall t = 1, \ldots, N.$$

This constrained optimization problem can be reformulated in the form of a standard QP problem as follows. Define

$$Z := \begin{bmatrix} z(1) \\ z(2) \\ \vdots \\ z(N) \end{bmatrix}, \quad \Phi := \begin{bmatrix} \phi^T(1) \\ \phi^T(2) \\ \vdots \\ \phi^T(N) \end{bmatrix},$$

we reformulate the loss function in (4) as

$$
\begin{aligned}
& \frac{1}{N}\sum_{t=1}^{N}\left(z(t)-\phi^T(t)\theta\right)^2, \\
= \quad & \frac{1}{N}\left(Z-\Phi\theta\right)^T\left(Z-\Phi\theta\right) \\
= \quad & \frac{1}{N}\left(Z^TZ-Z^T\Phi\theta-\theta^T\Phi^TZ+\theta^T\Phi^T\Phi\theta\right) \\
= \quad & \frac{1}{N}\left[\begin{array}{cc}\theta^T & Z^T\end{array}\right]\left[\begin{array}{cc}\Phi^T\Phi & -\Phi^T \\ -\Phi & I\end{array}\right]\left[\begin{array}{c}\theta \\ Z\end{array}\right] \\
= \quad & :\frac{1}{N}X^THX
\end{aligned}
$$

where

$$
X := \left[\begin{array}{c}\theta \\ Z\end{array}\right], H := \left[\begin{array}{cc}\Phi^T\Phi & -\Phi^T \\ -\Phi & I\end{array}\right].
$$

The original estimation problem in (4) is then equivalent to a standard QP problem,

$$
\begin{aligned}
& \min_X X^THX, \\
& \text{s.t. } X_L \leq X \leq X_U.
\end{aligned} \tag{5}
$$

Here X_L and X_U are lower and upper bounds of X,

$$
X_L := \left[\begin{array}{c}\theta_L \\ \alpha\left(y\left(1\right)\right) \\ \vdots \\ \alpha\left(y\left(N\right)\right)\end{array}\right], \quad X_U := \left[\begin{array}{c}\theta_U \\ \beta\left(y\left(1\right)\right) \\ \vdots \\ \beta\left(y\left(N\right)\right)\end{array}\right] \tag{6}
$$

where θ_L and θ_U are some assumed lower and upper bounds of θ, respectively. If no such bounds on θ are available, then they can be ignored in the above QP problem. In the sequel, this way of estimating θ is referred to as the QP-based method. Note that both $z(t)$ and θ are estimated by the QP-based method; hence the number of unknowns to be estimated increases with the data length N.

3 The complete Input Condition

This section introduces a so-called *complete input condition* to ensure unique estimation of the parameter vector θ and the non quantized output samples in Z.

The matrix H in (5) is positive semi-definite, not strictly positive definite, since

$$
H = \left[\begin{array}{cc}\Phi^T\Phi & -\Phi^T \\ -\Phi & I\end{array}\right] = \left[\begin{array}{cc}\Phi & -I\end{array}\right]^T\left[\begin{array}{cc}\Phi & -I\end{array}\right]
$$

is rank-deficient. Thus, the QP problem (5) is convex, but not strictly convex. As a result, it is possible that the global minimum is reached at more than one point. The condition on Φ specified in Definition 1 below, referred to as the complete input condition, ensures that the global minimum is unique.

If the matrix $\Phi \in \mathbb{R}^{N \times n}$ has full column rank, then its QR decomposition is

$$\Phi = [Q_0 \; Q_1] \begin{bmatrix} R_0 \\ 0 \end{bmatrix} \tag{7}$$

with a unitary matrix $[Q_0 \;\; Q_1]$ composed of $Q_0 \in \mathbb{R}^{N \times n}, Q_1 \in \mathbb{R}^{N \times (N-n)}$, and a non singular matrix $R_0 \in \mathbb{R}^{n \times n}$.

Definition 1 *If the matrix Φ and the corresponding matrices Q_0, Q_1 defined in* (7) *are such that*

1. *the matrix $\Phi \in \mathbb{R}^{N \times n}$ has full column rank;*
2. *the set $\left\{Z : Z_L \leq Z < Z_U, Q_1^T Z = 0\right\}$ is empty or has a singleton element, where Z_L and Z_U are as X_L and X_U defined in* (6)*, after removing the components θ_L and θ_U;*
3. *none of the rows of Q_1 is solely filled with zeros, nor is any row of Q_0.*

then the sequence of $u(t)$ constituting Φ is called a complete input*.*

Proposition 1 *If the matrix Φ satisfies the complete input condition defined in Definition 1, then the minimum of the QP problem* (5) *is reached at a single point X.*

Condition *1* is easy to understand, as it is also required to ensure the uniqueness of the least squares solution from non quantized data. Conditions *2* prevents that the unconstrained minimum of the quadratic criterion has a non singleton intersection with the constrained area of Z, and Conditions *3* excludes degenerate observations.

The formal proof of Proposition 1 is quite lengthy and will be presented in the next section.

4 Proofs related to the complete input condition

The proof of Proposition 1 will be made in several steps, first considering an equivalent QP problem by eliminating θ, then the QP problem ignoring the lower and upper bounds of θ, before considering the original QP problem (5).

4.1 The QP problem after minimization w.r.t. θ

Let us first minimize the quadratic criterion of the QP problem (5) with respect to θ for any fixed value of Z:

$$\min_\theta X^T H X = \min_\theta (Z - \Phi\theta)^T (Z - \Phi\theta). \tag{8}$$

This minimization is being considered by ignoring the possibly available lower and upper bounds of θ, which will be considered later in Section 4.5.

Since Φ has full column rank (Condition *1* of Definition 1), this minimization amounts to replacing θ by $\Phi(\Phi^T\Phi)^{-1}\Phi^T Z$, then

$$\begin{aligned}\min_{\theta} X^T H X &= Z^T \left(I - \Phi(\Phi^T\Phi)^{-1}\Phi^T\right)^2 Z \\ &= Z^T \left(I - \Phi(\Phi^T\Phi)^{-1}\Phi^T\right) Z \end{aligned} \tag{9}$$

In what follows, let us first consider the reduced QP problem

$$\min_{Z} Z^T \left(I - \Phi(\Phi^T\Phi)^{-1}\Phi^T\right) Z \tag{10a}$$

$$\text{s.t. } Z_L \leq Z \leq Z_U \tag{10b}$$

whose relationship with the original QP problem (5) is obvious.

4.2 Some notations

The following notations will be used in the proofs.

The loss function

$$V(Z) \triangleq Z^T \left(I - \Phi(\Phi^T\Phi)^{-1}\Phi^T\right) Z \tag{11}$$

The constrained area

$$C \triangleq \{Z \in \mathbb{R}^N : Z_L \leq Z \leq Z_U\} \tag{12}$$

The level set for a given value $\bar{V} > 0$

$$L(\bar{V}) \triangleq \{Z \in \mathbb{R}^N : V(Z) = \bar{V}\} \tag{13}$$

For two points $Z^\dagger, Z^\ddagger \in \mathbb{R}^N$, $l(Z^\dagger, Z^\ddagger)$ denotes the straight line segment joining $Z^\dagger$ and $Z^\ddagger \in \mathbb{R}^N$, i.e.,

$$l(Z^\dagger, Z^\ddagger) \triangleq \{Z \in \mathbb{R}^N : Z = \rho Z^\dagger + (1-\rho)Z^\ddagger, \rho \in [0,1]\} \tag{14}$$

Remind also the QR decomposition of a full column rank Φ:

$$\Phi = [Q_0 \ Q_1] \begin{bmatrix} R_0 \\ 0 \end{bmatrix} \tag{15}$$

with $Q_0 \in \mathbb{R}^{N\times n}, Q_1 \in \mathbb{R}^{N\times(N-n)}$ such that

$$[Q_0 \ Q_1][Q_0 \ Q_1]^T = I,$$

and a non singular matrix $R_0 \in \mathbb{R}^{n\times n}$. Then it is easy to check that

$$\begin{aligned} V(Z) &\triangleq Z^T \left(I - \Phi(\Phi^T\Phi)^{-1}\Phi^T\right) Z \\ &= Z^T(I - Q_0 Q_0^T)Z \\ &= Z^T Q_1 Q_1^T Z \end{aligned} \tag{16}$$

4.3 Some lemmas

Lemma 1 *The constrained area $C = \{Z \in \mathbb{R}^N : Z_L \leq Z \leq Z_U\}$ is an N-dimensional hyperrectangle with each of its faces(of dimension $1, 2, \ldots, N-1$) parallel to at least one of the Z-axes, and also orthogonal to at least one of the Z-axes.*

These facts are immediate consequences of the definition of the hyperrectangle C.

Lemma 2 *If for two given vectors $p, q \in \mathbb{R}^N$, the straight line $\rho p + q$ parametrized by the free real scalar ρ partly belongs to the level set $L(\bar{V}) = \{Z \in \mathbb{R}^N : V(Z) = \bar{V}\}$, or more precisely, if there exits an interval $[\rho_1, \rho_2]$ with $\rho_1 < \rho_2$ such that*

$$Z \in \{\xi \in \mathbb{R}^N : \xi = \rho p + q, \rho \in [\rho_1, \rho_2]\} \Longrightarrow Z \in L(\bar{V}),$$

then $Q_1^T p = 0$.

To prove this lemma, notice that $V(\xi) = V(\rho p + q) = (\rho p + q)^T Q_1 Q_1^T (\rho p + q)$, and

$$(\rho p + q)^T Q_1 Q_1^T (\rho p + q) = \bar{V}$$

for all $\rho \in [\rho_1, \rho_2]$, hence the polynomial in ρ

$$(p^T Q_1 Q_1^T p)\rho^2 + 2(p^T Q_1 Q_1^T q)\rho + q^T Q_1 Q_1^T q - \bar{V} = 0$$

for all $\rho \in [\rho_1, \rho_2]$. It then yields $p^T Q_1 Q_1^T p = 0$, and consequently $Q_1^T p = 0$.

Lemma 3 *The converse of Lemma 2 also holds: for two given vectors $p, q \in \mathbb{R}^N$, if $Q_1^T p = 0$, then the straight line $\rho p + q$ parametrized by the free real scalar ρ belongs to the level set $L(q^T Q_1 Q_1^T q)$.*

The proof of this lemma can be made by inverting the steps of the proof of Lemma 2.

4.4 Complete input condition for the reduced QP problem

Now we are ready to prove the reduced version of Proposition 1, stated as follows.

Proposition 2 *If Φ satisfies the complete input condition defined in Definition 1, the minimum of the reduced QP problem* (10) *is reached at a single point Z.*

Proof of Proposition 2.

For a proof by contradiction, first assume that there are at least two different values $Z^\dagger \neq Z^\ddagger$ in the constrained area C where the *constrained* minimum V^* of the loss function is reached, i.e.,

$$V^* = V(Z^\dagger) = V(Z^\ddagger).$$

Because the considered QP problem is convex, V^* is certainly the constrained global minimum of $V(Z)$.

Because of Condition *2* specified in Definition 1, $V^* > 0$, hence $L(V^*)$ is not the lowest level set of $L(0)$.

By the convexity of the constrained area C, all the points on the straight line segment between $Z^\dagger$ and $Z^\ddagger$, namely $l(Z^\dagger, Z^\ddagger)$, are also within C. Because of the convexity of the loss function $V(Z)$, any point $Z \in l(Z^\dagger, Z^\ddagger)$ satisfies $V(Z) \leq V^*$. On the other hand, because V^* is the global constrained minimum, $V(Z) \geq V^*$, hence certainly $V(Z) = V^*$ for any point $Z \in l(Z^\dagger, Z^\ddagger)$. Therefore the segment $l(Z^\dagger, Z^\ddagger)$ joining $Z^\dagger$ and $Z^\ddagger$ belongs to the level set $L(V^*)$ and also to the constrained area C.

$L(V^*)$ and C have at least a common straight line $l(Z^\dagger, Z^\ddagger)$ joining two distinct points $Z^\dagger$ and $Z^\ddagger$. Depending on the positions of $Z^\dagger$ and $Z^\ddagger$ in C (inside or on the border), the level set $L(V^*)$ (an $(N-1)$-dimensional manifold) either cuts C or is a tangent hypersurface outside C. More precisely, there are three possible cases regarding $L(V^*)$ and C:

1. $L(V^*)$ cuts C into at least two subsets of non empty volumes, or
2. $L(V^*)$ partly coincides with an edge (one dimensional face) of C, or
3. $L(V^*)$ partly coincides with a two or higher dimensional face of C.

For case (1), because $V^* > 0$, the loss function $V(Z) < V^*$ would hold for some Z at one side of the level set $L(V^*)$ inside one of the separated subsets of C. This case is impossible, since V^* is the global constrained minimum value of $V(Z)$.

For case (2), $l(Z^\dagger, Z^\ddagger)$ belongs to both $L(V^*)$ and an edge of C. This edge of C is parallel to one of the Z-axes (Lemma 1), say Z_i, so is $l(Z^\dagger, Z^\ddagger)$. Then any point $Z \in l(Z^\dagger, Z^\ddagger)$ can be parametrized as $Z = \rho e_i + Z^\dagger$ with e_i being the i-th standard basis vector of $\mathbb{R}^N$ (a vector filled with zeros except the i-th component equal to 1).

It then follows from Lemma 2 that $Q_1^T e_i = 0$. This result implies that the i-th row of Q_1 is filled with zeros, that is in contradiction with Condition *3*. Hence this case is impossible.

For case (3), one of the faces of C, say the face F, shares the common line segment $l(Z^\dagger, Z^\ddagger)$ with $L(V^*)$. The face F can be of dimension $m = 2, 3, \ldots$, or $N-1$. The line segment $l(Z^\dagger, Z^\ddagger)$ does not coincide with any border of the face F, as any border of F is a face of C of dimension lower than m, which would be considered at the place of F.

Let $Z^* \in l(Z^\dagger, Z^\ddagger)$ be a point off any border of the face F. Let also q denote any column of Q_0, then $Q_1^T q = 0$ (remind that $[Q_0 \; Q_1]$ is a unitary matrix). By Lemma 3, the straight line $Z = \rho q + Z^*$ belongs to the same level set $L(Z^{*T} Q_1 Q_1^T Z^*) = L(V^*)$, because $Z^* \in l(Z^\dagger, Z^\ddagger) \subset L(V^*)$.

This straight line $Z = \rho q + Z^*$ belonging to $L(V^*)$ cannot cross the face F of C, otherwise $L(V^*)$ would cut C as treated in case (1). Hence q is parallel to the face F, which is orthogonal to one of the Z-axes, say Z_i, therefore q is also orthogonal to Z_i. Then $q^T e_i = 0$ with e_i the i-th standard basis vector of $\mathbb{R}^N$. As q is any column of Q_0, then $Q_0^T e_i = 0$. This result implies that the i-th row of Q_0 is filled with zeros, that is in contradiction with Condition *3*. Hence this case is impossible.

As the cases (1-3) are exhaustive, the assumption made at the begging of this proof, namely there are at least two different Z values where the constrained minimum is reached, is impossible. Hence the minimum must be unique under the assumed complete input condition.

4.5 Back to the original QP problem

Now let us consider the QP prolem (5), but ignore for the moment the possibly available bounds θ_L, θ_U on θ. This QP problem can be solved by minimizing first in θ for any fixed value of Z, and by solving the the resulting the reduced QP problem (10). It has been shown that, if Φ satisfies the complete input condition, then the reduced QP problem (10) has a unique solution in Z, say Z^* then $\theta^* \triangleq (\Phi^T\Phi)^{-1}\Phi^T Z^*$ is also unique. The pair θ^*, Z^* then constitutes the unique solution of the QP prolem (5).

Finally, let us consider the QP prolem (5) with the possibly available bounds on θ, namely $\theta_L \leq \theta \leq \theta_U$. This has the effect of further constraining the previously considered problem into a convex subset, which preserves the uniqueness of the solution, under the complete input condition. Hence Proposition 1 is proved.

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