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Abstract

The present paper continues the study of infinite dimensional calculus via regularization, started by C. Di Girolami and the second named author, introducing the notion of *weak Dirichlet process* in this context. Such a process $\mathbb{X}$, taking values in a Hilbert space H, is the sum of a local martingale and a suitable *orthogonal* process.

The new concept is shown to be useful in several contexts and directions. On one side, the mentioned decomposition appears to be a substitute of an Itô's type formula applied to $f(t, \mathbb{X}(t))$ where $f : [0,T] \times H \to \mathbb{R}$ is a $C^{0,1}$ function and, on the other side, the idea of weak Dirichlet process fits the widely used notion of *mild solution* for stochastic evolution equations on infinite dimensional Hilbert spaces, including several classes of stochastic partial differential equations (SPDEs).

As a specific application, we provide a verification theorem for stochastic optimal control problems whose state equation is an infinite dimensional stochastic evolution equation.

KEY WORDS AND PHRASES: Covariation and Quadratic variation; Calculus via regularization; Infinite dimensional analysis; Tensor analysis; Dirichlet processes; Generalized Fukushima decomposition; Stochastic partial differential equations; Stochastic control theory.

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1 Introduction

Stochastic calculus constitutes one of the basic tools for stochastic optimal control theory; in particular the classical Itô formula enables to relate the solution of a control problem in closed loop form with the (smooth enough) solutions of the related Hamilton-Jacobi-Bellman (HJB) equation via some suitable *verification theorems*.

For the finite dimensional systems the literature presents quite precise and general results, see e.g. [26, 65]. If the system is infinite dimensional, for instance if it is driven by a stochastic partial differential (SPDEs) or a stochastic delay differential equation, the situation is more complex especially when the value function of the problem is not regular enough.

This paper contributes to the subject providing an efficient (infinite dimensional) stochastic calculus which fits the structure of a mild solution of a stochastic evolution equation in infinite dimension and gives the possibility to prove a verification theorem for a class of stochastic optimal control problems with non-regular value function, refining previous results.

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The contributions of the paper can be ascribed to the following three “labels”: infinite dimensional stochastic calculus, stochastic evolution equations in Hilbert spaces and dynamic programming. In the next subsection we describe the state of the art, while in the following we will concentrate on the new results.

State of the art

Stochastic calculus.

Stochastic calculus via regularization for real processes was initiated in [56] and [57]. It is an efficient calculus for non-semimartingales whose related literature is surveyed in [60].

Let $T > 0$. The processes will be indexed by $[0,T]$ adopting the convention described Notation 2.5. In the whole paper s will be a real in $[0,T]$. All the considered processes will be considered as measurable from $[s,T] \times \Omega$ (equipped with the product of the Borel σ-field of $[s,T]$ and the σ-algebra of all events $\mathscr{F}$) and the Borel σ-field of the value space, for instance $\mathbb{R}$. Given an a.s. bounded [resp. continuous] real process Y [resp. X], the forward integral of Y with respect to X and the covariation between Y and X are defined as follows. Suppose that, for every $t \in [s,T]$, the limit $I(t)$ [resp. $C(t)$] in probability exists:

$$\begin{aligned} I(t): &= \lim_{\epsilon\to 0^+} \int_s^t Y(r) \left(\frac{X(r+\epsilon) - X(r)}{\epsilon} \right) \mathrm{d}r, t \in [s,T], \\ C(t): &= \lim_{\epsilon\to 0^+} \int_s^t \frac{(X(r+\epsilon) - X(r))\,(Y(r+\epsilon) - Y(r))}{\epsilon} dr, t \in [s,T]. \end{aligned} \tag{1}$$

If the random function I [resp. C] admits a continuous version, this will be denoted by $\int_s^{\cdot} Y d^- X$ [resp. $[X,Y]$]. It is the *forward integral* of Y with respect to X [resp. the *covariation* of X and Y]. If X is a real continuous semimartingale and Y is a càdlàg process which is progressively measurable [resp. a semimartingale], the integral $\int_s^{\cdot} Y d^- X$ [resp. the covariation $[X,Y]$] is the same as the classical Itô's integral [resp. covariation].

The definition of $[X,Y]$ given above is slightly more general (weaker) than the one in [60]. There the authors supposed that the convergence in (1) holds in the ucp (uniformly convergence in probability) topology.[1] In this work we use the weak definition for the real case, i.e. when both X and Y are real, and the strong definition, via ucp convergence, when either X or Y is not one-dimensional. When $X = Y$ the two definitions are equivalent taking into account Lemma 2.1 of [60].

Real processes X for which $[X,X]$ exists are called *finite quadratic variation processes*. A rich class of finite quadratic variation processes is provided by Dirichlet processes. Let $(\mathscr{F}_t, t \in [0,T])$ be a fixed filtration, fulfilling the usual conditions. A real process X is said to be *Dirichlet* (or *Föllmer-Dirichlet*) if it is the sum of a local martingale M and a *zero quadratic variation process* A, i.e. such that $[A,A] = 0$. Those processes were defined by H. Föllmer [27] using limits of discrete sums. A significant generalization, due to [25, 40], is the notion of *weak Dirichlet process*, extended to the case of jump processes in [8].

Definition 1.1. *A real process* $X\colon [s,T] \times \Omega \to \mathbb{R}$ *is called weak Dirichlet process if it can be written as*

$$X = M + A, \tag{2}$$

where

(i) M *is a local martingale,*

(ii) A *is a process such that* $[A,N] = 0$ *for every continuous local martingale* N *and* $A(s) = 0$.

[1] Given a Banach space B and a probability space $(\Omega, \mathbb{P})$ a family of processes $\mathbb{X}^\epsilon \colon \Omega \times [s,T] \to B$ is said to converge in the ucp sense to $\mathbb{X}\colon \Omega \times [s,T] \to B$, when ϵ goes to zero, if $\lim_{\epsilon\to 0} \sup_{t\in[s,T]} |\mathbb{X}^\epsilon_t - \mathbb{X}_t|_B = 0$, in probability.

Obviously a semimartingale is a weak Dirichlet process. In Remark 3.5 of [40], appears the following important statement.

Proposition 1.2. *The decomposition described in Definition 1.1 is unique.*

Elements of calculus via regularization were extended to Banach space valued processes in a series of papers, see e.g. [18, 15, 16, 17]. We go on introducing two classical notions of stochastic calculus in Banach spaces, which appear in [49] and [20]: the scalar and tensor quadratic variations. We propose here a regularization approach, even though, originally they appeared in a discretization framework. The two monographs above use the term *real* instead of *scalar*; we have decided to change it to avoid confusion with the quadratic variation of real processes.

Definition 1.3. *Consider a separable Banach spaces B. We say that a process $\mathbb{X}\colon [s,T]\times\Omega \to B$, a.s. square integrable, admits a* **scalar quadratic variation** *if, for any $t \in [s,T]$, the limit, for $\epsilon \searrow 0$ of*

$$[\mathbb{X},\mathbb{X}]^{\epsilon,\mathbb{R}}(t) := \int_s^t \frac{|\mathbb{X}(r+\epsilon)-\mathbb{X}(r)|_B^2}{\epsilon} dr$$

exists in probability and it admits a continuous version. The limit process is called scalar quadratic variation of $\mathbb{X}$ and it is denoted by $[\mathbb{X},\mathbb{X}]^{\mathbb{R}}$.

Remark 1.4. *The definition above is equivalent to the one contained in [18]. In fact, previous convergence in probability implies the ucp convergence, since the ϵ-approximation processes are increasing and so Lemma 3.1 in [59] can be applied.*

Proposition 1.5. *Let B be a separable Banach space. A continuous B-valued process with bounded variation admits a zero scalar quadratic variation. In particular, a process $\mathbb{X}$ of the type $\mathbb{X}(t) = \int_s^t b(r)dr$, where b is a B-valued measurable process has a zero scalar quadratic variation.*

Proof. The proof is very similar to the one related to the case when $B = \mathbb{R}$, which was the object of Proposition 1 part 7-b, see [60]. □

From [18] we borrow the following definition.

Definition 1.6. *Consider two separable Banach spaces B_1 and B_2. Suppose that either B_1 or B_2 is different from $\mathbb{R}$. Let $\mathbb{X}\colon [s,T]\times\Omega \to B_1$ and $\mathbb{Y}\colon [s,T]\times\Omega \to B_2$ be two a.s. square integrable processes. We say that $(\mathbb{X},\mathbb{Y})$ admits a* **tensor covariation** *if the limit, for $\epsilon \searrow 0$ of the $B_1\hat{\otimes}_\pi B_2$-valued processes*

$$[\mathbb{X},\mathbb{Y}]^{\otimes,\epsilon} := \int_s^{\cdot} \frac{(\mathbb{X}(r+\epsilon)-\mathbb{X}(r))\otimes(\mathbb{Y}(r+\epsilon)-\mathbb{Y}(r))}{\epsilon} dr,$$

exists ucp. The limit process is called **tensor covariation** *of $(\mathbb{X},\mathbb{Y})$ and is denoted by $[\mathbb{X},\mathbb{Y}]^{\otimes}$. The tensor covariation $[\mathbb{X},\mathbb{X}]^{\otimes}$ is called* **tensor quadratic variation** *of $\mathbb{X}$ and is denoted by $[\mathbb{X}]^{\otimes}$.*

The concepts of scalar and tensor quadratic variation are however too strong for the applications: indeed several interesting examples of Banach (or even Hilbert) space valued processes have no tensor quadratic variation. A Banach space valued example is the $C([-\tau,0])$-valued process $\mathbb{X}$ defined as the frame (or window) of a standard Brownian motion W: $\mathbb{X}(t)(x) := W(t+x)$, $x \in [-\tau,0]$. It is neither a semimartingale, nor a process with scalar quadratic variation process, see considerations after Remarks 1.9 and Proposition 4.5 of [18]. A second example, that indeed constitutes a main motivation for the present paper, is given by mild solutions of classical stochastic evolution equations in infinite dimensions (including SPDEs): they have no scalar quadratic variation even if driven by a one-dimensional Brownian motion.

The idea of Di Girolami and Russo was to introduce a suitable space χ continuously embedded into the dual of the projective tensor space $B_1\hat{\otimes}_\pi B_2$, called **Chi-subspace**.
χ is a characteristics of their notion of quadratic variation, recalled in Section 4.1. When χ is the

full space $(B_1\hat{\otimes}_\pi B_2)^*$ the χ-quadratic variation is called **global quadratic variation**. Following the approach of Di Girolami and Russo, see for instance Definition 3.4 of [17], we make use of a the notion of χ-covariation $[\mathbb{X},\mathbb{Y}]_\chi$ when $\chi \subseteq (B_1\hat{\otimes}_\pi B_2)^*$ for two processes $\mathbb{X}$ and $\mathbb{Y}$ with values respectively in separable Hilbert spaces B_1 and B_2. That notion is recalled in Definition 4.4.

[15] introduces a (real valued) forward integral, denoted by $\int_s^t {}_{B^*}\langle \mathbb{Y}(r), d^-\mathbb{X}(r)\rangle_B$, in the case when the integrator $\mathbb{X}$ takes values in a Banach space B and the integrand $\mathbb{Y}$ is B^*-valued. This appears as a natural generalization of the first line of (1). That notion is generalized in Definition 3.1 for operator-valued integrands; in that case, this produces a Hilbert valued forward integral.

The Itô formula for processes $\mathbb{X}$ having a χ-quadratic variation is given in Theorem 5.2 of [18]. Let $F:[0,T]\times B\to\mathbb{R}$ or class $C^{1,2}$ such that $\partial^2_{xx}F\in C([0,T]\times B;\chi)$. Then, for every $t\in[s,T]$,

$$F(t,\mathbb{X}(t)) = F(s,\mathbb{X}(s)) + \int_s^t \partial_r F(r,\mathbb{X}(r))ds + \int_s^t {}_{B^*}\langle \partial_x F(r,\mathbb{X}(r)), d^-\mathbb{X}(r)\rangle_B + \frac{1}{2}\int_s^t {}_{\chi}\langle \partial^2_{xx}F(r,\mathbb{X}(r)), d\widetilde{[\mathbb{X},\mathbb{X}]}_r\rangle_{\chi^*} \ a.s. \quad (3)$$

We will introduce the notation $\widetilde{[\mathbb{X},\mathbb{X}]}$ in Section 3.

Stochastic evolution equations in Hilbert spaces.

The class of stochastic evolution equations in infinite dimension that we consider can be seen as the abstract and unified formulation of several classes of stochastic partial differential differential equations (SPDEs) and stochastic functional differential equations. They model a significant range of systems modeling phenomena arising in very different fields as physics, economics, physiology, population growth and migration etc.

The abstract formulation of the stochastic evolution equation introduced in Section 5 is characterized by an abstract generator of a C_0-semigroup A and Lipschitz coefficients b and σ. It appears as

$$\begin{cases} \mathrm{d}\mathbb{X}(t) = (A\mathbb{X}(t) + b(t,\mathbb{X}(t)))\ \mathrm{d}t + \sigma(t,\mathbb{X}(t))\,\mathrm{d}\mathbb{W}_Q(t) \\ \mathbb{X}(s) = x, \end{cases} \quad (4)$$

where $\mathbb{W}$ is a Q-Wiener process with respect to some covariance operator Q. As a particular case, when Q is the identity operator, $\mathbb{W}_Q$ represents a space-time white noise.

An SPDE is a partial differential equation with random forcing terms or coefficients. As described for example in Part III of [11], several families of SPDEs can be reformulated as stochastic evolution equations and then can be studied in the general abstract setting; of course for any of them the specification of the generator A and of the functions b and σ are different. The reformulation can be done easier when the SPDE is in the form of deterministic PDE perturbed by a Gaussian noise involving an infinite dimensional Wiener process $\mathbb{W}_Q$ as a multiplicative factor and/or an additive term. Among the SPDEs that can be expressed in the formalism of stochastic evolution equations in infinite dimension we recall the stochastic heat and parabolic equations (even with boundary noise), wave equations, reaction-diffusion equations, first order equations, Burgers equations, Navier-Stokes equations and Duncan-Mortensen-Zakai equations; however not all of them fulfill the assumpions of Section 5. In the same way the abstract formulation in the form of stochastic evolution equation in infinite dimension can be shown to include classes of stochastic delay differential equations and neutral differential equations, see again [11] and the contained references. We will recall some examples in Remark 5.1 of Section 5.

There are several different possible ways to define solutions of stochastic infinite dimensional evolutions equation and SPDEs: strong solutions (see e.g. [10] Section 6.1), variational solutions (see e.g. [54]), martingale solutions, see [51] and so on.

We will make use of the notion of *mild solution* (see [10] Chapter 7 or [34] Chapter 3) where the solution of (4) is defined (using, formally, a "variations of parameters" arguments) as the solution of the integral equation

$$\mathbb{X}(t) = e^{(t-s)A}x + \int_s^t e^{(t-r)A} b(r,\mathbb{X}(r))\,\mathrm{d}r + \int_s^t e^{(t-r)A}\sigma(r,\mathbb{X}(r))\,\mathrm{d}\mathbb{W}_Q(r).$$

This concept is widely used in the literature.

Optimal control: dynamic programming and verification theorems.

As in the study of finite-dimensional stochastic (and non-stochastic) optimal control problem, the dynamic programming approach connects the study of the minimization problem with the analysis of the related Hamilton-Jacobi-Bellman (HJB) equation: given a solution of HJB and a certain number of hypotheses, the optimal control can be found in feedback form (i.e. as a function of the state) through a so called *verification theorem*. The idea is to identify a solution of the HJB equation with the value function of the control problem. When the state equation of the optimal control problem is an infinite dimensional stochastic evolution equation, the related HJB is of second order and infinite dimensional. The simplest procedure for establishing a verification theorem consists in considering the *regular case* where the solution is assumed to have all the regularity needed to give meaning to all the terms appearing in the HJB in the classical sense: it needs to be C^1 in the time variable and C^2 in the state variable. Since in many interesting cases the value function does not have all the required regularity, several definitions of (more general) solutions have been introduced for the HJB equation. The various possibilities can be classified as follows.

> *Strong solutions*: in this approach, first introduced in [3], the solution is defined as a proper limit of solutions of regularized problems. Verification results in this framework are given for example in [36] and [38], see [5, 6], for the reaction-diffusion case.
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> *Viscosity solutions*: in this case the solution is defined using test functions that locally "touch" the candidate solution. The viscosity solution approach was first adapted to the second order Hamilton Jacobi equation in Hilbert space in [44, 45, 46] and then, for the "unbounded" case (i.e. including the unbounded operator A appearing in (4)) in [63]. As far as we know, differently from the finite-dimensional case there are no verification theorems available for the infinite-dimensional case.
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> L^2_μ *approach*: it was introduced in [2, 35], see also [7, 1]. In this case the value function is found in the space $L^2_\mu(H)$, where μ is an invariant measure for an associated uncontrolled process. The paper [35] contains as well an excellent literature survey.
>
> *Backward approach*: it can be applied when the mild solution of the HJB can be represented using the solution of a forward-backward system and makes it possible to find an optimal control in feedback form. It was introduced in [55] and developed in [29, 32, 33], see [13, 31] for other particular cases.

The method we use in the present work to prove the verification theorem does not belong to any of the previous categories even if we use a strong solution approach to define the solution of the HJB. In the sequel of this introduction, we will be more precise.

The contributions of the work

The novelty of the present paper arises at the three levels mentioned above: stochastic calculus, infinite dimensional stochastic differential equations and stochastic optimal control. Indeed stochastic optimal control for infinite dimensional problems is also a motivation to complete the theory of calculus via regularizations.

The stochastic calculus part starts (Sections 3) with a natural extension (Definition 3.1) of the notion of forward integral in Hilbert (and even Banach) spaces introduced in [15] and with the proof of its equivalence with the classical notion of integral when we integrate a predictable process w.r.t. a Q-Wiener process (Theorem 3.4) and w.r.t. a general local martingale (Theorem 3.6).

In Section 4, we extend the notion of Dirichlet process to infinite dimension. Let $\mathbb{X}$ be an H-valued stochastic process. According to the literature, $\mathbb{X}$ can be naturally considered to be an *infinite dimensional Dirichlet process* if it is the sum of a local martingale and a zero energy process. A *zero* *energy process* (with some light sophistications) $\mathbb{X}$ is a process such that the expectation of the quantity in Definition 1.3 converges to zero when ε goes to zero. This happens for instance in [14], even though that decomposition also appears in [47] Chapter VI Theorem 2.5, for processes $\mathbb{X}$ associated with an infinite-dimensional Dirichlet form.

Extending Föllmer's notion of Dirichlet process to infinite dimension, a process $\mathbb{X}$ taking values in a Hilbert space H, could be called *Dirichlet* if it is the sum of a local martingale $\mathbb{M}$ plus a process $\mathbb{A}$ having a zero scalar quadratic variation. However that natural notion is not suitable for an efficient stochastic calculus for infinite dimensional stochastic differential equations.

Similarly to the notion of χ-finite quadratic variation process we introduce the notion of χ-Dirichlet process as the sum of a local martingale $\mathbb{M}$ and a process $\mathbb{A}$ having a zero χ-quadratic variation.

A completely new notion in the present paper is the one of Hilbert valued *ν-weak Dirichlet process* which is again related to a Chi-subspace ν of the dual projective tensor product $H\hat{\otimes}_\pi H_1$ where H_1 is another Hilbert space, see Definition 4.22. It is of course an extension of the notion of real-valued weak Dirichlet process, see Definition 1.1. We illustrate that notion in the simple case when $H_1 = \mathbb{R}$, $\nu = \nu_0 \hat{\otimes}_\pi \mathbb{R} \equiv \nu_0$ and ν_0 is a Banach space continuously embedded in H^*: a process $\mathbb{X}$ is called ν-weak Dirichlet process if it is the sum of a local martingale $\mathbb{M}$ and a process $\mathbb{A}$ such that $[\mathbb{A}, N]_\nu = 0$ for every real continuous local martingale N. This happens e.g. under the following assumptions.

(i) There is a family $(R(\epsilon), \epsilon > 0)$ of non-negative random variables converging in probability, such that $Z(\epsilon) \le R(\epsilon), \epsilon > 0$, where

$$Z(\epsilon) := \frac{1}{\epsilon} \int_0^T |\mathbb{A}(r+\epsilon) - \mathbb{A}(r)|_{\nu_0^*} |N(r+\epsilon) - N(r)| \, \mathrm{d}r.$$

(ii) For all $h \in \nu_0$, $\lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_0^t {}_{\nu_0}\langle \mathbb{A}(r+\epsilon) - \mathbb{A}(r), h \rangle_{\nu_0^*} (N(r+\epsilon) - N(r)) \, \mathrm{d}r = 0, \ \forall t \in [0, T]$.

Remark 1.7. *We remark that, when condition (i) is fulfilled, the sequence $(Z(\epsilon))$ is bounded in the F-space of random variables, with metric defined by $d(X, Y) = E[|X - Y|_B \wedge 1]$; this distance governs the convergence in probability. The notion of bounded subset of an F-space is given in Section II.1 of [23].*

At the level of pure stochastic calculus, the most important result, is Theorem 4.31. It generalizes to the Hilbert values framework, Proposition 3.10 of [40] which states that given $f : [0, T] \times \mathbb{R} \to \mathbb{R}$ of class $C^{0,1}$ and X is a weak Dirichlet process with finite quadratic variation then $Y(t) = f(t, X(t))$ is a real weak Dirichlet process. This result is a Fukushima decomposition in the spirit of Dirichlet forms, which is the natural extension of Doob-Meyer decomposition for semimartingales. It can also be seen as a substitution-tool of Itô's formula if f is not smooth. Besides Theorem 4.31, an interesting general Itô's formula in the application to mild solutions of infinite dimensional stochastic differential equations is Theorem 5.5. The stochastic calculus theory developed in Sections 3 and 4, makes it possible to prove that a mild solution of an equation of type (4) is a χ-Dirichlet process and a ν-weak-Dirichlet process; this is done in Corollary 5.4.

As far as stochastic control is concerned, the main issue is the verification result stated in Theorem 6.11. As we said, the method we used does not belong to any of the described families even if we define the solution of the HJB in line with a strong solution approach. Since the solution $v : [0, T] \times H \to \mathbb{R}$ of the HJB equation is only of class $C^{0,1}$ (with derivative in $C(H, D(A^*))$), we cannot apply a Itô formula of class $C^{1,2}$. The substitute of such a formula is given in Theorem 6.8, which is based on the uniqueness character of the decomposition of the real weak Dirichlet process $v(t, \mathbb{X}(t))$, where $\mathbb{X}$ is a solution of the state equation $\mathbb{X}$. The fact that $v(t, \mathbb{X}(t))$ is weak Dirichlet follows by Theorem 4.31 because $\mathbb{X}$ is a ν-weak Dirichlet process for some suitable space ν. This is the first work that employs this method in infinite dimensions. A similar approach was used to deal with the finite dimensional case in [39] but of course in the infinite-dimensional case the situation is much more complicated since the state equation is not a semimartingale and so it indeed requires the introduction of the concept of ν-weak Dirichlet process.

For the reasons listed below, Theorem 6.11 is more general than the results obtained with the classical strong solutions approach, see e.g. [36, 38], and, in a context slightly different than ours, [5, 6].

(1) The state equation is more general.

(a) In equation (71) the coefficient σ depends on time and on the state while in classical strong solutions literature, it is constant and equal to identity.

(b) In classical strong solutions contributions, the coefficient b appearing in equation (71) is of the particular form $b(t, \mathbb{X}, a) = b_1(\mathbb{X}) + a$ so it "separates" the control and the state parts.

(2) We only need the Hamiltonian to be well-defined and continuous without any particular differentiability, differently to what happens in the classical strong solutions literature.

(3) We use a milder definition of solution than in [36, 38]; indeed we work with a bigger set of approximating functions: in particular (a), our domain $D(\mathscr{L}_0)$, does not require the functions and their derivatives to be uniformly bounded; (b), the convergence of the derivatives $\partial_x v_n \to \partial_x v$ in (6.7) is not necessary and it is replaced by the weaker condition (83).

However, we have to pay a price: we assume that the gradient of the solution of the HJB $\partial_x v$, is continuous from H to $D(A^*)$, instead of simply continuous from H to H.

In comparison to the strong solutions approach, the L^2_μ method used in [35] permits to use weaker assumptions on the data and enlarges the range of possible applications. However, the authors still require σ to be the identity, the Hamiltonian to be Lipschitz and the coefficient b to be in a "separated" form as in (1)(b) above. In the case treated by [2], the terms containing the control in the state variable is more general but the author assumes that A and Q have the same eigenvectors. So, in both cases, the assumptions on the state equation are for several aspects more demanding than ours.

The backward approach, used e.g. in [29, 32, 33], allows to treat degenerate cases in which the transition semigroup has no smoothing properties. Still, in the verification results proved in this context, the Hamiltonian has to be differentiable, the dependence on the control in the state equation is assumed to be linear and its coefficient needs to have a precise relation with $\sigma(t, \mathbb{X}(t))$. All those hypotheses are stronger than ours.

One important feature of Theorem 6.11 is that we do not need to assume any hypothesis to ensure the integrability of the target. This is due to the fact that we apply the expectation operator only at the last moment. We stress that, as far as we know, all the available verification theorems for optimal control problems driven by infinite dimensional stochastic evolution equations existing in the literature, obtained with any of the described method, require at least the C^1-regularity of the value function w.r.t. the state variable. Our method is not an exception in this sense but, as described below, we can avoid a series of assumptions needed with other approaches. It remains true that, in many interesting cases, the value function fails even to be C^1 w.r.t. the state variable.

A different approach to infinite dimensional stochastic optimal control problems is given by the use of maximum principle. Even recently, a series of interesting contributions on the subject appeared, see for example [21, 22, 28, 30, 52]. As in finite dimension (see e.g. [65]), if the Hamiltonian is not convex, the maximum principle approach only gives necessary conditions but it does not ensure the sufficiency. Moreover, at this stage, maximum principle results for stochastic infinite dimensional problems need a strong regularity on the coefficients and they allow to study mainly the case of finite-dimensional noise.

The scheme of the work is the following. After some preliminaries in Section 2, in Section 3 we introduce the definition of forward integral with values in Banach spaces and we discuss the relation with the Da Prato-Zabczyk integral in the Hilbert framework; for simplicity we do not treat the Banach case. Section 4, devoted to stochastic calculus, is the core of the paper: we introduce the concepts of χ-Dirichlet processes, ν-weak-Dirichlet processes and we study their general properties. In Section 5, the developed theory is applied to the case of mild solutions of stochastic PDEs and more in general of infinite dimensional stochastic differential equations, while Section 6 contains the application to stochastic optimal control problems in Hilbert spaces.

2 Preliminaries and notations

2.1 Basic functional analysis

Let us consider two separable Banach spaces B_1 and B_2. We denote by $C(B_1; B_2)$ the set of the continuous functions from B_1 to B_2. This linear space is a topological vector space if equipped with topology of the uniform convergence on compact sets.

If $B_2 = \mathbb{R}$ we will often simply use the notation $C(B_1)$ instead of $C(B_1; \mathbb{R})$. Similarly, given a real interval I, typically $I = [0, T]$ or $I = [0, T)$, we use the notation $C(I \times B_1; B_2)$ for the set of the continuous B_2-valued functions defined on $I \times B_1$ while we use the lighter notation $C(I \times B_1)$ when $B_2 = \mathbb{R}$. $C^1(I \times B_1$ will denote the space of Fréchet continuous differentiable functions $u : I \times B_1 \to \mathbb{R}$. For a function $u : I \times B_1 \to \mathbb{R}$, $(t, x) \mapsto u(t, x)$, we denote by $(t, x) \mapsto \partial_x u(t, x)$ [resp. $(t, x) \mapsto \partial^2_{xx} u(t, x)$] (if it exists) the first [resp. second] Fréchet derivative w.r.t. the variable $x \in B_1$). Eventually a function $(t, x) \mapsto u(t, x) \in C(I \times B_1)$ [resp. $u \in C^1(I \times B_1)$] will be said to belong to $C^{0,1}(I \times B_1)$ [resp. $C^{1,2}(I \times B_1)$] if $\partial_x u$ exists and it is continuous, i.e. it belongs to $C(I \times B_1; {B_1}^*)$ [resp. $\partial^2_{xx} u(t, x]$ exists for any $(t, x) \in [0, T] \times B_1$ and it is continuous, i.e. it belongs to $C(I \times B_1; \mathcal{B}i(B_1, B_1))$, where $\mathcal{B}i(B_1, B_2))$ is the linear topological space of bilinear bounded forms on $B_1 \times B_2$.
By convention all the continuous functions defined on an interval I are naturally extended by continuity to $\mathbb{R}$.

We denote by $\mathcal{L}(B_1; B_2)$ the space of linear bounded maps from B_1 to B_2 and by $\|\cdot\|_{\mathcal{L}(B_1;B_2)}$ the corresponding norm. We will often indicate in the sequel by a double bar, i.e. $\|\cdot\|$, the norm of an operator.

Often we will consider the case of two separable Hilbert spaces U and H. We denote $|\cdot|$ and $\langle \cdot, \cdot \rangle$ [resp. $|\cdot|_U$ and $\langle \cdot, \cdot \rangle_U$] the norm and the inner product on H [resp. U].

Notation 2.1. *If H is a Hilbert space, in order to argue more transparently, we often distinguish between H and its dual H^* and with every element $h \in H$ we associate $h^* \in H^*$ through Riesz Theorem.*

If $U = H$, we set $\mathcal{L}(U) := \mathcal{L}(U; U)$. $\mathcal{L}_2(U; H)$ will be the set of *Hilbert-Schmidt* operators from U to H and $\mathcal{L}_1(H)$ [resp. $\mathcal{L}_1^+(H)$] will be the space of [resp. non-negative] *nuclear* operators on H. For details about the notions of Hilbert-Schmidt and nuclear operator, the reader may consult [61], Section 2.6 and [10] Appendix C. If $T \in \mathcal{L}_2(U; H)$ and $T^* : H \to U$ is the adjoint operator, then $TT^* \in \mathcal{L}_1(H)$ and the Hilbert-Schmidt norm of T gives $\|T\|^2_{\mathcal{L}_2(U;H)} = \|TT^*\|_{\mathcal{L}_1(H)}$. We recall that, for a generic element $T \in \mathcal{L}_1(H)$ and given a basis $\{e_n\}$ of H, the sum $\sum_{n=1}^{\infty} \langle Te_n, e_n \rangle$ is absolutely convergent and independent of the chosen basis $\{e_n\}$. It is called *trace* of T and denoted by $\mathrm{T}r(T)$. $\mathcal{L}_1(H)$ is a Banach space and we denote by $\|\cdot\|_{\mathcal{L}_1(H)}$ the corresponding norm. If T is non-negative then $\mathrm{T}r(T) = \|T\|_{\mathcal{L}_1(H)}$ and in general we have the inequalities

$$|\mathrm{T}r(T)| \le \|T\|_{\mathcal{L}_1(H)}, \quad \sum_{n=1}^{\infty} |\langle Te_n, e_n \rangle| \le \|T\|_{\mathcal{L}_1(H)}, \tag{5}$$

see Proposition C.1, [10]. As a consequence, if T is a non-negative operator, the relation below

$$\|T\|^2_{\mathcal{L}_2(U;H)} = \mathrm{T}r(TT^*), \tag{6}$$

will be very useful in the sequel.

2.2 Reasonable norms on tensor products

Consider two real Banach spaces B_1 and B_2. Denote, for $i = 1, 2$, with $|\cdot|_i$ the norm on B_i. $B_1 \otimes B_2$ stands for the *algebraic tensor product* i.e. the set of the elements of the form $\sum_{i=1}^{n} x_i \otimes y_i$ where x_i and y_i are respectively elements of B_1 and B_2. On $B_1 \otimes B_2$ we identify all the expressions we need in order to ensure that the product $\otimes$: $B_1 \times B_2 \to B_1 \otimes B_2$ is bilinear.

On $B_1 \otimes B_2$ we introduce the projective norm π defined, for all $u \in B_1 \otimes B_2$, as

$$\pi(u) := \inf \left\{ \sum_{i=1}^n |x_i|_{B_1} |y_i|_{B_2} \ : \ u = \sum_{i=1}^n x_i \otimes y_i \right\}.$$

The *projective tensor product of* B_1 *and* B_2, $B_1 \hat{\otimes}_\pi B_2$, is the Banach space obtained as completion of $B_1 \otimes B_2$ for the norm π, see [61] Section 2.1, or [15] for further details.

For $u \in B_1 \otimes B_2$ of the form $u = \sum_{i=1}^n x_i \otimes y_i$ we define

$$\varepsilon(u) := \sup \left\{ \left| \sum_{i=1}^n \Phi(x_i) \Psi(y_i) \right| \ : \ \Phi \in B_1^*, \ \Psi \in B_2^*, \ |\Phi|_{B_1^*} = |\Psi|_{B_2^*} = 1 \right\}$$

and denote by $B_1 \hat{\otimes}_\epsilon B_2$ the completion of $B_1 \otimes B_2$ for such a norm: it is the *injective tensor product of* B_1 *and* B_2. We remind that $\epsilon(u)$ does not depend on the representation of u and that, for any $u \in B_1 \otimes B_2$, $\varepsilon(u) \leq \pi(u)$.

A norm α on $B_1 \otimes B_2$ is said to be *reasonable* if for any $u \in B_1 \otimes B_2$,

$$\varepsilon(u) \leq \alpha(u) \leq \pi(u). \tag{7}$$

We denote by $B_1 \hat{\otimes}_\alpha B_2$ the completion of $B_1 \otimes B_2$ w.r.t. the norm α. For any reasonable norm α on $B_1 \otimes B_2$, for any $x \in B_1$ and $y \in B_2$ one has $\alpha(x \otimes y) = |x|_{B_1} |y|_{B_2}$. See [61] Chapter 6.1 for details.

Lemma 2.2. *Let* B_1 *and* B_2 *be two real Banach spaces and* α *a reasonable norm on* $B_1 \otimes B_2$*. We denote* $B := B_1 \hat{\otimes}_\alpha B_2$*. Choose* $a^* \in B_1^*$ *and* $b^* \in B_2^*$*. One can associate to* $a^* \otimes b^*$ *the elements* $i(a^* \otimes b^*)$ *of* B^* *acting as follows on a generic element* $u = \sum_{i=1}^n x_i \otimes y_i \in B_1 \otimes B_2$*:*

$$\langle i(a^* \otimes b^*), u \rangle = \sum_i^n \langle a^*, x_i \rangle \langle b^*, y_i \rangle .$$

Then $i(a^* \otimes b^*)$ *extends by continuity to the whole* B *and*

$$|i(a^* \otimes b^*)|_B = |a^*|_{B_1^*} |b^*|_{B_2^*}. \tag{8}$$

Proof. We first prove the $\leq$ inequality in (8). Setting $\tilde{a}^* = \frac{a^*}{|a^*|}, \tilde{b}^* = \frac{b^*}{|b^*|}$ we write

$$\langle i(a^* \otimes b^*), u \rangle = \left\langle i(\tilde{a}^* \otimes \tilde{b}^*), u \right\rangle |a^*||b^*| \leq \varepsilon(u) |a^*|_{B_1^*} |b^*|_{B_2^*}. \tag{9}$$

The latter inequality comes from the definition of injective tensor norm ε, considering first $\Phi = a^*, \Psi = B^*$. By (7) $\langle i(a^* \otimes b^*), u \rangle \leq \alpha(u) |a^*|_{B_1^*} |b^*|_{B_2^*}$ and the $\leq$ inequality of (8) is proved.

Concerning the converse inequality, we have

$$|a^*|_{B_1^*} = \sup_{|\phi|_{B_1} = 1} {}_{B_1^*}\langle a^*, \phi \rangle_{B_1}$$

and similarly for b^*. So, chosen $\delta > 0$, there exist $\phi_1 \in B_1$ and $\phi_2 \in B_2$ with $|\phi_1|_{B_1} = |\phi_2|_{B_2} = 1$ and

$$|a^*|_{B_1^*} \leq \delta + {}_{B_1^*}\langle a^*, \phi_1 \rangle_{B_1}, \qquad |b^*|_{B_2^*} \leq \delta + {}_{B_2^*}\langle b^*, \phi_2 \rangle_{B_2}.$$

We set $u := \phi_1 \otimes \phi_2$. We obtain

$$\begin{aligned} |i(a^* \otimes b^*)|_{B^*} \geq \frac{{}_{B^*}\langle i(a^* \otimes b^*), u \rangle_B}{|u|_B} &= \frac{{}_{B^*}\langle i(a^* \otimes b^*), u \rangle_B}{|\phi_1|_{B_1} |\phi_2|_{B_2}} \\ &= {}_{B_1^*}\langle a^*, \phi_1 \rangle_{B_1} \ {}_{B_2^*}\langle b^*, \phi_2 \rangle_{B_2} \geq (|a^*|_{B_1^*} - \delta)(|b^*|_{B_2^*} - \delta). \end{aligned} \tag{10}$$

Since $\delta > 0$ is arbitrarily small we finally obtain

$$|i(a^* \otimes b^*)|_{B^*} \geq |a^*|_{B_1^*} |b^*|_{B_2^*}.$$

This gives the second inequality and concludes the proof. □

Notation 2.3. *When $B_1 = B_2$ and $x \in B_1$ we denote by $x\otimes^2$ the element $x \otimes x \in B_1 \otimes B_1$.*

The dual of the projective tensor product $B_1\hat{\otimes}_\pi B_2$, denoted by $(B_1\hat{\otimes}_\pi B_2)^*$, can be identified isomorphically with the linear space of bounded bilinear forms on $B_1 \times B_2$ denoted by $\mathcal{B}i(B_1, B_2)$. If $u \in (B_1\hat{\otimes}_\pi B_2)^*$ and ψ_u is the associated form in $\mathcal{B}i(B_1, B_2)$, we have

$$|u|_{(B_1\hat{\otimes}_\pi B_2)^*} = \sup_{|a|_{B_1}\leq 1, |b|_{B_2}\leq 1} |\psi_u(a, b)|.$$

See for this [61] Theorem 2.9 Section 2.2, page 22 and also the discussion after the proof of the theorem, page 23.

Every element $u \in H\hat{\otimes}_\pi H$ is isometrically associated with an element T_u in the space of nuclear operators $\mathcal{L}_1(H, H)$, defined, for u of the form $\sum_{i=1}^{\infty} a_n \otimes b_n$, as follows:

$$T_u(x) := \sum_{i=1}^{\infty} \langle x, a_n \rangle \, b_n,$$

see for instance [61] Corollary 4.8 Section 4.1 page 76.

Since T_u is nuclear, in particular (see Appendix C of [10]), there exists a sequence of real numbers (λ_n) and an orthonormal basis (h_n) of H such that T_u can be written as

$$T_u(x) = \sum_{n=1}^{+\infty} \lambda_n \langle h_n, x \rangle \, h_n, \qquad \text{for all } x \in H; \tag{11}$$

in particular $T_u(h_n) = \lambda_n h_n$ for each n. Moreover u can be written as

$$u = \sum_{n=1}^{+\infty} \lambda_n h_n \otimes h_n. \tag{12}$$

To each element ψ of $(H\hat{\otimes}_\pi H)^*$ we associate a bilinear continuous operator B_ψ and a linear continuous operator $L_\psi : H \to H$ (see [61] page 24, the discussion before Proposition 2.11 Section 2.2) such that

$$\langle L_\psi(x), y \rangle = B_\psi(x, y) = \psi(x \otimes y) \qquad \text{for all } x, y \in H. \tag{13}$$

Proposition 2.4. *Let $u \in H\hat{\otimes}_\pi H$ and $\psi \in (H\hat{\otimes}_\pi H)^*$ with associated maps $T_u \in \mathcal{L}_1(H), L_\psi \in \mathcal{L}(H)$. Then*

$$_{(H\hat{\otimes}_\pi H)^*}\langle \psi, u \rangle_{H\hat{\otimes}_\pi H} = \operatorname{Tr}(L_\psi T_u).$$

Proof. The claim follows from what we have recalled above. Indeed, using (12) and (13) we have

$$_{(H\hat{\otimes}_\pi H)^*}\langle \psi, u \rangle_{H\hat{\otimes}_\pi H} = \psi(u) = \psi\left(\sum_{i=1}^{+\infty} \lambda_n h_n \otimes h_n\right) = \sum_{n=1}^{+\infty} \langle L_\psi(\lambda_n h_n), h_n \rangle$$

and the last expression is exactly $\operatorname{Tr}(L_\psi T_u)$ when we compute it using the basis h_n. □

2.3 Probability and stochastic processes

In the whole paper, there will be an underlying complete probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Fix $T > 0$ and $s \in [0, T)$. Let $\{\mathscr{F}_t^s\}_{t\geq s}$ be a filtration satisfying the usual conditions. Given a subset $\tilde{\Omega} \in \mathscr{F}$ we denote by $I_{\tilde{\Omega}} : \Omega \to \{0, 1\}$ the characteristic function of the set $\tilde{\Omega}$, i.e. $I_{\tilde{\Omega}}(\omega) = 1$ if and only if $\omega \in \tilde{\Omega}$.

Given a real Banach space B we denote by $\mathscr{B}(B)$ the Borel σ-field on B.

By default we assume that all the processes $\mathbb{X} : [s, T] \times \Omega \to B$ are measurable functions with respect to the product σ-algebra $\mathscr{B}([s, T]) \otimes \mathscr{F}$ with values in $(B, \mathscr{B}(B))$. The dependence of a process on the variable $\omega \in \Omega$ is emphasized only if needed by the context. When we say that a process is continuous [resp. left continuous, right continuous, càdlàg, càglàd ...] we mean that almost all its paths are continuous [resp. left-continuous, right-continuous, càdlàg, càglàd...].

Let $\mathscr{G}$ be a sub-σ-field of $\mathscr{B}([s,T]) \otimes \mathscr{F}$. We say that such a process $\mathbb{X} : ([s,T] \times \Omega, \mathscr{G}) \to B$ is measurable with respect to $\mathscr{G}$ if it is the measurable in the usual sense. It is said *strongly (Bochner) measurable* (with respect to $\mathscr{G}$) if it is the limit of $\mathscr{G}$-measurable countable-valued functions. If $\mathbb{X}$ is measurable and $\mathbb{X}$ is càdlàg, càglàd or if B is separable then $\mathbb{X}$ is strongly measurable. The σ-field $\mathscr{G}$ will not be mentioned when it is clearly designated. We denote by $\mathscr{P}$ the predictable σ-field on $[s,T] \times \Omega$. The processes $\mathbb{X}$ measurable on $(\Omega \times [s,T], \mathcal{P})$ are also called *predictable* processes. All those processes will be considered as strongly measurable, with respect to $\mathcal{P}$. Each time we use expressions as "adapted", "predictable" etc... we will always mean "with respect to the filtration $\{\mathscr{F}_t^s\}_{t \geq s}$".

Notation 2.5. *The* blackboard bold *letters $\mathbb{X}$, $\mathbb{Y}$, $\mathbb{M}$... are used for Banach (or Hilbert)-space valued) processes, while notations X (or Y, M...) are reserved for real valued processes.*

Notation 2.6. *We always assume the following convention: when needed all the Banach space càdlàg processes (or functions) indexed by $[s,T]$ are extended setting $\mathbb{X}(t) = \mathbb{X}(s-)$ for $t \leq s$ and $\mathbb{X}(t) = \mathbb{X}(T)$ for $t \geq T$.*

3 Stochastic integrals

We adopt the notations introduced in Section 2.

Definition 3.1. *Let $\mathbb{X}\colon \Omega \times [s,T] \to \mathcal{L}(U,H)$ and $\mathbb{Y}\colon \Omega \times [s,T] \to U$ be two stochastic processes. Assume that $\mathbb{Y}$ is continuous and $\mathbb{X}$ is Bochner integrable.*

If for almost every $t \in [s,T]$ the following limit (in the norm of the space H) exists in probability

$$\int_s^t \mathbb{X}(r)\, \mathrm{d}^- \mathbb{Y}(r) := \lim_{\epsilon \to 0^+} \int_s^t \mathbb{X}(r) \left(\frac{\mathbb{Y}(r+\epsilon) - \mathbb{Y}(r)}{\epsilon} \right) \mathrm{d}r$$

and the process $t \mapsto \int_s^t \mathbb{X}(r)\, \mathrm{d}^- \mathbb{Y}(r)$ admits a continuous (in H) version, we say that $\mathbb{X}$ is forward integrable with respect to $\mathbb{Y}$. That version of $\int_s^\cdot \mathbb{X}(r)\, \mathrm{d}^- \mathbb{Y}(r)$ is called forward integral of $\mathbb{X}$ with respect to $\mathbb{Y}$.

Remark 3.2.
1. *The definition above is a natural generalization of that given in [15] Definition 3.4; there the forward integral is a real valued process.*
2. *Previous integral definition extends to the case when U and H are separable Banach spaces.*

3.1 The case of Q-Wiener process

Consider a positive and self-adjoint operator $Q \in \mathcal{L}(U)$. Even if not necessary, we assume Q to be injective; this allows us to avoid formal complications. However Theorem 3.4 below holds without this restriction.

Define $U_0 := Q^{1/2}(U)$: U_0 is a Hilbert space for the inner product $\langle x, y \rangle_{U_0} := \langle Q^{-1/2}x, Q^{-1/2}y \rangle_U$ and, clearly $Q^{1/2}\colon U \to U_0$ is an isometry, see e.g. [10] Section 4.3 for details. We remind that, given $A \in \mathcal{L}_2(U_0, H)$, we have $\|A\|^2_{\mathcal{L}_2(U_0,H)} = Tr\left(AQ^{1/2}(AQ^{1/2})^*\right)$.

Let $\mathbb{W}_Q = \{\mathbb{W}_Q(t) : s \leq t \leq T\}$ be an U-valued $\mathscr{F}_t^s$-Q-Wiener process with $\mathbb{W}_Q(s) = 0$, $\mathbb{P}$ a.s. The definition and properties of Q-Wiener processes are presented for instance in [34] Chapter 2.1. If $\mathbb{Y}$ is predictable with some integrability properties, $\int_s^t \mathbb{Y}(r)\, \mathrm{d}\mathbb{W}(r)$ denotes the classical Itô-type integral with respect to $\mathbb{W}$, defined e.g. in [10]. In the sequel such an integral will be shown to be equal to the forward integral so that the forward integral happens to be an extension of the Itô integral. In the next subsection we introduce the Itô integral $\int_s^t \mathbb{Y}(r)\, \mathrm{d}\mathbb{M}(r)$ with respect to a local martingale $\mathbb{M}$. If $H = \mathbb{R}$, previous integral will also denoted $\int_s^t \langle \mathbb{Y}(r),\, \mathrm{d}\mathbb{M}(r) \rangle_U$.

Definition 3.3. *We say that a sequence of $\mathscr{F}_t^s$-stopping times $\tau_n \colon \Omega \to [0, +\infty]$ is* suitable *if, denoted with Ω_n the set*

$$\Omega_n := \{\omega \in \Omega \ : \ \tau_n(\omega) > T\},$$

we have

$$\Omega_n \subseteq \Omega_{n+1} \qquad \textit{a.s. for all } n$$

and

$$\bigcup_{n\in\mathbb{N}} \Omega_n = \Omega \qquad \textit{a.s.}$$

In the sequel, we will use the terminology "stopping times" without mentioning the underlying filtration $(\mathscr{F}_t^s)$.

Theorem 3.4. *Let $\mathbb{X} \colon [s,T] \times \Omega \to \mathcal{L}_2(U_0, H)$ be a predictable process satisfying*

$$\int_s^T \|\mathbb{X}(r)\|^2_{\mathcal{L}_2(U_0,H)} \, \mathrm{d}r < +\infty \qquad \textit{a.s.} \tag{14}$$

Then, the forward integral

$$\int_s^\cdot \mathbb{X}(r) \, \mathrm{d}^- \mathbb{W}_Q(r).$$

exists and coincides with the classical Itô integral (defined for example in [10] Chapter 4)

$$\int_s^\cdot \mathbb{X}(r) \, \mathrm{d}\mathbb{W}_Q(r).$$

Proof. We fix $t \in [s,T]$. In the proof we follow the arguments related to the finite-dimensional case, see Theorem 2 of [60]. As a first step we consider $\mathbb{X}$ with

$$\mathbb{E}\left(\int_s^T \|\mathbb{X}(r)\|^2_{\mathcal{L}_2(U_0,H)} \, \mathrm{d}r \right) < +\infty. \tag{15}$$

This fact ensures that the hypotheses in the stochastic Fubini Theorem 4.18 of [10] are satisfied. We have

$$\int_s^t \mathbb{X}(r) \frac{\mathbb{W}_Q(r+\epsilon) - \mathbb{W}_Q(r)}{\epsilon} \, \mathrm{d}r = \int_s^t \mathbb{X}(r) \frac{1}{\epsilon} \left(\int_r^{r+\epsilon} \mathrm{d}\mathbb{W}_Q(\theta) \right) \mathrm{d}r;$$

applying the stochastic Fubini Theorem, the expression above is equal to

$$\int_s^t \left(\frac{1}{\epsilon} \int_{\theta-\epsilon}^{\theta} \mathbb{X}(r) \, \mathrm{d}r \right) \mathrm{d}\mathbb{W}_Q(\theta) + R_\epsilon(t)$$

where $R_\epsilon(t)$ is a boundary term that converges to 0 in probability, for any $t \in [s,T]$, so that we can ignore it. We can apply now the maximal inequality stated in [62], Theorem 1: there exists a universal constant $C > 0$ such that, for every $f \in L^2([s,t];\mathbb{R})$,

$$\int_s^t \left(\sup_{\epsilon\in(0,1]} \left| \frac{1}{\epsilon} \int_{(r-\epsilon)}^r f(\xi) \, \mathrm{d}\xi \right| \right)^2 \mathrm{d}r \le C \int_s^t f^2(r) \, \mathrm{d}r. \tag{16}$$

According to the vector valued version of the Lebesgue differentiation Theorem (see Theorem II.2.9 in [19]), the following quantity

$$\frac{1}{\epsilon} \int_{(r-\epsilon)}^r \mathbb{X}(\xi) \, \mathrm{d}\xi$$

converges $\mathrm{d}\mathbb{P}\otimes\mathrm{d}r$ a.e. to $\mathbb{X}(r)$. Consequently (16) and dominated convergence theorem imply

$$\mathbb{E}\int_s^t \left\| \left(\frac{1}{\epsilon}\int_{\theta-\epsilon}^{\theta} \mathbb{X}(r)\,\mathrm{d}r \right) - \mathbb{X}(\theta) \right\|^2_{\mathcal{L}_2(U_0,H)} \mathrm{d}\theta \xrightarrow{\epsilon\to 0} 0.$$

Finally, the convergence

$$J_\epsilon := \int_s^t \left(\frac{1}{\epsilon}\int_{\theta-\epsilon}^{\theta} \mathbb{X}(r)\,\mathrm{d}r \right) \mathrm{d}\mathbb{W}_Q(\theta) \xrightarrow[L^2(\Omega,H)]{\epsilon\to 0} J := \int_s^t \mathbb{X}(\theta)\,\mathrm{d}\mathbb{W}_Q(\theta), \tag{17}$$

justifies the claim.

If (15) is not satisfied we proceed by localization. Denote again by J_ϵ and J the processes defined in (17). Call τ_n the stopping times given by

$$\tau_n := \inf\left\{ t\in[s,T] \ :\ \int_s^t \|\mathbb{X}(r)\|^2_{\mathcal{L}_2(U_0,H)}\,\mathrm{d}r \geq n \right\}$$

(and $+\infty$ if the set is void) and call Ω_n the sets

$$\Omega_n := \{\omega\in\Omega \ :\ \tau_n(\omega) > T\}.$$

It is easy to see that the stopping times τ_n are suitable in the sense of Definition 3.3.

For each fixed n, the process $I_{[s,\tau_n]}\mathbb{X}$ verifies (15) and from the first step

$$J_n^\epsilon := \int_s^t \left(\frac{1}{\epsilon}\int_{\theta-\epsilon}^{\theta} I_{[s,\tau_n]}(r)\mathbb{X}(r)\,\mathrm{d}r \right) \mathrm{d}\mathbb{W}(\theta) \xrightarrow[L^2(\Omega,H)]{\epsilon\to 0} J_n := \int_s^t I_{[s,\tau_n]}(\theta)\mathbb{X}(\theta)\,\mathrm{d}\mathbb{W}(\theta).$$

So

$$I_{\Omega_n}J_\epsilon = I_{\Omega_n}J_n^\epsilon \xrightarrow[L^2(\Omega,H)]{\epsilon\to 0} I_{\Omega_n}J_n = I_{\Omega_n}J.$$

Consequently, for all n, $I_{\Omega_n}J_\epsilon$ converges to $I_{\Omega_n}J$ in probability and finally J_ϵ converges to J in probability as well. This fact concludes the proof. □

3.2 The semimartingale case

Consider now the case when the integrator is a more general local martingale. Let H and U be again two separable Hilbert spaces; we adopt the notations introduced in Section 2.

An U-valued measurable process $\mathbb{M}\colon [s,T]\times\Omega\to U$ is called martingale if, for all $t\in[s,T]$, $\mathbb{M}$ is $\mathscr{F}_t^s$-adapted with $\mathbb{E}\left[|\mathbb{M}(t)|\right] < +\infty$ and $\mathbb{E}\left[\mathbb{M}(t_2)|\mathscr{F}^s_{t_1}\right] = \mathbb{M}(t_1)$ for all $s\leq t_1\leq t_2\leq T$. The concept of (conditional) expectation for B-valued processes, for a separable Banach space B, is recalled for instance in [10] Section 1.3. All the considered martingales will be continuous.

We denote by $\mathcal{M}^2(s,T;H)$ the linear space of square integrable martingales equipped with the norm

$$|\mathbb{M}|_{\mathcal{M}^2(s,T;U)} := \left(\mathbb{E}\sup_{t\in[s,T]} |\mathbb{M}(t)|^2 \right)^{1/2}.$$

It is a Banach space as stated in [10],Proposition 3.9.

An U-valued measurable process $\mathbb{M}\colon [s,T]\times\Omega\to U$ is called local martingale if there exists a non-decreasing sequence of stopping times $\tau_n\colon\Omega\to[s,T]\cup\{+\infty\}$ such that $\mathbb{M}(t\wedge\tau_n)$ for $t\in[s,T]$ is a martingale and $\mathbb{P}\left[\lim_{n\to\infty}\tau_n = +\infty\right] = 1$. All the considered local martingales are continuous.

Given a continuous local martingale $\mathbb{M}\colon [s,T]\times\Omega \to U$, the process $|\mathbb{M}|^2$ is a real local sub-martingale, see Theorem 2.11 in [42]. The increasing predictable process, vanishing at zero, appearing in the Doob-Meyer decomposition of $|\mathbb{M}|^2$ will be denoted by $[\mathbb{M}]^{\mathbb{R},cl}(t), t \in [s,T]$. It is of course uniquely determined and continuous.

We remind some properties of the Itô stochastic integral with respect to a local martingale $\mathbb{M}$. Call $\mathcal{I}_{\mathbb{M}}(s,T;H)$ the set of the processes $\mathbb{X}\colon [s,T]\times\Omega \to \mathcal{L}(U;H)$ that are strongly measurable from $([s,T]\times\Omega,\mathscr{P})$ to $\mathcal{L}(U;H)$ and such that

$$|\mathbb{X}|_{\mathcal{I}_{\mathbb{M}}(s,T;H)} := \left(\mathbb{E}\int_s^T \|\mathbb{X}(r)\|^2_{\mathcal{L}(U;H)}\,\mathrm{d}[\mathbb{M}]^{\mathbb{R},cl}(r)\right)^{1/2} < +\infty.$$

$\mathcal{I}_{\mathbb{M}}(s,T;H)$ endowed with the norm $|\cdot|_{\mathcal{I}_{\mathbb{M}}(s,T;H)}$ is a Banach space.

The linear map

$$\begin{cases} I\colon \mathcal{I}_{\mathbb{M}}(s,T;H) \to \mathcal{M}^2(s,T;H) \\ \mathbb{X} \mapsto \int_s^{\cdot} \mathbb{X}(r)\,\mathrm{d}\mathbb{M}(r) \end{cases}$$

is a contraction, see e.g. [48] Section 20.4 (above Theorem 20.5). As illustrated in [42] Section 2.2 (above Theorem 2.14), the stochastic integral w.r.t. $\mathbb{M}$ extends to the integrands $\mathbb{X}$ which are measurable from $([s,T]\times\Omega,\mathscr{P})$ to $\mathcal{L}(U;H)$ and such that

$$\int_s^T \|\mathbb{X}(r)\|^2_{\mathcal{L}(U;H)}\,\mathrm{d}[\mathbb{M}]^{\mathbb{R},cl}(r) < +\infty \qquad a.s. \tag{18}$$

We denote by $\mathcal{J}^2(s,T;U,H)$ such a family of integrands w.r.t. $\mathbb{M}$. Actually, the integral can be even defined for a wider class of integrands, see e.g. [49]. For instance, according to Section 4.7 of [10], let

$$\mathbb{M}_t = \int_s^t \mathbb{A}(r)\,\mathrm{d}\mathbb{W}_Q(r), t \in [s,T], \tag{19}$$

and $\mathbb{A}$ be an $\mathcal{L}(U,H)$-valued predictable process such that $\int_s^T \mathrm{Tr}[\mathbb{A}(r)Q^{1/2}(\mathbb{A}(r)Q^{1/2})^*]\,\mathrm{d}r < \infty$ a.s.

If $\mathbb{X}$ is an H-valued (or H^*-valued using Riesz identification) predictable process such that

$$\int_s^T \langle \mathbb{X}(r), \mathbb{A}(r)Q^{1/2}(\mathbb{A}Q^{1/2})^*\mathbb{X}(r)\rangle_H\,\mathrm{d}r < \infty, \text{ a.s.}, \tag{20}$$

then, as argued in Section 4.7 of [10],

$$N(t) = \int_s^t \langle \mathbb{X}(r),\,\mathrm{d}\mathbb{M}(r)\rangle_H, t \in [s,T], \tag{21}$$

is well-defined and it equals $N(t) = \int_s^t \langle \mathbb{X}(r), \mathbb{A}(r)\,\mathrm{d}\mathbb{W}_Q(r)\rangle_H$ for $t \in [s,T]$.

We recall in the following proposition some significant properties of the stochastic integral with respect to local martingales.

Proposition 3.5. *Let $\mathbb{M}$ be a continuous $(\mathcal{F}^s_t)$-local martingale, $\mathbb{X}$ verifying* (18). *We set $\mathbb{N}(t) = \int_s^t \mathbb{X}(r)d\mathbb{M}(r)$.*

- *(i) $\mathbb{N}$ is an $(\mathcal{F}^s_t)$-local martingale.*
- *(ii) Let $\mathbb{K}$ be an $(\mathcal{F}^s_t)$- predictable process such that $\mathbb{K}\mathbb{X}$ fulfills* (18). *Then the Itô-type stochastic integral $\int_s^t \mathbb{K}d\mathbb{N}$ for $t \in [s,T]$ is well-defined and it equals $\int_s^t \mathbb{K}\mathbb{X}d\mathbb{M}$.*

(iii) *If $\mathbb{M}$ is a Q-Wiener process W_Q, then, whenever $\mathbb{X}$ is such that*

$$\int_s^T Tr\left[\left(\mathbb{X}(s)Q^{1/2}\right)\left(\mathbb{X}(s)Q^{1/2}\right)^*\right] \mathrm{d}s < +\infty \text{ a.s.}, \tag{22}$$

then $\mathbb{N}(t) = \int_s^t \mathbb{X}(s) d\mathbb{W}_Q(s)$ is a local martingale and

$$[N]^{\mathbb{R},cl}(t) = \int_s^t \left(\mathbb{X}(s)Q^{1/2}\right)\left(\mathbb{X}(s)Q^{1/2}\right)^* \mathrm{d}s.$$

(iv) *If in item (iii), the expectation of the quantity* (22) *is finite, then $\mathbb{N}(t) = \int_s^t \mathbb{X}(s) d\mathbb{W}_Q(s)$ is square integrable continuous martingale.*

(v) *If $\mathbb{M}$ is defined as in* (19) *and $\mathbb{X}$ fulfills* (20)*, then $\mathbb{M}$ is a real local martingale. If moreover, the expectation of* (20) *is finite, then N, defined in (21), is a square integrable martingale.*

Proof. For (i) see [42] Theorem 2.14 page 14-15. For (ii) see [49], proof of Proposition 2.2 Section 2.4. (iii) and (iv) are contained in [10] Theorem 4.12 Section 4.4. (v) is a consequence of (iii) and (iv) and of the considerations before the statement of Proposition 3.5. □

Theorem 3.6. *Let us consider a continuous local martingale $\mathbb{M}\colon [s,T]\times\Omega \to U$ and a càglàd process predictable $\mathcal{L}(U,H)$-valued process. Then, the forward integral*

$$\int_s^\cdot \mathbb{X}(r)\,\mathrm{d}^-\mathbb{M}(r),$$

defined in Definition 3.1 exists and coincides with the Itô integral

$$\int_s^\cdot \mathbb{X}(r)\,\mathrm{d}\mathbb{M}(r).$$

Remark 3.7. *Any càglàd adapted process is a.s. bounded and therefore it belongs to $J^2(s,T;U,H)$.*

Proof of Theorem 3.6. Without loss of generality (replacing if necessary $\mathbb{X}$ with $(\mathbb{X} - \mathbb{X}(s))$) we can suppose that $\mathbb{X}(s) = 0$.

The proof follows partially the lines of Theorem 3.4. Similarly we first localize the problem using the suitable sequence of stopping times defined by

$$\tau_n := \inf\left\{t \in [s,T] \ : \ \|\mathbb{X}(t)\|^2_{\mathcal{L}(U,H)} + [\mathbb{M}]^{\mathbb{R},cl}(t) \geq n\right\}$$

(and $+\infty$ if the set is void); the localized process belongs to $\mathcal{I}_{\mathbb{M}}(s,T;H)$ and satisfies the hypotheses of the stochastic Fubini theorem in the form given in [43]. Since the integral is a contraction from $\mathcal{I}_{\mathbb{M}}(s,T;H)$ to $\mathcal{M}^2(s,T;H)$, it only remains to show that

$$\mathbb{E}\int_s^t \left\| \left(\frac{1}{\epsilon}\int_{\theta-\epsilon}^{\theta} \mathbb{X}(r)\,\mathrm{d}r\right) - \mathbb{X}(\xi)\right\|^2_{\mathcal{L}(U,H)} \mathrm{d}[\mathbb{M}]^{\mathbb{R},cl}(\xi) \xrightarrow{\epsilon\to 0} 0, \tag{23}$$

when $\mathbb{X}$ belongs to $\mathcal{I}_{\mathbb{M}}(s,T;H)$. (23) holds, taking into account the Lebesgue dominated convergence theorem, because $\mathbb{X}$ is left continuous and both $\mathbb{X}$ and $[M]^{\mathbb{R},cl}$ are bounded. □

An easier but still important statement concerns the integration with respect bounded variation processes.

Proposition 3.8. *Let us consider a continuous bounded variation process $\mathbb{V}\colon [s,T]\times\Omega\to U$ and let $\mathbb{X}$ be a càglàd measurable process $[s,T]\times\Omega\to\mathcal{L}(U,H)$. Then the forward integral*

$$\int_s^{\cdot} \mathbb{X}(r)\,\mathrm{d}^-\mathbb{V}(r),$$

defined in Definition 3.1 exists and coincides with the Lebesgue-Bochner integral

$$\int_s^{\cdot} \mathbb{X}(r)\,\mathrm{d}\mathbb{V}(r).$$

Proof. The proof is similar to the one of Theorem 3.6; one proceeds via Fubini theorem. □

4 χ-quadratic variation and χ-Dirichlet processes

4.1 χ-quadratic variation processes

Denote by $\mathscr{C}([s,T])$ the space of the real continuous processes equipped with the ucp (uniform convergence in probability) topology. Consider two real Banach spaces B_1 and B_2 with the same notations as in Section 2.

Following [15, 18] a **Chi-subspace** (of $(B_1\hat{\otimes}_\pi B_2)^*$) is defined as any Banach subspace $(\chi, |\cdot|_\chi)$ which is continuously embedded into $(B_1\hat{\otimes}_\pi B_2)^*$: in other words, there is some constant C such that

$$|\cdot|_{(B_1\hat{\otimes}_\pi B_2)^*}\le C|\cdot|_\chi.$$

Lemma 4.1. *Let us consider a Banach space ν_1 [resp. ν_2] continuously embedded in B_1^* [resp. B_2^*]. Then $\bar\chi:=\nu_1\hat{\otimes}_\pi\nu_2$ can be continuously embedded in $(B_1\hat{\otimes}_\pi B_2)^*$. In particular there exists a constant $C>0$ such that, for all $u\in\bar\chi$,*

$$|u|_{(B_1\hat{\otimes}_\pi B_2)^*}\le C|u|_{\bar\chi}, \tag{24}$$

after having identified an element of $\bar\chi$ with an element of $(B_1\hat{\otimes}_\pi B_2)^*$, as indicated in Lemma 2.2. In other words $\bar\chi$ is a Chi-subspace of $(B_1\hat{\otimes}_\pi B_2)^*$.

Remark 4.2. *In particular $B_1^*\hat{\otimes}_\pi B_2^*$ is a Chi-subspace of $(B_1\hat{\otimes}_\pi B_2)^*$.*

Proof of Lemma 4.1. To simplify the notations assume the norm of the injections $\nu_1\hookrightarrow B_1^*$ and $\nu_2\hookrightarrow B_2^*$ to be less or equal than 1. We remind that $(B_1\hat{\otimes}_\pi B_2)^*$ is isometrically identified with the Banach space of the bilinear bounded forms from $B_1\times B_2$ to $\mathbb{R}$, denoted by $\mathcal{B}i(B_1,B_2)$.

Consider first an element $u\in\bar\chi$ of the form $u=\sum_{i=1}^n a_i^*\otimes b_i^*$ for some $a_i^*\in\nu_1$ and $b_i^*\in\nu_2$. u can be identified with an element of $\mathcal{B}i(B_1,B_2)$ acting as

$$u(\phi,\psi):=\sum_{i=1}^n\langle a_i^*,\phi\rangle\langle b_i^*,\psi\rangle.$$

We can choose $a_i^*\in\nu_1$ and $b_i^*\in\nu_2$ such that $u=\sum_{i=1}^n a_i^*\otimes b_i^*$ and

$$|u|_{\bar\chi}=\inf\left\{\sum_{i=1}^n|x_i|_{\nu_1}|y_i|_{\nu_2}\ :\ u=\sum_{i=1}^n x_i\otimes y_i,\qquad x_i\in\nu_1,\ y_i\in\nu_2\right\}>-\epsilon+\sum_{i=1}^n|a_i^*|_{\nu_1}|b_i^*|_{\nu_2}.$$

Using such an expression for u we have

$$\|u\|_{\mathcal{B}i(B_1,B_2)}=\sup_{|\phi|_{B_1},|\psi|_{B_2}\le1}\left|\sum_{i=1}^n\langle a_i^*,\phi\rangle\langle b_i^*,\psi\rangle\right|\le\sum_{i=1}^n|a_i^*|_{B_1^*}|b_i^*|_{B_2^*}\le\sum_{i=1}^n|a_i^*|_{\nu_1}|b_i^*|_{\nu_2}\le\epsilon+|u|_{\bar\chi}.$$

Since ϵ is arbitrary, we conclude that $\|u\|_{\mathcal{B}i(B_1,B_2)}\le|u|_{\bar\chi}$.

Since this proves that the mapping that associates to $u\in\nu_1\hat{\otimes}_\pi\nu_2$ its corresponding element in $\mathcal{B}i(B_1,B_2)$, has norm 1 on the dense subset $\nu_1\hat{\otimes}_\pi\nu_2$, then the claim is proved. □

Remark 4.3. *Even though the Chi-subspaces of tensor product type, described in Lemma 4.1 are natural, there are examples of Chi-subspace not of that form, see e.g. Section 2.6 in [15].*

Let χ be a generic Chi-subspace. We introduce the following definition.

Definition 4.4. *Given $\mathbb{X}$ [resp. $\mathbb{Y}$] a B_1-valued [resp. B_2-valued] process, we say that $(\mathbb{X},\mathbb{Y})$ admits a χ-covariation if the two following conditions are satisfied.*

H1 *For any sequence of positive real numbers $\epsilon_n \searrow 0$ there exists a subsequence ϵ_{n_k} such that*

$$\sup_k \int_s^T \frac{|(J(\mathbb{X}(r+\epsilon_{n_k})-\mathbb{X}(r))\otimes(\mathbb{Y}(r+\epsilon_{n_k})-\mathbb{Y}(r)))|_{\chi^*}}{\epsilon_{n_k}} ds \ < \infty \ a.s. \tag{25}$$

H2 *If we denote by $[X,\mathbb{Y}]^{\epsilon}_{\chi}$ the application*

$$\begin{cases} [\mathbb{X},\mathbb{Y}]^{\epsilon}_{\chi} : \chi \longrightarrow \mathscr{C}([s,T]) \\ \phi \mapsto \int_s^{\cdot} {}_{\chi}\left\langle \phi, \frac{J((\mathbb{X}(r+\epsilon)-\mathbb{X}(r))\otimes(\mathbb{Y}(r+\epsilon)-\mathbb{Y}(r)))}{\epsilon} \right\rangle_{\chi^*} dr, \end{cases} \tag{26}$$

where $J : B_1\hat{\otimes}_{\pi}B_2 \longrightarrow (B_1\hat{\otimes}_{\pi}B_2)^{**}$ is the canonical injection between a space and its bidual, the following two properties hold.

(i) There exists an application, denoted by $[\mathbb{X},\mathbb{Y}]_{\chi}$, defined on χ with values in $\mathscr{C}([s,T])$, satisfying

$$[\mathbb{X},\mathbb{Y}]^{\epsilon}_{\chi}(\phi) \xrightarrow[\epsilon\longrightarrow 0_+]{ucp} [\mathbb{X},\mathbb{Y}]_{\chi}(\phi), \tag{27}$$

for every $\phi \in \chi \subset (B_1\hat{\otimes}_{\pi}B_2)^*$.

(ii) There exists a strongly measurable process $\widetilde{[\mathbb{X},\mathbb{Y}]}_{\chi} : \Omega\times[s,T] \longrightarrow \chi^*$, such that

- *for almost all $\omega\in\Omega$, $\widetilde{[\mathbb{X},\mathbb{Y}]}_{\chi}(\omega,\cdot)$ is a (càdlàg) bounded variation process,*
- *$\widetilde{[\mathbb{X},\mathbb{Y}]}_{\chi}(\cdot,t)(\phi) = [\mathbb{X},\mathbb{Y}]_{\chi}(\phi)(\cdot,t)$ a.s. for all $\phi\in\chi$, $t\in[s,T]$.*

Remark 4.5. *Since, $(B_1\hat{\otimes}_{\pi}B_2)^{**}$ is continuously embedded in χ^*, then $J(a\otimes b)$ can be considered as an element of χ^*. Therefore we have*

$$|J(a\otimes b)|_{\chi^*} = \sup_{\phi\in\chi, |\phi|_{\chi}\le 1} \langle J(a\otimes b),\phi\rangle = \sup_{\phi\in\chi, |\phi|_{\chi}\le 1} |\phi(a\otimes b)|.$$

We can apply this fact to the expression (25) *considering $a=\mathbb{X}(r+\epsilon_{n_k})-\mathbb{X}(r)$ and $b=\mathbb{Y}(r+\epsilon_{n_k})-\mathbb{Y}(r)$.*

Remark 4.6. *An easy consequence of Remark 3.10 and Lemma 3.18 in [18] is the following. We set*

$$A(\varepsilon) := \int_s^T \frac{|(J(\mathbb{X}(r+\epsilon)-\mathbb{X}(r))\otimes(\mathbb{Y}(r+\epsilon)-\mathbb{Y}(r)))|_{\chi^*}}{\epsilon} dr. \tag{28}$$

1. *If $\lim_{\epsilon\to 0} A(\epsilon)$ exists in probability then Condition* **H1** *of Definition 4.4 is verified.*
2. *If $\lim_{\epsilon\to 0} A(\epsilon) = 0$ in probability then $(\mathbb{X},\mathbb{Y})$ admits a χ-covariation and $\widetilde{[\mathbb{X},\mathbb{Y}]}$ vanishes.*

If $(\mathbb{X},\mathbb{Y})$ admits a χ-covariation we call χ-covariation of $(\mathbb{X},\mathbb{Y})$ the χ^*-valued process $\widetilde{[\mathbb{X},\mathbb{Y}]}_{\chi}$ defined for every $\omega\in\Omega$ and $t\in[s,T]$ by $\phi\mapsto\widetilde{[\mathbb{X},\mathbb{Y}]}_{\chi}(\omega,t)(\phi) = [\mathbb{X},\mathbb{Y}]_{\chi}(\phi)(\omega,t)$. By abuse of notation, $[\mathbb{X},\mathbb{Y}]_{\chi}$ will also be often called χ-covariation and it will be confused with $\widetilde{[\mathbb{X},\mathbb{Y}]}_{\chi}$.

Definition 4.7. *If $\chi = (B_1 \hat{\otimes}_\pi B_2)^*$ the χ-covariation is called* global covariation. *In this case we omit the index $(B_1 \hat{\otimes}_\pi B_2)^*$ using the notations $[\mathbb{X}, \mathbb{Y}]$ and $\widetilde{[\mathbb{X}, \mathbb{Y}]}$.*

Remark 4.8. *The notions of scalar and tensor covariation have been defined in Definitions 1.3 and 1.6.*

1. *Suppose that $\mathbb{X}$ and $\mathbb{Y}$ admits a scalar quadratic variation and $(\mathbb{X}, \mathbb{Y})$ has a tensor covariation, denoted by $[\mathbb{X}, \mathbb{Y}]^{\otimes}$. Then $(\mathbb{X}, \mathbb{Y})$ admits a global covariation $[\mathbb{X}, \mathbb{Y}]$. In particular, recalling that $B_1 \hat{\otimes}_\pi B_2$ is embedded in $(B_1 \hat{\otimes}_\pi B_2)^{**}$, we have $\widetilde{[\mathbb{X}, \mathbb{Y}]} = [\mathbb{X}, \mathbb{Y}]^{\otimes}$. The proof is a slight adaptation of the one of Proposition 3.14 in [18]. In particular condition* **H1** *holds using Cauchy-Schwarz inequality.*

2. *If $\mathbb{X}$ admits a scalar zero quadratic variation then, by definition, the tensor covariation of $(\mathbb{X}, \mathbb{X})$ also vanishes. Consequently, by item (i) $\mathbb{X}$ also admits a global quadratic variation, which is also zero.*

Remark 4.9. *If $(\mathbb{X}, \mathbb{Y})$ admits a global covariation then it admits a χ-covariation for any Chi-subspace χ. Moreover $[\mathbb{X}, \mathbb{Y}]_\chi(\phi) = [\mathbb{X}, \mathbb{Y}](\phi)$ for all $\phi \in \chi$.*

We say that a process $\mathbb{X}$ admits a χ-quadratic variation if $(\mathbb{X}, \mathbb{X})$ admits a χ-covariation. The process $\widetilde{[\mathbb{X}, \mathbb{X}]}_\chi$, often denoted by $\widetilde{[\mathbb{X}]}_\chi$, is also called χ-quadratic variation of $\mathbb{X}$.

Remark 4.10. *For the global covariation case (i.e. for $\chi = (B_1 \hat{\otimes}_\pi B_2)^*$) the condition **H1** reduces to*

$$\sup_k \int_s^T \frac{1}{\epsilon_{n_k}} \left|(\mathbb{X}(r + \epsilon_{n_k}) - \mathbb{X}(r))\right|_{B_1} \left|\mathbb{Y}(r + \epsilon_{n_k}) - \mathbb{Y}(r))\right|_{B_2} ds \; < \infty \; a.s.$$

In fact the embedding of $(B_1 \hat{\otimes}_\pi B_2)$ in its bi-dual is isometric and, for $x \in B_1$ and $y \in B_2$, $|x \otimes y|_{(B_1 \hat{\otimes}_\pi B_2)} = |x|_1 |y|_2$.

The product of a real finite quadratic variation process and a zero real quadratic variation process is again a zero quadratic variation processes. Under some conditions this can be generalized to the infinite dimensional case.

Proposition 4.11. *Let $i = 1, 2$ and ν_i be a real Banach space continuously embedded in the dual B_i^* of a real Banach space B_i. Let consider the Chi-subspace of the type $\chi_1 = \nu_1 \hat{\otimes}_\pi B_2^*$ and $\chi_2 = B_1^* \hat{\otimes}_\pi \nu_2$, $\hat{\chi}_i = \nu_i \hat{\otimes}_\pi \nu_i$, $i = 1, 2$. Let $\mathbb{X}$ [resp. $\mathbb{Y}$] be a process with values in B_1 [resp. B_2].*

1. *Suppose that $\mathbb{X}$ admits a $\hat{\chi}_1$-quadratic variation and $\mathbb{Y}$ a zero scalar quadratic variation. Then $[\mathbb{X}, \mathbb{Y}]_{\chi_1} = 0$.*

2. *Similarly suppose that $\mathbb{Y}$ admits a $\hat{\chi}_2$-quadratic variation and $\mathbb{X}$ a zero scalar quadratic variation. Then $[\mathbb{X}, \mathbb{Y}]_{\chi_2} = 0$.*

Proof. We remark that Lemma 4.1 imply that χ_i and $\hat{\chi}_i$, $i = 1, 2$ are indeed Chi-subspaces. By item 2. of Remark 4.6, it is enough to show that $A(\varepsilon)$ defined in (28) converge to zero, with $\chi = \chi_i, i = 1, 2$. By symmetry it is enough to show item 1.
We set $\chi = \chi_1$. The Banach space B_i is isometrically embedded in its bidual $B_i^{**}, i = 1, 2$, so, since $\nu_1 \subseteq B_1^*$ with continuous inclusion, we have $B_1 \subseteq B_1^{**} \subset \nu_1^*$ where the inclusion are continuous.

Moreover, since $\chi = \nu_1 \hat{\otimes}_\pi B_2^* \subseteq B_1^* \otimes_\pi B_2^* \subset (B_1 \hat{\otimes}_\pi B_2)^*$, with continuous inclusions, taking into account Remark 4.2, we have

$$J(B_1 \hat{\otimes}_\pi B_2) \subset (B_1 \hat{\otimes}_\pi B_2)^{**} \subset \chi^*.$$

Let $a \in B_1$ and $b \in B_2$. We have

$$|J(a \otimes b)|_{\chi^*} = \sup_{|\varphi|_{\nu_1} \le 1, |\psi|_{B_2^*} \le 1} |_{\chi_1^*} \langle J(a \otimes b), \varphi \otimes \psi \rangle_{\chi_1}|$$
$$= \sup_{|\varphi|_{\nu_1} \le 1} |_{\nu_1} \langle \varphi, a \rangle_{\nu_1^*}| \sup_{|\psi|_{B_2^*} \le 1} |_{B_2^*} \langle \psi, b \rangle_{B_2^{**}}| = |a|_{\nu_1^*} |b|_{B_2^{**}} = |a|_{\nu_1^*} |b|_{B_2}. \quad (29)$$

Consequently, with $a = \mathbb{X}(r+\varepsilon) - \mathbb{X}(r)$ and $b = \mathbb{Y}(r+\varepsilon) - \mathbb{Y}(r)$ for $r \in [s,T]$, we have

$$A(\varepsilon) = \int_s^T \frac{|(J(\mathbb{X}(r+\epsilon) - \mathbb{X}(r)) \otimes (\mathbb{Y}(r+\epsilon) - \mathbb{Y}(r)))|_{\chi^*}}{\epsilon} dr = \int_s^T |\mathbb{X}(r+\varepsilon) - \mathbb{X}(r)|_{\nu_1^*} |\mathbb{Y}(r+\varepsilon) - \mathbb{Y}(r)|_{B_2} \frac{dr}{\varepsilon}$$
$$\le \left(\int_s^T |\mathbb{X}(r+\varepsilon) - \mathbb{X}(r)|^2_{\nu_1^*} \frac{dr}{\varepsilon} \int_s^T |\mathbb{Y}(r+\varepsilon) - \mathbb{Y}(r)|^2_{B_2} \frac{dr}{\varepsilon} \right)^{1/2}$$
$$= \left(\int_s^T \frac{|(J(\mathbb{X}(r+\epsilon) - \mathbb{X}(r)) \otimes (\mathbb{X}(r+\epsilon) - \mathbb{X}(r)))|_{\hat{\chi}_1^*}}{\epsilon} dr \right)^{1/2} \left(\int_s^T \frac{|\mathbb{Y}(r+\epsilon) - \mathbb{Y}(r))|^2_{B_2}}{\epsilon} dr \right)^{1/2}. \quad (30)$$

The last equality is obtained using an argument similar to (29). The condition **H1** related to the $\hat{\chi}_1$-quadratic variation of $\mathbb{X}$ and the zero scalar quadratic variation of $\mathbb{Y}$, imply that previous expression converges to zero. □

When one of the processes is real the formalism of global covariation can be simplified as shown in the following proposition.

Here, and in the sequel, we consider the case of a real separable Hilbert space H instead of a general real Banach space B. According to our conventions, $|\cdot|$ represents both the norm in H and the absolute value in $\mathbb{R}$.

Proposition 4.12. *Let H be a real separable Hilbert space. Let be $\mathbb{X}\colon [s,T] \times \Omega \to H$ a Bochner integrable process and $Y\colon [s,T] \times \Omega \to \mathbb{R}$ a real valued process. Suppose the following.*

(a) For any ϵ, $\frac{1}{\epsilon}\int_s^T |\mathbb{X}(r+\epsilon) - \mathbb{X}(r)||Y(r+\epsilon) - Y(r)|\, \mathrm{d}r$ is bounded by a r.v. $A(\epsilon)$ such that $A(\epsilon)$ converges in probability when $\epsilon \to 0$.

(b) For every $h \in H$ the following limit

$$C(t)(h) := \lim_{\epsilon \to 0^+} \frac{1}{\epsilon} \int_s^t \langle h, \mathbb{X}(r+\epsilon) - \mathbb{X}(r) \rangle (Y(r+\epsilon) - Y(r))\, \mathrm{d}r$$

exists ucp and there exists a continuous process $\tilde{C}\colon [s,T] \times \Omega \to H$ s.t.

$$\left\langle \tilde{C}(t,\omega), h \right\rangle = C(t)(h)(\omega) \qquad \text{for } \mathbb{P}\text{-a.s. } \omega \in \Omega,$$

for all $t \in [s,T]$ and $h \in H$.

If we identify H with $(H \hat{\otimes}_\pi \mathbb{R})^*$, then $\mathbb{X}$ and Y admit a global covariation and $\tilde{C} = \widetilde{[\mathbb{X}, Y]}$.

Proof. Taking into account the identification of H with $(H \hat{\otimes}_\pi \mathbb{R})^*$ the result is a consequence of Corollary 3.26 of [18]. In particular condition **H1** follows from Remark 4.10.

□

4.2 Relations with the tensor covariation and the classical tensor covariation

The notions of tensor covariation recalled in Definition 1.6 concerns general processes. In the specific case when H_1 and H_2 are two separable Hilbert spaces and $\mathbb{M}\colon [s,T]\times\Omega \to H_1$, $\mathbb{N}\colon [s,T]\times\Omega\to H_2$ are two continuous local martingales, another (classical) notion of tensor covariation is defined, see for instance in Section 23.1 of [48]. This will be denoted by $[\mathbb{M},\mathbb{N}]^{cl}$. Recall that the notion introduced in Definition 1.6 is denoted by $[\mathbb{M},\mathbb{N}]^{\otimes}$.

Remark 4.13. *We observe the following facts.*

(i) According to Chapter 22 and 23 in [48], given an H_1-valued [resp. H_2-valued] continuous local martingale $\mathbb{M}$ [resp. $\mathbb{N}$], $[\mathbb{M},\mathbb{N}]^{cl}$ is an $(H_1\hat{\otimes}_\pi H_2)$-valued process. Recall that $(H_1\hat{\otimes}_\pi H_2)\subseteq (H_1\hat{\otimes}_\pi H_2)^{**}$.

(ii) Taking into account Lemma 2.2 we know that, given $h\in H_1$ and $k\in H_2$, $h^*\otimes k^*$ can be considered as an element of $(H_1\hat{\otimes}_\pi H_2)^*$. One has

$$[\mathbb{M},\mathbb{N}]^{cl}(t)(h^*\otimes k^*) = [\langle \mathbb{M},h\rangle\,\langle \mathbb{N},k\rangle]^{cl}(t), \tag{31}$$

where h^* [resp. k^*] is associated with h [resp. k] via Riesz theorem. This property characterizes $[\mathbb{M},\mathbb{N}]^{cl}$, see e.g. [10], Section 3.4 after Proposition 3.11.

(iii) If $H_2=\mathbb{R}$ and $\mathbb{N}=N$ is a real continuous local martingale then, identifying $H_1\hat{\otimes}_\pi H_2$ with H_1, $[\mathbb{M},\mathbb{N}]^{cl}$ can be considered as an H_1-valued process. The characterization (31) can be translated into

$$[\mathbb{M},\mathbb{N}]^{cl}(t)(h^*) = [\langle \mathbb{M},h\rangle\,,N]^{cl}(t), \forall h\in H_1. \tag{32}$$

By inspection, this allows us to see that the classical covariation between $\mathbb{M}$ and N can be expressed as

$$[\mathbb{M},N]^{cl}(t) := \mathbb{M}(t)N(t) - \mathbb{M}(s)N(s) - \int_s^t N(r)\,\mathrm{d}\mathbb{M}(r) - \int_s^t \mathbb{M}(r)\,\mathrm{d}N(r). \tag{33}$$

In the sequel H will denote a separable Hilbert space.

Remark 4.14. *The following properties hold.*

1. *If $\mathbb{M}$ is a continuous local martingale with values in H then $\mathbb{M}$ has a scalar quadratic variation, see Proposition 1.7 in [18].*
2. *If $\mathbb{M}$ is a continuous local martingale with values in H then $\mathbb{M}$ has a tensor quadratic variation. This fact is proved in Proposition 1.6 of [18]. Using similar arguments one can see that if $\mathbb{M}_1$ [resp. $\mathbb{M}_2$] is a continuous local martingale with values in H_1 [resp. H_2] then $(\mathbb{M}_1,\mathbb{M}_2)$ admits a tensor covariation.*

Lemma 4.15. *Let H be a separable Hilbert space. Let $\mathbb{M}$ [resp. $\mathbb{N}$] be a continuous local martingale with values in H. Then $(\mathbb{M},\mathbb{N})$ admits a tensor covariation and*

$$[\mathbb{M},\mathbb{N}]^{\otimes} = [\mathbb{M},\mathbb{N}]^{cl}. \tag{34}$$

In particular $(\mathbb{M},\mathbb{N})$ admits a global covariation and

$$\widetilde{[\mathbb{M},\mathbb{N}]} = [\mathbb{M},\mathbb{N}]^{cl}. \tag{35}$$

Proof. Thanks to Remark 4.14 $\mathbb{M}$ and $\mathbb{N}$ admit a scalar quadratic variation and $(\mathbb{M},\mathbb{N})$ a tensor covariation. By Remark 4.8 they admit a global covariation. It is enough to show that they are equal as elements of $(H_1\hat{\otimes}_\pi H_2)^{**}$, so one needs to prove that

$$[\mathbb{M},\mathbb{N}]^{\otimes}(\phi) = [\mathbb{M},\mathbb{N}]^{cl}(\phi), \tag{36}$$

for every $\phi \in (H_1 \hat{\otimes}_\pi H_2)^*$.

Given $h \in H_1$ and $k \in H_2$, we consider (via Lemma 2.2) $h^* \otimes k^*$ as an element of $(H_1 \hat{\otimes}_\pi H_2)^*$. According to Lemma 4.16 below, $H_1^* \hat{\otimes}_\pi H_2^*$ is sequentially dense in $(H_1 \hat{\otimes}_\pi H_2)^*$ in the weak-* topology. Therefore, taking into account item (ii) of Remark 4.13 we only need to show that

$$[\mathbb{M}, \mathbb{N}]^{\otimes}(h^* \otimes k^*) = [\langle \mathbb{M}, h\rangle \langle \mathbb{N}, k\rangle]^{cl}, \tag{37}$$

for every $h \in H_1, k \in H_2$. By the usual properties of Bochner integral the left-hand side of (37) is the limit of

$$\frac{1}{\epsilon} \int_s^{\cdot} (M(r+\epsilon) - M(r)) \otimes (N(r+\epsilon) - N(r))(h^* \otimes k^*)\, \mathrm{d}r$$
$$= \frac{1}{\epsilon} \int_s^{\cdot} \langle (M(r+\epsilon) - M(r)), h\rangle \langle \otimes (N(r+\epsilon) - N(r)), k\rangle\, \mathrm{d}r. \tag{38}$$

Since $\langle \mathbb{M}, h\rangle$ and $\langle \mathbb{N}, k\rangle$ are real local martingales, the covariation $[\langle \mathbb{M}, h\rangle \langle \mathbb{N}, k\rangle]$ exists and equals the classical covariation of local martingales because of Proposition 2.4(3) of [58]. □

Lemma 4.16. *Let H_1, H_2 be two separable Hilbert spaces. Then $H_1^* \hat{\otimes}_\pi H_2^*$ is sequentially dense in $(H_1 \hat{\otimes}_\pi H_2)^*$ in the weak-* topology.*

Proof. Let (e_i) and (f_i) be respectively two orthonormal bases of H_1 and H_2. We denote by $\mathcal{D}$ the linear span of finite linear combinations of $e_i \otimes f_i$. Let $T \in (H_1 \hat{\otimes}_\pi H_2)^*$, which is a linear continuous functional on $H_1 \hat{\otimes}_\pi H_2$. Using the identification of $(H_1 \hat{\otimes}_\pi H_2)^*$ with $\mathcal{B}i(H_1, H_2)$, for each $n \in \mathbb{N}$, we define the bilinear form

$$T_n(a, b) := \sum_{i=1}^n \langle a, e_i\rangle_{H_1} \langle b, f_i\rangle_{H_2} T(e_i, f_i).$$

It defines an element of $H_1^* \hat{\otimes}_\pi H_2^* \subset (H_1 \hat{\otimes}_\pi H_2)^*$. It remains to show that

$${}_{(H_1 \hat{\otimes}_\pi H_2)^*}\langle T_n, l\rangle_{H_1 \hat{\otimes}_\pi H_2} \xrightarrow{n \to \infty} {}_{(H_1 \hat{\otimes}_\pi H_2)^*}\langle T, l\rangle_{H_1 \hat{\otimes}_\pi H_2}, \quad \text{for all } l \in H_1 \hat{\otimes}_\pi H_2.$$

We show now the following.

(i) $T_n(a, b) \xrightarrow{n \to \infty} T(a, b)$ for all $a \in H_1$, $b \in H_2$.

(ii) For a fixed $l \in H_1 \hat{\otimes}_\pi H_2$, the sequence $T_n(l)$ is bounded.

Let us prove first *(i)*. Let $a \in H_1$ and $b \in H_2$. We write

$$T_n(a, b) = T\left(\sum_{i=1}^n \langle a, e_i\rangle_{H_1} e_i, \quad \sum_{i=1}^n \langle b, f_i\rangle_{H_2} f_i\right). \tag{39}$$

Since

$$\sum_{i=1}^n \langle a, e_i\rangle e_i \xrightarrow{n \to +\infty} a \quad \text{in } H_1,$$
$$\sum_{i=1}^n \langle b, f_i\rangle f_i \xrightarrow{n \to +\infty} b \quad \text{in } H_2,$$

and T is a bounded bilinear form the point (i) follows.

Let us prove now *(ii)*. Let $\epsilon > 0$ fixed and $l_0 \in \mathcal{D}$ such that $|l - l_0|_{H_1 \hat{\otimes}_\pi H_2} \le \epsilon$. Then

$$|T_n(l)| \le |T_n(l - l_0)| + |T_n(l_0)| \le |T_n|_{(H_1 \hat{\otimes}_\pi H_2)^*} |l - l_0|_{H_1 \hat{\otimes}_\pi H_2} + |T_n(l_0)|. \tag{40}$$

So (40) is bounded by

$$\sup_{|a|_{H_1},|b|_{H_2}\le 1} \sum_{i=1}^{n} |\langle a, e_i\rangle e_i|_{H_1} |\langle b, f_i\rangle f_i|_{H_2} |T|_{(H_1\hat{\otimes}_\pi H_2)^*}\epsilon + \sup_n |T_n(l_0)| \le |T|_{(H_1\hat{\otimes}_\pi H_2)^*}\epsilon + \sup_n |T_n(l_0)|,$$

recalling that the sequence $(T_n(l_0))$ is bounded, since it is convergent. Finally (ii) is also proved. At this point (i) implies that

$${}_{(H_1\hat{\otimes}_\pi H_2)^*}\big\langle T_n, l\big\rangle_{H_1\hat{\otimes}_\pi H_2} \xrightarrow{n\to\infty} {}_{(H_1\hat{\otimes}_\pi H_2)^*}\big\langle T, l\big\rangle_{H_1\hat{\otimes}_\pi H_2}, \quad \text{for all } l \in \mathcal{D}.$$

Since $\mathcal{D}$ is dense in $H_1\hat{\otimes}_\pi H_2$, the conclusion follows by Banach-Steinhaus theorem, see Theorem 18, Chapter II in [24]. □

We recall the following fact that concerns the classical tensor covariation.

Lemma 4.17. *Let $\mathbb{W}_Q$ be a Q-Wiener process as in Subsection 3.1. Let $\Psi\colon ([s,T]\times\Omega, \mathscr{P}) \to \mathcal{L}_2(U_0, H)$ be a strongly measurable process satisfying condition (14), with $X = \Psi$. Consider the local martingale*

$$\mathbb{M}(t) := \int_s^t \Psi(r)\,\mathrm{d}\mathbb{W}_Q(r).$$

Then

$$[\mathbb{M},\mathbb{M}]^{cl}(t) = \int_s^t g(r)\,\mathrm{d}r,$$

where $g(r)$ is the element of $H\hat{\otimes}_\pi H$ associated with the nuclear operator $G_g(r) := \Big(\Psi(r)Q^{1/2}\Big)\Big(\Psi(r)Q^{1/2}\Big)^*$.

Proof. See [10] Section 4.7. □

Lemma 4.18. *Let $\mathbb{M}\colon [s,T]\times\Omega \to H$ be a continuous local martingale and $\mathbb{Z}$ a measurable process from $([s,T]\times\Omega, \mathscr{P})$ to H and such that $\int_s^T \|\mathbb{Z}(r)\|^2\,\mathrm{d}[\mathbb{M}]^{\mathbb{R},cl}(r) < +\infty$ a.s. Of course $\mathbb{Z}$ can be Riesz-identified with an element of $\mathcal{J}^2(s,T;H^*,\mathbb{R})$. We define*

$$X(t) := \int_s^t \langle \mathbb{Z}(r),\, \mathrm{d}\mathbb{M}(r)\rangle\,. \tag{41}$$

Then, X is a real continuous local martingale and for every continuous real local martingale N, the (classical, one-dimensional) covariation process $[X,N]^{cl}$ is given by

$$[X,N]^{cl}(t) = \int_s^t \big\langle \mathbb{Z}(r),\, \mathrm{d}[\mathbb{M},N]^{cl}(r)\big\rangle\,; \tag{42}$$

in particular the integral in the right-side is well-defined.

Proof. The fact that X is a local martingale is part of the result of Theorem 2.14 in [42]. For the other claim we can reduce, using a sequence of suitable stopping times as in the proof of Theorem 3.6, to the case in which $\mathbb{Z}$, $\mathbb{M}$ and N are square integrable martingales.Taking into account the characterization (32) and the discussion developed in [50], page 456, (42) follows. □

Proposition 4.19. *If $\mathbb{M}\colon [s,T]\times\Omega \to H$ and $N\colon [s,T]\times\Omega \to \mathbb{R}$ are continuous local martingales. Then $\mathbb{M}$ and N admit a global covariation and $\widetilde{[\mathbb{M},N]} = [\mathbb{M},N]^{cl}$.*

Proof. We have to check the conditions stated in Proposition 4.12 for $\tilde{C}$ equal to the right side of (33). Concerning (a), by Cauchy-Schwarz inequality we have

$$\frac{1}{\epsilon}\int_s^T |N(r+\epsilon)-N(r)||\mathbb{M}(r+\epsilon)-\mathbb{M}(r)|\,\mathrm{d}r \leq [N,N]^{\epsilon,\mathbb{R}}[\mathbb{M},\mathbb{M}]^{\epsilon,\mathbb{R}}.$$

Since both N and $\mathbb{M}$ are local martingales they admit a scalar quadratic variation (as recalled in Remark 4.14), the result is established. Concerning (b), taking into account (32), we need to prove that for any $h \in H$

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int_s^{\cdot} (M^h(r+\epsilon)-M^h(r))(N(r+\epsilon)-N(r))\,\mathrm{d}r = [\langle M,h\rangle, N]^{cl} \tag{43}$$

ucp, where M^h is the real local martingale $\langle \mathbb{M}, h\rangle$. (43) follows by Proposition 2.4(3) of [58]. □

4.3 χ-Dirichlet and ν-weak Dirichlet processes

We have now at our disposal all the elements we need to introduce the concept of χ-Dirichlet process and ν-weak Dirichlet process.

Definition 4.20. *Let $\chi \subseteq (H\hat{\otimes}_\pi H)^*$ be a Chi-subspace. A continuous H-valued process $\mathbb{X}\colon ([s,T]\times\Omega,\mathscr{P}) \to H$ is called* χ-Dirichlet process *if there exists a decomposition $\mathbb{X} = \mathbb{M} + \mathbb{A}$ where*

(i) $\mathbb{M}$ *is a continuous local martingale,*

(ii) $\mathbb{A}$ *is a continuous χ-zero quadratic variation process,*

(iii) $\mathbb{A}(0) = 0$.

Definition 4.21. *Let H and H_1 be two separable Hilbert spaces. Let $\nu \subseteq (H\hat{\otimes}_\pi H_1)^*$ be a Chi-subspace. A continuous adapted H-valued process $\mathbb{A}\colon [s,T]\times\Omega \to H$ is said to be $\mathscr{F}^s_t$-ν-*martingale-orthogonal *if*

$$[\mathbb{A},\mathbb{N}]_\nu = 0,$$

for any H_1-valued continuous local martingale $\mathbb{N}$.

As we have done for the expressions “stopping time”, ”adapted”, “predictable”... since we always use the filtration $\mathscr{F}^s_t$, we simply write *ν-martingale-orthogonal* instead of $\mathscr{F}^s_t$-ν-martingale-orthogonal.

Definition 4.22. *Let H and H_1 be two separable Hilbert spaces. Let $\nu \subseteq (H\hat{\otimes}_\pi H_1)^*$ be a Chi-subspace. A continuous H-valued process $\mathbb{X}\colon [s,T]\times\Omega \to H$ is called* ν-weak-Dirichlet process *if it is adapted and there exists a decomposition $\mathbb{X} = \mathbb{M} + \mathbb{A}$ where*

(i) $\mathbb{M}$ *is an H-valued continuous local martingale,*

(ii) $\mathbb{A}$ *is an ν-martingale-orthogonal process,*

(iii) $\mathbb{A}(0) = 0$.

Remark 4.23. *The sum of two ν-martingale-orthogonal processes is again a ν-martingale-orthogonal process.*

Proposition 4.24. *1. Any process admitting a zero scalar quadratic variation (for instance a bounded variation process) is a ν-martingale-orthogonal process.*

2. Let $\mathbb{Q}$ be an equivalent probability to $\mathbb{P}$. Any ν-weak Dirichlet process under $\mathbb{P}$ is a ν-weak Dirichlet process under $\mathbb{Q}$.

Proof. 1. follows from Proposition 4.11 1. setting $\nu_1 = H_1^*$. In fact, any local martingale has a global quadratic variation because of Remark 4.8 and Remark 4.14. So it has a $\nu_1 \hat{\otimes}_\pi \nu_1$-quadratic variation by Remark 4.2 and Remark 4.9.
Concerning 2., by Theorem in Section 30.3, page 208 of [48], a local martingale under $\mathbb{P}$ is a local martingale under $\mathbb{Q}$ plus a bounded variation process. The result follows by Proposition 1.5 item 1. and Remark 4.23 since the ν-covariation remains unchanged under an equivalent probability measure. □

We recall that the decomposition of a real weak Dirichlet process is unique, see Remark 3.5 of [40]. For the infinite dimensional case we now establish the uniqueness of the decomposition of a ν-weak-Dirichlet process in two cases: when $H_1 = H$ and when $H_1 = \mathbb{R}$.

Proposition 4.25. *Let $\nu \subseteq (H\hat{\otimes}_\pi H)^*$ be a Chi-subspace. Suppose that ν is dense in $(H\hat{\otimes}_\pi H)^*$. Then any decomposition of a ν-weak-Dirichlet process $\mathbb{X}$ is unique.*

Proof. Assume that $\mathbb{X} = \mathbb{M}^1 + \mathbb{A}^1 = \mathbb{M}^2 + \mathbb{A}^2$ are two decompositions where $\mathbb{M}^1$ and $\mathbb{M}^2$ are continuous local martingales and $\mathbb{A}^1, \mathbb{A}^2$ are ν-martingale-orthogonal processes. If we call $\mathbb{M} := \mathbb{M}^1 - \mathbb{M}^2$ and $\mathbb{A} := \mathbb{A}^1 - \mathbb{A}^2$ we have $0 = \mathbb{M} + \mathbb{A}$.

By Lemma 4.15, $\mathbb{M}$ has a global quadratic variation. In particular it also has a ν-quadratic variation and, thanks to the bilinearity of the ν-covariation,

$$0 = [\mathbb{M}, 0]_\nu = [\mathbb{M}, \mathbb{M} + \mathbb{A}]_\nu = [\mathbb{M}, \mathbb{M}]_\nu + [\mathbb{M}, \mathbb{A}]_\nu = [\mathbb{M}, \mathbb{M}]_\nu + 0 = [\mathbb{M}, \mathbb{M}]_\nu .$$

We prove now that $\mathbb{M}$ has also zero global quadratic variation. We have denoted by $\mathcal{C}([s,T])$ the space of the real continuous processes defined on $[s,T]$. We introduce, for $\epsilon > 0$, the operators

$$\begin{cases} [\mathbb{M}, \mathbb{M}]^\epsilon \colon (H\hat{\otimes}_\pi H)^* \to \mathcal{C}([s,T]) \\ ([\mathbb{M}, \mathbb{M}]^\epsilon(\phi))(t) := \dfrac{1}{\epsilon} \displaystyle\int_s^t {}_{(H\hat{\otimes}_\pi H)}\big\langle (\mathbb{M}_{r+\epsilon} - \mathbb{M}_r)\otimes^2, \phi \big\rangle_{(H\hat{\otimes}_\pi H)^*} \, \mathrm{d}r. \end{cases} \tag{44}$$

Observe the following

(a) $[\mathbb{M}, \mathbb{M}]^\epsilon$ are linear and bounded operators.

(b) For $\phi \in (H\hat{\otimes}_\pi H)^*$ the limit $[\mathbb{M}, \mathbb{M}](\phi) := \lim_{\epsilon \to 0} [\mathbb{M}, \mathbb{M}]^\epsilon(\phi)$ exists.

(c) If $\phi \in \nu$ we have $[\mathbb{M}, \mathbb{M}](\phi) = 0$.

Thanks to (a) and (b) and Banach-Steinhaus theorem (see Theorem 17, Chapter II in [24]) we know that $[\mathbb{M}, \mathbb{M}]$ is linear and bounded. Thanks to (c) and the fact that the inclusion $\nu \subseteq (H\hat{\otimes}_\pi H)^*$ is dense, it follows $[\mathbb{M}, \mathbb{M}] = 0$. By Lemma 4.15 $[\mathbb{M}, \mathbb{M}]$ coincides with the classical quadratic variation $[\mathbb{M}, \mathbb{M}]^{cl}$ and it is characterized by

$$0 = [\mathbb{M}, \mathbb{M}]^{cl}(h^*, k^*) = [\langle \mathbb{M}, h \rangle, \langle \mathbb{M}, k \rangle]^{cl},$$

by Remark 4.13 (ii). Since $\mathbb{M}(0) = 0$ and therefore $\langle \mathbb{M}, h \rangle(0) = 0$ it follows that $\langle \mathbb{M}, h \rangle \equiv 0$ for any $h \in H$. Finally $\mathbb{M} \equiv 0$, which concludes the proof. □

Proposition 4.26. *Let H be a separable Hilbert space. Let $\nu \subseteq H^* \equiv (H\hat{\otimes}_\pi \mathbb{R})^*$ be a Banach space with continuous and dense inclusion. Then any decomposition of a ν-weak-Dirichlet process $\mathbb{X}$ with values in H is unique.*

Remark 4.27. *Taking into account the identification of H^* with $(H\hat{\otimes}_\pi \mathbb{R})^*$ it is possible to consider ν as a dense subset of $(H\hat{\otimes}_\pi \mathbb{R})^*$ which is a Chi-subspace.*

Proof of Proposition 4.26. We denote again the inner product on H by $\langle \cdot, \cdot \rangle$. We show that the unique decomposition of the 0 process is trivial. Assume that $0 = \mathbb{X} = \mathbb{M} + \mathbb{A}$. Since ν is dense in H^* it is possible to choose an orthonormal basis (e_i^*) in ν. We introduce $M^i := \langle \mathbb{M}, e_i\rangle$, they are continuous real local martingales and then, thanks to the properties of $\mathbb{A}$ we have

$$0 = [\mathbb{X}, M^i]_\nu = [\mathbb{M}, M^i]_\nu + [\mathbb{A}, M^i]_\nu = [\mathbb{M}, M^i]_\nu.$$

By Remark 4.9, Proposition 4.19 and Remark 4.13 (iii) we know that

$$\left\langle [\mathbb{M}, M^i]_\nu, e_i \right\rangle = \left\langle [\mathbb{M}, M^i]^{cl}, e_i \right\rangle = [M^i, M^i]^{cl};$$

so $M^i = 0$ for all i and then $\mathbb{M} = 0$. This concludes the proof. □

Proposition 4.28. *Let H and H_1 be two separable Hilbert spaces. Let $\chi = \chi_0 \hat{\otimes}_\pi \chi_0$ for some χ_0 Banach space continuously embedded in H^*. Define $\nu = \chi_0 \hat{\otimes}_\pi H_1^*$. Then an H-valued continuous zero χ-quadratic variation process $\mathbb{A}$ is a ν-martingale-orthogonal process.*

Proof. Taking into account Lemma 4.1, χ is a Chi-subspace of $(H\hat{\otimes}_\pi H)^*$ and ν is a Chi-subspace of $(H\hat{\otimes}_\pi H_1)^*$. Let $\mathbb{N}$ be a continuous local martingale with values in H_1. We need to show that $[\mathbb{A}, \mathbb{N}]_\nu = 0$. We consider the random maps $T^\epsilon \colon \nu \times \Omega \to \mathcal{C}([s, T])$ defined by

$$T^\epsilon(\phi) := [\mathbb{A}, \mathbb{N}]^\epsilon_\nu(\phi) = \frac{1}{\epsilon}\int_s^{\cdot} {}_{\nu^*}\langle (\mathbb{A}(r+\epsilon) - \mathbb{A}(r)) \otimes (\mathbb{N}(r+\epsilon) - \mathbb{N}(r)), \phi\rangle_\nu \,\mathrm{d}r,$$

for $\phi \in \nu$.

Step 1:

Suppose that $\phi = h^* \otimes k^*$ for $h^* \in \chi_0$ and $k \in H_1$. Then

$$\begin{aligned} T^\epsilon(\phi)(t) &= \frac{1}{\epsilon}\int_s^t {}_{\chi_0^*}\langle (\mathbb{A}(r+\epsilon) - \mathbb{A}(r)), h^*\rangle_{\chi_0} \, \langle (\mathbb{N}(r+\epsilon) - \mathbb{N}(r)), k\rangle_{H_1} \,\mathrm{d}r \\ &\le \left[\frac{1}{\epsilon}\int_s^t {}_{\chi_0^*}\langle (\mathbb{A}(r+\epsilon) - \mathbb{A}(r)), h^*\rangle^2_{\chi_0} \,\mathrm{d}r \frac{1}{\epsilon}\int_s^t \langle (\mathbb{N}(r+\epsilon) - \mathbb{N}(r)), k\rangle^2_{H_1} \,\mathrm{d}r\right]^{1/2} \\ &= \left[\frac{1}{\epsilon}\int_s^t {}_{\chi^*}\left\langle (\mathbb{A}(r+\epsilon) - \mathbb{A}(r))\otimes^2, h^* \otimes h^*\right\rangle_\chi \,\mathrm{d}r\right]^{\frac{1}{2}} \\ &\qquad \times \left[\frac{1}{\epsilon}\int_s^t \langle (\mathbb{N}(r+\epsilon) - \mathbb{N}(r)), k\rangle^2_{H_1} \,\mathrm{d}r\right]^{\frac{1}{2}}, \end{aligned} \tag{45}$$

that converges ucp to

$$\left([\mathbb{A}, \mathbb{A}](t)(h^* \otimes h^*)[\mathbb{N}, \mathbb{N}]^{\mathrm{cl}}(t)(k^* \otimes k^*)\right)^{1/2} = 0,$$

since the quadratic quadratic variation of a local martingale is the classical one and taking into account item (ii) of Remark 4.13.

Step 2:

We denote by $\mathcal{D}$ the linear combinations of elements of the form $h^* \otimes k^*$ for $h^* \in \chi_0$ and $k \in H_1$. We remark that $\mathcal{D}$ is dense in ν. From the convergence found in *Step 1*, it follows that, for every $\phi \in \mathcal{D}$, ucp we have

$$T^\epsilon(\phi) \xrightarrow{\epsilon \to 0} 0.$$

Step 3:

We consider a generic $\phi \in \nu$. By Lemma 2.2, for $t \in [s,T]$ it follows

$$
\begin{aligned}
|T^\epsilon(\phi)(t)| \le |\phi|_\nu \int_s^t \frac{|(\mathbb{A}(r+\epsilon)-\mathbb{A}(r)) \otimes (\mathbb{N}(r+\epsilon)-\mathbb{N}(r))|_{\nu^*}}{\epsilon} \mathrm{d}r \\
= |\phi|_\nu \frac{1}{\epsilon} \int_s^t |(\mathbb{N}(r+\epsilon)-\mathbb{N}(r))|_{H_1} |(\mathbb{A}(r+\epsilon)-\mathbb{A}(r))|_{\chi_0^*} \mathrm{d}r \\
\le |\phi|_\nu \left(\frac{1}{\epsilon} \int_s^t |(\mathbb{N}(r+\epsilon)-\mathbb{N}(r))|^2_{H_1} \mathrm{d}r \, \frac{1}{\epsilon} \int_s^t |(\mathbb{A}(r+\epsilon)-\mathbb{A}(r))|^2_{\chi_0^*} \mathrm{d}r \right)^{\frac{1}{2}} \\
= |\phi|_\nu \left(\frac{1}{\epsilon} \int_s^t |(\mathbb{N}(r+\epsilon)-\mathbb{N}(r))|^2_{H_1} \mathrm{d}r \times \frac{1}{\epsilon} \int_s^t \left|(\mathbb{A}(r+\epsilon)-\mathbb{A}(r))\otimes^2\right|_{\chi^*} \mathrm{d}r \right)^{\frac{1}{2}}. \quad (46)
\end{aligned}
$$

To prove that $[\mathbb{A},\mathbb{N}]_\nu = 0$ we check the corresponding conditions **H1** and **H2** of the Definition 4.4. By Lemma 4.15 we know that $\mathbb{N}$ admits a global quadratic variation i.e. a $(H_1 \otimes H_1)^*$-quadratic variation. By condition **H1** of the Definition 4.4 related to $(H_1 \otimes H_1)^*$-quadratic variation for the process $\mathbb{N}$ and the χ-quadratic variation of $\mathbb{A}$, for any sequence (ϵ_n) converging to zero, there is a subsequence (ϵ_{n_k}) such that the sequence $T^{\epsilon_{n_k}}(\phi)$ is bounded for any ϕ in the $\mathcal{C}[s,T]$ metric a.s. condition **H1** of the ν-covariation. By Banach-Steinhaus for F-spaces (Theorem 17, Chapter II in [24]) it follows that $T^\epsilon(\phi) \xrightarrow{\epsilon \to 0} 0$ ucp for all $\phi \in \nu$ and so condition **H2** and the final result follows. □

Corollary 4.29. *Assume that the hypotheses of Proposition 4.28 are satisfied. If $\mathbb{X}$ is a χ-Dirichlet process then we have the following.*

(i) $\mathbb{X}$ is a ν-weak-Dirichlet process.

(ii) $\mathbb{X}$ is a χ-weak Dirichlet process.

(iii) $\mathbb{X}$ is a χ-finite-quadratic-variation process.

Proof. (i) follows by Proposition 4.28.
As far as (ii) is concerned, let $\mathbb{X} = \mathbb{M} + \mathbb{A}$ be a χ-Dirichlet process decomposition, where $\mathbb{M}$ is a local martingale. Setting $H_1 = H$, then χ is included in ν, so Proposition 4.28 implies that $\mathbb{A}$ is a χ-orthogonal process and so (ii) follows.
We prove now (iii). By Lemma 4.15 and Remark 4.9 $\mathbb{M}$ admits a χ-quadratic variation. By the bilinearity of the χ-covariation, it is enough to show that $[\mathbb{M},\mathbb{A}]_\chi = 0$. This follows from item (ii). □

Proposition 4.30. *Let B_1 and B_2 be two Banach spaces and χ a Chi-subspace of $(B_1 \hat{\otimes}_\pi B_2)^*$. Let $\mathbb{X}$ and $\mathbb{Y}$ be two stochastic processes with values respectively in B_1 and B_2 such that $(\mathbb{X},\mathbb{Y})$ admits a χ-covariation. Let G be a continuous measurable process $G : [s,T] \times \Omega \to \mathcal{K}$ where $\mathcal{K}$ is a closed separable subspace of χ. Then for every $t \in [s,T]$*

$$
\int_s^t {}_{\chi}\langle G(\cdot,r), [\mathbb{X},\mathbb{Y}]^\epsilon(\cdot,r)\rangle_{\chi^*} \mathrm{d}r \xrightarrow[\epsilon \to 0]{} \int_s^t {}_{\chi}\langle G(\cdot,r), d\widetilde{[\mathbb{X},\mathbb{Y}]}(\cdot,r)\rangle_{\chi^*} \quad (47)
$$

in probability.

Proof. See [17] Proposition 3.7. □

We state below the most important result related to the stochastic calculus part of the paper. It generalizes the finite dimensional result contained in [40] Theorem 4.14. The definition of real weak Dirichlet process is recalled in Definition 1.1.

Theorem 4.31. *Let ν_0 be a Banach subspace continuously embedded in H^*. Define $\nu := \nu_0 \hat{\otimes}_\pi \mathbb{R}$ and $\chi := \nu_0 \hat{\otimes}_\pi \nu_0$. Let $F\colon [s,T] \times H \to \mathbb{R}$ be a $C^{0,1}$-function. Denote with $\partial_x F$ the Frechet derivative of F w.r.t. x and assume that the mapping $(t,x) \mapsto \partial_x F(t,x)$ is continuous from $[s,T] \times H$ to ν_0. Let $\mathbb{X}(t) = \mathbb{M}(t) + \mathbb{A}(t)$ for $t \in [s,T]$ be an ν-weak-Dirichlet process with finite χ-quadratic variation. Then $Y(t) := F(t, \mathbb{X}(t))$ is a (real) weak Dirichlet process with local martingale part*

$$R(t) = F(s, \mathbb{X}(s)) + \int_s^t \langle \partial_x F(r, \mathbb{X}(r)),\, \mathrm{d}\mathbb{M}(r) \rangle .$$

Remark 4.32. *Indeed the condition $\mathbb{X}$ having a χ-quadratic variation may be replaced with the weaker condition of $(\mathbb{X}, M)$ having a $\nu_0 \hat{\otimes}_\pi \mathbb{R}$-covariation for any real continuous local martingale M.*

Proof of Theorem 4.31. By definition $\mathbb{X}$ can be written as the sum of a continuous local martingale $\mathbb{M}$ and a ν-martingale-orthogonal process $\mathbb{A}$.

Let N be a real-valued local martingale. Taking into account Lemma 4.18 and that the covariation of two real local martingales defined in (1), coincides with the classical covariation, it is enough to prove that

$$[F(\cdot, \mathbb{X}(\cdot)), N](t) = \int_s^t \left\langle \partial_x F(r, \mathbb{X}(r)),\, \mathrm{d}[\mathbb{M}, N]^{cl}(r) \right\rangle , \qquad \text{for all } t \in [s,T].$$

Let $t \in [s,T]$. We evaluate the ϵ-approximation of the covariation, i.e.

$$\frac{1}{\epsilon} \int_s^t \left(F(r+\epsilon, \mathbb{X}(r+\epsilon)) - F(r, \mathbb{X}(r)) \right) \left(N(r+\epsilon) - N(r) \right) \mathrm{d}r.$$

It equals

$$I_1(t,\epsilon) + I_2(t,\epsilon),$$

where

$$I_1(t,\epsilon) = \int_s^t \left(F(r+\epsilon, \mathbb{X}(r+\epsilon)) - F(r+\epsilon, \mathbb{X}(r)) \right) \frac{(N(r+\epsilon) - N(r))}{\epsilon} \,\mathrm{d}r$$

and

$$I_2(t,\epsilon) = \int_s^t \left(F(r+\epsilon, \mathbb{X}(r)) - F(r, \mathbb{X}(r)) \right) \frac{(N(r+\epsilon) - N(r))}{\epsilon} \,\mathrm{d}r.$$

We prove now that

$$I_1(t,\epsilon) \xrightarrow{\epsilon \to 0} \int_s^t \left\langle \partial_x F(r, \mathbb{X}(r)),\, \mathrm{d}[\mathbb{M}, N]^{cl}(r) \right\rangle \tag{48}$$

in probability; in fact

$$I_1(t,\epsilon) = I_{11}(t,\epsilon) + I_{12}(t,\epsilon)$$

where

$$I_{11}(t,\epsilon) := \int_s^t \frac{1}{\epsilon} \langle \partial_x F(r, \mathbb{X}(r)), \mathbb{X}(r+\epsilon) - \mathbb{X}(r) \rangle \left(N(r+\epsilon) - N(r) \right) \mathrm{d}r,$$

$$\begin{aligned} I_{12}(t,\epsilon) := \int_0^1 \int_s^t \frac{1}{\epsilon} \langle \partial_x F(r+\epsilon, a\mathbb{X}(r) + (1-a)\mathbb{X}(r+\epsilon)) - \partial_x F(r, \mathbb{X}(r)), \\ \mathbb{X}(r+\epsilon) - \mathbb{X}(r) \rangle \left(N(r+\epsilon) - N(r) \right) \mathrm{d}r \,\mathrm{d}a. \end{aligned}$$

Now we apply Proposition 4.30 with $B_1 = H$, $B_2 = \mathbb{R}$, $\mathbb{X} = \mathbb{M}$, $\mathbb{Y} = N$, $\chi = \nu$ so that

$$I_{11}(t,\epsilon) \xrightarrow{\epsilon \to 0} \int_s^t {}_{\nu}\left\langle \partial_x F(r, \mathbb{X}(r)),\, \mathrm{d}\widetilde{[\mathbb{X}, N]}(r) \right\rangle_{\nu^*} . \tag{49}$$

Recalling that $\mathbb{X} = \mathbb{M} + \mathbb{A}$, we remark that $[\mathbb{X}, N]_\nu$ exists and the ν^*-valued process $\widetilde{[\mathbb{X}, N]}_\nu$ equals

$$\widetilde{[\mathbb{M}, N]}_\nu + \widetilde{[\mathbb{A}, N]}_\nu = \widetilde{[\mathbb{M}, N]}_\nu,$$

since $\mathbb{A}$ is a ν-martingale orthogonal process. Taking into account the formalism of Proposition 4.12, Remark 4.9 and Proposition 4.19 if $\Phi \in H \equiv H^*$, we have

$$\begin{aligned} {}_\nu\left\langle \Phi, \widetilde{[\mathbb{M}, N]}_\nu \right\rangle_{\nu^*} &= {}_{\nu_0}\left\langle \Phi, \widetilde{[\mathbb{M}, N]}_\nu \right\rangle_{\nu_0^*} = {}_{H^*}\left\langle \Phi, \widetilde{[\mathbb{M}, N]} \right\rangle_{H^{**}} \\ &= {}_{H^*}\left\langle \Phi, [\mathbb{M}, N]^{cl} \right\rangle_H . \end{aligned}$$

Consequently, it is not difficult to show that the right-hand side of (49) gives

$$\int_s^t {}_{H^*}\left\langle \partial_x F(r, \mathbb{X}(r)),\, \mathrm{d}[\mathbb{M}, N]^{cl}(r) \right\rangle_H .$$

For a fixed $\omega \in \Omega$ we consider the function $\partial_x F$ restricted to $[s, T] \times K$ where K is the (compact) subset of H obtained as convex hull of $\{a\mathbb{X}(r_1) + (1-a)\mathbb{X}(r_2) \ : \ r_1, r_2 \in [s, T]\}$. $\partial_x F$ restricted to $[s, T] \times K$ is uniformly continuous with values in ν_0. Consequently, for ω-a.s.

$$|I_{12}(t,\epsilon)| \leq \int_s^T \delta\left(\partial_x F_{|[s,T]\times K}; \epsilon + \sup_{|r-t|\leq\epsilon} |\mathbb{X}(r) - \mathbb{X}(t)|_{\nu_0^*} \right) \times |\mathbb{X}(r+\epsilon) - \mathbb{X}(r)|_{\nu_0^*} \frac{1}{\epsilon} |N(r+\epsilon) - N(r)| \,\mathrm{d}r, \tag{50}$$

where, for a uniformly continuous function $g : [s, T] \times K \to \nu_0$, $\delta(g; \epsilon)$ is the modulus of continuity $\delta(g;\epsilon) := \sup_{|x-y|\leq\epsilon} |g(x) - g(y)|_{\nu_0}$. In previous formula we have identified H with H^{**} so that $|x|_H \leq |x|_{\nu_0^*}, \forall x \in H$. So (50) is lower than

$$\begin{aligned} &\delta\left(\partial_x F_{|[s,T]\times K}; \epsilon + \sup_{|s-t|\leq\epsilon} |\mathbb{X}(s) - \mathbb{X}(t)|_{\nu_0^*} \right) \times \left(\int_s^T \frac{1}{\epsilon} |N(r+\epsilon) - N(r)|^2 \,\mathrm{d}r \int_s^T \frac{1}{\epsilon} |(\mathbb{X}(r+\epsilon) - \mathbb{X}(r))|^2_{\nu_0^*} \,\mathrm{d}r \right)^{1/2} \\ &= \delta\left(\partial_x F_{|[s,T]\times K}; \epsilon + \sup_{|s-t|\leq\epsilon} |\mathbb{X}(s) - \mathbb{X}(t)|_{\nu_0^*} \right) \times \left(\int_s^T \frac{1}{\epsilon} |N(r+\epsilon) - N(r)|^2 \,\mathrm{d}r \int_s^T \frac{1}{\epsilon} |(\mathbb{X}(r+\epsilon) - \mathbb{X}(r)) \otimes^2 |_{\chi^*} \,\mathrm{d}r \right)^{1/2}, \end{aligned} \tag{51}$$

when $\varepsilon \to 0$, where we have used Lemma 2.2, for $\alpha = \pi$ with the usual identification. The right-hand side of (51), of course converges to zero, since $\mathbb{X}$ [resp. N] is a χ-finite quadratic variation process [resp. a real finite quadratic variation process] and X is also continuous as a ν_0^*-valued process.

To conclude the proof of the proposition we only need to show that $I_2(t, \epsilon) \xrightarrow{\epsilon\to 0} 0$. This is relatively simple since

$$I_2(t,\epsilon) = \frac{1}{\epsilon} \int_s^t \Gamma(u,\epsilon) \,\mathrm{d}N(u) + R(t,\epsilon),$$

where $R(t, \epsilon)$ is a boundary term s.t. $R(t, \epsilon) \xrightarrow{\epsilon\to 0} 0$ in probability and

$$\Gamma(u,\epsilon) = \frac{1}{\epsilon} \int_{(u-\epsilon)_+}^u \left(F(r+\epsilon, \mathbb{X}(r)) - F(r, \mathbb{X}(r)) \right) \mathrm{d}r.$$

Since

$$\int_s^T (\Gamma(u,\epsilon))^2 \,\mathrm{d}[N](u) \to 0,$$

in probability, Problem 2.27, chapter 3 of [41] implies that $I_2(\cdot, \epsilon) \to 0$ ucp. The result finally follows. $\square$

5 The case of stochastic infinite dimensional stochastic differential equations

This section concerns applications of the stochastic calculus via regularization to mild solutions of infinite dimensional stochastic differential equations.

Assume, as in Subsection 3.1 that H and U are real separable Hilbert spaces, $Q \in \mathcal{L}(U)$, $U_0 := Q^{1/2}(U)$. Assume that $\mathbb{W}_Q = \{\mathbb{W}_Q(t) : s \leq t \leq T\}$ is an U-valued $\mathscr{F}_s^t$-Q-Wiener process (with $\mathbb{W}_Q(s) = 0$, $\mathbb{P}$ a.s.) and denote by $\mathcal{L}_2(U_0, H)$ the Hilbert space of the Hilbert-Schmidt operators from U_0 to H. We adopt the conventions of the mentioned subsection.

We denote by $A\colon D(A) \subseteq H \to H$ the generator of the C_0-semigroup e^{tA} (for $t \geq 0$) on H. The reader may consult for instance [4] Part II, Chapter 1 for basic properties of C_0-semigroups. A^* denotes the adjoint of A, $D(A)$ and $D(A^*)$ are Banach (even Hilbert) spaces when endowed with the graph norm.

Let b be a predictable process with values in H and σ be a predictable process with values in $\mathcal{L}_2(U_0, H)$ such that

$$\mathbb{P}\left[\int_0^T |b(t)| + \|\sigma(t)\|^2_{\mathcal{L}_2(U_0,H)}\,\mathrm{d}t < +\infty\right] = 1. \tag{52}$$

We introduce the process

$$\mathbb{X}(t) = e^{(t-s)A}x + \int_s^t e^{(t-r)A} b(r)\,\mathrm{d}r + \int_s^t e^{(t-r)A}\sigma(r)\,\mathrm{d}\mathbb{W}_Q(r). \tag{53}$$

Remark 5.1. *1. A mild solution to an equation of type* (4) *is a particular case of* (53)*. Indeed, once existence and uniqueness of the solution* $\mathbb{X}(\cdot)$ *of* (53) *is proved, we can simply take* $b(r) = b(r, \mathbb{X}(r))$ *and* $\sigma(r) = \sigma(r, \mathbb{X}(r))$.

2. *Typical examples of SPDEs that can be rewritten in the form* (4) *(see e.g. [11] Part III) are for example stochastic heat (and more general parabolic) equations of the form* $dy(s,\xi) = [\Delta_\xi y(s,\xi) + b(s, y(s,\xi))]\,\mathrm{d}s + \sigma(s, y(s,\xi))dW_Q(s)(\xi)$ *with zero Dirichlet or Neumann boundary conditions (and a suitable initial datum). There are also infinite-dimensional reformulations for heat equations with time-dependent boundary terms and/or boundary noise. Here we use the abstract and general formulation with a generic generator of a* C_0*-semigroup* A*, so we do not use regularizing properties that are typical of the heat semigroup. Other classes of SPDE can be rewritten in the same setting.*

3. *A different example arises for instance from stochastic delay differential equations. The following simple example (but more general setting can be studied, see also Chapter 10 in [11]), is*

$$\mathrm{d}y(s) = \left(a_0 y(s) + \int_{-R}^0 a_1(r) y(s+r)\,\mathrm{d}r\right)\,\mathrm{d}s + \sigma\,\mathrm{d}W_0(s), \tag{54}$$

(coupled with the initial datum) where $R > 0$ *is a positive real constant,* a_0 *and* σ *are real numbers,* $a_1(r)$ *an element of* $L^2(-R, 0)$ *and* W_0 *a real Brownian motion. It is reformulated for instance in the infinite dimension abstract setting in [37].*

We define

$$\mathbb{Y}(t) := \mathbb{X}(t) - \int_s^t b(r)\,\mathrm{d}r - \int_s^t \sigma(r)\,\mathrm{d}\mathbb{W}_Q(r) - x. \tag{55}$$

Lemma 5.2. *Let* b *[resp.* σ*] be a predictable process with values in* H *[resp. with values in* $\mathcal{L}_2(U_0, H)$*] such that (52) is satisfied. Let* $\mathbb{X}(t)$ *be defined by (53) and* $\mathbb{Y}$ *defined by (55). If* $z \in D(A^*)$ *we have*

$$\langle \mathbb{Y}(t), z \rangle = \int_s^t \langle \mathbb{X}(r), A^* z\rangle\,\mathrm{d}r. \tag{56}$$

Proof. See [53] Theorem 12. □

We want now to prove that $\mathbb{Y}$ has zero-χ-quadratic variation for a suitable space χ. We will see that the space

$$\bar{\chi} := D(A^*)\hat{\otimes}_\pi D(A^*). \tag{57}$$

does the job. We set $\bar{\nu}_0 := D(A^*)$ which is clearly continuously embedded into H^*.

By Lemma 4.1, $\bar{\chi}$ is a Chi-subspace of $(H\hat{\otimes}_\pi H)^*$.

Proposition 5.3. *The process $\mathbb{Y}$ has zero $\bar{\chi}$-quadratic variation.*

Proof. Observe that, thank to Lemma 3.18 in [18] it will be enough to show that

$$I(\epsilon) := \frac{1}{\epsilon}\int_s^T |(\mathbb{Y}(r+\epsilon) - \mathbb{Y}(r)) \otimes^2 |_{\bar{\chi}^*} \, \mathrm{d}r \xrightarrow{\epsilon \to 0} 0, \qquad \text{in probability.}$$

In fact, identifying $\bar{\chi}^*$ with $B(\bar{\nu}_0, \bar{\nu}_0; \mathbb{R})$, we get

$$\begin{aligned} I(\epsilon) = \frac{1}{\epsilon}\int_s^T & \sup_{|\phi|_{\bar{\nu}_0}, |\psi|_{\bar{\nu}_0} \le 1} |\langle (\mathbb{Y}(r+\epsilon) - \mathbb{Y}(r)), \phi\rangle \, \langle (\mathbb{Y}(r+\epsilon) - \mathbb{Y}(r)), \psi\rangle| \, \mathrm{d}r \\ & \le \frac{1}{\epsilon}\int_s^T \sup_{|\phi|_{\bar{\nu}_0}, |\psi|_{\bar{\nu}_0} \le 1} \left\{ \left| \int_r^{r+\epsilon} \langle (\mathbb{X}(\xi), A^*\phi\rangle \, \mathrm{d}\xi \right| \left| \int_r^{r+\epsilon} \langle (\mathbb{X}(\xi), A^*\psi\rangle \, \mathrm{d}\xi \right| \right\} \mathrm{d}r, \end{aligned} \tag{58}$$

where we have used Lemma 5.2. This is smaller than

$$\frac{1}{\epsilon}\int_s^T \left| \int_r^{r+\epsilon} |\mathbb{X}(\xi)| \, \mathrm{d}\xi \right|^2 \mathrm{d}r \le \epsilon \sup_{\xi \in [s,T]} |\mathbb{X}(\xi)|^2,$$

which converges to zero almost surely. □

Corollary 5.4. *The process $\mathbb{X}$ is a $\bar{\chi}$-Dirichlet process. Moreover it is also a $\bar{\chi}$ finite quadratic variation process and a $\bar{\nu}_0\hat{\otimes}_\pi\mathbb{R}$-weak-Dirichlet process.*

Proof. For $t \in [s,T]$, we have $\mathbb{X}(t) = \mathbb{M}(t) + \mathbb{A}(t)$, where

$$\begin{aligned} \mathbb{M}(t) &= x + \int_s^t \sigma(r) \, \mathrm{d}\mathbb{W}_Q(r), \\ \mathbb{A}(t) &= \mathbb{V}(t) + \mathbb{Y}(t), \\ \mathbb{V}(t) &= \int_s^t b(r) dr. \end{aligned}$$

$\mathbb{M}$ is a local martingale by Proposition 3.5 (i) and $\mathbb{V}$ is a bounded variation process. By Proposition 4.11 and Remark 4.9, we get

$$[\mathbb{V}, \mathbb{V}]_{\bar{\chi}} = [\mathbb{V}, \mathbb{Y}]_{\bar{\chi}} = [\mathbb{Y}, \mathbb{V}]_{\bar{\chi}} = 0.$$

By Proposition 5.3 and the bilinearity of the $\bar{\chi}$-covariation, it yields that $\mathbb{A}$ has a zero $\bar{\chi}$-quadratic variation and so $\mathbb{X}$ is a $\bar{\chi}$-Dirichlet process. The second part of the statement is a consequence of Corollary 4.29. □

In the sequel we will denote by $UC([s,T] \times H; D(A^*))$ the F-space of the functions $G : [s,T] \times H \to D(A^*)$ which are uniformly continuous on each closed ball, equipped with the topology of the uniform convergence on closed balls.

The theorem below generalizes for some aspects the Itô formula of [18], i.e. their Theorem 5.2, to the case when the second derivatives do not necessarily belong to the Chi-subspace χ.

Theorem 5.5. *Let $F\colon [s,T]\times H \to \mathbb{R}$ of class $C^{1,2}$ which belongs to $UC([s,T]\times H; D(A^*))$.*

Let $\mathbb{X}$ be an H-valued process process admitting a $\bar{\chi}$-finite quadratic variation. We suppose the following.

(i) There exists a (càdlàg) bounded variation process $C\colon [s,T]\times\Omega \to (H\hat{\otimes}_\pi H)$ such that, for all t in $[s,T]$ and $\phi\in\bar{\chi}$,

$$C(t,\cdot)(\phi) = [\mathbb{X},\mathbb{X}]_{\bar{\chi}}(\phi)(t,\cdot) \qquad a.s.$$

(ii) For every continuous function $\Gamma\colon [s,T]\times H \to D(A^*)$ the integral

$$\int_s^t \left\langle \Gamma(r,\mathbb{X}(r)),\, \mathrm{d}^-\mathbb{X}(r)\right\rangle \tag{59}$$

exists.

Then

$$F(t,\mathbb{X}(t)) = F(s,\mathbb{X}(s)) + \int_s^t \left\langle \partial_r F(r,\mathbb{X}(r)),\, \mathrm{d}^-\mathbb{X}(r)\right\rangle \\ + \frac{1}{2}\int_s^t {}_{(H\hat{\otimes}_\pi H)^*}\left\langle \partial^2_{xx}F(r,\mathbb{X}(r)),\, \mathrm{d}C(s)\right\rangle_{H\hat{\otimes}_\pi H} + \int_s^t \partial_r F(r,\mathbb{X}(r))\,\mathrm{d}r. \tag{60}$$

Before the proof of the theorem we make some comments.

Remark 5.6. *A consequence of assumption (i) of Theorem 5.5 is the existence of a $\mathbb{P}$-null set O such that, for every $t\in[s,T]$, $\omega\notin O$,*

$$\widetilde{[\mathbb{X},\mathbb{X}]}_{\bar{\chi}}(\phi)(t,\omega) = C(t,\omega)(\phi),$$

for every $\phi\in D(A^*)\hat{\otimes}_\pi D(A^*)$. In other words the $\bar{\chi}$-quadratic variation of $\mathbb{X}$ coincides with C.

Remark 5.7. *The conditions (i) and (ii) of Theorem 5.5 are verified if for instance $\mathbb{X} = \mathbb{M}+\mathbb{V}+\mathbb{S}$, where $\mathbb{M}$ is a local martingale, $\mathbb{V}$ is an H-valued bounded variation process, and $\mathbb{S}$ is a process verifying*

$$\langle \mathbb{S}, h\rangle(t) = \int_s^t \langle \mathbb{Z}(r), A^*h\rangle\,\mathrm{d}r, \qquad \text{for all } h\in D(A^*),$$

for some measurable process $\mathbb{Z}$ with $\int_s^T |\mathbb{Z}(r)|^2\,\mathrm{d}r < +\infty$ a.s.

Indeed, by Lemma 4.15, $\mathbb{M}$ admits a global quadratic quadratic variation which can be identified with $[\mathbb{M},\mathbb{M}]^{cl}$.
On the other hand $\mathbb{A}=\mathbb{V}+\mathbb{S}$ has a zero $\bar{\chi}$-quadratic variation, by Proposition 4.11 and the bilinearity character of the $\bar{\chi}$-covariation. $\mathbb{X}$ is therefore a $\bar{\chi}$-Dirichlet process. By Corollary 4.29 and again the bilinearity of the $\bar{\chi}$-covariation, we obtain that $\mathbb{X}$ has a finite $\bar{\chi}$-quadratic variation. Taking also into account Lemma 4.15 and Remark 4.9, we get $\widetilde{[\mathbb{X},\mathbb{X}]}_{\bar{\chi}}(\Phi)(\cdot) = \langle [\mathbb{M},\mathbb{M}]^{cl},\Phi\rangle$ if $\Phi\in\bar{\chi}$. Consequently, we can set $C=[\mathbb{M},\mathbb{M}]^{cl}$ and condition (i) is verified.
To prove (ii) consider a continuous function $\Gamma\colon [s,T]\times H\to D(A^*)$. The integral of $(\Gamma(r,\mathbb{X}(r)))$ w.r.t. the semimartingale $\mathbb{M}+\mathbb{V}$ where $\mathbb{M}(t) = x+\int_s^t \sigma(r)\,\mathrm{d}W_Q(r)$ exists and equals the classical Itô integral, by Proposition 3.6 and Proposition 3.8. Therefore, we only have to prove that

$$\int_s^t \left\langle \Gamma(r,\mathbb{X}(r)),\, \mathrm{d}^-\mathbb{S}(r)\right\rangle, \qquad t\in[s,T]$$

exists. For every $t\in[s,T]$ the ϵ-approximation of such an integral gives, up to a remainder boundary term $C(\epsilon,t)$ which converges in probability to zero,

$$\frac{1}{\epsilon}\int_s^t \langle \Gamma(r,\mathbb{X}(r)), \mathbb{S}(r+\epsilon)-\mathbb{S}(r)\rangle\,\mathrm{d}r = \frac{1}{\epsilon}\int_s^t\int_r^{r+\epsilon} \langle \mathbb{Z}(u), A^*\Gamma(r,\mathbb{X}(r))\rangle\,\mathrm{d}u\,\mathrm{d}r \\ = \frac{1}{\epsilon}\int_s^t\int_{u-\epsilon}^{u} \langle \mathbb{Z}(u), A^*\Gamma(r,\mathbb{X}(r))\rangle\,\mathrm{d}r\,\mathrm{d}u \xrightarrow{\epsilon\to 0} \int_s^t \langle \mathbb{Z}(u), A^*\Gamma(u,\mathbb{X}(u))\rangle\,\mathrm{d}u, \tag{61}$$

in probability by classical Lebesgue integration theory. The right-hand side of (61) has obviously a continuous modification so (59) *exists by definition and condition (ii) is fulfilled.*
In particular we have proved that

$$\begin{aligned}\int_s^t \langle \Gamma(r,\mathbb{X}(r)),\, \mathrm{d}^-\mathbb{X}(r)\rangle &= \int_s^t \langle \Gamma(r,\mathbb{X}(r)),\, \mathrm{d}\mathbb{M}(r)\rangle + \int_s^t \langle \Gamma(r,\mathbb{X}(r)),\, \mathrm{d}\mathbb{V}(r)\rangle \\ &+ \int_s^t \langle \mathbb{Z}(u), A^*\Gamma(u,\mathbb{X}(u))\rangle\, \mathrm{d}u\end{aligned}$$

Proof of Theorem 5.5.
Step 1.
Let $\{e_i^*\}_{i\in\mathbb{N}}$ be an orthonormal basis of H^* made of elements of $D(A^*) \subseteq H^*$. This is always possible since $D(A^*) \subseteq H^*$ densely embedded, via a Gram-Schmidt orthogonalization procedure.
For $N \geq 1$ we denote by $P_N\colon H \to H$ the orthogonal projection on the span of the vectors $\{e_1, .., e_N\}$. $P_\infty\colon H \to H$ will simply denote the identity.
Let us for a moment omit the time dependence on F, which is supposed to be of class C^2 from H to $\mathbb{R}$. We define $F_N\colon H \to \mathbb{R}$ as $F_N(x) := F(P_N(x))$. We have

$$\partial_x F_N(x) = P_N \partial_x F(P_N X(x)) \tag{62}$$

and

$$\partial^2_{xx} F_N(x) = (P_N \otimes P_N)\partial^2_{xx} F(P_N X(x)),$$

where the latter equality has to be understood as

$$_{(H\hat{\otimes}_\pi H)^*}\left\langle \partial^2_{xx} F_N(x), h_1 \otimes h_2 \right\rangle_{(H\hat{\otimes}_\pi H)} = {}_{(H\hat{\otimes}_\pi H)^*}\left\langle \partial^2_{xx} F(P_N(x)), (P_N(h_1)) \otimes (P_N(h_2)) \right\rangle_{(H\hat{\otimes}_\pi H)}, \tag{63}$$

for all $h_1, h_2 \in H$. $\partial^2_{xx} F_N(x)$ is an element of $(H\hat{\otimes}_\pi H)^*$ but it belongs to $(D(A^*)\hat{\otimes}_\pi D(A^*))$ as well; indeed it can be written as

$$\sum_{i,j=1}^N {}_{(H\hat{\otimes}_\pi H)^*}\left\langle \partial^2_{xx} F(P_N(x)), e_i \otimes e_j \right\rangle_{(H\hat{\otimes}_\pi H)} \left(e_i^* \otimes e_j^*\right)$$

and $e_i^* \otimes e_j^*$ are in fact elements of $D(A^*)\hat{\otimes}_\pi D(A^*)$.

We come back now again to the time dependence notation $F(t,x)$. We can apply the Itô formula proved in [18], Theorem 5.2, and with the help of Assumption (i), we find

$$\begin{aligned}F_N(t,\mathbb{X}(t)) = F_N(s,\mathbb{X}(s)) &+ \int_s^t \left\langle \partial_x F_N(r,\mathbb{X}(r)),\, \mathrm{d}^-\mathbb{X}(r)\right\rangle \\ &+ \frac{1}{2}\int_s^t \left\langle \partial^2_{xx} F_N(r,\mathbb{X}(r)),\, \mathrm{d}C(s)\right\rangle + \int_s^t \partial_r F_N(r,\mathbb{X}(r))\,\mathrm{d}r.\end{aligned} \tag{64}$$

Step 2. We consider, for fixed $\epsilon > 0$, the map

$$\begin{cases} T_\epsilon\colon UC([s,T]\times H; D(A^*)) \to L^0(\Omega) \\ T_\epsilon\colon G \mapsto \displaystyle\int_s^t \left\langle G(r,\mathbb{X}(r)), \frac{\mathbb{X}(r+\epsilon) - \mathbb{X}(r)}{\epsilon} \right\rangle \mathrm{d}r, \end{cases}$$

where the set $L^0(\Omega)$ of all real random variables is equipped with the topology of the convergence in probability. Assumption (ii) implies that $\lim_{\epsilon\to 0} T_\epsilon G$ exists for every G. By Banach-Steinhaus for F-spaces (see Theorem 17, Chapter II in [24]) it follows that the map

$$\begin{cases} UC([s,T]\times H; D(A^*)) \to L^0(\Omega) \\ G \mapsto \displaystyle\int_s^t \left\langle G(r,\mathbb{X}(r)),\, \mathrm{d}^-\mathbb{X}(r)\right\rangle \end{cases}$$

is linear and continuous.

Step 3.

If $K \subseteq H$ is a compact set then the set

$$P(K) := \{P_N(y) \ :\ y \in K,\ N \in \mathbb{N}\cup +\infty\}$$

is compact as well. Indeed, consider $\{P_{N_l}(y_l)\}_{l\geq 1}$ be a sequence in $P(K)$. We look for a subsequence convergence to an element of $P(K)$.

Since K is compact we can assume, without loss of generality, that y_l converges, for $l\to +\infty$, to some $y\in K$. If $\{N_l\}$ assumes only a finite number of values then (passing if necessary to a subsequence) $N_l \equiv \bar{N}$ for some $\bar{N}\in\mathbb{N}\cup +\infty$ and then $P_{N_l}(y_l) \xrightarrow{l\to+\infty} P_{\bar{N}}(y)$. Otherwise we can assume (passing if necessary to a subsequence) that $N_l \xrightarrow{l\to+\infty} +\infty$ and then it is not difficult to prove that $P_{N_l}(y_l) \xrightarrow{l\to+\infty} y$, which belongs to $P(K)$ since $y = P_\infty y$.

In particular, being $\partial_x F$ continuous,

$$\mathcal{D} := \{\partial_x F(P_N(x)) \ :\ x\in K,\ N\in\mathbb{N}\cup\{+\infty\}\}$$

is compact in $D(A^*)$. Since the sequence of maps $\{P_N\}$ is uniformly continuous, it follows that

$$\sup_{x\in\mathcal{D}} |(P_N - I)(x)| \xrightarrow{N\to\infty} 0. \tag{65}$$

Step 4.

We show now that

$$\lim_{N\to\infty} \int_s^t \left\langle \partial_x F_N(r,\mathbb{X}(r)),\, \mathrm{d}^-\mathbb{X}(r)\right\rangle = \int_s^t \left\langle \partial_x F(r,\mathbb{X}(r)),\, \mathrm{d}^-\mathbb{X}(r)\right\rangle. \tag{66}$$

holds in probability for every $t\in[s,T]$. Let K be a compact subset of H. In fact

$$\sup_{t\in[0,T],x\in K} |\partial_x F(t,P_N(x)) - \partial_x F(t,x)| \xrightarrow{N\to\infty} 0,$$

since $\partial_x F$ is continuous. On the other hand

$$\sup_{t\in[0,T],x\in K} |(P_N - I)(\partial_x F(t,P_N x))| \xrightarrow{N\to\infty} 0,$$

because of (65). Consequently, by (62),

$$\partial_x F_N \to \partial_x F, \tag{67}$$

uniformly on each compact, with values in H. This yields that ω-a.s.

$$\partial_x F_N(r,\mathbb{X}(r)) \to \partial_x F(r,\mathbb{X}(r)),$$

uniformly on each compact. By Step 2, then (66) follows.

Step 5.
Finally, we prove that

$$\lim_{N\to\infty} \frac{1}{2}\int_s^t \left\langle \partial^2_{xx} F_N(r,\mathbb{X}(r)),\, \mathrm{d}C(s)\right\rangle = \frac{1}{2}\int_s^t \left\langle \partial^2_{xx} F(r,\mathbb{X}(r)),\, \mathrm{d}C(s)\right\rangle. \tag{68}$$

For a fixed $\omega \in \Omega$ we define $K(\omega)$ the compact set as

$$K(\omega) := \{\mathbb{X}(t)(\omega) \;:\; t \in [s,T]\}.$$

We write

$$\begin{aligned}&\left|\int_s^t \left\langle \partial^2_{xx} F_N(r,\mathbb{X}(x)) - \partial^2_{xx} F(r,\mathbb{X}(x)),\, \mathrm{d}C(r)\right\rangle\right|(\omega)\\ &\qquad\qquad\le \sup_{\substack{y\in K(w)\\ t\in[s,T]}} \left\|\partial^2_{xx} F_N(t,y) - \partial^2_{xx} F(t,y)\right\|_{(H\hat{\otimes}_\pi H)^*} \int_s^T \mathrm{d}|C(r)|(\omega).\end{aligned} \tag{69}$$

Using arguments similar to those used in proving (67) one can see that

$$\partial^2_{xx} F_N \xrightarrow{N\to\infty} \partial^2_{xx} F$$

uniformly on each compact. Consequently

$$\sup_{r\in[s,T]} |(\partial^2_{xx} F_N - \partial^2_{xx} F)(r,\mathbb{X}(r))|_{(H\hat{\otimes}_\pi H)^*} \xrightarrow{N\to\infty} 0.$$

Since C has bounded variation, finally (68) holds.

Step 6.
Since F_N [resp. $\partial_r F_N$] converges uniformly on each compact to F [resp. $\partial_r F$], when $N \to \infty$, then

$$\int_s^t \partial_r F_N(r,\mathbb{X}(r))\,\mathrm{d}r \xrightarrow{N\to\infty} \int_s^t \partial_r F(r,\mathbb{X}(r))\,\mathrm{d}r.$$

Taking the limit when $N \to \infty$ in (64), finally provides (60). $\square$

Next result can be considered a Itô formula for *mild type processes*, essentially coming out from mild solutions of infinite dimensional stochastic differential equations. An interesting contribution in this direction, but in a different spirit appears in [9].

Corollary 5.8. *Assume that b is a predictable process with values in H and σ is a predictable process with values in $\mathcal{L}_2(U_0,H)$ satisfying (52). Define $\mathbb{X}$ as in (53). Let x be an element of H. Assume that $f \in C^{1,2}([0,T]\times H)$ with $\partial_x f \in UC([0,T]\times H, D(A^*))$. Then, $\mathbb{P}-a.s.$,*

$$\begin{aligned}f(t,\mathbb{X}(t)) = f(s,x) + \int_s^t \partial_s f(r,\mathbb{X}(r))\,\mathrm{d}r + \int_s^t \langle A^*\partial_x f(r,\mathbb{X}(r)),\mathbb{X}(r)\rangle\,\mathrm{d}r + \int_s^t \langle \partial_x f(r,\mathbb{X}(r)), b(r)\rangle\,\mathrm{d}r\\ + \frac{1}{2}\int_s^t Tr\left[\left(\sigma(r)Q^{1/2}\right)\left(\sigma(r)Q^{1/2}\right)^* \partial^2_{xx} f(r,\mathbb{X}(r))\right]\mathrm{d}r + \int_s^t \langle \partial_x f(r,\mathbb{X}(r)), \sigma(r)\,\mathrm{d}\mathbb{W}_Q(r)\rangle.\end{aligned} \tag{70}$$

Remark 5.9. *We remark that in* (70)*, the partial derivative $\partial^2_{xx} f(r,x)$ for any $r \in [s,T]$ and $x \in H$ stands in fact for its associated linear bounded operator in the sense of* (13)*. From now on we will make this natural identification.*

Proof. It is a consequence of Theorem 5.5 taking into account Remark 5.7: we have $\mathbb{M}(t) = x + \int_s^t \sigma(r)\,\mathrm{d}\mathbb{W}_Q(r), t \in [0,T]$, $\mathbb{V}(t) = \int_s^t b(r)dr$, $\mathbb{S} = \mathbb{Y}$ with $\mathbb{Z}(r) = \mathbb{X}(r)$. According to that Remark, in Theorem 5.5 we set $C = [\mathbb{M},\mathbb{M}]^{cl}$. We also use the chain rule for Itô's integrals in Hilbert spaces, see the considerations before Proposition 3.5, together with Lemma 4.17. The fourth integral in the right-hand side of (70) appears from the second integral in (60) using Proposition 2.4 and again Lemma 4.17. □

6 The optimal control problem

In this section we illustrate the utility of the tools of stochastic calculus via regularization in the study of optimal control problems driven by stochastic PDEs or more in general by infinite dimensional stochastic differential equations. We will prove a decomposition result for the strong solutions of the Hamilton-Jacobi-Bellman equation related to the optimal control problem and we use that decomposition to derive a verification theorem.

U, U_0, $\mathbb{W}_Q = \{\mathbb{W}_Q(t) : t \le s \le T\}$, $\mathcal{L}_2(U_0, H)$ and $A\colon D(A) \subseteq H \to H$ were defined as in Section 5.

6.1 The setting of the problem

We consider Λ a Polish space (i.e. a complete and separable metric space) that will be our control space. In other words we will try to minimize our functional on a class of Λ-valued processes. We formulate the following standard assumptions that will ensure existence and uniqueness for the solution of the state equation.

Hypothesis 6.1. *$b\colon [0,T] \times H \times \Lambda \to H$ is a continuous function and satisfies, for some $C > 0$,*

$$\begin{aligned} &|b(s,x,a) - b(s,y,a)| \le C|x-y|,\\ &|b(s,x,a)| \le C(1+|x|), \end{aligned}$$

for all $x,y \in H$, $s \in [0,T]$, $a \in \Lambda$. $\sigma\colon [0,T] \times H \to \mathcal{L}_2(U_0,H)$ is continuous and, for some $C > 0$,

$$\begin{aligned} &\|\sigma(s,x) - \sigma(s,y)\|_{\mathcal{L}_2(U_0,H)} \le C|x-y|,\\ &\|\sigma(s,x)\|_{\mathcal{L}_2(U_0,H)} \le C(1+|x|), \end{aligned}$$

for all $x,y \in H$, $s \in [0,T]$.

Let us fix for the moment a predictable process $a = a(\cdot) : [s,T] \times \Omega \to \Lambda$, where the dot refers to the time variable. a will indicate in the sequel an admissible control in a sense to be specified.

We consider the state equation

$$\begin{cases} \mathrm{d}\mathbb{X}(t) = (A\mathbb{X}(t) + b(t,\mathbb{X}(t),a(t)))\ \mathrm{d}t + \sigma(t,\mathbb{X}(t))\,\mathrm{d}\mathbb{W}_Q(t)\\ \mathbb{X}(s) = x. \end{cases} \tag{71}$$

The solution of (71) is understood in the mild sense, so an H-valued adapted strongly measurable process $\mathbb{X}(\cdot)$ is a solution if

$$\mathbb{P}\left\{ \int_s^T \Big(|\mathbb{X}(r)| + |b(r,\mathbb{X}(r),a(r))| + \|\sigma(r,\mathbb{X}(r))\|^2_{\mathcal{L}_2(U_0,H)} \Big)\ \mathrm{d}r < +\infty \right\} = 1$$

and

$$\mathbb{X}(t) = e^{(t-s)A}x + \int_s^t e^{(t-r)A} b(r,\mathbb{X}(r),a(r))\,\mathrm{d}r + \int_s^t e^{(t-r)A}\sigma(r,\mathbb{X}(r))\,\mathrm{d}\mathbb{W}_Q(r) \tag{72}$$

$\mathbb{P}$-a.s. for every $t \in [s,T]$.

Thanks to Theorem 3.3 of [34], given Hypothesis 6.1, there exists a unique solution $\mathbb{X}(\cdot; s, x, a(\cdot))$ of (71), which admits a continuous modification. So for us $\mathbb{X}$ can always be considered as a continuous process.

Remark 6.2. *(i) When the operator Q is trace class, the norm $\mathcal{L}_2(U_0, H)$ that appears in Hypothesis 6.1 (assumptions on σ) can be controlled using the norm $\mathcal{L}(U, H)$ so the Lipschitz and the growing conditions imposed on σ follows from similar hypotheses involving the norm of $\mathcal{L}(U, H)$.*

(ii) When the operator Q is the identity, $\mathbb{W}_Q$ is naturally associated to a space time white noise, see also [64] and [12]. In this case the norm $\mathcal{L}(U, H)$ does not control anymore the norm $\mathcal{L}_2(U_0, H)$ but still something can be done, in certain cases, to weaken the Lipschitz and the growing conditions imposed on σ, asking something more on the semigroup. For example (see [34] Section 3.10), when σ only depends on $\mathbb{X}$, if we only ask for linear growth and Lipschitz continuity w.r.t. the norm of $\mathcal{L}(U, H)$, we have existence, uniqueness and regularity results for the stochastic evolution equation similar to the ones we use in this work if the following condition is satisfied: $\|S(t)\sigma(x)\|_{\mathcal{L}_2(U,H)} \le K(t)C(1 + |x|)$ and $\|S(t)\sigma(x) - S(t)\sigma(y)\|_{\mathcal{L}_2(U,H)} \le K(t)C(|x - y|)$ where K is a real function s.t. $\int_0^1 K^2(r)r^{-2\alpha}\,\mathrm{d}r < +\infty$ for some $\alpha \in (0, 1/2)$.

A simple case in which such a condition is verified (see [11] Section 11.2) is given by a one-dimensional heat equation on the segment $(0, 1)$ of the following form:

$$dX(t,\xi) = \left[\frac{\partial^2 X}{\partial \xi^2}(t,\xi) + f(\xi, X(t,\xi))\right] dt + b(\xi, X(t,\xi))dW(t,\xi),$$

with zero Dirichlet boundary conditions and some square integrable initial datum where W is cylindrical Wiener process on $U = L^2(0, 1)$.

(iii) In the framework described by item (i), all our results do not extend automatically. Indeed when Q is the identity and σ has linear growth w.r.t. the norm of $\mathcal{L}(U, H)$, the stochastic integral appearing in the definition $\mathbb{Y}$ given in (55) is not defined in H but only in a larger space. This means in particular that the decomposition of $\mathbb{X}$ described in Theorem 5.5 is nomore a decomposition in H-valued addenda. To include the "purely cylindrical" case we need to describe the evolution of the solution of the stochastic equation in a larger Hilbert space and introduce there consistent notions. This will the object of a paper in preparation.

Setting $b := b(\cdot, \mathbb{X}(\cdot; s, x, a(\cdot)), a(\cdot))$, $\sigma := \sigma(\cdot, \mathbb{X}(\cdot; s, x, a(\cdot)))$ then $\mathbb{X}$ fulfills (52) and it is of type (53). The following corollary is just a particular case of Corollary 5.8, which is reformulated here for the reader convenience.

Corollary 6.3. *Let Λ be a Polish space and assume that b and σ satisfy the Hypothesis 6.1. Let $a : [s, T] \times \Omega \to \Lambda$ be predictable and $x \in H$. Let $\mathbb{X}(\cdot)$ denote $\mathbb{X}(\cdot; s, x, a(\cdot))$. Assume that $f \in C^{1,2}([0, T] \times H)$ with $\partial_x f \in UC([0, T] \times H, D(A^*))$. Then, $\mathbb{P} - a.s.$,*

$$\begin{aligned} f(t,\mathbb{X}(t)) &= f(s,x) + \int_s^t \partial_r f(r,\mathbb{X}(r))\,\mathrm{d}r + \int_s^t \langle A^*\partial_x f(r,\mathbb{X}(r)), \mathbb{X}(r)\rangle\,\mathrm{d}r + \int_s^t \langle \partial_x f(r,\mathbb{X}(r)), b(r,\mathbb{X}(r),a(r))\rangle\,\mathrm{d}r \\ &+ \frac{1}{2}\int_s^t Tr\left[\left(\sigma(r,\mathbb{X}(r))Q^{1/2}\right)\left(\sigma(r,\mathbb{X}(r))Q^{1/2}\right)^* \partial^2_{xx} f(r,\mathbb{X}(r))\right]\mathrm{d}r + \int_s^t \langle \partial_x f(r,\mathbb{X}(r)), \sigma(r,\mathbb{X}(r))\,\mathrm{d}\mathbb{W}_Q(r)\rangle. \end{aligned} \tag{73}$$

Let $l\colon [0, T] \times H \times \Lambda \to \mathbb{R}$ be a measurable function and $g\colon H \to \mathbb{R}$ a continuous function. l is called the running cost and g the terminal cost.
We introduce now the class $\mathcal{U}_s$ of admissible controls. It is constituted by $a : [s, T] \times \Omega \to \Lambda$ such that for ω a.s. $(r, \omega) \mapsto l(r, \mathbb{X}(r, s, x, a(\cdot)), a(r)) + g(\mathbb{X}(T, s, x, a(\cdot)))$ is $\mathrm{d}r \otimes \mathrm{d}\mathbb{P}$- is quasi-integrable. This means that, either its positive or negative part are integrable.
We want to determine a *minimum* over all $a(\cdot) \in \mathcal{U}_s$, of the cost functional

$$J(s, x; a(\cdot)) = \mathbb{E}\left[\int_s^T l(r, \mathbb{X}(r; s, x, a(\cdot)), a(r))\,\mathrm{d}r + g(\mathbb{X}(T; s, x, a(\cdot)))\right]. \tag{74}$$

The value function of this problem is defined as

$$V(s,x) = \inf_{a(\cdot)\in\mathcal{U}_s} J(s,x;a(\cdot)). \tag{75}$$

Definition 6.4. *If $a^*(\cdot) \in \mathcal{U}_s$ minimizes (74) among the controls in $\mathcal{U}_s$, i.e. if $J(s,x;a^*(\cdot)) = V(s,x)$, we say that the control $a^*(\cdot)$ is* optimal *at (s,x). In this case the pair $(a^*(\cdot), \mathbb{X}^*(\cdot))$, where $\mathbb{X}^*(\cdot) := \mathbb{X}(\cdot; s,x,a^*(\cdot))$, is called an* optimal couple *(or optimal pair) at (s,x).*

6.2 The HJB equation

The HJB equation associated with the minimization problem above is

$$\begin{cases} \partial_s v + \langle A^*\partial_x v, x\rangle + \frac{1}{2} Tr\left[\sigma(s,x)\sigma^*(s,x)\partial^2_{xx} v\right] \\ \qquad\qquad + \inf_{a\in\Lambda}\left\{ \langle \partial_x v, b(s,x,a)\rangle + l(s,x,a)\right\} = 0, \\ v(T,x) = g(x). \end{cases} \tag{76}$$

In the above equation $\partial_x v$ [resp. $\partial^2_{xx} v$] is the [resp. second] Fréchet derivatives of v w.r.t. the x variable; it is identified with elements of H [resp. with a symmetric bounded operator on H]. $\partial_s v$ is the derivative w.r.t. the time variable. For $(t,x,p,a) \in [0,T]\times H \times H \times \Lambda$, the term

$$F_{CV}(t,x,p,a) := \langle p, b(t,x,a)\rangle + l(t,x,a), \tag{77}$$

is called the *current value Hamiltonian* of the system and its infimum over $a \in \Lambda$

$$F(t,x,p) := \inf_{a\in\Lambda}\{\langle p, b(t,x,a)\rangle + l(t,x,a)\} \tag{78}$$

is called the *Hamiltonian*. Using this notation the HJB equation, (76) can be rewritten as

$$\begin{cases} \partial_s v + \langle A^*\partial_x v, x\rangle + \frac{1}{2} Tr\left[\sigma(s,x)\sigma^*(s,x)\partial^2_x v\right] + F(s,x,\partial_x v) = 0, \\ v(T,x) = g(x). \end{cases} \tag{79}$$

Hypothesis 6.5. *The value function is always finite and the Hamiltonian $F(t,x,p)$ is well-defined and finite for all $(t,x,p) \in [0,T]\times H\times H$. Moreover it is supposed to be continuous.*

We introduce the operator $\mathscr{L}_0$ on $C([0,T]\times H)$ defined as

$$\begin{cases} D(\mathscr{L}_0) := \left\{\varphi \in C^{1,2}([0,T]\times H) \ : \ \partial_x\varphi \in C([0,T]\times H; D(A^*))\right\} \\ \mathscr{L}_0(\varphi)(s,x) := \partial_s\varphi(s,x) \\ \qquad\qquad + \langle A^*\partial_x\varphi(s,x), x\rangle + \frac{1}{2} Tr\left[\sigma(s,x)\sigma^*(s,x)\partial^2_{xx}\varphi(s,x)\right]. \end{cases} \tag{80}$$

The HJB equation (79) can be rewritten as

$$\begin{cases} \mathscr{L}_0(v)(s,x) + F(s,x,\partial_x v(s,x)) = 0 \\ v(T,x) = g(x). \end{cases}$$

6.3 Strict and strong solutions

For some $h \in C([0,T]\times H)$ and $g \in C(H)$ we consider the following Cauchy problem

$$\begin{cases} \mathscr{L}_0(v)(s,x) = h(s,x) \\ v(T,x) = g(x). \end{cases} \tag{81}$$

Definition 6.6. *We say that $v \in C([0,T] \times H)$ is a* strict solution *of (81) if $v \in D(\mathscr{L}_0)$ and (81) is satisfied.*

Definition 6.7. *Given $h \in C([0,T] \times H)$ and $g \in C(H)$ we say that $v \in C^{0,1}([0,T] \times H)$ with $\partial_x v \in UC([0,T) \times H; D(A^*))$ is a* strong solution *of (81) if there exist three sequences: $\{v_n\} \subseteq D(\mathscr{L}_0)$, $\{h_n\} \subseteq C([0,T] \times H)$ and $\{g_n\} \subseteq C(H)$ fulfilling the following.*

(i) For any $n \in \mathbb{N}$, v_n is a strict solution of the problem

$$\begin{cases} \mathscr{L}_0(v_n)(s,x) = h_n(s,x) \\ v_n(T,x) = g_n(x). \end{cases} \tag{82}$$

(ii) The following convergences hold:

$$\begin{cases} v_n \to v & \text{in } C([0,T] \times H) \\ h_n \to h & \text{in } C([0,T] \times H) \\ g_n \to g & \text{in } C(H). \end{cases}$$

6.4 Decomposition for solutions of the HJB equation

Theorem 6.8. *Consider $h \in C([0,T] \times H)$ and $g \in C(H)$. Assume that Hypothesis 6.1 is satisfied. Suppose that $v \in C^{0,1}([0,T] \times H)$ with $\partial_x v \in UC([0,T) \times H; D(A^*))$ is a strong solution of (81). Let $\mathbb{X}(\cdot) := \mathbb{X}(\cdot; t, x, a(\cdot))$ be the solution of (71) starting at time s at some $x \in H$ and driven by some control $a(\cdot) \in \mathcal{U}_s$. Assume that b is of the form*

$$b(t,x,a) = b_g(t,x,a) + b_i(t,x,a), \tag{83}$$

where b_g and b_i satisfy the following conditions.

(i) $\sigma(t, \mathbb{X}(t))^{-1} b_g(t, \mathbb{X}(t), a(t))$ is bounded (being $\sigma(t, \mathbb{X}(t))^{-1}$ the pseudo-inverse of σ);

(ii) b_i satisfies

$$\lim_{n\to\infty} \int_s^{\cdot} \langle \partial_x v_n(r, \mathbb{X}(r)) - \partial_x v(r, \mathbb{X}(r)), b_i(r, \mathbb{X}(r), a(r)) \rangle \, \mathrm{d}r = 0 \qquad ucp, \tag{84}$$

on $[s, T_0]$ for each $s < T_0 < T$.

Then

$$v(t, \mathbb{X}(t)) - v(s, \mathbb{X}(s)) = v(t, \mathbb{X}(t)) - v(s, x) = \int_s^t h(r, \mathbb{X}(r)) \, \mathrm{d}r$$
$$+ \int_s^t \langle \partial_x v(r, \mathbb{X}(r)), b(r, \mathbb{X}(r), a(r)) \rangle \, \mathrm{d}r + \int_s^t \langle \partial_x v(r, \mathbb{X}(r)), \sigma(r, \mathbb{X}(r)) \, \mathrm{d}\mathbb{W}_Q(r) \rangle . \tag{85}$$

Example 6.9. *Hypothesis (i) and (ii) of Theorem 6.8 are satisfied if the approximating sequence v_n converges to v in a stronger way. For example if v is a strong solution of the HJB in the sense of Definition 6.7 and, moreover, $\partial_x v_n$ converges to $\partial_x v$ in $C([0,T] \times H)$, then the convergence in point (ii) can be easily checked. The convergence of the spatial partial derivative is the typical assumption required in the standard strong solutions literature.*

Example 6.10. *The assumptions of Theorem 6.8 are fulfilled if the following assumption is satisfied.*

$$\sigma(t, \mathbb{X}(t))^{-1} b(t, \mathbb{X}(t), a(t)) \text{ is bounded}$$

for all choice of admissible controls $a(\cdot)$. In this case we apply Theorem 6.8 with $b_i = 0$ and $b = b_g$.

Proof of Theorem 6.8. We fix T_0 in (s,T). We denote by v_n the sequence of smooth solutions of the approximating problems prescribed by Definition 6.7, which converges to v. Thanks to Corollary 6.3, every v_n verifies for $t \in [s,T_0]$,

$$\begin{aligned}
v_n(t,\mathbb{X}(t)) &= v_n(s,x) + \int_s^t \partial_r v_n(r,\mathbb{X}(r))\,\mathrm{d}r \\
&+ \int_s^t \langle A^* \partial_x v_n(r,\mathbb{X}(r)), \mathbb{X}(r)\rangle \,\mathrm{d}r + \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)), b(r,\mathbb{X}(r),a(r))\rangle \,\mathrm{d}r \\
&+ \frac{1}{2}\int_s^t \mathrm{Tr}\left[\left(\sigma(r,\mathbb{X}(r))Q^{1/2}\right)\left(\sigma(r,\mathbb{X}(r))Q^{1/2}\right)^* \partial^2_{xx} v_n(r,\mathbb{X}(r))\right]\,\mathrm{d}r \\
&+ \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)), \sigma(r,\mathbb{X}(r))\,\mathrm{d}\mathbb{W}_Q(r)\rangle\,,\ t\in[s,T]. \qquad \mathbb{P}-\text{a.s.}
\end{aligned} \tag{86}$$

Using Girsanov's Theorem (see [10] Theorem 10.14) we can observe that

$$\beta_Q(t) := W_Q(t) + \int_s^t \sigma(r,\mathbb{X}(r))^{-1} b_g(r,\mathbb{X}(r),a(r))\,\mathrm{d}r,$$

is a Q-Wiener process w.r.t. a probability $\mathbb{Q}$ equivalent to $\mathbb{P}$ on the whole interval $[s,T]$. We can rewrite (86) as

$$\begin{aligned}
v_n(t,\mathbb{X}(t)) = v_n(s,x) &+ \int_s^t \partial_r v_n(r,\mathbb{X}(r))\,\mathrm{d}r \\
&+ \int_s^t \langle A^* \partial_x v_n(r,\mathbb{X}(r)), \mathbb{X}(r)\rangle \,\mathrm{d}r + \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)), b_i(r,\mathbb{X}(r),a(r))\rangle \,\mathrm{d}r, \\
&+ \frac{1}{2}\int_s^t \mathrm{Tr}\left[\left(\sigma(r,\mathbb{X}(r))Q^{1/2}\right)\left(\sigma(r,\mathbb{X}(r))Q^{1/2}\right)^* \partial^2_{xx} v_n(r,\mathbb{X}(r))\right]\,\mathrm{d}r \\
&+ \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)), \sigma(r,\mathbb{X}(r))\,\mathrm{d}\beta_Q(r)\rangle\,. \qquad \mathbb{P}-a.s.
\end{aligned} \tag{87}$$

Since v_n is a strict solution of (82), the expression above gives

$$\begin{aligned}
v_n(t,\mathbb{X}(t)) = v_n(s,x) &+ \int_s^t h_n(r,\mathbb{X}(r))\,\mathrm{d}r \\
&+ \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)), b_i(r,\mathbb{X}(r),a(r))\rangle \,\mathrm{d}r + \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)), \sigma(r,\mathbb{X}(r))\,\mathrm{d}\beta_Q(r)\rangle\,.
\end{aligned} \tag{88}$$

Since we wish to take the limit for $n \to \infty$, we define

$$M_n(t) := v_n(t,\mathbb{X}(t)) - v_n(s,x) - \int_s^t h_n(r,\mathbb{X}(r))\,\mathrm{d}r - \int_s^t \langle \partial_x v_n(r,\mathbb{X}(r)), b_i(r,\mathbb{X}(r),a(r))\rangle \,\mathrm{d}r. \tag{89}$$

$\{M_n\}_{n\in\mathbb{N}}$ is a sequence of real $\mathbb{Q}$-local martingales converging ucp, thanks to the definition of strong solution and Hypothesis (84), to

$$M(t) := v(t,\mathbb{X}(t)) - v(s,x) - \int_s^t h(r,\mathbb{X}(r))\,\mathrm{d}r - \int_s^t \langle \partial_x v(r,\mathbb{X}(r)), b_i(r,\mathbb{X}(r),a(r))\rangle \,\mathrm{d}r. \tag{90}$$

Since the space of real continuous local martingales equipped with the ucp topology is closed (see e.g. Proposition 4.4 of [40]) then M is a continuous $\mathbb{Q}$-local martingale.

We have now gathered all the ingredients to conclude the proof. As in Section 5, we set $\bar{\nu}_0 = D(A^*)$, $\nu = \bar{\nu}_0 \hat{\otimes}_\pi \mathbb{R}$, $\bar{\chi} = \bar{\nu}_0 \hat{\otimes}_\pi \bar{\nu}_0$.

Corollary 5.4 ensures that $\mathbb{X}(\cdot)$ is a ν-weak Dirichlet process admitting a $\bar{\chi}$-quadratic variation with decomposition $\mathbb{M} + \mathbb{A}$ where $\mathbb{M}$ is the local martingale (with respect to $\mathbb{P}$) defined by $\mathbb{M}(t) = x + \int_s^t \sigma(r, \mathbb{X}(r))\, \mathrm{d}\mathbb{W}_Q(r)$ and $\mathbb{A}$ is a ν-martingale-orthogonal process. Now

$$\mathbb{X}(t) = \tilde{\mathbb{M}}(t) + \mathbb{V}(t) + \mathbb{A}(t), t \in [s, T_0],$$

where $\tilde{\mathbb{M}}(t) = x + \int_s^t \sigma(r, \mathbb{X}(r))\, \mathrm{d}\beta_Q(r)$ and $\mathbb{V}(t) = -\int_s^t b_g(r, \mathbb{X}(r), a(r))dr,\ t \in [s, T_0]$ is a bounded variation process. So by Proposition 3.5 (i), $\tilde{\mathbb{M}}$ is a $\mathbb{Q}$-local martingale and by Proposition 4.24 1., $\mathbb{V}$ is a $\mathbb{Q} - \nu$-martingale orthogonal process. By Remark 4.23 $\mathbb{V} + \mathbb{A}$ is a $\mathbb{Q} - \nu$-martingale orthogonal process and $\mathbb{X}$ is a ν-weak Dirichlet process with local martingale part $\tilde{M}$, with respect to $\mathbb{Q}$. Still under $\mathbb{Q}$, Theorem 4.31 ensures that the process $v(\cdot, \mathbb{X}(\cdot))$ is a real weak Dirichlet process whose local martingale part being equal to

$$N(t) = \int_s^t \langle \partial_x v(r, \mathbb{X}(r)), \sigma(r, \mathbb{X}(r))\, \mathrm{d}\beta_Q(r) \rangle .$$

On the other hand, with respect to $\mathbb{Q}$, (90) implies that

$$v(t, \mathbb{X}(t)) \;=\; \left[v(s, x) \;+\; \int_s^t h(r, \mathbb{X}(r))\, \mathrm{d}r \;+\; \int_s^t \langle \partial_x v(r, \mathbb{X}(r)), b_i(r, \mathbb{X}(r), a(r)) \rangle\, \mathrm{d}r \right] \;+\; N(t), \quad (91)$$

is a decomposition of $v(\cdot, \mathbb{X}(\cdot))$ as $\mathbb{Q}$- semimartingale, which is also in particular, a $\mathbb{Q}$-weak Dirichlet process. By Proposition 1.2 such a decomposition is unique and so

$$\begin{aligned} M(t) = N(t) &= \int_s^t \langle \partial_x v(r, \mathbb{X}(r)), \sigma(r, \mathbb{X}(r))\, \mathrm{d}\beta_Q(r) \rangle \\ &= \int_s^t \langle \partial_x v(r, \mathbb{X}(r)), b_g(r, \mathbb{X}(r), a(r))\, \mathrm{d}r \rangle + \int_s^t \langle \partial_x v(r, \mathbb{X}(r)), \sigma(r, \mathbb{X}(r))\, \mathrm{d}\mathbb{W}_Q(r) \rangle . \quad (92) \end{aligned}$$

This shows (85) for $t \in [s, T_0]$. Letting T_0 go to T allows to conclude the proof of Theorem 6.8. □

6.5 Verification Theorem

Theorem 6.11. *Assume that Hypotheses 6.1 and 6.5 are satisfied. Let $v \in C^{0,1}([0,T] \times H)$ with $\partial_x v \in UC([0,T] \times H; D(A^*))$ be a strong solution of (76). Assume that for all initial data $(s,x) \in [0,T] \times H$ and every control $a(\cdot) \in \mathcal{U}_s$ b can be written as $b(t,x,a) = b_g(t,x,a) + b_i(t,x,a)$ with b_i and b_g satisfying hypotheses (i) and (ii) of Theorem 6.8. Let v such that $\partial_x v$ has most polynomial growth in the x variable. Then we have the following.*

(i) $v \le V$ on $[0,T] \times H$.

(ii) Suppose that, for some $s \in [0,T)$, there exists a predictable process $a(\cdot) = a^*(\cdot) \in \mathcal{U}_s$ such that, denoting $\mathbb{X}\,(\cdot; s, x, a^*(\cdot))$ simply by $\mathbb{X}^*(\cdot)$, we have

$$F\left(t, \mathbb{X}^*\left(t\right), \partial_x v\left(t, \mathbb{X}^*\left(t\right)\right)\right) = F_{CV}\left(t, \mathbb{X}^*\left(t\right), \partial_x v\left(t, \mathbb{X}^*\left(t\right)\right); a^*\left(t\right)\right), \quad (93)$$

$dt \otimes \mathrm{d}\mathbb{P}$ a.e. Then $a^*(\cdot)$ is optimal at (s,x); moreover $v\,(s,x) = V\,(s,x)$.

Proof. We choose a control $a(\cdot) \in \mathcal{U}_s$ and call $\mathbb{X}$ the related trajectory. Thanks to Theorem 6.8 we can write

$$\begin{aligned} g(\mathbb{X}(T)) = v(T, \mathbb{X}(T)) = v(s,x) - \int_s^T F(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r)))\, \mathrm{d}r \\ + \int_s^T \langle \partial_x v(r, \mathbb{X}(r)), b(r, \mathbb{X}(r), a(r)) \rangle\, \mathrm{d}r + \int_s^T \langle \partial_x v(r, \mathbb{X}(r)), \sigma(r, \mathbb{X}(r))\, \mathrm{d}\mathbb{W}_Q(r) \rangle . \quad (94) \end{aligned}$$

Since both sides of (94) are a. s. finite, we can add $\int_s^T l(r, \mathbb{X}(r), a(r))\,\mathrm{d}r$ to them, obtaining

$$
\begin{aligned}
g(\mathbb{X}(T)) + \int_s^T l(r, \mathbb{X}(r), a(r))\,\mathrm{d}r &= v(s,x) + \int_s^T \langle \partial_x v(r, \mathbb{X}(r)), \sigma(r, \mathbb{X}(r))\,\mathrm{d}\mathbb{W}_Q(r)\rangle \\
&+ \int_s^T \left(-F(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r))) + F_{CV}(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r)))\right)\mathrm{d}r. \quad (95)
\end{aligned}
$$

Observe now that, by definition of F and F_{CV} we know that

$$
-F(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r))) + F_{CV}(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r)))
$$

is always positive. So its expectation always exists even if it could be $+\infty$, but not $-\infty$ on an event of positive probability. This shows a posteriori that $\int_s^T l(r, \mathbb{X}(r), a(r))\,\mathrm{d}r$ cannot be $-\infty$ on a set of positive probability.
By [10] Theorem 7.4, all the momenta of $\sup_{r\in[s,T]} |\mathbb{X}(r)|$ are finite. On the other hand, σ is Lipschitz-continuous, $v(s,x)$ is deterministic and, since $\partial_x v$ has polynomial growth, then

$$
\mathbb{E}\int_s^T \left\langle \partial_x v(r, \mathbb{X}(r)), \left(\sigma(r, \mathbb{X}(r))Q^{1/2}\right)\left(\sigma(r, \mathbb{X}(r))Q^{1/2}\right)^* \partial_x v(r, \mathbb{X}(r))\right\rangle \mathrm{d}r
$$

is finite. Consequently, by Proposition 3.5 (v)

$$
\int_s^{\cdot} \langle \partial_x v(r, \mathbb{X}(r)), \sigma(r, \mathbb{X}(r))\,\mathrm{d}\mathbb{W}_Q(r)\rangle
$$

is a true martingale vanishing at s. Consequently, its expectation is zero. So the expectation of the right-hand side of (95) exists even if it could be $+\infty$; consequently the same holds for the left-hand side. By definition of J, we have

$$
\begin{aligned}
J(s,x,a(\cdot)) = \mathbb{E}\left[g(\mathbb{X}(T)) + \int_s^T l(r, \mathbb{X}(r), a(r))\,\mathrm{d}r\right] &= v(s,x) \\
+ \mathbb{E}\int_s^T \Big(-F(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r))) &+ F_{CV}(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r)), a(r))\Big)\,\mathrm{d}r. \quad (96)
\end{aligned}
$$

So minimizing $J(s,x,a(\cdot))$ over $a(\cdot)$ is equivalent to minimize

$$
\mathbb{E}\int_s^T \Big(-F(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r))) + F_{CV}(r, \mathbb{X}(r), \partial_x v(r, \mathbb{X}(r)), a(r))\Big)\,\mathrm{d}r, \quad (97)
$$

which is a non-negative quantity. As mentioned above, the integrand of such an expression is always nonnegative and then a lower bound for (97) is 0. If the conditions of point (ii) are satisfied such a bound is attained by the control $a^*(\cdot)$, that in this way is proved to be optimal.

Concerning the proof of (i), since the integrand in (97) is nonnegative, (96) gives

$$
J(s,x,a(\cdot)) \geq v(s,x).
$$

Taking the inf over $a(\cdot)$ we get $V(s,x) \geq v(s,x)$, which concludes the proof. □

Remark 6.12. *1. The first part of the proof does not make use that a belongs to $\mathcal{U}_s$, but only that $r \mapsto l(r, \mathbb{X}(\cdot, s, x, a(\cdot)), a(\cdot))$ is a.s. strictly bigger then $-\infty$. Under that only assumption, $a(\cdot)$ is forced to be admissible, i.e. to belong to $\mathcal{U}_s$.*

2. Let v be a strong solution of HJB equation. Observe that the condition (93) can be rewritten as

$$a^*(t) \in \arg\min_{a\in\Lambda} \Big[F_{CV}\left(t, \mathbb{X}^*(t), \partial_x v\left(t, \mathbb{X}^*(t)\right); a\right) \Big].$$

Suppose that for any $(t,y) \in [0,T]\times H$, $\phi(t,y) = \arg\min_{a\in\Lambda} \big(F_{CV}\left(t, y, \partial_x v(t,y); a\right) \big)$ is measurable and single-valued. Suppose moreover that

$$\int_s^T l(r, \mathbb{X}^*(r), a^*(r))dr > -\infty \text{ a.s.} \tag{98}$$

Suppose that the equation

$$\begin{cases} \mathrm{d}\mathbb{X}(t) = (A\mathbb{X}(t) + b(t, \mathbb{X}(t), \phi(t, \mathbb{X}(t))\ \mathrm{d}t + \sigma(t, \mathbb{X}(t))\,\mathrm{d}\mathbb{W}_Q(t) \\ \mathbb{X}(s) = x. \end{cases} \tag{99}$$

admits a unique mild solution $\mathbb{X}^*$. Now (98) *and Remark 6.12 imply that $a(\cdot)^*$ is admissible. Then $\mathbb{X}^*$ is the optimal trajectory of the state variable and $a^*(t) = \phi(t, \mathbb{X}^*(t)), t \in [0,T]$ is the optimal control. The function ϕ is the* optimal feedback *of the system since it gives the optimal control as a function of the state.*

Remark 6.13. *Observe that, using exactly the same arguments we used in this section one could treat the (slightly) more general case in which b has the form:*

$$b(t,x,a) = b_0(t,x) + b_g(t,x,a) + b_i(t,x,a).$$

where b_g and b_i satisfy condition of Theorem 6.8 and $b_0 : [0,T] \times H \to H$ is continuous. In this case the addendum b_0 can be included in the expression of $\mathscr{L}_0$ that becomes the following

$$\begin{cases} D(\mathscr{L}_0^{b_0}) := \left\{ \varphi \in C^{1,2}([0,T]\times H)\ :\ \partial_x\varphi \in C([0,T]\times H; D(A^*)) \right\} \\ \mathscr{L}_0^{b_0}(\varphi) := \partial_s\varphi + \langle A^*\partial_x\varphi, x\rangle + \langle \partial_x\varphi, b_0(t,x)\rangle + \frac{1}{2}Tr\left[\sigma(s,x)\sigma^*(s,x)\partial^2_{xx}\varphi\right]. \end{cases} \tag{100}$$

Consequently in the definition of regular solution the operator $\mathscr{L}_0^{b_0}$ appears instead $\mathscr{L}_0$.

Remark 6.14. *In the definition of strong solution given in Definition 6.7 one could substitute the assumption $v \in C^{0,1}([0,T]\times H)$ with $v \in C^{0,1}([0,T)\times H) \cap C^0([0,T]\times H)$ such that $\partial_x v$ has polynomial growth. This, together with the fact that $\mathbb{X}$ has all moments (see Theorem 7.4, Chapter 7 of [10]), would permit to pass to the limit $T_0 \to T$ in the last step of the proof of Theorem 6.8.*

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References

[1] N. U. Ahmed. Optimal control of ∞-dimensional stochastic systems via generalized solutions of HJB equations. *Discuss. Math. Differ. Incl. Control Optim.*, 21(1):97–126, 2001.

[2] N. U. Ahmed. Generalized solutions of HJB equations applied to stochastic control on Hilbert space. *Nonlinear Anal.*, 54(3):495–523, 2003.

[3] V. Barbu and G. Da Prato. *Hamilton-Jacobi equations in Hilbert spaces*, volume 86 of *Research Notes in Mathematics.* Pitman (Advanced Publishing Program), Boston, MA, 1983.

[4] A. Bensoussan, G. Da Prato, M. C. Delfour, and S. K. Mitter. *Representation and control of infinite dimensional systems*. Systems & Control: Foundations & Applications. Birkhäuser Boston Inc., Boston, MA, second edition, 2007.

[5] S. Cerrai. Optimal control problems for stochastic reaction-diffusion systems with non-Lipschitz coefficients. *SIAM J. Control Optim.*, 39(6):1779–1816, 2001.

[6] S. Cerrai. Stationary Hamilton-Jacobi equations in Hilbert spaces and applications to a stochastic optimal control problem. *SIAM J. Control Optim.*, 40(3):824–852, 2001.

[7] P. L. Chow and J. L. Menaldi. Infinite-dimensional Hamilton-Jacobi-Bellman equations in Gauss-Sobolev spaces. *Nonlinear Anal.*, 29(4):415–426, 1997.

[8] F. Coquet, A. Jakubowski, J. Mémin, and L. Słomiński. Natural decomposition of processes and weak Dirichlet processes. In *In memoriam Paul-André Meyer: Séminaire de Probabilités XXXIX*, volume 1874 of *Lecture Notes in Math.*, pages 81–116. Springer, Berlin, 2006.

[9] G. Da Prato, A. Jentzen, and M. Röckner. A mild Itô formula for SPDEs. *Preprint Arxiv*, http://arxiv.org/pdf/1009.3526.pdf, 2011.

[10] G. Da Prato and J. Zabczyk. *Stochastic equations in infinite dimensions*, volume 44 of *Encyclopedia of Mathematics and its Applications*. Cambridge University Press, Cambridge, 1992.

[11] G. Da Prato and J. Zabczyk. *Ergodicity for infinite-dimensional systems*, volume 229 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 1996.

[12] R. C. Dalang and L. Quer-Sardanyons. Stochastic integrals for spde's: a comparison. *Expo. Math.*, 29(1):67–109, 2011.

[13] A. Debussche, M. Fuhrman, and G. Tessitore. Optimal control of a stochastic heat equation with boundary-noise and boundary-control. *ESAIM Control Optim. Calc. Var.*, 13(1):178–205 (electronic), 2007.

[14] L. Denis. Solutions of stochastic partial differential equations considered as Dirichlet processes. *Bernoulli*, 10(5):783–827, 2004.

[15] C. Di Girolami and F. Russo. Infinite dimensional stochastic calculus via regularization. *Preprint HAL-INRIA*, http://hal.archives-ouvertes.fr/inria-00473947/fr/, 2010.

[16] C. Di Girolami and F. Russo. Clark-Ocone type formula for non-semimartingales with finite quadratic variation. *C. R. Acad. Sci. Paris Sér. I Math.*, 349(3-4):209 – 214, 2011.

[17] C. Di Girolami and F. Russo. Generalized covariation and extended Fukushima decomposition for Banach space-valued processes. Applications to windows of Dirichlet processes. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.*, 15(2), 2012.

[18] C. Di Girolami and F. Russo. Generalized covariation for Banach space valued processes and Itô formula. *Osaka journal of mathematics.*, 51(3), 2014.

[19] J. Diestel and J. J. Uhl. *Vector measures*. Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I., 1977.

[20] N. Dinculeanu. *Vector integration and stochastic integration in Banach spaces*. Wiley, New York, NY, 2000.

[21] K. Du and Q. Meng. Maximum principle for infinite dimensional stochastic control systems. Preprint.

[22] Kai Du and Qingxin Meng. A maximum principle for optimal control of stochastic evolution equations. *SIAM J. Control Optim.*, 51(6):4343–4362, 2013.

[23] N. Dunford and J. T. Schwartz. *Linear Operators. I. General Theory*. With the assistance of W. G. Bade and R. G. Bartle. Pure and Applied Mathematics, Vol. 7. Interscience Publishers, Inc., New York, 1958.

[24] N. Dunford and J. T. Schwartz. *Linear operators. Part I*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1988. General theory, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1958 original, A Wiley-Interscience Publication.

[25] M. Errami and F. Russo. n-covariation, generalized Dirichlet processes and calculus with respect to finite cubic variation processes. *Stochastic Process. Appl.*, 104(2):259–299, 2003.

[26] W. H. Fleming and R. W. Rishel. *Deterministic and stochastic optimal control*. Applications of Mathematics, No. 1. Springer-Verlag, Berlin, 1975.

[27] H. Föllmer. Dirichlet processes. In *Stochastic integrals (Proc. Sympos., Univ. Durham, Durham, 1980)*, volume 851 of *Lecture Notes in Math.*, pages 476–478. Springer, Berlin, 1981.

[28] M. Fuhrman, Y. Hu, and G. Tessitore. Stochastic maximum principle for optimal control of spdes (extended version). Preprint, http://hal.inria.fr/docs/00/78/36/15/PDF/SmpPDE.pdf.

[29] M. Fuhrman, Y. Hu, and G. Tessitore. Stochastic control and BSDEs with quadratic growth. In *Control theory and related topics*, pages 80–86. World Sci. Publ., Hackensack, NJ, 2007.

[30] M. Fuhrman, Y. Hu, and G. Tessitore. Stochastic maximum principle for optimal control of spdes. *Comptes Rendus Mathematique*, 350(13-14):683–688, 2012.

[31] M. Fuhrman, F. Masiero, and G. Tessitore. Stochastic equations with delay: optimal control via BSDEs and regular solutions of Hamilton-Jacobi-Bellman equations. *SIAM J. Control Optim.*, 48(7):4624–4651, 2010.

[32] M. Fuhrman and G. Tessitore. Nonlinear Kolmogorov equations in infinite dimensional spaces: the backward stochastic differential equations approach and applications to optimal control. *Ann. Probab.*, 30(3):1397–1465, 2002.

[33] M. Fuhrman and G. Tessitore. Infinite horizon backward stochastic differential equations and elliptic equations in Hilbert spaces. *Ann. Probab.*, 32(1B):607–660, 2004.

[34] L. Gawarecki and V Mandrekar. *Stochastic differential equations in infinite dimensions with applications to stochastic partial differential equations*. Probability and its Applications (New York). Springer, Heidelberg, 2011.

[35] B. Goldys and F. Gozzi. Second order parabolic Hamilton-Jacobi-Bellman equations in Hilbert spaces and stochastic control: L^2_μ approach. *Stochastic Process. Appl.*, 116(12):1932–1963, 2006.

[36] F. Gozzi. Global regular solutions of second order hamilton-jacobi equations in hilbert spaces with locally lipschitz nonlinearities. *J. Math. Anal. App.*, 198(2):399–443, 1996.

[37] F. Gozzi, C. Marinelli, and S. Savin. On controlled linear diffusions with delay in a model of optimal advertising under uncertainty with memory effects. *Journal of Optimization Theory and Applications*, 142(2):291–321, 2009.

[38] F. Gozzi and E. Rouy. Regular solutions of second-order stationary Hamilton Jacobi equations. *J. Differ. Equations*, 130, 1996.

[39] F. Gozzi and F. Russo. Verification theorems for stochastic optimal control problems via a time dependent Fukushima-Dirichlet decomposition. *Stochastic Process. Appl.*, 116(11):1530–1562, 2006.

[40] F. Gozzi and F. Russo. Weak Dirichlet processes with a stochastic control perspective. *Stochastic Process. Appl.*, 116(11):1563–1583, 2006.

[41] I. Karatzas and S. E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1988.

[42] N. V. Krylov and B. L. Rozovskii. Stochastic evolution equations. In P. H. Baxendale and S. V. Lototsky, editors, *Stochastic differential equations: theory and applications*, volume 2 of *Interdisciplinary Mathematical Sciences*, pages 1–70. World Scientific, 2007. Translated from Itogi Naukii Tekhniki, Seriya Sovremennye Problemy Matematiki.

[43] J. A. Leon. Stochastic Fubini theorem for semimartingales in Hilbert space. *Can. J. Math.*, 42(5):890–901, 1990.

[44] P. L. Lions. Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. I. The dynamic programming principle and applications. *Comm. Partial Differential Equations*, 8(10):1101–1174, 1983.

[45] P. L. Lions. Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. II. Viscosity solutions and uniqueness. *Comm. Partial Differential Equations*, 8(11):1229–1276, 1983.

[46] P. L. Lions. Optimal control of diffusion processes and Hamilton-Jacobi-Bellman equations. III. Regularity of the optimal cost function. In *Nonlinear partial differential equations and their applications. Collège de France seminar, Vol. V (Paris, 1981/1982)*, volume 93 of *Res. Notes in Math.*, pages 95–205. Pitman, Boston, MA, 1983.

[47] Z.M. Ma and M. Röckner. *Introduction to the theory of (nonsymmetric) Dirichlet forms*. Universitext. Springer-Verlag, Berlin, 1992.

[48] M. Métivier. *Semimartingales: a course on stochastic processes*, volume 2 of *De Gruyter Studies in Mathematics*. Walter de Gruyter, Berlin, 1982.

[49] M. Métivier and J. Pellaumail. *Stochastic integration.* Probability and Mathematical Statistics. Academic Press [Harcourt Brace Jovanovich Publishers], New York, 1980.

[50] P.A. Meyer. Notes sur les integrales stochastiques. I Integrales hilbertiennes. In C. Dellacherie, P. Meyer, and M. Weil, editors, *Séminaire de Probabilites XI*, volume 581 of *Lecture Notes in Mathematics*, pages 446–481. Springer, 1977.

[51] R. Mikulevicius and BL Rozovskii. Martingale problems for stochastic pde's. In R. Carmona and B. Rozovskii, editors, *Stochastic partial differential equations: six perspectives*, volume 64 of *Math. Surveys and Monographs*, pages 243–326. 1999.

[52] B. Oksendal, A. Sulem, and T. Zhang. Singular control of spdes and backward spdes with reflection. preprint.

[53] M. Ondreját. Uniqueness for stochastic evolution equations in Banach spaces. *Dissertationes Math. (Rozprawy Mat.)*, 426:63 p., 2004.

[54] C. Prévôt and M. Röckner. *A concise course on stochastic partial differential equations*, volume 1905 of *Lecture Notes in Mathematics*. Springer, Berlin, 2007.

[55] M. C. Quenez. Stochastic control and BSDEs. In *Backward stochastic differential equations*, volume 364, pages 83–99. Longman. Pitman Res. Notes Math. Ser. 364, 2007.

[56] F. Russo and P. Vallois. Intégrales progressive, rétrograde et symétrique de processus non adaptés. *C. R. Acad. Sci. Paris Sér. I Math.*, 312(8):615–618, 1991.

[57] F. Russo and P. Vallois. Forward, backward and symmetric stochastic integration. *Probab. Theory Related Fields*, 97(3):403–421, 1993.

[58] F. Russo and P. Vallois. Noncausal stochastic integration for làd làg processes. In *Stochastic analysis and related topics (Oslo, 1992)*, volume 8 of *Stochastics Monogr.*, pages 227–263. Gordon and Breach, Montreux, 1993.

[59] F. Russo and P. Vallois. Stochastic calculus with respect to continuous finite quadratic variation processes. *Stochastics Stochastics Rep.*, 70(1-2):1–40, 2000.

[60] F. Russo and P. Vallois. Elements of stochastic calculus via regularization. In *Séminaire de Probabilités XL*, volume 1899 of *Lecture Notes in Math.*, pages 147–185. Springer, Berlin, 2007.

[61] R. A. Ryan. *Introduction to tensor products of Banach spaces.* Springer Monographs in Mathematics. Springer-Verlag London Ltd., London, 2002.

[62] E. M. Stein. *Singular integrals and differentiability properties of functions.* Princeton Mathematical Series, No. 30. Princeton University Press, Princeton, N.J., 1970.

[63] A. Święch. "Unbounded" second order partial differential equations in infinite-dimensional Hilbert spaces. *Comm. Partial Differential Equations*, 19(11-12):1999–2036, 1994.

[64] J. B. Walsh. An introduction to stochastic partial differential equations. In *École d'été de probabilités de Saint-Flour, XIV—1984*, volume 1180 of *Lecture Notes in Math.*, pages 265–439. Springer, Berlin, 1986.

[65] J. Yong and X. Y. Zhou. *Stochastic controls, Hamiltonian systems and HJB equations*, volume 43 of *Applications of Mathematics*. Springer-Verlag, New York, 1999.