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ABSTRACT

Motivation – Our objective was to design a Boolean algebra allowing both a diagrammatic and sentential representation of logical propositions in an intuitive manner. The purpose of this notation is to support inferential activity without heavy deductive procedure to follow.

Research approach – This research is founded on the notions of logical space proposed by Wittgenstein and of hypercube proposed by Pólya.

Findings/Design – Complex propositions in propositional logic can be depicted by hypercube within a coordinate system, and by sequences of lexical symbols allowing operations on hypercube with more than three dimensions.

Research limitations/Implications – Empirical studies are now required to validate the intuitiveness of this notation. Its scope of application must also be delimited.

Originality/Value – Contrary to classical diagrammatic notations based on Euler topological diagrams in logic, the hypercube algebra involves a coordinate-based representation combining diagrams and lexical symbols. This is a new form of notation.

Take away message – Rather than the format, the important variables in designing a representational system are those specifying the cognitive activity induced, e.g. straightforward inferences, abstraction level of symbols.

Keywords
Logic, reasoning, diagram, Boolean operator, logical space, hypercube

INTRODUCTION
Research on cognition in the field of Human Factors and Ergonomics aims at improving performance and human well-being through the design of adapted information systems (Dul et al., 2012). Improving cognitive performance often means the design of cognitive tools that extend our mind capabilities with the purpose of processing a larger scope of information and increasing the power of cognition by thinking faster and further (Dror and Harnad, 2008; Gorayska, Mey, 1996). Improving the "cognitive well-being" of individuals relies on the development of support systems, like graphical forms on user interfaces or documents that facilitate the understanding of task and domain complexity and support the matching between the work domain information processes by a system or a medium to the mental decision process of an operator (Rasmussen, 1986; Burns and Hajdukiewicz, 2004).

An intellectual task particularly sensitive to cognitive performance and cognitive support is reasoning in formal logic. Numerous studies in the literature show that reasoning on logical problems is prone to error and dissatisfaction from participants who have difficulties in understanding the principles governing logic, even among scientists (e.g. Kern, Mirels, and Hinshaw, 1983). The cognitive issue posed by formal reasoning and the possible cognitive aids that this issue can involve have been particularly well addressed by Lindsay (1988, 1992). According to him, reasoning consists in producing inferences that can be defined as "making explicit information that was implicit in sets of inputs" (Lindsay, 1988). To reach this goal, logic is basically based on sentential representation of knowledge that requires a chain of inferences with multiple possibilities producing a significant cognitive workload (Johnson-Laird, 2001). A solution to support inference is to develop notations involving non-deductive procedure. Different forms of knowledge representations inducing a non-deductive procedure during inferences exist. The process of hierarchical inheritance in semantics networks allows direct inferences on properties shared by classes, subclasses and instances of objects without effort of explicit deduction (Collins and Quillian, 1969). Another form of direct inference is heuristics that take advantage of specialised knowledge to permit inferential short-cuts (Cheng and Holyoak, 1985; Gigerenzer, 2000). Mental imagery also provides with detailed representation of objects that conveys implicit information without deductive effort (Denis, 1979). Similarly, the metaphor is a representation modality that bypasses deductive

procedure. The source of the metaphor brings a structure exhibiting explicit and implicit attributes of the target. The metaphor allows direct inferences and learning about new things (target) by extending what it is already known about the source (Blackwell, 2001; Indurkhya, 2006).

In logic, diagrammatic notation is the main form of knowledge representation that excludes a deductive procedure through the process of spatial inference. Lindsay (1988) illustrated this process with the simple case of a discrete grid in which each cell could be occupied by a single point labelled by a letter. Now, consider the case where "*b is one grid point due right of a, c is directly above b, and d is one grid point due left of c*". The conclusion that "*d is directly above a*" could be done by a set of appropriate rules of deduction with a sentential notation. Alternatively, spatial inference leads to the construction of an imaging system representing information. This construction allows a retrieval process of information, just by scanning the grid. Diagrammatic notations can also benefit from metaphor to facilitate direct inferences on abstract mathematical concepts and properties (Lakof and Núñez, 2000). In logic, a long story of development of diagrammatic notations exists, at least since the Middle Age, with the purpose to facilitate the understanding of formal relations between statements (Baron, 1969).

In the next section, we will present Euler diagrams as the classical notation based on spatial inference to depict formal relations. We will show some limitations of such a kind of diagrams to describe certain formal relations. We will also describe the evolution of Euler diagrams towards recent notations that bring additional conventional symbols extending the descriptiveness of diagrams but restricting their intuitiveness. On the base of this analysis, section 3 presents the principles for a new algebra, named hypercube algebra, based on non-deductive procedure for the understanding of connectives in propositional logic. This algebra can be conveyed through a diagrammatic notation but also through sequences of lexical symbols on which Boolean operations can be easily performed without deductive effort. Section 4 discusses perspectives on the hypercube algebra.

EULER DIAGRAMS AND THEIR EVOLUTION

Euler diagrams describe logical propositions through topological relationships. Circles, intersections and exclusions allow an intuitive and straightforward description of assertions (Gurr, 1999; Shin and Lemon, 2008). For instance, figure 1 (left side) shows an Euler diagram expressing the assertion that there is no intersection between sets *a* and *b*, i.e. the proposition: "not (*a* and *b*) is always true». Euler diagrams look so natural because they are based on a metaphor of categories as containers with objects inside or outside these containers (Lakof and Núñez, 2000). But, by depicting assertions literally, Euler diagrams can be ambiguous. Venn diagrams solved this problem both by introducing a depiction of the whole semantic domain in

which the proposition is asserted, that is to say all the possibilities of combination between propositions depicted on the diagrams, and a graphical syntax (or token syntax) expressing the assertion concretely (Howse, Molina, Shin, and Taylor, 2001).
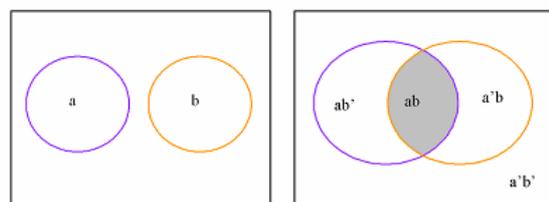

Fig. 1. The proposition "not (*a* and *b*) is always true" represented by Euler Diagram (left side) and Venn Diagram (right side). The symbol' means "negation"

The token syntax allows for locating the assertion within the whole domain depicted. Figure 1 (rigt side) shows a Venn diagram representation of the proposition, "not(*a* and *b*)". In this case, the token syntax consists in shading topological zones that are not concerned with the assertion. Current studies mainly focus on the improvement of token syntax by augmenting Venn diagrams with graphical symbols, as in the case of Venn-Peirce diagrams and Spider diagrams (e.g. Stapleton, 2005 for a review).

However, the effort of formalizing and expanding the scope of diagrammatic representation, coupled with the improvement of their expressiveness, has also reduced their readability. Figure 2 shows an example of a Peirce diagram, that constitues an evolution of Venn diagrams. Specific symbols, such as "x" to describe the existential quantifier, the linear symbol "-" for disjunction or the symbol "o"for emptiness convey extra conventions that reduce comprehensibility. This evolution reduced the concreteness of diagrams which allowed direct spatial inferences in the scope of the "container" metaphor originally involved in Euler diagrams.
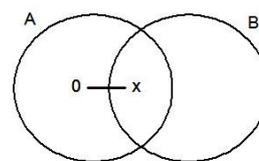

Fig. 2. A Peirce diagram representing the statement "All A are B or some A is B"

Coordinate-based representation constitutes an alternative to topological representation. This kind of diagram locates sets within a coordinate system. Like Venn diagrams, it allows the representation of the whole domain and the use of a token syntax to specify an assertion. A first attempt at using coordinate-based diagrams can be found in the notes of one of Peirce's students from 1880 that reported a figure depicting the combination of truth values for two variables within a coordinate system, but did not provide any syntax to specify an assertion (Anellis, 2004). Lewis Carroll

(1958) proposed a notation based on regions equivalent to Euler sets, separated by orthogonal axes, with a specific conventional notation to define regions occupied by syllogistic statements, but without a reference to a true geometric system.

THE HYPERCUBE ALGEBRA

To depict propositional relations, we propose an alternative approach of diagrammatic representation based on coordinate-based diagrams. Wittgenstein (1921) and Pólya (1940), have respectively posed the foundations for considering the notions of logical space and of hypercube. According to their theoretical works, we present an algebra operating on hypercube, i.e. *n*-dimensions square. This algebra provides the possibility of representing propositional connectives and abstract notions spatially, like contradiction and tautology, and to simplify complex propositions through operations on sequences of lexical symbols. It purpose is to facilitate inferences through spatial inferences when diagrammatic representation can be used and/or through elementary operations on sequences of lexical symbols, when sentential notation is used.

Principles of hypercube algebra

In propositional logic, a complex proposition is composed of connectives (e.g. "and" for conjunction, "or" for disjunction, "non" for negation, "if/then" for conditional relation, etc), and of atomic propositions labelled {p, q, r, s,…}. Now, we can spatially represent a complex proposition through a coordinate system in which axes correspond to the atomic propositions of the complex proposition that we want to depict. On each axis, we can assign a coordinate to a point as the function of the truth value that can take the atomic proposition represented by the axis considered. The value "true" assigned to the atomic proposition "p" will correspond to the coordinate (+ 1) on the axis representing "p". The value "false" assigned to the atomic proposition "p" will correspond to the coordinate (-1) on its axis. Figure 3 shows a space composed of two axes represented respectively the atomic propositions "p" and "q", with four vertices corresponding to the four possibilities of truth values that can take the combination of the atomic propositions "p" and "q" in a given complex proposition.
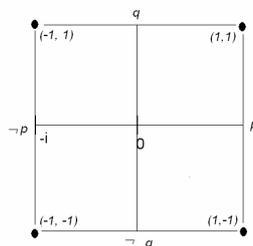

Fig. 3. A 2D Pólya's hypercube (a square), with its four vertices. Vertices coordinate correspond to the possible truth values of the atomic propositions *p* and *q* depicted by the axes of the coordinate system.

The space defined here is a hypercube (Pólya, 1940). A space representing two atomic propositions will be a square, three propositions, a cube, etc... On the base of this definition of a hypercube, we add the following formal properties defining what we call after Wittgenstein (1921): *"logical spaces"* and *"logical spaces occupied by a proposition"*.

Firstly, under each point of a hypercube, a subspace between this point and the origin of the axes can be identified. Each subspace of a hypercube will be defined as a logical space. It can be noticed by a label, a capital letter {A, B, C, D, ...}. Additionally, each logical space can be considered as a set that the following Boolean operators can operate on:

- $\cup$: The union of the logical spaces X and Y will be written X $\cup$ Y or conventionally XY or YX;

- $\cap$: The intersection of the logical spaces X and Y will be written X $\cap$ Y;

- ' : the negation operation of the algebra or complement space. The complement space of the logical space X will be written X';

- $\varnothing$: the zero element of the algebra meaning an empty D-space, or "nothing" class;

- *U*: the unity element of the algebra or "universe" class, i.e. all the space covered by a hypercube. The hypercube algebra operates on the parts of *U*.

Secondly, each logical space in a given hypercube can represent a combination of truth values providing either the value true or the value false to the complex proposition, according the connective(s) included in this complex proposition. The logical spaces representing the combination leading to a true value for the complex proposition are models that will be called "logical spaces occupied" by the complex proposition. In the next section, we illustrate these notions through a depiction of propositional connectives within a 2D hypercube.

Propositional connectives as logical spaces in a 2D hypercube

Consider the 2 hypercube *U*=ABCD, with the dimensions (p, q) and a conventional assignment of logical spaces A, B, C, and D within the hypercube (figure 4).

The literals (negated or non-negated proposition) of the atomic propositions can be rewritten as p=AB, ¬p=CD (i.e. NOT-p=CD), q=DA, ¬q=BC. Therefore, we can rewrite the propositional connectives as below ($\models$ means "equivalent to"):

- conjunction: p AND q, i.e. (p $\wedge$ q) $\models$ A

- disjunction: p OR q OR BOTH, i.e. (p $\vee$ q) $\models$ ABD

- conditional relation: IF p THEN q, i.e. (p $\rightarrow$ q) $\models$ CDA = B'

- biconditional relation IF AND ONLY IF p THEN q, i.e. (p $\leftrightarrow$ q) $\models$ AC

- exclusive disjunction, p OR q, i.e. (p $\otimes$ q) $\models$ BD

Note that a particularity of hypercube algebra resides in the relative definitions of literals and consequently the relative definition of complex propositions with connectives. In a 2D hypercube, a literal will be denoted by two capital letters representing two logical spaces (e.g. $\neg p \models CD$), while in 3D hypercube, the same literal will required four capital letters (e.g. $\neg p \models EFGH$) and in a 4D hypercube, eight logical spaces (e.g. $\neg p \models IJKLMNOP$), etc.
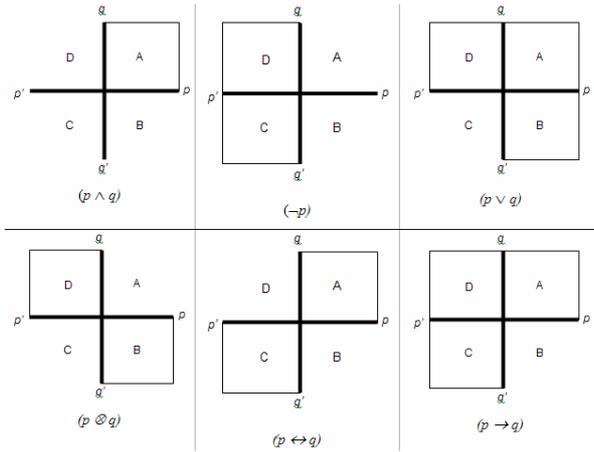

Fig. 4. From the top-left to the bottom-right, four connectives respectively: conjunction, negation, disjunction, exclusive disjunction, biconditional and conditional relations. Bordered logical spaces correspond to combinations of *p* and *q* giving a true value to the complex propositions. The symbols *p'* and *q'* represented the negation of *p* and *q*.

Example of spatial inferences
Through the depiction of connectives within a 2D hypercube, inference can be supported by spatial relationship (figure 4). For example, one can directly perceive that the subspace AC is the complement space of BD, i.e. AC = BD'. This relation translated in propositional terms means that $(p \leftrightarrow q) \models \neg (p \otimes q)$. We can also conclude that $((p \leftrightarrow q) \wedge (p \to q)) \models (p \leftrightarrow q)$, since $AC \cap CDA = AC$. Another possible spatial inference is that $(p \to q) \models (\neg p \vee q)$. Tautology and contradiction can also visually emerge. If a set of logical spaces covers all the hypercube space *U*, then it is a tautology, like $DC \cup BDA$, which is equivalent to $(\neg p) \vee (\neg p \vee q)$. If a set of logical spaces corresponds to the empty set, then it is a contradiction, like $DA \cap BC = \varnothing$, which is equivalent to excluded middle $(q \wedge \neg q)$. This definition of tautology and contradiction are in accordance with their definitions by Wittgenstein (1921).

Operations on the lexical symbols of logical spaces
In the hypercube algebra, capital letters as labels representing the logical spaces are subject themselves to a set of lexical processes reflecting the Boolean operations of the algebra. The concatenation of letter

represents the inclusion operator (e.g. $A \cup B = AB$). The identification of one or more letters shared by sequences corresponds to the intersection between logical spaces (e.g. $AC \cap ACD = AC$). The identification of missing letters in the sequence represents the negation operation (e.g. $ACD' = B$). This lexical processing offers the opportunity to perform straight-forward calculi without diagram, which is difficult to produce in the case of hypercubes made of more than 3 dimensions.

Example: Consider the simplification of the complex proposition ϕ represented below and comprising the three atomic propositions, p, q and r. Its writing requires 8 subspaces (2^3), where conventionally *U*={A, B, C, D, E, F, G, H} where A = $(p \wedge q \wedge r)$; B = $(p \wedge \neg q \wedge r)$; C = $(p \wedge q \wedge \neg r)$; D = $(p \wedge \neg q \wedge \neg r)$; E = $(\neg p \wedge q \wedge \neg r)$; F = $(\neg p \wedge \neg q \wedge \neg r)$; G = $(\neg p \wedge q \wedge r)$; and H = $(\neg p \wedge \neg q \wedge r)$. Consequently, the literals in a 3D-space can be specified as: p = ABCD, $\neg p$ = EFGH, q = ACEG, $\neg q$ = BDFH, r = ABHG, $\neg r$ = ECDF.

ϕ : $(q \wedge (p \vee r)) \vee (p \wedge \neg q) \vee (q \wedge \neg r) \vee r$

$(ACEG \cap (ABCD \cup ABHG)) \cup (ABCD \cap BDFH) \cup (ACEG \cap ECDF) \cup ABHG$

$(ACEG \cap (ABCDHG)) \cup (ABCD \cap BDFH) \cup (ACEG \cap ECDF) \cup ABHG$

$(ACG) \cup (ABCD \cap BDFH) \cup (ACEG \cap ECDF) \cup ABHG$

$(ACG) \cup (BD) \cup (CE) \cup ABHG$

$ABCDEHG = F'$

{The union of all the literals which do not contains the logical space F represent the complex proposition, that is to say the union of the literals p, q, r}

Conclusion: $ABCDEHG \models p \vee q \vee r$

DISCUSSION
Traditionally, knowledge representations are classified according to their representation formats: propositional, visual-spatial or pictorial formats. In terms of cognitive ergonomics, this classification is not very useful. Relevant criteria reside in cognitive properties of the representation, and at this level, the representational format is not so much significant. In logic, a diagram or a propositional format can be more or less adapted depending on whether the task is a task of proof generation, presentation, or understanding, requiring either focused or diffused attention. A diagram facilitates rather diffused attention, but in certain cases, can also facilitate focused attention (Coppin, Burton, and Hockema, 2010). Another important criterion is the level of abstraction afforded by the representation. This criterion called "representation specificity" distinguishes representations with low levels of abstraction aiding information processing, as many diagrams, from those involving abstract symbols aiding their expressiveness (Stenning and Oberlander, 1997). Thus, we can mention

Peirce's distinction between index, icon and symbol (Burks, 1949), a distinction reused in cognitive ergonomics by Rasmussen (1986). On this criterion, we can also distinguish the format from the level of abstractiveness. The example of Peirce diagram shown in Figure 3 combines signs both at the icon level and symbol level of abstraction. Finally, knowledge representation can more or less involve the necessity of deductive procedures to make inferences (Lindsay, 1988).

Through the hypercube algebra, we propose a notation that goes beyond a format distinction and constructed with the purpose to facilitate inferences either on a diagrammatic representation, or a lexical one. The latter particularly allows operations on complex propositions within hypercube with more than three dimensions, mentally difficult to imagine. This algebra, by posting directly configurations in which a complex proposition is true, can explicitly represent all mental models, in the sense of Johnson-Laird (2001), necessary for the apprehension of a proposition in all its complexity. We think that this algebra could represent a tool for assisting work activities that involve inductive reasoning, like case-based reasoning in medical diagnosis (Eshach and Bitterman, 2003) or fault diagnosis task (Rouse, 1978). Particularly, troubleshooting on logical networks could be an area of application (Sanderson and Murtagh, 1989)

Now, though the hypercube algebra seems to be a straight-forward method to depict the logical relations in propositional logic, further research must be planned to go beyond this first approach of the notation. Future studies ought to validate empirically the contribution of this algebra to the understanding of logical relationships, notably in educational settings. The intuitive character of this algebra to apprehend logical relations must be tested in comparison with classical methods of learning. The second perspective will attempt at broadening the scope of this notation, which is currently confined in the simplification of propositions in propositional logic. The possibility of assisting the proof process in propositional logic requires further examination and the application in predicate logic representation remains to be explored.

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REFERENCES
Anellis, I. (2004). The genesis of the truth-table device. *Russell: the journal of Bertrand Russell*, 24, 55-70.

Baron, M.E. (1969). A Note on the historical development of logica diagrams: Leibniz, Euler and Venn. *The Mathematical Gazette*, 53, 113-125.

Blackwell, A. (2001). *Thinking with diagrams*. Dordrecht: Kluwer Academic Publishers.

Burns, C.M. and Hajdukiewicz, J.R., 2004. *Ecological interface design*. Boca Raton: CRC Press.

Burks, A.W. (1949). Icon, Index, and Symbol. *Philosophy and Phenomenological Research*, 9, 673-689.

Carroll, L (1958) *Symbolic Logic*. New-York: Dover.

Cheng P., Holyoak K. (1985). Pragmatic reasoning schemas. *Cognitive Psychology*, 17, 391-416.

Collins, A. M., Quillian, M.R. (1969). Retrieval Time From Semantic Memory. *Journal of Verbal Behavior and Verbal Learning*, 8, 240-247.

Coppin P., Burton, J., Hockema, S. (2010). An attention based theory to explore affordances of textual and diagrammatic proofs. In A.K. Goel, M. Jamnik, N.H. Narayanan (Eds). *Diagrams 2010*. LNAI 6170, pp. 271-278. Heildeberg: Springer-Verlag.

Denis, M. (1979). *Les images mentales*. Paris: Presses Universitaires de France.

Dul, J., Bruder, R., Buckle, P., Carayon P., Falzon P., Marras W.S., Wilson, J.R., Van der Doelen B. (2012). A strategy for human factors/ergonomics: developing the discipline and profession. *Ergonomics*, iFirst article, 1-27.

Dror, I.E., Harnad, S. (2008). Offloading cognition onto cognitive technology. In I. E. Dror & S. Harnad (Eds), *Distributed Cognition: How Cognitive Technology Extends Our Minds* (pp. 1–23). Amsterdam: John Benjamins.

Eshach, H, Bitterman, H. (2003). From case-based reasoning to problem-based learning. *Academic Medicine*, 78, 491-496.

Gigerenzer, G. (2000). *Adaptive thinking: rationality in the real world*. Oxford: Oxford University Press.

Gorayska, B., Mey, J.L. (1996). Cognitive technology. In search of Human Interface. North Holland: Elsevier.

Gurr, C.A. (1999). Effective diagrammatic communication: syntactic, semantic and pragmatic issues. *Journal of Visual Languages and Computing,* 10, 317-342.

Howse, J., Molina, F., Shin, S., Taylor, J. (2001). Type syntax and token syntax in diagrammatic systems, *Proceedings of the international conference on Formal Ontology in Information Systems*, 174-185.

Indurkhya, B. (2006). Emergent representations, interaction theory, and the cognitive force of metaphor. *New Ideas in Psychology*, 24, 133-162.

Johnson-Laird, P.N. (2001). Mental models and deduction. *Trends in Cognitive Science*, 10, 434-442.

Kern, L.H., Mirels, H.L., Hinshaw, V.G. (1983). Scientist's understanding of propositional logic: an

experimental investigation. *Social Studies of Science*, 13, 131-146.

Lakof, G., Núňez, R.E. (2000). *Where mathematics comes from*. Basic Books.

Lindsay, R.K. (1988). Images and inference. *Cognition*. 29, 229-250.

Lindsay, R.K. (1992). Diagrammatic reasoning by simulation. *AAAI Symposium on Reasoning with Diagrammatic Representations*.

Pólya, G. (1940). Sur les types des propositions composées. *Journal of Symbolic Logic*, 5, 98-103.

Rasmussen, J. (1986) Information processing and human-machine interaction: An approach to cognitive engineering. New York: North-Holland

Rouse, W.B. (1978). Human problem solving performance in a fault diagnosis task. IEEE Transactions on Systems, Man, and Cybernetics, SMC-8, 258-271.

Sanderson, P.M., Murtagh, J.M. (1989). Troubleshooting with an inaccurate mental model. *IEEE International Conference on Systems, Man, and Cybernetics*, 1238-1243.

Shin, S.-J., Lemon, O. (2008). Diagrams In: E. N. Zalta (Ed.), The Standford Encyclopedia of Philosophy, Available as http://plato.standford.edu/archives/win2008/entries/diagrams/.

Stapleton, G. (2005). A survey of reasoning systems based on Euler diagrams. *Electronic Notes in Theoretical Computer Sciences*, 134, 127–151.

Stenning, K., Oberlander, J. (1997). A cognitive theory of graphical and linguistic reasoning: logic and implementation. *Cognitive Science*, 19, 97-140