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# A note on extreme values and kernel estimators of sample boundaries

Stéphane Girard<sup>1,\*</sup> and Pierre Jacob<sup>2</sup>

INRIA Rhône-Alpes, team Mistis,  
655, avenue de l'Europe, Montbonnot,  
38334 Saint-Ismier Cedex, France.  
`Stephane.Girard@inrialpes.fr`

<sup>2</sup> EPS/I3M, Université Montpellier 2,  
Place Eugène Bataillon, 34095 Montpellier Cedex 5, France.  
`jacob@math.univ-montp2.fr`

## Abstract

In a previous paper [3], we studied a kernel estimate of the upper edge of a two-dimensional bounded set, based upon the extreme values of a Poisson point process. The initial paper [1] on the subject treats the frontier as the boundary of the support set for a density and the points as a random sample. We claimed in [3] that we are able to deduce the random sample case from the point process case. The present note gives some essential indications to this end, including a method which can be of general interest.

**Keywords and phrases:** support estimation, asymptotic normality, kernel estimator, extreme values.

## 1 Introduction and main results

As in the early paper of Geffroy [1], we address the problem of estimating a subset  $D$  of  $\mathbb{R}^2$  given a sequence of random points  $\Sigma_n = \{Z_1, \dots, Z_n\}$  where the  $Z_i = (X_i, Y_i)$  are independent and uniformly distributed on  $D$ . The problem is reduced to functional estimation by defining

$$D = \{(x, y) \in \mathbb{R}^2 / 0 \leq x \leq 1; 0 \leq y \leq f(x)\},$$

where  $f$  is a strictly positive function. Given an increasing sequence of integers  $0 < k_n < n$ ,  $k_n \uparrow \infty$ , for  $r = 1, \dots, k_n$ , let  $I_{n,r} = [(r-1)/k_n, r/k_n[$  and

$$U_{n,r} = \max \{Y_i / (X_i, Y_i) \in \Sigma_n; X_i \in I_{n,r}\}, \quad (1)$$

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\*Corresponding author

where it is conveniently understood that  $\max \emptyset = 0$ . Now, let  $K$  be a bounded density, which has a support within a compact interval  $[-A, A]$ , a bounded first derivative and which is piecewise  $C^2$ , and  $h_n \downarrow 0$  a sequence of positive numbers. Following [3], Section 6, we define the estimate

$$\hat{f}_n(x) = \frac{1}{k_n} \sum_{r=1}^{k_n} K_n(x - x_r) \left( U_{n,r} + \frac{1}{n - k_n} \sum_{s=1}^{k_n} U_{n,s} \right), x \in \mathbb{R}, \quad (2)$$

where  $x_r$  is the center of  $I_{n,r}$  and, as usually,

$$K_n(t) = \frac{1}{h_n} K\left(\frac{t}{h_n}\right), t \in \mathbb{R}.$$

The perhaps curious second term in brackets in formula (2) is designed for reducing the bias (see [3], Lemma 8). Note that  $\hat{f}_n$  can be rewritten as a linear combination of extreme values

$$\hat{f}_n(x) = \frac{1}{k_n} \sum_{r=1}^{k_n} \beta_{n,r}(x) U_{n,r},$$

where

$$\beta_{n,r}(x) = \frac{1}{k_n} K_n(x - x_r) + \frac{1}{k_n(n - k_n)} \sum_{s=1}^{k_n} K_n(x - x_s).$$

In the sequel, we suppose that  $f$  is  $\alpha$ -Lipschitzian,  $0 < \alpha \leq 1$ , and strictly positive. Our result is the following:

**Theorem 1** *If  $h_n k_n \rightarrow \infty$ ,  $n = o(k_n^{1/2} h_n^{-1/2-\alpha})$ ,  $n = o(k_n^{5/2} h_n^{3/2})$  and  $k_n = o(n/\ln n)$ , then for every  $x \in ]0, 1[$ ,*

$$\left( n h_n^{1/2} / k_n^{1/2} \right) \left( \hat{f}_n(x) - f(x) \right) \Rightarrow \mathcal{N}(0, \sigma^2),$$

with  $\sigma = \|K\|_2/c$ .

## 2 Proofs

If formally the definition of  $\hat{f}_n$  is identical here and in [3] the fundamental difference lies in the fact that in [3] the sample is replaced by a homogeneous Poisson point process with a mean measure  $\mu_n = nc\lambda_{1D}$  where  $\lambda$  is the Lebesgue measure of  $\mathbb{R}^2$  and  $c^{-1} = \lambda(D)$ . Here we denote by  $\Sigma_{0,n}$  this point process and we need, for the sake of approximation, two further Poisson point processes  $\Sigma_{1,n}$  and  $\Sigma_{2,n}$ . The point processes  $\Sigma_{j,n}$  are constructed as in [2], extending an original idea of J. Geffroy. Given a sequence  $\gamma_n \downarrow 0$ , consider independent Poisson random variables  $N_{1,n}$ ,  $M_{1,n}$ ,  $M_{2,n}$ , independent of the sequence  $(Z_n)$ , with parameters  $\mathbb{E}(N_{1,n}) = n(1 - \gamma_n)$  and  $\mathbb{E}(M_{1,n}) = \mathbb{E}(M_{2,n}) = n\gamma_n$ . Define  $N_{0,n} = N_{1,n} + M_{1,n}$ ,  $N_{2,n} = N_{0,n} + M_{2,n}$  and take  $\Sigma_{j,n} = \{Z_1, \dots, Z_{N_{j,n}}\}$ ,  $j = 0, 1, 2$ . For  $j = 0, 1, 2$  we define  $U_{j,n,r}$  and  $\hat{f}_{j,n}$  by imitating (1) and (2). Finally, let us introduce the event  $E_n = \{\Sigma_{1,n} \subseteq \Sigma_n \subseteq \Sigma_{2,n}\}$ . The following lemma is the starting point of our "random sandwiching" technique.

**Lemma 1** *One always has  $\hat{f}_{1,n} \leq \hat{f}_{0,n} \leq \hat{f}_{2,n}$ . Moreover, if  $E_n$  holds,  $\hat{f}_{1,n} \leq \hat{f}_n \leq \hat{f}_{2,n}$ .*

**Proof :** The definition of the random sets  $\Sigma_{j,n}$ ,  $j = 0, 1, 2$  implies that  $\Sigma_{1,n} \subseteq \Sigma_{0,n} \subseteq \Sigma_{2,n}$ . Thus, since  $\beta_{n,r}(x) \geq 0$  for all  $r = 1, \dots, k_n$ , we have  $\hat{f}_{1,n} \leq \hat{f}_{0,n} \leq \hat{f}_{2,n}$ . Similarly,  $E_n$  implies that  $\hat{f}_{1,n} \leq \hat{f}_n \leq \hat{f}_{2,n}$ . ■

The success of the approximation between  $\hat{f}_n$  and  $\hat{f}_{0,n}$  is based upon two lemmas. The first one shows how large is the probability of the event  $E_n$ .

**Lemma 2** *For  $n$  large enough,*

$$\mathbb{P}(\Omega \setminus E_n) \leq 2 \exp\left(-\frac{1}{8}n\gamma_n^2\right).$$

**Proof :** Using the Laplace transform of a Poisson random variable  $X$  with parameter  $\lambda > 0$ , we get for  $\varepsilon/2\lambda$  small enough,

$$\mathbb{P}(|X - \lambda| > \varepsilon) < \exp(-\varepsilon^2/4\lambda),$$

see for instance Lemma 1 in [2]. Clearly,  $\Omega \setminus E_n = \{N_{1,n} > n\} \cup \{N_{2,n} < n\}$  and thus

$$\mathbb{P}(\Omega \setminus E_n) \leq \exp\left(-\frac{n\gamma_n^2}{4(1-\gamma_n)}\right) + \exp\left(-\frac{n\gamma_n^2}{4(1+\gamma_n)}\right).$$

The lemma follows. ■

The second lemma is essential to control the approximation obtained when the event  $E_n$  holds.

**Lemma 3** *If  $k_n = o(n/\log n)$  and  $n = O(k_n^{1+\alpha})$ , then uniformly on  $r = 1, \dots, k_n$ ,*

$$\mathbb{E}(U_{2,n,r} - U_{1,n,r}) = O\left(\frac{k_n\gamma_n}{n}\right).$$

**Proof :** Let us define  $m_{n,r} = \min_{x \in I_{n,r}} f(x)$  and  $M_{n,r} = \max_{x \in I_{n,r}} f(x)$ . Then,

$$\begin{aligned} & \mathbb{E}(U_{2,n,r} - U_{1,n,r}) \\ &= \int_0^{M_{n,r}} (\mathbb{P}(U_{2,n,r} > y) - \mathbb{P}(U_{1,n,r} > y)) dy \\ &= \int_0^{m_{n,r}} (\mathbb{P}(U_{2,n,r} > y) - \mathbb{P}(U_{1,n,r} > y)) dy + \int_{m_{n,r}}^{M_{n,r}} (\mathbb{P}(U_{2,n,r} > y) - \mathbb{P}(U_{1,n,r} > y)) dy \\ &\stackrel{def}{=} A_{n,r} + B_{n,r}. \end{aligned}$$

Introducing  $\lambda_{n,r} = \int_{I_{n,r}} f(x)dx$ , we can write  $A_{n,r}$  as

$$A_{n,r} = \int_0^{m_{n,r}} \exp\left(\frac{n(1-\gamma_n)}{k_n}(y - k_n\lambda_{n,r})\right) dy - \exp\left(\frac{n(1+\gamma_n)}{k_n}(y - k_n\lambda_{n,r})\right) dy.$$

Now,  $A_{n,r}$  is expanded as a sum  $A_{1,n,r} + A_{2,n,r}$  with

$$\begin{aligned} A_{1,n,r} &= \frac{k_n}{n(1-\gamma_n)} \exp\left(\frac{n(1-\gamma_n)}{k_n}(m_{n,r} - k_n\lambda_{n,r})\right) - \frac{k_n}{n(1+\gamma_n)} \exp\left(\frac{n(1+\gamma_n)}{k_n}(m_{n,r} - k_n\lambda_{n,r})\right), \\ A_{2,n,r} &= \frac{k_n}{n(1+\gamma_n)} \exp(-n(1+\gamma_n)\lambda_{n,r}) - \frac{k_n}{n(1-\gamma_n)} \exp(-n(1-\gamma_n)\lambda_{n,r}). \end{aligned}$$

The part  $A_{2,n,r}$  is easily seen to be a  $o(n^{-s})$  where  $s$  is a arbitrarily large exponent under the condition  $k_n = o(n/\log n)$ . Now, If  $a, b, x, y$  are real numbers such that  $x < y < 0 < b < a$ , we have  $0 < ae^y - be^x < (a-b) + b(y-x)$ . Applying to  $A_{1,n,r}$  this inequality yields

$$A_{1,n,r} \leq \frac{k_n}{n} \frac{2\gamma_n}{(1-\gamma_n^2)} + (M_{n,r} - m_{n,r}) \frac{2\gamma_n}{(1+\gamma_n)}.$$

Under the hypothesis that  $f$  is  $\alpha$ -Lipschitzian, and the condition  $n = O(k_n^{1+\alpha})$ , we have  $(M_{n,r} - m_{n,r}) = O(k_n/n)$ , so that  $A_{n,r} = A_{1,n,r} + A_{2,n,r} = O(k_n\gamma_n/n)$ . Now, for  $m_{n,r} \leq y \leq M_{n,r}$ , it is easily seen that

$$\mathbb{P}(U_{2,n,r} > y) - \mathbb{P}(U_{1,n,r} > y) \leq 2\gamma_n \frac{n}{k_n} (M_{n,r} - m_{n,r}),$$

and thus

$$B_{n,r} \leq 2\gamma_n \frac{n}{k_n} (M_{n,r} - m_{n,r})^2 = O\left(\frac{k_n}{n} \gamma_n\right).$$

Clearly, the bounds on  $A_{n,r}$  and  $B_{n,r}$  are uniform in  $r = 1, \dots, k_n$ , and thus we obtain the result.  $\blacksquare$

We quote a technical lemma.

**Lemma 4** *If  $k_n = o(n)$  and  $h_n k_n \rightarrow \infty$  when  $n \rightarrow \infty$ ,*

$$\lim_{n \rightarrow \infty} \sum_{r=1}^{k_n} \beta_{n,r}(x) = 1.$$

**Proof :** Remarking that

$$\sum_{r=1}^{k_n} \beta_{n,r}(x) = \frac{n}{n - k_n} \frac{1}{k_n} \sum_{r=1}^{k_n} K_n(x - x_r),$$

the result follows from the well-known property

$$\lim_{n \rightarrow \infty} \frac{1}{k_n} \sum_{r=1}^{k_n} K_n(x - x_r) = 1,$$

see for instance [3], Corollary 2.  $\blacksquare$

The next proposition is the key tool to extend the results obtained on Poisson processes to samples.

**Proposition 1** *If  $k_n = o(n/\log n)$ ,  $h_n k_n \rightarrow \infty$ , and  $n = O(k_n^{1+\alpha})$ , then, for every  $x \in ]0, 1[$ ,*

$$(nh_n^{1/2}/k_n^{1/2}) \mathbb{E}\left(\left|\hat{f}_n(x) - \hat{f}_{0,n}(x)\right|\right) \rightarrow 0.$$

**Proof :** From Lemma 1, we have

$$\begin{aligned} \mathbb{E}\left(\left|\hat{f}_n(x) - \hat{f}_{0,n}(x)\right| \mathbf{1}_{E_n}\right) &\leq \mathbb{E}\left(\hat{f}_{2,n}(x) - \hat{f}_{1,n}(x)\right) \\ &= \sum_{r=1}^{k_n} \beta_{n,r}(x) \mathbb{E}(U_{2,n,r} - U_{1,n,r}) \\ &\leq \sum_{r=1}^{k_n} \beta_{n,r}(x) \max_{1 \leq s \leq k_n} \mathbb{E}(U_{2,n,s} - U_{1,n,s}) \\ &= O\left(\frac{k_n \gamma_n}{n}\right), \end{aligned}$$

in view of Lemma 3 and Lemma 4. As a consequence,

$$(nh_n^{1/2}/k_n^{1/2})\mathbb{E}\left(\left|\hat{f}_n(x) - \hat{f}_{0,n}(x)\right|\mathbf{1}_{E_n}\right) = O\left(k_n^{1/2}h_n^{1/2}\gamma_n\right). \quad (3)$$

Now, let  $M = \sup\{f(x), x \in [0, 1]\}$ . Then, applying Lemma 4 again,

$$\max\left\{\hat{f}_n(x), \hat{f}_{0,n}(x)\right\} \leq M \sum_{r=1}^{k_n} \beta_{n,r}(x) = O(1),$$

and therefore, from Lemma 2,

$$\begin{aligned} (nh_n^{1/2}/k_n^{1/2})\mathbb{E}\left(\left|\hat{f}_n(x) - \hat{f}_{0,n}(x)\right|\mathbf{1}_{\Omega \setminus E_n}\right) &= (nh_n^{1/2}/k_n^{1/2})O(1)\mathbb{P}(\Omega \setminus E_n) \\ &= o(n) \exp\left(-\frac{1}{8}n\gamma_n^2\right) \\ &= o(1) \exp\left(-\frac{n}{k_n}\left(\frac{1}{8}k_n\gamma_n^2 - \frac{k_n}{n}\log n\right)\right). \end{aligned} \quad (4)$$

From (3) and (4) it suffices to take  $\gamma_n = k_n^{-1/2}$  to obtain the desired result. ■

The main theorem is now obtained without difficulty.

**Proof of Theorem 1.** Under the conditions  $h_n k_n \rightarrow \infty$ ,  $n = o(k_n^{1/2} h_n^{-1/2-\alpha})$ ,  $n = o(k_n^{5/2} h_n^{3/2})$  and  $k_n = o(n/\ln n)$ , Theorem 5 of [3] asserts that

$$\left(nh_n^{1/2}/k_n^{1/2}\right)\left(\hat{f}_{0,n}(x) - f(x)\right) \Rightarrow \mathcal{N}(0, \sigma^2),$$

while from Proposition 1,

$$\left(nh_n^{1/2}/k_n^{1/2}\right)\left(\hat{f}_{0,n}(x) - \hat{f}_n(x)\right) \xrightarrow{\mathbb{P}} 0.$$

Thus, the result is an immediate application of Slutsky's theorem. ■

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