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A note on extreme values and kernel estimators of sample boundaries

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Abstract

In a previous paper [3], we studied a kernel estimate of the upper edge of a two-dimensional bounded set, based upon the extreme values of a Poisson point process. The initial paper [1] on the subject treats the frontier as the boundary of the support set for a density and the points as a random sample. We claimed in [3] that we are able to deduce the random sample case from the point process case. The present note gives some essential indications to this end, including a method which can be of general interest.

Keywords and phrases: support estimation, asymptotic normality, kernel estimator, extreme values.

1 Introduction and main results

As in the early paper of Geffroy [1], we address the problem of estimating a subset D of \mathbb{R}^2 given a sequence of random points $\Sigma_n = \{Z_1, \dots, Z_n\}$ where the $Z_i = (X_i, Y_i)$ are independent and uniformly distributed on D . The problem is reduced to functional estimation by defining

$$D = \{(x, y) \in \mathbb{R}^2 / 0 \leq x \leq 1; 0 \leq y \leq f(x)\},$$

where f is a strictly positive function. Given an increasing sequence of integers $0 < k_n < n$, $k_n \uparrow \infty$, for $r = 1, \dots, k_n$, let $I_{n,r} = [(r-1)/k_n, r/k_n[$ and

$$U_{n,r} = \max \{Y_i / (X_i, Y_i) \in \Sigma_n; X_i \in I_{n,r}\}, \quad (1)$$

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where it is conveniently understood that $\max \emptyset = 0$. Now, let K be a bounded density, which has a support within a compact interval $[-A, A]$, a bounded first derivative and which is piecewise C^2 , and $h_n \downarrow 0$ a sequence of positive numbers. Following [3], Section 6, we define the estimate

$$\hat{f}_n(x) = \frac{1}{k_n} \sum_{r=1}^{k_n} K_n(x - x_r) \left(U_{n,r} + \frac{1}{n - k_n} \sum_{s=1}^{k_n} U_{n,s} \right), x \in \mathbb{R}, \quad (2)$$

where x_r is the center of $I_{n,r}$ and, as usually,

$$K_n(t) = \frac{1}{h_n} K\left(\frac{t}{h_n}\right), t \in \mathbb{R}.$$

The perhaps curious second term in brackets in formula (2) is designed for reducing the bias (see [3], Lemma 8). Note that \hat{f}_n can be rewritten as a linear combination of extreme values

$$\hat{f}_n(x) = \frac{1}{k_n} \sum_{r=1}^{k_n} \beta_{n,r}(x) U_{n,r},$$

where

$$\beta_{n,r}(x) = \frac{1}{k_n} K_n(x - x_r) + \frac{1}{k_n(n - k_n)} \sum_{s=1}^{k_n} K_n(x - x_s).$$

In the sequel, we suppose that f is α -Lipschitzian, $0 < \alpha \leq 1$, and strictly positive. Our result is the following:

Theorem 1 *If $h_n k_n \rightarrow \infty$, $n = o(k_n^{1/2} h_n^{-1/2-\alpha})$, $n = o(k_n^{5/2} h_n^{3/2})$ and $k_n = o(n/\ln n)$, then for every $x \in]0, 1[$,*

$$\left(n h_n^{1/2} / k_n^{1/2} \right) \left(\hat{f}_n(x) - f(x) \right) \Rightarrow \mathcal{N}(0, \sigma^2),$$

with $\sigma = \|K\|_2/c$.

2 Proofs

If formally the definition of \hat{f}_n is identical here and in [3] the fundamental difference lies in the fact that in [3] the sample is replaced by a homogeneous Poisson point process with a mean measure $\mu_n = nc\lambda_{1D}$ where λ is the Lebesgue measure of \mathbb{R}^2 and $c^{-1} = \lambda(D)$. Here we denote by $\Sigma_{0,n}$ this point process and we need, for the sake of approximation, two further Poisson point processes $\Sigma_{1,n}$ and $\Sigma_{2,n}$. The point processes $\Sigma_{j,n}$ are constructed as in [2], extending an original idea of J. Geffroy. Given a sequence $\gamma_n \downarrow 0$, consider independent Poisson random variables $N_{1,n}$, $M_{1,n}$, $M_{2,n}$, independent of the sequence (Z_n) , with parameters $\mathbb{E}(N_{1,n}) = n(1 - \gamma_n)$ and $\mathbb{E}(M_{1,n}) = \mathbb{E}(M_{2,n}) = n\gamma_n$. Define $N_{0,n} = N_{1,n} + M_{1,n}$, $N_{2,n} = N_{0,n} + M_{2,n}$ and take $\Sigma_{j,n} = \{Z_1, \dots, Z_{N_{j,n}}\}$, $j = 0, 1, 2$. For $j = 0, 1, 2$ we define $U_{j,n,r}$ and $\hat{f}_{j,n}$ by imitating (1) and (2). Finally, let us introduce the event $E_n = \{\Sigma_{1,n} \subseteq \Sigma_n \subseteq \Sigma_{2,n}\}$. The following lemma is the starting point of our "random sandwiching" technique.

Lemma 1 *One always has $\hat{f}_{1,n} \leq \hat{f}_{0,n} \leq \hat{f}_{2,n}$. Moreover, if E_n holds, $\hat{f}_{1,n} \leq \hat{f}_n \leq \hat{f}_{2,n}$.*

Proof : The definition of the random sets $\Sigma_{j,n}$, $j = 0, 1, 2$ implies that $\Sigma_{1,n} \subseteq \Sigma_{0,n} \subseteq \Sigma_{2,n}$. Thus, since $\beta_{n,r}(x) \geq 0$ for all $r = 1, \dots, k_n$, we have $\hat{f}_{1,n} \leq \hat{f}_{0,n} \leq \hat{f}_{2,n}$. Similarly, E_n implies that $\hat{f}_{1,n} \leq \hat{f}_n \leq \hat{f}_{2,n}$. ■

The success of the approximation between \hat{f}_n and $\hat{f}_{0,n}$ is based upon two lemmas. The first one shows how large is the probability of the event E_n .

Lemma 2 *For n large enough,*

$$\mathbb{P}(\Omega \setminus E_n) \leq 2 \exp\left(-\frac{1}{8}n\gamma_n^2\right).$$

Proof : Using the Laplace transform of a Poisson random variable X with parameter $\lambda > 0$, we get for $\varepsilon/2\lambda$ small enough,

$$\mathbb{P}(|X - \lambda| > \varepsilon) < \exp(-\varepsilon^2/4\lambda),$$

see for instance Lemma 1 in [2]. Clearly, $\Omega \setminus E_n = \{N_{1,n} > n\} \cup \{N_{2,n} < n\}$ and thus

$$\mathbb{P}(\Omega \setminus E_n) \leq \exp\left(-\frac{n\gamma_n^2}{4(1-\gamma_n)}\right) + \exp\left(-\frac{n\gamma_n^2}{4(1+\gamma_n)}\right).$$

The lemma follows. ■

The second lemma is essential to control the approximation obtained when the event E_n holds.

Lemma 3 *If $k_n = o(n/\log n)$ and $n = O(k_n^{1+\alpha})$, then uniformly on $r = 1, \dots, k_n$,*

$$\mathbb{E}(U_{2,n,r} - U_{1,n,r}) = O\left(\frac{k_n\gamma_n}{n}\right).$$

Proof : Let us define $m_{n,r} = \min_{x \in I_{n,r}} f(x)$ and $M_{n,r} = \max_{x \in I_{n,r}} f(x)$. Then,

$$\begin{aligned} & \mathbb{E}(U_{2,n,r} - U_{1,n,r}) \\ &= \int_0^{M_{n,r}} (\mathbb{P}(U_{2,n,r} > y) - \mathbb{P}(U_{1,n,r} > y)) dy \\ &= \int_0^{m_{n,r}} (\mathbb{P}(U_{2,n,r} > y) - \mathbb{P}(U_{1,n,r} > y)) dy + \int_{m_{n,r}}^{M_{n,r}} (\mathbb{P}(U_{2,n,r} > y) - \mathbb{P}(U_{1,n,r} > y)) dy \\ &\stackrel{def}{=} A_{n,r} + B_{n,r}. \end{aligned}$$

Introducing $\lambda_{n,r} = \int_{I_{n,r}} f(x)dx$, we can write $A_{n,r}$ as

$$A_{n,r} = \int_0^{m_{n,r}} \exp\left(\frac{n(1-\gamma_n)}{k_n}(y - k_n\lambda_{n,r})\right) dy - \exp\left(\frac{n(1+\gamma_n)}{k_n}(y - k_n\lambda_{n,r})\right) dy.$$

Now, $A_{n,r}$ is expanded as a sum $A_{1,n,r} + A_{2,n,r}$ with

$$\begin{aligned} A_{1,n,r} &= \frac{k_n}{n(1-\gamma_n)} \exp\left(\frac{n(1-\gamma_n)}{k_n}(m_{n,r} - k_n\lambda_{n,r})\right) - \frac{k_n}{n(1+\gamma_n)} \exp\left(\frac{n(1+\gamma_n)}{k_n}(m_{n,r} - k_n\lambda_{n,r})\right), \\ A_{2,n,r} &= \frac{k_n}{n(1+\gamma_n)} \exp(-n(1+\gamma_n)\lambda_{n,r}) - \frac{k_n}{n(1-\gamma_n)} \exp(-n(1-\gamma_n)\lambda_{n,r}). \end{aligned}$$

The part $A_{2,n,r}$ is easily seen to be a $o(n^{-s})$ where s is a arbitrarily large exponent under the condition $k_n = o(n/\log n)$. Now, If a, b, x, y are real numbers such that $x < y < 0 < b < a$, we have $0 < ae^y - be^x < (a-b) + b(y-x)$. Applying to $A_{1,n,r}$ this inequality yields

$$A_{1,n,r} \leq \frac{k_n}{n} \frac{2\gamma_n}{(1-\gamma_n^2)} + (M_{n,r} - m_{n,r}) \frac{2\gamma_n}{(1+\gamma_n)}.$$

Under the hypothesis that f is α -Lipschitzian, and the condition $n = O(k_n^{1+\alpha})$, we have $(M_{n,r} - m_{n,r}) = O(k_n/n)$, so that $A_{n,r} = A_{1,n,r} + A_{2,n,r} = O(k_n\gamma_n/n)$. Now, for $m_{n,r} \leq y \leq M_{n,r}$, it is easily seen that

$$\mathbb{P}(U_{2,n,r} > y) - \mathbb{P}(U_{1,n,r} > y) \leq 2\gamma_n \frac{n}{k_n} (M_{n,r} - m_{n,r}),$$

and thus

$$B_{n,r} \leq 2\gamma_n \frac{n}{k_n} (M_{n,r} - m_{n,r})^2 = O\left(\frac{k_n}{n} \gamma_n\right).$$

Clearly, the bounds on $A_{n,r}$ and $B_{n,r}$ are uniform in $r = 1, \dots, k_n$, and thus we obtain the result. \blacksquare

We quote a technical lemma.

Lemma 4 *If $k_n = o(n)$ and $h_n k_n \rightarrow \infty$ when $n \rightarrow \infty$,*

$$\lim_{n \rightarrow \infty} \sum_{r=1}^{k_n} \beta_{n,r}(x) = 1.$$

Proof : Remarking that

$$\sum_{r=1}^{k_n} \beta_{n,r}(x) = \frac{n}{n - k_n} \frac{1}{k_n} \sum_{r=1}^{k_n} K_n(x - x_r),$$

the result follows from the well-known property

$$\lim_{n \rightarrow \infty} \frac{1}{k_n} \sum_{r=1}^{k_n} K_n(x - x_r) = 1,$$

see for instance [3], Corollary 2. \blacksquare

The next proposition is the key tool to extend the results obtained on Poisson processes to samples.

Proposition 1 *If $k_n = o(n/\log n)$, $h_n k_n \rightarrow \infty$, and $n = O(k_n^{1+\alpha})$, then, for every $x \in]0, 1[$,*

$$(nh_n^{1/2}/k_n^{1/2}) \mathbb{E}\left(\left|\hat{f}_n(x) - \hat{f}_{0,n}(x)\right|\right) \rightarrow 0.$$

Proof : From Lemma 1, we have

$$\begin{aligned} \mathbb{E}\left(\left|\hat{f}_n(x) - \hat{f}_{0,n}(x)\right| \mathbf{1}_{E_n}\right) &\leq \mathbb{E}\left(\hat{f}_{2,n}(x) - \hat{f}_{1,n}(x)\right) \\ &= \sum_{r=1}^{k_n} \beta_{n,r}(x) \mathbb{E}(U_{2,n,r} - U_{1,n,r}) \\ &\leq \sum_{r=1}^{k_n} \beta_{n,r}(x) \max_{1 \leq s \leq k_n} \mathbb{E}(U_{2,n,s} - U_{1,n,s}) \\ &= O\left(\frac{k_n \gamma_n}{n}\right), \end{aligned}$$

in view of Lemma 3 and Lemma 4. As a consequence,

$$(nh_n^{1/2}/k_n^{1/2})\mathbb{E}\left(\left|\hat{f}_n(x) - \hat{f}_{0,n}(x)\right|\mathbf{1}_{E_n}\right) = O\left(k_n^{1/2}h_n^{1/2}\gamma_n\right). \quad (3)$$

Now, let $M = \sup\{f(x), x \in [0, 1]\}$. Then, applying Lemma 4 again,

$$\max\left\{\hat{f}_n(x), \hat{f}_{0,n}(x)\right\} \leq M \sum_{r=1}^{k_n} \beta_{n,r}(x) = O(1),$$

and therefore, from Lemma 2,

$$\begin{aligned} (nh_n^{1/2}/k_n^{1/2})\mathbb{E}\left(\left|\hat{f}_n(x) - \hat{f}_{0,n}(x)\right|\mathbf{1}_{\Omega \setminus E_n}\right) &= (nh_n^{1/2}/k_n^{1/2})O(1)\mathbb{P}(\Omega \setminus E_n) \\ &= o(n) \exp\left(-\frac{1}{8}n\gamma_n^2\right) \\ &= o(1) \exp\left(-\frac{n}{k_n}\left(\frac{1}{8}k_n\gamma_n^2 - \frac{k_n}{n}\log n\right)\right). \end{aligned} \quad (4)$$

From (3) and (4) it suffices to take $\gamma_n = k_n^{-1/2}$ to obtain the desired result. ■

The main theorem is now obtained without difficulty.

Proof of Theorem 1. Under the conditions $h_n k_n \rightarrow \infty$, $n = o(k_n^{1/2} h_n^{-1/2-\alpha})$, $n = o(k_n^{5/2} h_n^{3/2})$ and $k_n = o(n/\ln n)$, Theorem 5 of [3] asserts that

$$\left(nh_n^{1/2}/k_n^{1/2}\right)\left(\hat{f}_{0,n}(x) - f(x)\right) \Rightarrow \mathcal{N}(0, \sigma^2),$$

while from Proposition 1,

$$\left(nh_n^{1/2}/k_n^{1/2}\right)\left(\hat{f}_{0,n}(x) - \hat{f}_n(x)\right) \xrightarrow{\mathbb{P}} 0.$$

Thus, the result is an immediate application of Slutsky's theorem. ■

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