Bocop - A collection of examples

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Project-Teams Commands

Abstract: In this document we present a collection of classical optimal control problems which have been implemented and solved with Bocop. We recall the main features of the problems and of their solutions, and describe the numerical results obtained.

Key-words: optimal control, direct method, nonlinear programming, automatic differentiation, singular arc, state constraint

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BOCOP - Un catalogue d'exemples

Résumé : Ce document présente un catalogue d'exemples de problèmes de contrôle optimal et leur résolution avec Bocop. On rappelle les principales caractéristiques des problèmes avec leur solutions, et on décrit les résultats numériques obtenus.

Mots-clés : contrôle optimal, méthode directe, optimisation non linéaire, différentiation automatique, arc singulier, contrainte d'état

Contents

1   Overview

The Bocop project aims to develop an open-source toolbox for solving optimal control problems, with collaborations involving industrial and academic partners. Optimal control (optimization of dynamical systems governed by differential equations) has numerous applications in the fields of transportation, energy, process optimization, and biology. Bocop is developed in the framework of the Inria-Saclay initiative for an open source optimal control toolbox (2010-2012), and is supported by the Commands team.

Please visit the website for the latest news and updates. You are welcome to suscribe to the trimestrial newsletter, and join the forum as well.

The Bocop team hopes you have a nice experience with the software !

Website: www.bocop.org
Contact: Pierre Martinon (`pierre.martinon@inria.fr`).

Platform

The native platform for Bocop is Linux, but Bocop can also be compiled under MacOS X and Windows. We are considering the possibility of providing precompiled binaries for the thirdparty softwares.

Interface

The current GUI for Bocop is written in Scilab, and a standalone GUI is planned for future versions. Bocop can be used in command line mode, especially for experienced users. However, we recommend using the GUI, at least for the first steps. Also, some graphical features such as interpolation for the initial point are much easier to use with the GUI.

Core

The core files for Bocop are written in C++ and released under the Eclipse Public License. User supplied functions can be written in plain C, and do not require advanced programming skills. We plan to provide import guidelines for models already written in different languages, such as Fortran.

Thirdparty

Bocop currently uses Ipopt (with MUMPS as linear solver) for solving the nonlinear programming problem resulting from the direct transcription of the optimal control problem. Bocop relies on ADOL-C (with ColPack for the sparsity) to compute derivatives of the objective and constraints by automatic differentation.

In this document we present a collection of classical optimal control problems which have been implemented and solved with Bocop. We recall the main features of the problems and of their solutions, and describe the numerical results obtained. Our numerical tests use generally 100 time steps or so, with initialization of the control and state variables by appropriate constants. The solution is computed in a few seconds.

Users are encouraged to experiment with the data in these problems in order to get acquainted with the use of Bocop. It is interesting to observe how the convergence is affected by changes in the initialisation of the control and state, the number of time steps, or the discretization scheme. A further step might be to make changes in the dynamics or cost function.

We hope that providing these documented examples will help users to write and solve their own applications with Bocop. The following problems are sorted in four general categories: integrator systems, process control, mechanical systems and aerospace, and PDE control of parabolic equations.

2   Integrator systems

2.1   Generic form

We consider *integrator systems* of the form

$$x^{(k)}(t) = u(t), \quad t \in [0, T], \tag{1}$$

for $k = 1$ to 3. The state variables are therefore $y_1 = x$, and for $k > 1$, $y_2 = \dot{x}, \ldots, y_k = x^{(k-1)}$. The cost function is $\int_0^T \ell(t, u(t), y(t))dt$, with

$$\ell(t, u(t), y(t)) := \alpha x(t) + \beta_1 x^2(t) + \beta_2 \dot{x}^2(t) + \gamma u(t) + \delta u^2(t). \tag{2}$$

Setting the constants $\alpha, \ldots, \delta$ allows for a wide variety of cost functions (note that of course $\beta_2 = 0$ when $k = 1$). We add the control and state constraints for all t

$$u(t) \in [-1, 1]; \quad y(t) \geq a. \tag{3}$$

2.2   First-order system

While these examples are very simple, they nevertheless show some typical behavior that will be extended later to higher order systems. Consider first the problem

$$\begin{cases} Min \displaystyle\int_0^T x^2(t) + \gamma u(t) + \delta u^2(t) \quad dt \\ \dot{x}(t) = u(t), \quad t \in [0, T], \quad x(0) = x^0. \end{cases} \tag{4}$$

If $(\gamma, \delta) = (1, 0)$, $x(0) = 1$, and $T > 1$, then the solution is $u(t) = -1$ for $t \in [0, 1]$, and $u(t) = 0$ for $t > 1$. In particular, the control is discontinuous but piecewise continuous. If we change δ to a small positive value, say 0.1, we see that the control is continuous, althought it varies sharply when the time comes close to 1.

The user may experiment what happens when the state constraint threshold a is positive: again the control is discontinuous when $\delta = 0$, and continuous when $\delta > 0$.

We next discuss the optimal control of two second order integrator systems.

2.3   Fuller problem

Here is a very classical example of a chattering phenomenon [6]:

$$Min \int_0^T x^2(t)dt; \quad \ddot{x}(t) = u(t) \in [-1, 1]. \tag{5}$$

The solution is, for large enough T, bang-bang (i.e., with values alternativement ± 1), the switching times geometrically converging to a value $\tau > 0$, and then the (trivial) *singular arc* $x = 0$ and $u = 0$. These switches are not easy to reproduce numerically. We display in figure 1 the control, with a zoom on the entry point of the singular arc.

• *Numerical simulations: problem **fuller***
Discretization: Gauss II with 1000 steps.
We take here $T = 3.5$, $x(0) = 0, \dot{x}(0) = 1$, $x(T) = \dot{x}(T) = 0$ and $u(t) \in [-10^{-2}, 10^{-2}]$.
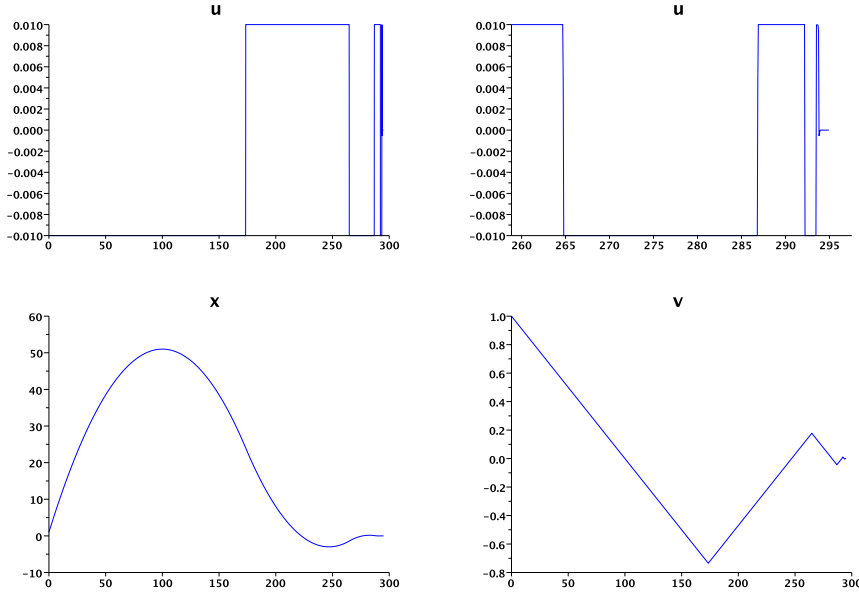
Figure 1: Fuller problem: chattering control (with zoom); x and v.

2.4 Second order singular regulator

We consider a second order singular regulator problem, see Aly [1], or [2]:

$$Min \int_0^T x^2(t) + \dot{x}^2(t) \quad dt; \quad \ddot{x}(t) = u(t) \in [-1, 1]. \tag{6}$$

The difference with Fuller's problem is that the cost function includes a penalization of the "speed" $\dot{x}(t)$. We observe in figure 2 the occurence of a singular arc, the optimal control being of the form *bang (-1) - singular*.

• *Numerical simulations: problem* **regulator**
Discretization: Runge-Kutta 4 with 1000 steps. We take here $T = 5$, $x(0) = 0, \dot{x}(0) = 1$.

2.5 Third order state constraints

Robbins [14] considered the following family of problems:

$$Min \frac{1}{2} \int_0^T \alpha y(t) + \frac{1}{2} u(t)^2 \quad dt; \quad y^{(3)}(t) = u(t); \quad y(t) \geq 0.$$

It has been proved by Robbins [14] that, for appropriate initial conditions, the exact solution has infinitely many isolated contact points, such that the length of unconstrained arcs decreases geometrically. Detailed computations can be found in [8]. Therefore the isolated contact points have an accumulation point; the latter is followed by the trivial singular arc $u = 0$, $y = 0$. It is not easy to reproduce numerically this behavior, since the unconstrained arcs rapidly become
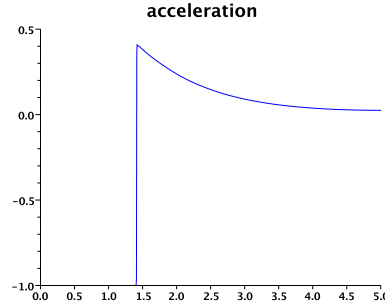
Figure 2: regulator problem: control.

too small to be captured by a given time discretization. We display in Figure 3 the value of the first state component and of the control, computed with Bocop.

• *Numerical simulations: problem* ***state_constraint_3***
Discretization: Runge-Kutta 4 with 100 steps.
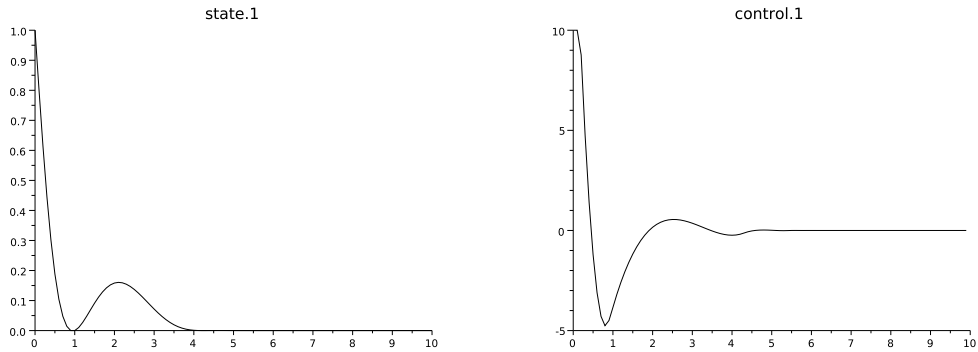*We take here* $\alpha = 3$, $T = 10$, $y(0) = (1, -2, 0)$.

Figure 3: Robbins example: first order state constraint and control.

It seems that no "generic" (stable with respect to a perturbation) example of a third order state constraint with a regular entry/exit point for a singular arc is known. It is conjectured that no such point exists.

Jacobson et al. [11] considered the following example:

$$Min \frac{1}{2} \int_0^T u(t)^2 \, dt; \ \ y^{(3)}(t) = u(t); \ \ y(t) \le y_{max}.$$

with initial condition for which there is no boundary arc, and one or two touch points.

Fourth order state constraints
No example with a nontrivial boundary arc is known, and it is conjectured that this does not

occur. Let us mention the example studied by Jacobson et al. [11]:

$$Min\frac{1}{2}\int_0^T u(t)^2\ dt;\ \ y^{(4)}(t) = u(t);\ \ |y(t)| \le y_{max}.$$

3 Process control

3.1 Jackson problem

Consider the model in [10], also discussed in Biegler [4], of reactions $A \Leftrightarrow B \Rightarrow C$. Here the control $u(t) \in [0, 1]$ is the fraction of catalyst, and we want to maximize the production of C. The problem is written as

$$
\begin{array}{rcl}
Max & c(T) \\
\dot{a}(t) & = & u(t)\,(k_1 a(t) - k_2 b(t)) \\
\dot{b}(t) & = & u(t)\,(k_1 a(t) - k_2 b(t)) - (1 - u(t))k_3 b(t) \\
\dot{c}(t) & = & u(t)(1 - u(t))k_3 b(t) \\
u(t) & \in & [0, 1]
\end{array}
\tag{7}
$$

Note that, $a(t) + b(t) + c(t)$ being an invariant, we could eliminate the third state variable. However, we decided to keep it in our implementation. We display on figure 4 the second state variable and control. We observe that, as expected, the control has a singular arc, with a *bang (1) - singular - bang(0)* structure.

• *Numerical simulations: problem* **jackson**
Discretization: Gauss II with 300 steps.
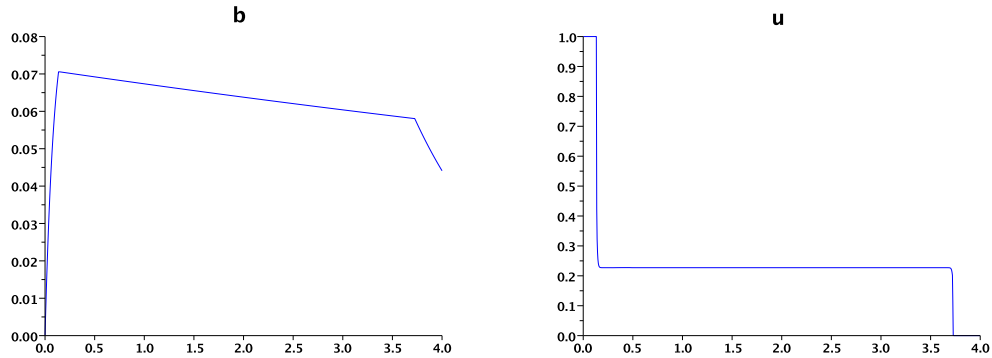*We take here $k_1 = k_3 = k_4 = 1, k_2 = 10, T = 4, a(0) = 1, b(0) = c(0) = 0$.*

Figure 4: Jackson problem: concentration of "B" and control.

3.2 Methane production

We study here a bio-reactor problem involving methane production with micro-algae. The objective is to maximize the methane production by the biomass. The dynamics of the biomass x, substrate s and micro-algae y are as follows:

$$
\begin{array}{rcl}
Max & \int_0^T x(t)u(t)\,dt \\
\dot{y}(t) & = & \frac{\mu(t)y}{1+y} - ry - uy \\
\dot{s}(t) & = & u\beta(\gamma y - s) - \mu_2(s)x \\
\dot{x}(t) & = & (\mu_2(s) - u\beta)x \\
u(t) & \in & [0, 1]
\end{array}
\tag{8}
$$

The growth rate of the algae depends on the day/night cycle and is of the form

$$\mu(t) = \left\{ \begin{array}{ll} \bar{\mu} & 0 \leq t < T/2 \\ 0 & T/2 \leq t < T \end{array} \right. \tag{9}$$

The metabolic activity of the biomass is

$$\mu_2(s) = \mu_2^m \frac{s}{K_s + s} \tag{10}$$

We have a periodicity constraint, with the value for the initial/final being free

$$y(T) = y(0), \ s(T) = s(0), \ x(T) = x(0). \tag{11}$$

We display the states and control below, and observe that the periodicity constraints are satisfied. The control seems to present a *bang (0) - singular - bang (0)* structure. The spikes at the day/night switch may be numerical artifacts.

• *Numerical simulations: problem* **methane**
Discretization: Gauss II with 500 steps.
*We take here $\bar{\mu} = 0.5, r = 0.005, T = 10, \mu_2^m = 0.1, K_s = 5, \beta = 1, \gamma = 100$.*
We also add the constraint $x(\cdot) \geq 50$.
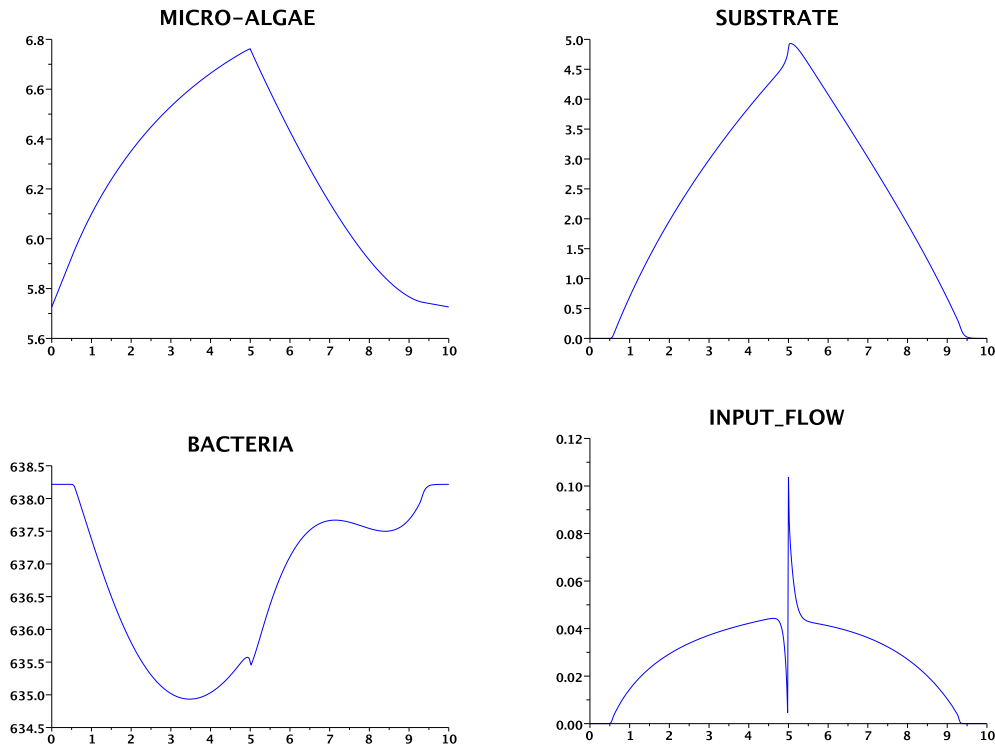
*Methane problem: states y, s, x and control.*

Comparison of discretization methods

We set here a final time $T = 50$ corresponding to five day/night cycles. It is worth noting that we obtain a periodic solution of period 10 (one cycle) while the constraint only appears at 0 and T. Let us compare several methods for 100 and 200 discretization steps.
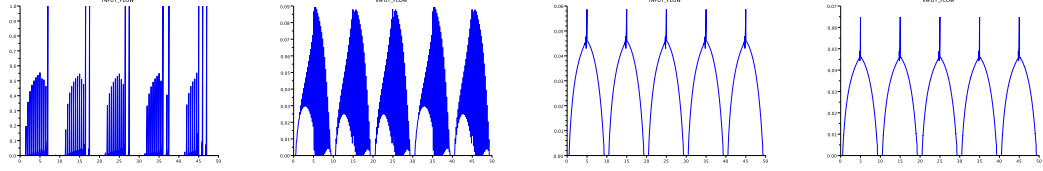
Methane problem (200 steps): RK4, Gauss II, Radau II A, Lobatto III C

We notice oscillations in several methods. A way to fix this behaviour is to force a parametrized control, for instance continuous piecewise linear. We set 10 control intervals per cycle (50 in total), and keep the 200 steps.
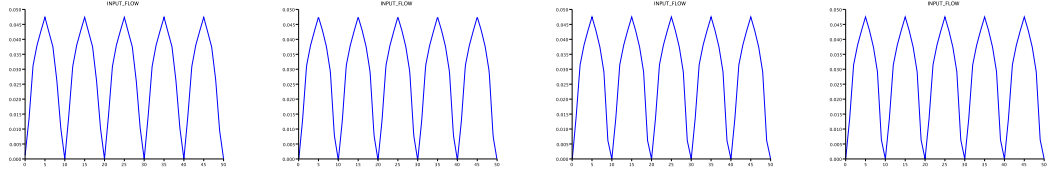
Continuous piecewise linear control: RK4, Gauss II, Radau II A, Lobatto III C

Now we obtain similar solutions regardless of the method. With this formulation, even low order methods (Euler, Midpoint) give a close solution.

4 Mechanical systems, aerospace

4.1 Clamped beam

The classical example of second-order state constraint is the one of the Euler-Bernoulli beam, see Bryson et al. [5]

$$Min \frac{1}{2} \int_0^1 u(t)^2 dt$$
$$\ddot{x}(t) = u(t); \quad x(t) \leq a$$
$$x(0) = x(1) = 0; \quad \dot{x}(0) = -\dot{x}(1) = 1.$$

The exact solution, for various values of a, is displayed in figure 5.
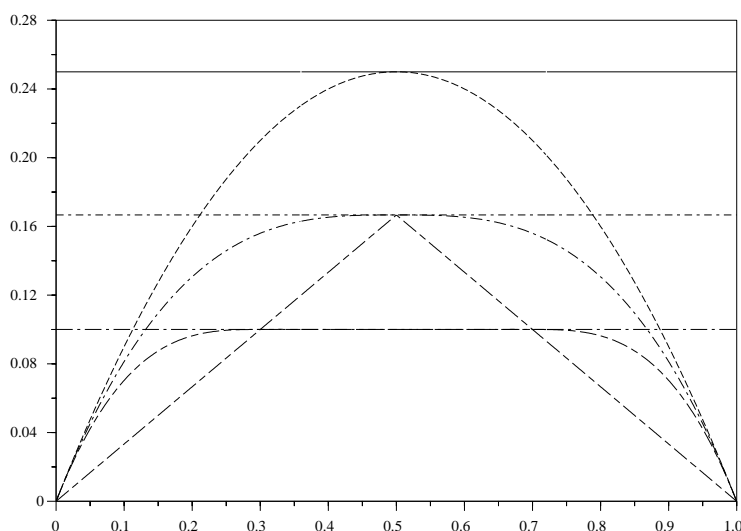

Figure 5: Shape of a beam: the three cases and the locus of junction points

The qualitative behavior is as follows:
If $a \geq 1/4$, the constraint is not active and the solution is $x(t) = t(1 - t)$.
If $a \in [1/6, 1/4]$, there is a touch point at $t = 1/2$.
If $a < 1/6$, there is a boundary arc without strict complementarity: the measure has its support at end points. The locus of switching points is piecewise affine.

Our numerical results are in accordance with the theory: we display in figure 6 the displacement and control when $a = 0.1$, i.e., when a boundary arc occurs.

• *Numerical simulations: problem **clamped_beam***
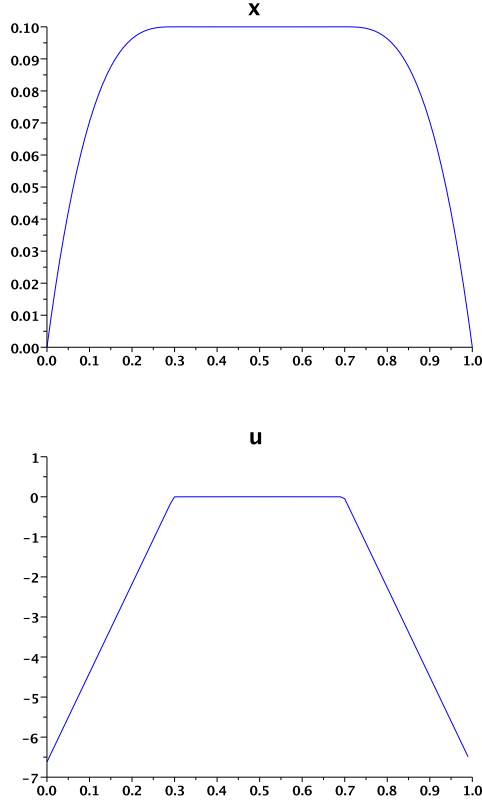*Discretization: Gauss II with 100 steps.*

Figure 6: Clamped beam: $a = 0.1$. Boundary arc and control.

4.2   Lagrange equations

We briefly recall the derivation of rational mechanics by the Lagrange approach [12]. Given generalized coordinates $q \in \mathbf{R}^N$, we note $E(q, \dot{q})$ and $U(q)$ the expression of cinetic and potential energy. The associated *Lagrangian function* and *action functional* are

$$L(q, \dot{q}) := E(q, \dot{q}) - U(q); \quad A(q, \dot{q}) := \int_0^T L(q(t), \dot{q}(t))dt, \tag{12}$$

where by $(q, \dot{q})$ we have denoted generalized coordinates as function of time, and their derivatives. The Lagrange equations are the Euler Lagrange equations of the classical calculus of variations, namely

$$0 = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = \frac{d}{dt} \left(\frac{\partial E(q, \dot{q})}{\partial \dot{q}} \right) - \frac{\partial E(q, \dot{q})}{\partial q} + U'(q). \tag{13}$$

Here by $U'(q)$ we denote the derivative of the potential function (opposite of the force deriving from the potential). The above relation must be understood as

$$\frac{d}{dt} \left(\frac{\partial E(q, \dot{q})}{\partial \dot{q}_i} \right) = \frac{\partial E(q, \dot{q})}{\partial q_i} - \frac{\partial U(q)}{\partial q_i}, \quad i = 1, \ldots, N. \tag{14}$$

The cinetic energy is usually of the form

$$E(q, \dot{q}) = \frac{1}{2} \dot{q}^\top M(q) \dot{q}, \tag{15}$$

where the $N \times N$ *mass matrix* $M(q)$ is symmetric, positive definite.
Since $\frac{\partial E(q, \dot{q})}{\partial \dot{q}_i} = M(q) \dot{q}_i$, the expression of the Lagrangian equations is then

$$\frac{d}{dt}(M(q)\dot{q})_i = \frac{1}{2}(\dot{q})^\top \frac{\partial M(q)}{\partial q_i} \dot{q} - \frac{\partial U(q)}{\partial q_i}, \quad i = 1, \dots, N. \tag{16}$$

For the simplest spring model, we have $E(q, \dot{q}) = \frac{1}{2} m \dot{q}^2$ and $U(q) = \frac{1}{2} k q^2$, where m and k are the mass and spring stiffness. The Lagrangian equations reduce to $m \ddot{q}(t) = -kq(t)$, as expected.

4.3 Holonomic constraints

A (vector) holonomic constraint $G(q) = 0$, with $G : \mathbf{R}^N \to \mathbf{R}^M$, generates (generalized) forces of the type $DG(q)^\top \lambda$, i.e., orthogonal to $KerDG(q)$. The simplest way to express the resulting equations is to apply the Euler-Lagrange equation to the "augmented" Lagrangian $L[\lambda](q, \dot{q}) := L(q, \dot{q}) + \lambda \cdot G(q)$. The resulting equations are, with the above notations

$$\frac{d}{dt}(M(q)\dot{q})_i = \frac{1}{2}(\dot{q})^\top \frac{\partial M(q)}{\partial q_i} \dot{q} + \lambda \cdot \frac{\partial G(q)}{\partial q_i} - \frac{\partial U(q)}{\partial q_i}, \quad i = 1, \dots, N \tag{17}$$

$$G(q) = 0. \tag{18}$$

This is an example of an algebraic differential system. The successive time derivatives of the algebraic constraint are

$$G^{(1)}(q) = DG(q)\dot{q}; \quad G^{(2)}(q) = D^2 G(q)(\dot{q})(\dot{q}) + DG(q)\ddot{q} \tag{19}$$

Substituting the expression of $\ddot{q}$ in (17), we obtain an expression of the form

$$G^{(2)}(q) = DG(q)M(q)^{-1}D^\top G(q)\lambda + F(q, \dot{q}) = 0. \tag{20}$$

If $DG(q)$ is onto, and $M(q)$ is positive definite, then $DG(q)M(q)^{-1}D^\top$ is invertible, meaning that we can eliminate the algebraic variable λ from the algebraic equation (20). This is a highly desirable property for the numerical schemes, and hence, the reader is advised to use the second derivative of the holonomic constraint in the discretizated problem, rather than the holonomic constraint itself.

Of course the intial condition $(q^0, \dot{q}^0)$ should be compatible with the holonomic constraint, i.e., it should satisfy

$$G(q^0) = G^{(1)}(q) = DG(q^0)\dot{q}^0 = 0. \tag{21}$$

4.4   Inverted pendulum
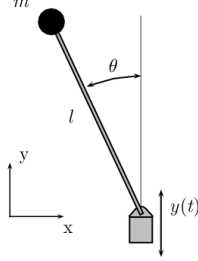
The inverted pendulum has Lagrangian $L = \frac{1}{2}m\dot{\theta}^2 - g\cos\theta$, and equation $m\ddot{\theta} = g\sin\theta$ where θ is the angle to the vertical. Alternatively, let (x, y) be the Cartesian coordinates of the position of the pendulum, subject to the constraint $G(x, y) = \frac{1}{2}(x^2 + y^2 - 1) = 0$. The Lagrangian is then

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + mgy + \frac{1}{2}\lambda(x^2 + y^2 - 1), \tag{22}$$

and the mechanical equations are

$$m\ddot{x} = \lambda x + u, \quad m\ddot{y} = \lambda y - mg. \tag{23}$$

where we have set an horizontal force as the control u.

We want to minimize the objective

$$Min \int_0^T x^2(t) + (y(t) - 1)^2 + u^2(t) \quad dt$$

Figure 7 shows the states x, y, the control u and multiplier λ.

• *Numerical simulations: problem* **pendulum**
Discretization: Runge Kutta 4 with 400 steps.
We take here $T = 12$, $m = 1$ and $g = 1$.
The final conditions are

$$\begin{array}{ll} x(T) = 0 & , \quad y(T) = 1 \\ \dot{x}(T) = 0 & , \quad \dot{y}(T) = 0 \end{array}$$

The initial conditions are

$$\begin{array}{ll} x(0) = -0.4794255 & , \quad y(0) = 0.8775826 \\ \dot{x}(0) = 1.0530991 & , \quad \dot{y}(0) = 0.5753106 \end{array}$$
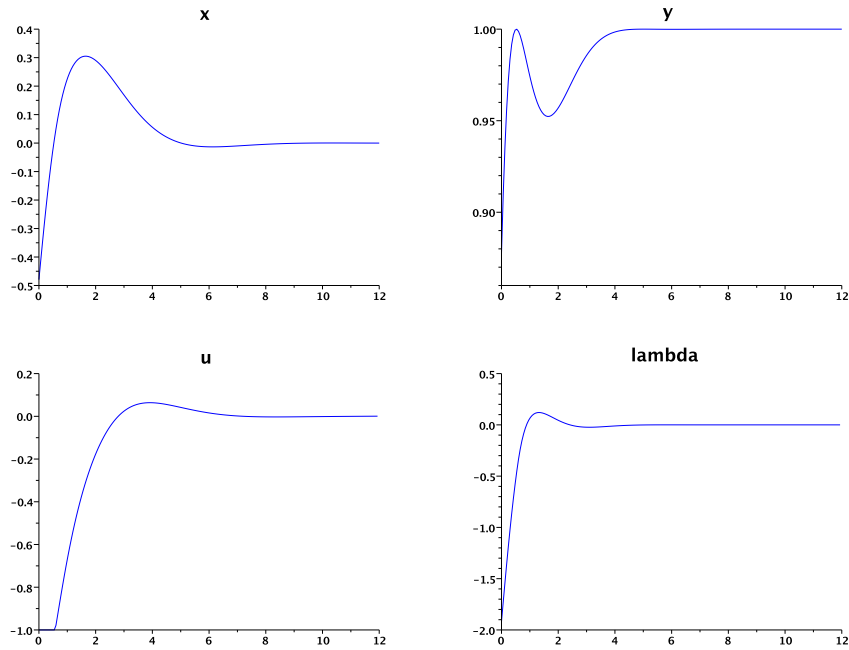
Figure 7: Inverted pendulum: states x, y, control u and multiplier λ.

4.5 Goddard problem

This well-known problem (see for instance [7, 15]) models the ascent of a rocket through the atmosphere, and we restrict here ourselves to vertical (monodimensional) trajectories. The state variables are the altitude, speed and mass of the rocket during the flight, for a total dimension of 3. The rocket is subject to gravity, thrust and drag forces. The final time is free, and the objective is to reach a certain altitude with a minimal fuel consumption, ie a maximal final mass.

All units are renormalized.

$$(P_3) \begin{cases} \max \ m(T) \\ \dot{r} = v \\ \dot{v} = -\frac{1}{r^2} + \frac{1}{m}(T_{max}u - D(r,v)) \\ \dot{m} = -bT_{max}u \\ u(\cdot) \in [0,1] \\ r(0) = 1, \ v(0) = 0, m(0) = 1, \\ r(T) = 1.01 \\ D(r(\cdot), v(\cdot)) \leq C \\ T \ free \end{cases}$$

The drag is $D(r,v) := Av^2\rho(r)$, with the volumic mass is $\rho(r) := exp(-k*(r-r0))$.
We use the parameters $b = 7$, $T_{max} = 3.5$, $A = 310$, $k = 500$ and $r0 = 1$.

The Hamiltonian is an affine function of the control, so singular arcs may occur. We consider here a path constraint limiting the value of the drag effect: $D(r,v) \leq C$. We will see that depending on the value of C, the control structure changes. In the unconstrained case, the optimal trajectory presents a singular arc with a non-maximal thrust. When C is set under the maximal value attained by the drag in the unconstrained case, a constrained arc appears. If C is small enough, the singular arc is completely replaced by the constrained arc.
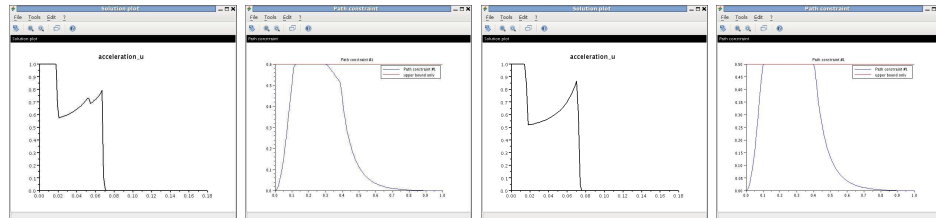
● *Numerical simulations: problem* **goddard**

Discretization: Heun with 100 steps.
Try to change the value of C and observe the resulting control structure after optimization. You can access this value at `Definition > Bounds > Paths Constraints`
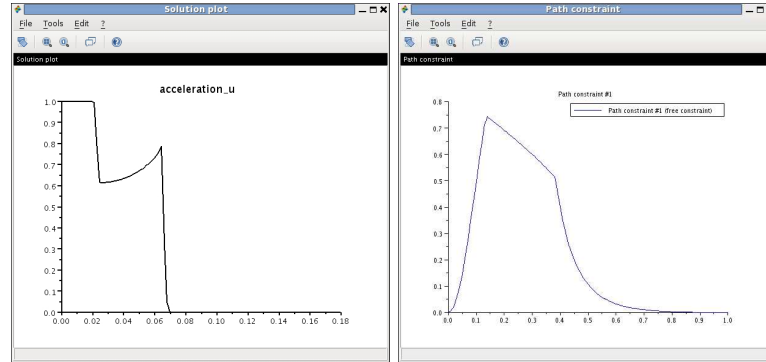
Goddard problem: drag constraint and control - unconstrained

Goddard problem: drag constraint and control - $C = 0.5, 0.6$

5 PDE control of parabolic equations

The space discretization of parabolic equations allows to obtain large scale, stiff ODE models for which an implicit Euler scheme is convenient. In the case of complex geometries, one should import the dynamics from finite elements libraries such as FreeFem (available on FreeFem.org). Relevent references on this subject are Barbu [3], Hinze et al. [9], Tröltzsch [16], and of course the pioneering book by J.L. Lions [13].

5.1 Control of the heat equation

We next give a simple example for the one dimensional heat equation, over the domain $\Omega = [0, 1]$. We set $Q = \Omega \times [0, T]$, where the final time is fixed. The control $u(t)$ is either (i) over a part of the domain, with Dirichlet conditions, or (ii) at the boundary by the Neumann condition. So the state equation is in case (i)

$$\frac{d}{dt}y(x, t) - c_0 \, y_{xx}(x, t) = \chi_{[0,a]}c_1 \, u(t), \quad (x, t) \in Q, \tag{24}$$

$$y(\cdot, 0) = y_0(x); \quad y(0, t) = y(1, t) = 0, \quad t \in [0, T], \tag{25}$$

where $0 < a \leq 1$, and $\chi_{[0,a]}$ is the characteristic function of $[0, a]$, and in case (ii)

$$\frac{d}{dt}y(x, t) - c_0 \, y_{xx}(x, t) = 0, \quad (x, t) \in Q, \tag{26}$$

$$y(\cdot, 0) = y_0(x); \quad y_x(0, t) = -c_1 u(t); \quad y_x(1, t) = 0, \quad t \in [0, T]. \tag{27}$$

The cost function is, for $\gamma \geq 0$ and $\delta \geq 0$:

$$\frac{1}{2}\int_Q y(x, t)^2 dx dt + \int_0^T \left(\gamma u(t) + \delta u(t)^2\right) dt. \tag{28}$$

We discretize in space by standard finite difference approximations.

● *Numerical simulations: problem **heat***
As an example, we take 50 space variables, with $c_0 = 0.02$, $c_1 = 20$, and a final time T=20. The discretization method is implicit Euler with 200 steps.
We set here $\gamma = \delta = 0$, which gives a singular arc for the control.
We display on Fig.8 the results in the case of the Dirichlet boundary condition ($a = 0$). Fig.9 shows the Neumann case, this time with c0 = 0.2.
We can clearly see the differences between the boundary conditions $y(1,t) = 0$ and $y_x(1,t) = 0$.
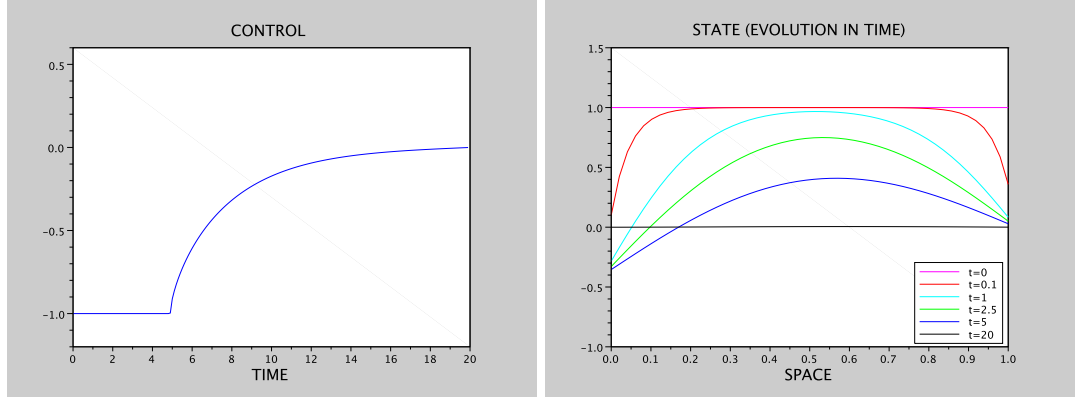

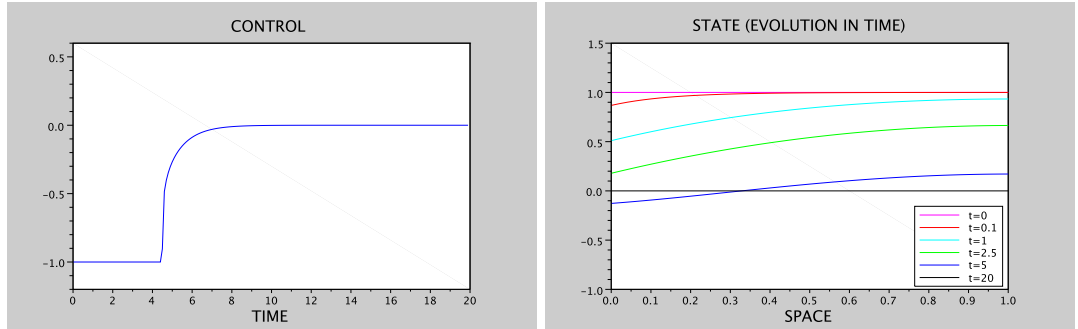Figure 8: Heat equation, Dirichlet condition, $u(t)$ and $y(\cdot, t)$.

Figure 9: Heat equation, Neumann condition, $u(t)$ and $y(\cdot, t)$.

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