On the inversion of computable functions

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September 26, 2012

Abstract

The question of the computability of diverse operators arising from mathematical analysis has received a lot of attention. Many classical operators are not computable, and the proof often does not resort to computability theory: the function under consideration is not computable simply because it is not continuous. A more challenging problem is then its computable invariance: is the image of every computable point computable? Very often it happens that it is not the case, and the proof is usually much more evolved, based on computability-theoretic constructions. This empirical observation raises the following question: is it a coincidence that discontinuous functions often happen to be non-computably invariant? Or do these functions share a strong form of discontinuity that prevents them to be computably invariant? A positive answer was brought by Pour-El and Richards through their so-called "First Main Theorem" [PER89]: for certain classes of closed linear operators continuity is equivalent to computable invariance.

In this paper, we focus on inverses of computable functions and introduce a discontinuity notion that prevents computable invariance. This result is applicable in many situations, unifies many ad hoc constructions and sheds light on the relationship between computability and continuity. The strength of this result lies in the fact that verifying that a function satisfies the property is much easier than disproving its computable invariance.

We present two applications of our main result. First it enables us to answer the following open question in the negative: if the sum of two shift-invariant ergodic measures is computable, must these measures be computable as well? Second, it enables to significantly improve Pour-El and Richards First Main Theorem by requiring the graph of the linear operator to be an effective G_δ-set instead of a closed set.

1 Introduction

When extending computability theory from the discrete to the continuum, topology comes into sight through the following fundamental principle: a computable function must be continuous, for suitable topologies. As an example, every Markov computable function on the computable reals numbers is continuous (Kreisel-Lacombe-Shœnfield and Tseitin theorem [KLS57, Tse62]); on represented spaces [Wei00], every computable function is continuous for the final topology of the representation.

This fact gives in many situations a simple way to show that a function is not computable: it is simply not continuous. But there is another possible reason for which a function f may fail to be computable: there may exist x such that $f(x)$ is not computable relative to x; even better, there may exist a computable x such that $f(x)$ is not computable, in which case we say that f is not computably invariant.

Thus we can identify two possible reasons (among others) for which a function fails to be computable: a topological one, a computability-theoretic one.

Example 1. The non-computability of the floor function is purely topological.

Example 2. The non-computability of the function mapping a real number to its binary expansion (the one with finitely many ones when the real number is a dyadic rational) is purely topological. Similarly, the function $x \mapsto 3x$ on binary sequences x interpreted as real numbers is not computable only for topological reasons ([Wei00]).

Example 3. The Turing jump $A \mapsto A'$ from Cantor space to itself is not computable due to both, a topological and a computability-theoretic issue: it is not continuous and for every A, A' is not computable relative to A.

Now, restricting to the 1-generic elements, the jump becomes continuous and computable relative to $\emptyset'$: for every 1-generic A, $A' \equiv_T A \oplus \emptyset'$, uniformly. So we could say that the non-computability of the jump restricted to the 1-generic elements is purely computability-theoretic.

Example 4. Braverman and Yampolsky have a number of results on the computability of the Julia sets [BY08]. Let us focus on the quadratic family $z \mapsto z^2 + \alpha$. They have a positive result: "the filled Julia set is always computable". More precisely, for every α, the filled Julia set K_α is computable relative to α. However the function $\alpha \mapsto K_\alpha$ is not computable just because it is not continuous. Hence the mapping $\alpha \mapsto K_\alpha$ is non-computable for purely topological reasons.

They also have a negative result: the Julia set is in general not computable. They first prove in [BY06] that there exists α such that the Julia set J_α is not computable relative to α. In [BY07] they strengthen this result by constructing a computable α such that J_α is not computable. It also happens that the mapping $\alpha \mapsto J_\alpha$ is not continuous. Hence its non-computability is both of a topological and of a computability-theoretic nature.

As these examples show, the topological and the computability-theoretic reasons are independent. However in many situations both occur at the same time, but proving that a function is discontinuous is often much easier than proving that it is not computably invariant. The following is quoted from [BHW08].

> Functions considered in analysis that are not computable are often not computable simply because they are discontinuous. But they may still be computably invariant, like the sign function. It is often, not only for real number functions, but for many other kinds of functions, an interesting task to show that some noncomputable function is not even computably invariant, i.e., that there exists a computable input element such that the output element is not computable. [...]
>
> a noncomputable function may nevertheless be computably invariant; i.e., map computable elements to computable elements, and if it is not computably invariant, it can be a challenge to construct a computable element of the domain that is mapped to a noncomputable element. Pour-El and Richards [80] have shown a general result that shows that for linear operators the situation is simpler.

The result of Pour-El and Richards ("First Main Theorem" in [PER89]) shows that in the case of linear operators with c.e. closed graph, computable invariance is equivalent to continuity (i.e., boundedness). Their result embodies many *ad hoc* constructions, such as Myhill's differentiable computable function whose derivative is not computable [Myh71]. A generalization of their theorem to certain algebraic structures was proved by Brattka [Bra99], applicable to operators on the set of compact subsets of $\mathbb{R}$.

In the same vein, our main result (Theorem 5.1) reduces the property of not being computably invariant to a strong discontinuity property, in the specific case of the inversion of a computable function. It can be seen as a partial answer to the open problem no. 7 in [PER89]. As an application we answer the following open question in the affirmative: are there non-computable shift-invariant ergodic measures whose sum is computable? We also apply our result to strengthen Pour-El and Richards First Main Theorem by requiring the graph of the linear operator to be a c.e. effective G_δ-set instead of a c.e. closed set.

The paper is organized as follows: in Section 2 we introduce basic notions of computable analysis, in particular simple results of independent interest about effective G_δ-sets in Polish spaces; in Section 3 we introduce a notion of continuous invertibility at a point and prove that for "almost" every point, if a function is computably invertible at that point then it is continuously invertible there (Theorem 3.1). In Section 4 we introduce the notion of an *irreversible function*, which in substance expresses that a function is topologically hard to inverse. In Section 5 we present our main result: a function that is topologically hard to inverse is computably hard to inverse, in particular it maps a non-computable point to a computable image. In Section 6 we present two applications of our main result: first to the ergodic decomposition, second, to improve Pour-El and Richards "First Main Theorem".

2 Background and notations

We assume familiarity with basic computability theory on the natural numbers. We implicitly use Weihrauch's notions of computability on effective topological spaces, based on the standard representation (see [Wei00] for more details), however we do not express them in terms of representations. The simple lemmas presented here are essentially classical results that we include in order for the paper to be self-contained.

2.1 Notations

In a metric space (X,d), if $x \in X$ and $r \in (0,+\infty)$ then we denote the open ball with center x and radius r by $B(x,r) = \{x' \in X : d(x,x') < r\}$. We denote the corresponding closed ball by $\overline{B}(x,r) = \{x' \in X : d(x,x') \leq r\}$. Observe that while the closed ball $\overline{B}(x,r)$ always contains the topological closure of the open ball $B(s,r)$, the inclusion may be strict[1]. In this paper, the overline will only be used to denote closed balls.

The Cantor space of infinite binary sequences, or equivalently subsets of $\mathbb{N}$, is denoted by $2^{\mathbb{N}}$.

W_e is a standard effective enumeration of the c.e. subsets of $\mathbb{N}$. W_e^A is a standard effective enumeration of the c.e. subsets of $\mathbb{N}$ relative to A.

If $A \in 2^{\mathbb{N}}$ then the Turing jump of A is $J(A) = A' = \{e \in \mathbb{N} : e \in W_e^A\}$. A' is c.e. relative to A. The halting set is $\emptyset'$.

[1] for instance, in the metric space $\mathbb{Z}$ of integers with the distance $d(n,p) = |n-p|$, $\overline{B}(n,1) = \{n-1,n,n+1\}$ is not the closure of $B(n,1) = \{n\}$.

2.2 Effective topology

An ***effective topological space*** $(X, \tau, \mathscr{B})$ consists of a topological space (X, τ) together with a countable basis $\mathscr{B} = \{B_0, B_1, \ldots\}$ numbered in such a way that the finite intersection operator is computable, i.e. there exists a total computable function $f : \mathbb{N}^2 \to \mathbb{N}$ such that $B_i \cap B_j = \bigcup_{k \in W_{f(i,j)}} B_k$. An open subset $U \subseteq X$ is ***effectively open*** if $U = \bigcup_{k \in W_e} B_k$ for some e. The enumeration of c.e. sets directly induces an enumeration of the effective open sets.

Let (X, τ) be a countably-based topological space. Two numbered basis $\mathscr{B} = \{B_0, B_1, \ldots\}$ and $\mathscr{B}' = \{B'_0, B'_1, \ldots\}$ are ***effectively equivalent*** if B'_n are uniformly effective open sets in $(X, \tau, \mathscr{B})$ and B_n are uniformly effective open sets in $(X, \tau, \mathscr{B}')$.

To a point $x \in X$ we associate $N(x) = \{n \in \mathbb{N} : x \in B_n\}$. A point x is ***computable*** if $N(x)$ is c.e.

2.3 Effective Polish spaces

An ***effective Polish space*** is a topological space such that there exists a dense sequence $s_0, s_1, \ldots$ of points, called ***simple*** points and a complete metric d inducing the topology, such that all the reals numbers $d(s_i, s_j)$ are computable uniformly in (i, j). Every effective Polish space can be made an effective topological space, taking as canonical basis the open balls $B(s, r)$ with s simple point and r positive rational together with a standard effective numbering. The intersection is computable as $B(s, r) \cap B(s', r')$ is the union of all the balls $B(s'', r'')$ with $d(s'', s) + r'' < r$ and $d(s'', s') + r'' < r'$, which can be effectively enumerated. Let X be an effective Polish space. An ***effective basis*** of X is a numbered basis of the topology on X which is effectively equivalent to the canonical basis of X.

In an effective Polish space, a point x is computable if and only if there exists a computable sequence of simple points converging to x exponentially fast.

In the Cantor space, constructing an infinite sequence is often achieved by progressively defining longer and longer initial segments of that sequence. For instance this technique is used in the finite extension method or priority methods. This type of construction has an analog in every Polish space.

Definition 1. A sequence of balls $(B_n)_{n \in \mathbb{N}}$ is ***shrinking*** if $\overline{B}_{n+1} \subseteq B_n$ and the radius of B_n converges to 0.

Lemma 2.1. *The intersection of a shrinking sequence of balls always contains one point. If the sequence is computable then so is that point.*

Proof. The centers of the balls form a Cauchy sequence, which converges to a point x by completeness. For each n, as $\overline{B}_{n+1}$ is closed and contains almost of the terms of the sequence, $x \in \overline{B}_{n+1}$. Hence $x \in \bigcap_n \overline{B}_{n+1} \subseteq \bigcap_n B_n$. The center of a ball of radius $< \varepsilon$ provides an ε-approximation of x, which enables to compute x up to any precision. □

Let X be an effective Polish space. $X' \subseteq X$ is an ***effective Polish subspace*** if it is an effective Polish space with the induced topology and such that the basis given by $\{X' \cap B_n : n \in \mathbb{N}\}$ (where $\{B_n : n \in \mathbb{N}\}$ is the canonical basis of X) is an effective basis of X'. Another way of expressing this technical condition is to require the canonical injection from X' to X to be a computable homeomorphism. Alexandrov theorem gives a way to obtain Polish subspaces of a Polish space, and has an effective version.

First we prove a result that is already well-known for closed sets and extends to effective G_δ-sets.

Lemma 2.2. *Let $D \subseteq X$ be an effective G_δ-set, i.e. an intersection of uniformly effective open sets. The following are equivalent:*

1. *the set $\{n \in \mathbb{N} : B_n \cap D \neq \emptyset\}$ is c.e.,*
2. *there exists a computable sequence $x_n \in D$ which is dense in D.*

Proof. $2 \Rightarrow 1$ is straightforward, as $B_n \cap D \neq \emptyset \iff \exists m, x_m \in B_n$. We prove $1 \Rightarrow 2$. Let $D = \bigcap_n U_n$ where U_n are uniformly effective open sets. Let V_0 be a ball intersecting D. Inductively define V_n as follows: assuming V_n intersects D, let V_{n+1} be a ball of radius $< 2^{-n}$ intersecting D and such that $\overline{V}_{n+1} \subseteq V_n \cap U_n$. Such a ball must exist and can be effectively found. The sequence V_n is a computable shrinking sequence, let x be the computable point in its intersection. As $V_{n+1} \subseteq U_n$, $x \in \bigcap_n U_n = D$. The construction can start from any ball intersecting D, uniformly, which enables to construct a computable sequence which is dense in D. □

We call such a set a ***c.e. effective G_δ-set***. Examples of such sets are given by the computable Baire theorem [YMT99, Bra01]: any dense effective G_δ-set is a c.e. effective G_δ-set.

Proposition 2.1 (Effective Alexandrov Theorem)**.** *Every c.e. effective G_δ-set is an effective Polish subspace of X.*

Proof. Let $D = \bigcap_n U_n$ be a c.e. effective G_δ-set. Let d be a complete computable metric on X. We slightly modify the classical proof (Theorem 3.11 in [Kec95]) to make it effective. Let $d_n : X \to [0, +\infty)$ be uniformly computable functions such that $d_n(x) > 0 \iff x \in U_n$ (such functions exist, as proved in [BP03]). d_n is a computable version of the distance to the complement of U_n. For $x, y \in D$, let

$$d'(x,y) = d(x,y) + \sum_{n=0}^{\infty} \min\left\{2^{-n-1}, \left|\frac{1}{d_n(x)} - \frac{1}{d_n(y)}\right|\right\}.$$

The function $d' : D \times D \to \mathbb{R}$ is computable. On D it is a complete metric that induces the same topology as d and the classical proof is effective in the sense that the corresponding bases are effectively equivalent. Lemma 2.2 provides a computable sequence $x_n \in D$ which is dense in D and can serve as special points in D. □

We will apply these concepts on the Polish space of Borel probability measures over the Cantor space to derive an important application of our main result.

Example 5. We consider the space $\mathscr{P}(2^{\mathbb{N}})$ of Borel probability measures over the Cantor space together with the complete metric

$$d(P,Q) = \sum_{w \in \{0,1\}^*} 2^{-|w|} |P[w] - Q[w]|.$$

The finite rational combination of Dirac measures are dense in $\mathscr{P}(2^{\mathbb{N}})$ and d is computable over them, so $\mathscr{P}(2^{\mathbb{N}})$ is an effective Polish space. A measure P is ***shift-invariant*** if $P[w] = P[0w] + P[1w]$ for every string w. The subset $\mathscr{I}$ of shift-invariant measures is closed so d is complete over $\mathscr{I}$ as well. $\mathscr{I}$ easily contains a dense computable sequence, so $\mathscr{I}$ is an effective Polish subspace of $\mathscr{P}(2^{\mathbb{N}})$. A shift-invariant measure P is ***ergodic*** if it cannot be written as $P = \frac{1}{2}(P_1 + P_2)$ with $P_1 \neq P_2$ both shift-invariant. The subset $\mathscr{E} \subseteq \mathscr{I}$ of ergodic shift-invariant measures is an effective G_δ-set which is dense, hence c.e., so it is an effective Polish subspace of $\mathscr{I}$ (see [Par61]).

We will be concerned with computability and Baire category, so we will naturally meet the notion of a 1-generic point: a point that does not belong to any "effectively meager set" in the following sense.

Definition 2. $x \in X$ is 1-***generic*** if x does not belong to the boundary of any effective open set. In other words, for every effective open set U, either $x \in U$ or there exists a neighborhood B of x disjoint from U.

By Baire theorem, every Polish space is a Baire space so 1-generic points exist and form a co-meager set.

2.4 Relative computability

Given a point x in an effective topological space X with basis $B_0, B_1, \ldots$, let $N(x) = \{n \in \mathbb{N} : x \in B_n\}$. By an ***enumeration of*** $N(x)$ we mean a total function $f : \mathbb{N} \to \mathbb{N}$ whose range is $N(x)$. It can be provided to a Turing machine *via* an oracle.

Given points x, y in effective topological spaces X, Y respectively, we say that y is ***computable relative to*** x if there is an oracle Turing machine M that, given any enumeration of $N(x)$ as oracle, outputs an enumeration of $N(y)$. We denote it by $M^x = y$. In other words, y is computable relative to x if $N(y)$ is enumeration reducible to $N(x)$. As proved by Selman [Sel71] and pointed out by Miller [Mil04], y is computable relative to x if and only if every enumeration of $N(x)$ computes an enumeration of $N(y)$ (uniformity is not explicitly required, but is a consequence).

Given a machine M, let

$$\text{Graph}_{X,Y}(M) := \{(x,y) \in X \times Y : M^x = y\}$$

A (possibly partial) function $f : X \to Y$ is ***computable*** if there is a machine M such that $\text{Graph}(f) \subseteq \text{Graph}_{X,Y}(M)$, i.e. for every $x \in \text{dom}(f)$, $M^x = f(x)$.

Relative computability can be formulated in terms of effective open sets. Only Corollary 2.1 and Lemma 2.6 will be used in the sequel, the rest of the section can be skipped at first reading.

From machines to open sets. To a machine M we associate a family of uniformly effective open sets $V_n \subseteq X$ in the following way: for each n, enumerate all the finite sequences of basic open sets of X on which the machine outputs B_n, and enumerate the intersection of the open sets from the finite sequence into V_n.

Lemma 2.3. *$x \in V_n$ if and only if there is an enumeration of $N(x)$ on which the machine outputs B_n.*

Lemma 2.4. *$\text{Graph}_{X,Y}(M) \subseteq \{(x,y) : \forall n, x \in V_n \iff y \in B_n\}$.*

The inclusion may be strict.

From open sets to machines. Conversely, to a family of uniformly effective open sets $V_n \subseteq X$ we associate the machine M that, on an enumeration of basic open sets of X, outputs all the numbers n such that one of the input open sets appear in the enumeration of V_n (the machine not only depends on the open sets V_n but also on their particular enumerations).

Lemma 2.5. *$\text{Graph}_{X,Y}(M) = \{(x,y) : \forall n, x \in V_n \iff y \in B_n\}$.*

Going from V_n to M and then from M to V'_n is an involution. Going from M to V_n and then from V_n to M' provides an extensional version of M in the sense that M' enumerates the same set on any enumeration of a given set, and M' computes what M computes where M is already extensional, i.e. $\text{Graph}_{X,Y}(M) \subseteq \text{Graph}_{X,Y}(M')$.

To summarize, we have the following characterization:

Corollary 2.1.
1. *$y \in Y$ is computable relative to $x \in X$ if and only if there exist uniformly effective open sets $V_n \subseteq X$ such that for all n, $y \in B_n \iff x \in V_n$.*
2. *$f : X \to Y$ is computable if and only if $f^{-1}(B_n)$ are effective open sets, uniformly.*

We may use this characterization as many arguments often become technically simpler and more direct using effective open sets rather than oracle Turing machines.

When a machine positively fails to compute a given point. Let X be an effective Polish space and Y an effective topological space. Let $x \in X$ and $y \in Y$. There are two possible ways in which a machine M may fail to compute x from y: either it diverges, or what it produces is incompatible with x. The latter can be checked in finite time. We write $M^y \perp x$ if there is some enumeration of $N(y)$ on which M outputs a ball B such that $x \notin \overline{B}$. Of course $M^y = x$ and $M^y \perp x$ cannot occur together. Observe that if $M^y = x$ then $M^y \perp x'$ for every $x' \neq x$ as X is Hausdorff.

Lemma 2.6. *Let $V_n \subseteq Y$ be the effective open sets associated to a machine M. One has*

$$\{(x,y) : M^y \perp x\} = \bigcup_n (X \setminus \overline{B}_n) \times V_n.$$

In particular, this set is effectively open.

Proof. $M^y \perp x$ if and only if there is an enumeration of $N(y)$ on which the machine outputs some B_n with $x \notin \overline{B}_n$ if and only if there exists n such that $y \in V_n$ and $x \notin \overline{B}_n$. □

In particular if $f : X \to Y$ is a computable function then

$$\begin{aligned}\{x : M^{f(x)} \perp x\} &= \{x : \exists n, f(x) \in V_n \text{ and } x \notin \overline{B}_n\} \\ &= \bigcup_n f^{-1}(V_n) \setminus \overline{B}_n\end{aligned}$$

is an effective open set.

2.5 Effective Banach space

In our applications we will consider the following particular class of effective Polish spaces.

Definition 3. An ***effective Banach space*** is a Banach space which is an effective Polish space with the metric induced by the norm, such that 0 is a computable point and the vector space operations are computable functions.

Many classical Banach spaces $\mathbb{R}$, $\mathscr{C}[0,1]$ (with the uniform norm) or $L^1[0,1]$ are effective Banach spaces.

2.6 Weihrauch reducibility

Definition 4. Let X,Y be effective topological space and $h : X \to Y$. A function $g : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is ***Weirauch reducible*** to h, written $g \leq_W h$, if there exist computable functions $\phi : 2^{\mathbb{N}} \to X$ and $\psi : 2^{\mathbb{N}} \times Y \to 2^{\mathbb{N}}$ such that $g(A) = \psi(A, h(\phi(A)))$.

Intuitively, g is Weihrauch reducible to h if g can be computed using one application of h. In the usual definition of Weihrauch reducibility, the function ϕ and ψ work on representations of points. However, as elements of the Cantor space are their own representations and using the fact that Y is implicitly endowed with the standard representation, it can be proved that the definition given here is equivalent to the usual one.

If the Turing Jump is reducible to h then h is not computably invariant. Indeed, let $x = \phi(\emptyset)$: x is computable but $h(x)$ is not, as it computes $\emptyset'$.

Recent advances on the structure of Weihrauch degrees can be found in [BdBP12].

3 A non-uniform result

Let X be an effective Polish space, Y an effective topological space and $f : X \to Y$ a computable function.

To introduce informally the results of this section, assume temporarily that f is one-to-one. If f^{-1} is computable, i.e. if x is uniformly computable relative to $f(x)$, then f^{-1} is continuous. As mentioned earlier uniformity is crucial here: if some x is computable relative to $f(x)$, f^{-1} need not be continuous at $f(x)$. Theorem 3.1 below surprisingly shows that a non-uniform version can still be obtained, valid for "almost every" x.

We now come to the formal definitions and results. We do not assume anymore that f is one-to-one.

We say that f is invertible at x if x is the only pre-image of $f(x)$; we say that f is locally invertible at x if x is isolated in the pre-image of $f(x)$. We consider stronger notions which not only require that x be uniquely determined by its image, but also that x can be recovered from the knowledge of its image. In a topological space, "knowing a point" means having access to a neighborhood basis of the point.

Definition 5. f is ***continuously invertible at*** x if the pre-images of the neighborhoods of $f(x)$ form a neighborhood basis of x, i.e. for every neighborhood U of x there exists a neighborhood V of $f(x)$ such that $f^{-1}(V) \subseteq U$.

f is ***locally continuously invertible at*** x if there exists a neighborhood B of x such that the restriction of f to B is continuously invertible at x, i.e. for every neighborhood U of x there exists a neighborhood V of $f(x)$ such that $B \cap f^{-1}(V) \subseteq U$.

The topological space Y is sequential, i.e. continuity notions can be expressed in terms of sequences, which may be more intuitive.

Proposition 3.1. *f is not continuously invertible at x if and only if there exist $\delta > 0$ and a sequence x_n such that $d(x,x_n) > \delta$ and $f(x_n)$ converge to $f(x)$.*

f is not locally continuously invertible at x if and only if for every $\varepsilon > 0$ there exist $\delta > 0$ and a sequence x_n such that $\varepsilon > d(x,x_n) > \delta$ and $f(x_n)$ converge to $f(x)$.

Let us illustrate these notions on a few examples.

Example 6. If f is one-to-one then f is continuously invertible at x if and only if f^{-1} is continuous at $f(x)$.

However if f is one-to-one and locally continuously invertible at x then f^{-1} may not be continuous at $f(x)$. For instance, let $f : [0,+\infty) \to S^1$, where $S^1 = [0,1]$ mod 1 is the unit circle, be defined by $f(x) = \frac{x}{1+x}$. f is bijective and locally continuously invertible at 0 (with $B = [0,1)$ e.g.) but f^{-1} is not continuous at 0.

Example 7. The real function $f(x) = x^2$ is continuously invertible exactly at 0, and locally continuously invertible everywhere (for $x \neq 0$ take for B an open interval avoiding 0).

Example 8. The two fundamental notions of computable and c.e. subset of $\mathbb{N}$ are underlay by two topologies on the Cantor space: the product topology induced by the cylinders; the Scott topology, induced by the sets $\{A \subseteq \mathbb{N} : F \subseteq A\}$ where F varies among the finite sets. The computable elements of the two effective topological spaces are the computable sets and the c.e. sets respectively.

The information about a set given by the Scott topology is usually strictly weaker than the information given by the product topology. In terms of computability, a manifestation of it is the existence of non-computable c.e. sets. In terms of topology, it is expressed by the fact that the identity from the product topology to the Scott topology, which we will call the enumeration operator and denote by Enum, is continuous and one-to-one but has a discontinuous inverse. More precisely, (i) Enum is continuously invertible exactly at $\mathbb{N}$: every cylinder containing $\mathbb{N}$ is a Scott open set; and (ii) Enum is locally continuously invertible exactly at the co-finite sets: if A is co-finite then let B be a cylinder specifying all the 0's in A, every cylinder containing A is the intersection of a Scott open set with B.

Intuitively, $\mathbb{N}$ is the only set that can be known entirely by simply enumerating it: if $A \subseteq \mathbb{N}$ does not contain some n then the question of the membership of n to A can never be answered from observing an enumeration of A (provided in an arbitrary order). For a co-finite set, a finite amount of additional information is needed to describe its characteristic function from its enumeration.

In general continuous invertibility at a point is strictly stronger than local continuous invertibility. It is not the case for linear operators, where a dichotomy appears. Following Pour-El and Richards [PER89], by a linear operator $T : X \to Y$ between Banach spaces we mean a linear function $T : \mathscr{D}(T) \to Y$ where $\mathscr{D}(T)$ is a subspace of X.

Proposition 3.2. *Let X, Y be Banach spaces and $T : X \to Y$ a one-to-one linear operator.*

- *If T^{-1} is bounded then T is continuously invertible everywhere.*
- *If T^{-1} is unbounded then T is nowhere locally continuously invertible.*

Proof. The first point simply follows from the fact that T^{-1} is continuous. Assume that T^{-1} is unbounded. There exists a sequence $a_n \in X$ such that $\|a_n\| = 1$ and $\|T(a_n)\| \to 0$. Let $x \in X$ and $\varepsilon > 0$. Take $\delta = \varepsilon/3$ and define $x_n = x + 2\delta a_n$: $T(x_n)$ converge to $T(x)$ and $\varepsilon > \|x - x_n\| > \delta$ for all n. □

Observe that in the case when T is not one-to-one, T is also nowhere locally continuously invertible, with exactly the same proof (one can take a single a with $\|a\| = 1$ and $\|T(a)\| = 0$).

We now come to our first result.

Theorem 3.1. *Let $f : X \to Y$ be a computable function and $x \in X$ a 1-generic point.*

If x is computable relative to $f(x)$ then f is locally continuously invertible at x.

Proof. Let M be a machine, $C_M = \{x : M^{f(x)} = x\}$ and $U_M = \{x : M^{f(x)} \perp x\}$. Of course, $C_M \cap U_M = \emptyset$. Let V_n be the effective open sets associated to M.

$$C_M = \{x : \forall n, x \in B_n \iff f(x) \in V_n\} \quad \text{by Lemma 2.5,}$$
$$U_M = \bigcup_n f^{-1}(V_n) \setminus \overline{B}_n \quad \text{by Lemma 2.6.}$$

In particular, U_M is an effective open set.

Let $x \in C_M$ be such that f is not locally continuously invertible at x. We prove that x belongs to the closure of U_M. Let B be some basic neighborhood of x and U_B witness that f is not locally continuously invertible at x, and let B_n be a neighborhood of x such that $\overline{B}_n \subseteq U_B$. As $x \in C_M$ and $x \in B_n$, V_n is a neighborhood of $f(x)$ so $B \cap f^{-1}(V_n) \nsubseteq U_B$ hence $B \cap f^{-1}(V_n) \setminus \overline{B}_n \neq \emptyset$. As a result, $B \cap U_M \neq \emptyset$. As it is true for every neighborhood B of x, x belongs to the closure of U_M. If x is 1-generic then $x \in U_M$, which contradicts $x \in C_M$. □

In the sequel we introduce a condition on f which roughly means that f is "almost nowhere" locally continuously invertible and that entails (i) the existence of an x that is not computable relative to $f(x)$ (Theorem 4.1) and, better, (ii) the existence of a non-computable x such that $f(x)$ is computable (Theorem 5.1).

4 Reversibility

We define two dual notions for a function: reversible (Section 4.1) and irreversible (Section 4.2). In the sense of Baire category, a reversible function is continuously invertible almost everywhere; an irreversible function is almost nowhere locally continuously invertible.

4.1 Reversible functions

Let X, Y be T_0 topological spaces. For a continuous function $f: X \to Y$, the following are equivalent:

- f is one-to-one and $f^{-1}: f(X) \to X$ is continuous,
- the initial topology of f is the topology of X, i.e. for every open set $U \subseteq X$ there exists an open set $V \subseteq Y$ such that $U = f^{-1}(V)$.

A function satisfying these conditions can be *reversed* in the sense that x can be recovered from $f(x)$ for every x: x is not only uniquely determined by $f(x)$, but a neighborhood basis of x can be progressively constructed from a neighborhood basis of $f(x)$.

We first consider a slight weakening of this notion.

Definition 6. We say that f is ***reversible*** if for every non-empty open set $U \subseteq X$ there is an open set $V \subseteq Y$ such that $\emptyset \neq f^{-1}(V) \subseteq U$.

This terminology is justified by the following proposition, saying that for almost every x, there is no "topological information" loss when applying f to x.

Proposition 4.1. *If f is reversible then it is continuously invertible at every point in a dense G_δ-set.*

Proof. For each basic ball $B \subseteq X$ there exists $V_B \subseteq Y$ such that $\emptyset \neq f^{-1}(V_B) \subseteq B$. Let U_n be the union of $f^{-1}(V_B)$ over all basic balls B of radius $< 2^{-n}$. U_n is a dense open set. If $x \in U_n$ for all n then f is continuously invertible at x. Indeed, for every n there exists a ball B of radius $< 2^{-n}$ such that $x \in f^{-1}(V_B) \subseteq B$. □

We say that f is ***effectively reversible*** if for every basic open set $B \subseteq X$ there exists a basic open set $V_B \subseteq Y$ such that $\emptyset \neq f^{-1}(V_B) \subseteq B$ and V_B can be computed from B. This notion is robust to a change of effective basis.

Proposition 4.1 has an effective version: if f is effectively reversible then for almost every x there is no "algorithmic information" loss when applying f to x.

Proposition 4.2. *If f is effectively reversible then there is a dense effective G_δ-set D such that $f_{|D}$ is one-to-one and its inverse is computable on $f(D)$, i.e. x is uniformly computable from $f(x)$ when $x \in D$.*

Proof. Let U_n be the union of all $f^{-1}(V_B)$ over all basic balls of radius $< 2^{-n}$. If $x \in \bigcap_n U_n$ then for each n there exists a basic ball B of radius $< 2^{-n}$ such that $f(x) \in V_B$, which can be found from any enumeration of $N(f(x))$. It gives an approximation of x within 2^{-n}. □

In particular if x is 1-generic then x is computable relative to $f(x)$.

4.2 Irreversible functions

We now consider the dual notion: an *irreversible* function is a function that is not reversible, even locally.

Definition 7. f is ***irreversible*** if for every open set $B \subseteq X$ the restriction $f_{|B}: B \to f(B)$ is not reversible.

Formally, f is irreversible if for every non-empty open set B there exists a non-empty open set $U_B \subseteq B$ such that there is no open set V satisfying $\emptyset \neq f^{-1}(V) \cap B \subseteq U_B$.

In other words, each pre-image of open set that intersects B does so outside U_B. If $x \in U_B$ then we will never know it from $f(x)$, even with the help of the advice $x \in B$.

Observe that one can assume w.l.o.g. that $f^{-1}(V) \cap B \nsubseteq \overline{U}_B$. Indeed, one can replace U_B by some ball $B(s,r)$ such that $\overline{B}(s,r) \subseteq U_B$.

An application of an irreversible function f to x comes with a loss of information about x, that can hardly be recovered. Being irreversible is orthogonal to not being one-to-one: the function $x \mapsto x^2$ is not one-to-one but not irreversible: x can be (continuously or computably) recovered from x^2; a one-to-one function can be irreversible if its inverse is dramatically discontinuous (examples of such functions will be encountered in the sequel).

In terms of sequences, f is irreversible if and only if for every B there exists a non-empty open set $U_B \subseteq B$ such that for every $x \in U_B$ there is a sequence $x_n \in B \setminus U_B$ such that $f(x_n)$ converge to $f(x)$.

As announced, the set of points at which an irreversible function is locally continuously invertible is small in the sense of Baire category.

Proposition 4.3. *Let f be irreversible. There is a dense G_δ-set D such that f is not locally continuously invertible at any $x \in D$.*

Proof. Let U_n be the union of U_B for all basic open sets B of radius $< 2^{-n}$. U_n is a dense open set. Let $x \in \bigcap_n U_n$. For each n there is a ball of radius $< 2^{-n}$ such that $x \in U_B$. For every neighborhood V of $f(x)$, $x \in f^{-1}(V) \cap B \neq \emptyset$ so $f^{-1}(V) \cap B \nsubseteq U_B$. □

In other words, for almost every x the application of f to x comes with a "topological information" loss.

The preceding proposition does not rule out the possibility that the restriction of f to a "large" set be continuously invertible (for instance, the characteristic function of the rational numbers is nowhere continuous, but its restriction to the co-meager set of irrational numbers is continuous). The next assertion shows that it is not possible.

Proposition 4.4. *Let f be irreversible and $C \subseteq X$ be such that $f_{|C}: C \to f(C)$ is an homeomorphism. C is nowhere dense.*

Proof. Assume the closure of C contains a ball B. $U_B \cap C$ is non-empty. Let $x \in U_B \cap C$. There exists a sequence $x_n \in B \setminus \overline{U_B}$ such that $f(x_n)$ converge to $f(x)$. By density of C in B, x_n can be taken in C. As $f_{|C}$ is an homeomorphism and $f(x_n)$ converge to $f(x)$, x_n should converge to x and eventually enter U_B, which gives a contradiction. □

Example 9. Let f be a constant function defined on the Polish space X. f is irreversible if and only if X is perfect, i.e. has no isolated point.

Example 10. Let $f : [0,1] \to [0,1]$ be the Cantor function (also called Devil's staircase). If the ternary expansion of x contains only 0's and 2's, $f(x)$ is obtained by replacing each 2 by 1 and interpreting the result as the binary expansion of $f(x)$. It defines f on the third-middle Cantor set and is extended as a constant function in the holes. f is almost-one-way: given an interval $B = (a,b)$, let U_B be an open interval (c,d) with $a < c < d < b$ on which f is constant. Knowing the value of $f(x)$, one cannot know that $x \in U_B$, even with the advice that x belongs to B. Indeed, $c \in B \setminus U_B$ has the same image as any $x \in U_B$.

In the definition of an irreversible function (Definition 7), B and U_B can be assumed w.l.o.g. to be basic balls.

Definition 8. f is ***effectively irreversible*** if U_B can be computed from B.

Remark 1. One easily checks that this notion is robust to a change of effective basis.

In the same way as Proposition 4.2 was the effective version of Proposition 4.1, the following result is the effective version of Proposition 4.4.

Theorem 4.1. *If f is effectively irreversible then for every 1-generic x, $x \not\leq_c f(x)$.*

Proof. The dense G_δ-set provided by Proposition 4.3 is effective when f is effectively irreversible so it contains every 1-generic point. Hence for every 1-generic x, f is not locally continuously invertible at x. We now apply Theorem 3.1. □

In other words, if x is 1-generic then the application of f to x comes with an "algorithmic information" loss. So if f is effectively irreversible then there exists some x that is not computable relative to $f(x)$.

4.3 Examples

Several well-known results in computability theory can be interpreted using Theorem 4.1 as consequences of the strong discontinuity of the inverse of some computable function.

Example 11. Consider the enumeration operator of Example 8. Enum is effectively irreversible: to each cylinder $B = [w]$ associate $U_B = [w0]$.

Applying Theorem 4.1 then gives: if A is 1-generic then A and $\mathbb{N} \setminus A$ have incomparable enumeration degrees. Such an A was first proved to exist in [Sel71], Theorem 2.9.

Example 12. Consider the projection $\pi_1 : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ which maps $A_1 \oplus A_2$ to A_1. π_1 is effectively irreversible: to $B = [w]$ associate $U_B = [w00]$. Applying Theorem 4.1 to π_1 and symmetrically to the second projection π_2 gives the classical result that if $A = A_1 \oplus A_2$ is 1-generic then A_1 and A_2 are Turing incomparable ([JP78]), which implies Kleene-Post theorem.

Again, linear operators provide a large class of examples.

Proposition 4.5. *Let X, Y be effective Banach spaces and $T : X \to Y$ a computable linear operator. Assume that either T is not one-to-one or T is one-to-one and T^{-1} is unbounded. T is effectively irreversible.*

Proof. To a ball $B = B(s,r)$ associate $U_B = B(s, r/2)$. According to the assumption about T, for every ε there exists a such that $\|a\| = r/2$ and $\|T(a)\| < \varepsilon$. For every $x \in U_B$ and $\varepsilon > 0$ there exists $\lambda \in \{-1, 1\}$ such that $r/2 \leq d(x + \lambda a, s) < r$, i.e. $x + \lambda a \in B \setminus U_B$. Indeed, $d(x+a,s) + d(x-a,s) \geq 2\|a\| = r$ and $d(x \pm a, s) < r$. Moreover, $d(T(x \pm \lambda a), T(x)) < \varepsilon$. □

Example 13. Applying Proposition 4.5 and Theorem 4.1 to the integration operator that maps $f \in \mathscr{C}[0,1]$ to $F : x \mapsto \int_0^x f(t)\,dt$ gives that if $f \in \mathscr{C}[0,1]$ is 1-generic then f is not computable relative to its primitive F that vanishes at 0.

Example 14. Applying Proposition 4.5 and Theorem 4.1 to the canonical injection from $\mathscr{C}[0,1]$ to $L^1[0,1]$ gives that if $f \in \mathscr{C}[0,1]$ is 1-generic then it is not computable relative to itself, as an element of $L^1[0,1]$. In other words, the description of f as an element of $L^1[0,1]$ contains strictly less algorithmic information than the description of f as an element of $\mathscr{C}[0,1]$.

Example 15. A function $f : \mathbb{N} \to \mathbb{N}$ can be described by enumerating its graph or by enumerating the complement of its graph. The former alternative gives in general strictly more information about the function than the latter.

On the Baire space $\mathbb{B}$, consider the product topology whose basic open sets are the cylinders: if F is a function with finite domain then the cylinder $[F]$ is the set of functions extending F. The negative topology is generated by the sets $\mathbb{B} \setminus [F]$ as a subbasis. The identity from the product topology to the negative topology is computable: from an enumeration of the graph of f one can enumerate the complement of the graph. The identity is also effectively irreversible: to a cylinder $B=[F]$ associate $U_B=[F\cup\{n\mapsto 0\}]$ where n is fresh, i.e. does not belong to the domain of F. Indeed, U_B does not contain a negative open set, even intersected with B.

By Theorem 4.1, if $f:\mathbb{N}\to\mathbb{N}$ is 1-generic then it is not computable relative to every co-enumeration of its graph.

5 The constructive result

We now present the main result of the paper. It is the constructive version of Theorem 4.1 as it makes $f(x)$ computable. The construction uses a priority argument with finite injury.

Theorem 5.1. *If f is effectively irreversible then there exists a non-computable x such that $f(x)$ is computable.*

If f is moreover one-to-one then $J\leq_W f^{-1}$ where $J: 2^{\mathbb{N}}\to 2^{\mathbb{N}}$ is the Turing jump.

The rest of the section is devoted to the proof of the theorem.

We fix any $A\in 2^{\mathbb{N}}$, relative to which the construction will be effective (the reader interested in the first part of the theorem only may take $A=\emptyset$). In particular we consider a one-to-one A-computable enumeration $n_0,n_1,\dots$ of A', the Turing jump of A. We construct $x\in X$ such that $f(x)$ is computable relative to A and A' is computable relative to the pair (A,x). We construct a shrinking sequence of balls B_n and define x as the unique member of their intersection. Of course, B_n must not be computable otherwise x would be computable. The sequence B_n is constructed in stages: at stage s we define $B_n[s]$ and for each n the sequence $B_n[s]$ is stationary, with limit B_n. For each s, the sequence $B_n[s]$ is shrinking, so the limiting sequence B_n will be shrinking as well. One may imagine, for each s, the sequence $B_n[s]$ as an infinite path in a tree. At stage $s+1$, n_s is enumerated into A' and the current path branches at depth n_s.

In order to make $f(x)$ computable relative to A we enumerate along the construction all its basic neighborhoods into a list L. L is the union of a computable growing sequence of finite lists L_s. At stage s, the current neighborhood of $f(x)$, denoted $\mathscr{V}_s$ is the (finite) intersection of the members of L_s. As $L_s\subseteq L_{s+1}$, $\mathscr{V}_{s+1}\subseteq\mathscr{V}_s$.

In order to construct the list L, we start with a technical point: in the space X, we make an effective change of simple points and basic open sets. We can assume w.l.o.g. that the radius of U_B is at most half the radius of B. Given a basic ball B, consider the computable sequence $U_B^{(n)}$ defined inductively by $U_B^{(0)}=B$ and $U_B^{(n+1)}=U_{U_B^{(n)}}$. $U_B^{(n)}$ is a computable shrinking sequence and the unique member a of $\bigcap_n U_B^{(n)}$ is computable, uniformly in B. The canonical enumeration B_j of basic balls induces a computable dense sequence a_j, which will serve as simple points.

We then change the basic open subsets of X. Let $(V_k)_{k\in\mathbb{N}}$ be the canonical enumeration of the basic open subsets of Y.

Lemma 5.1. *There is a double-sequence of open sets $O_{k,i}\subseteq X$ such that*

- $O_{k,i}\subseteq O_{k,i+1}$,
- $f^{-1}(V_k)=\bigcup_{i\in\mathbb{N}}O_{k,i}$,
- *the predicate $a_j\in O_{k,i}$ is decidable.*

Proof. By a standard diagonalization argument (computable Baire theorem on the real numbers), there exists a computable dense sequence of positive real numbers r_n such that $d(s_i,a_j)\neq r_n$ for all i,j,n. The metric balls $B(s_i,r_n)$ form an effective basis and the predicate $a_j\in B(s_i,r_n)$ is decidable. $f^{-1}(V_k)$ can be expressed as an effective union of such balls. Define $O_{k,i}$ as the union of the first i balls enumerated into $f^{-1}(V_k)$. □

We now proceed to the construction of the sequence $B_n[s]$ for each stage s. For each s, $B_n[s]$ will be a shrinking sequence, $x[s]$ will be defined as the unique member of their intersection and will be one of the points $\{a_j:j\in\mathbb{N}\}$.

Stage 0. We start with a ball $B_0[0]$ of radius 1, $B_{n+1}[0]=U_{B_n[0]}$ and $\{x[0]\}=\bigcap_n B_n[0]$. Start with $L_0=\emptyset$ and $\mathscr{V}_0=Y$. Observe that for each n, $B_n[0]\cap f^{-1}(\mathscr{V}_0)$ is non-empty as it contains $x[0]$.

Stage $s+1$. First, L_{s+1} is obtained by adding to L_s all the numbers $k\leq s$ such that $x[s]\in O_{k,s}$. Let $\mathscr{V}_{s+1}$ be the intersection of the open sets V_k with $k\in L_{s+1}$.

Let $n=n_s$ be the next element enumerated into A'. Let $B_{n+1}[s+1]$ be a ball satisfying $\overline{B}_{n+1}[s+1]\subseteq f^{-1}(\mathscr{V}_{s+1})\cap B_n[s]\setminus\overline{B}_{n+1}[s]$. Such a ball exists: $f^{-1}(\mathscr{V}_{s+1})\cap B_{n+1}[s]$ is non-empty as it contains $x[s]$, f is irreversible and $B_{n+1}[s]=U_{B_n[s]}$. For $n'\leq n$, let $B_{n'}[s+1]=B_{n'}[s]$. For $n'>n$ define by induction $B_{n'+1}[s+1]=U_{B_{n'}[s+1]}$. Let $\{x[s+1]\}=\bigcap_n B_n[s+1]$.

Verification. By construction one has $\overline{B}_{n+1}[s]\subseteq B_n[s]$ and $B_{n+1}[s]=U_{B_n[s]}$ for sufficiently large n so $B_n[s]$ is a shrinking sequence.

We call the *settling time* of n the minimal number s such that $n_{s'}\geq n$ for all $s'\geq s$.

We say that $n\in A'$ is a *forward element* if no element $m<n$ is enumerated into A' after the enumeration stage of n: in other words, the settling time of n coincides with its enumeration stage. As A' is infinite, it has infinitely many forward elements.

Claim 1. For each n, $B_n[s]$ is a stationary sequence.

Proof. Let s be the settling time of n: $B_n[s]=B_n[s_0]$ for all $s\geq s_0$. □

Let B_n be its limit. B_n is a shrinking sequence as well, let x be the member of its intersection. Observe that the sequence $x[s]$ converges to x: given ε, let n be such that B_n has radius $< \varepsilon$ and s_0 be the settling time n: for all $s \geq s_0$, $x[s] \in B_n[s] = B_n$ so $d(x[s], x) < \varepsilon$.

Claim 2. $f(x)$ is computable.

Proof. We prove that a basic open set V_k contains $f(x)$ if and only if k is enumerated into the list $L = \bigcup_s L_s$.

If $k \in L_s$ for some s, let n be a forward element which is enumerated at some stage $s' \geq s$. $x \in \overline{B}_{n+1} = \overline{B}_{n+1}[s'+1] \subseteq f^{-1}(\mathscr{V}_{s'+1}) \subseteq f^{-1}(\mathscr{V}_s) \subseteq f^{-1}(V_k)$.

Now let V_k be a basic neighborhood of $f(x)$. Let i_0 be such that $x \in O_{k,i_0}$. As $x[s]$ converge to x there is s such that $x[s] \in O_{k,i_0}$ for all $s' \geq s$. Let $t = \max(s, i_0)$: $x[t] \in O_{k,i_0} \subseteq O_{k,t}$ so k must be added to the list at stage $t+1$ or earlier. □

Claim 3. A' is computable relative to the pair (A, x).

Proof. Let p_i be the increasing sequence of forward elements. A' can be computed from the sequence p_i and the (A-computable) enumeration of A'.

From x and A one can inductively compute the sequence p_i. First, p_0 is the minimal n such that $x \notin B_{n+1}[0]$. Once p_i is known, let s be the stage at which p_i is enumerated into A', i.e. $n_s = p_i$. p_{i+1} is the minimal $n > p_i$ such that $x \notin B_{n+1}[s+1]$. □

6 Applications

We now present two applications of Theorem 5.1.

6.1 Ergodic decomposition

In the effective Polish space $\mathscr{I}$ of shift-invariant probability measures, the set $\mathscr{E}$ of ergodic measures is a dense effective G_δ-set, so it is an effective Polish subspace by Proposition 2.1. The space $\mathscr{E} \times \mathscr{E}$ of pairs of shift-invariant ergodic measures is an effective Polish space as well.

Theorem 6.1. *The function $(P, Q) \mapsto P + Q$ defined on $\mathscr{E} \times \mathscr{E}$ is effectively irreversible.*

Proof. On the space of probability measures over the Cantor space, we consider the complete metric

$$d(P, Q) = \sum_{w \in \{0,1\}^*} 2^{-|w|} |P[w] - Q[w]|.$$

The subspace $\mathscr{I}$ of shift-invariant measures is closed so d is a complete metric over $\mathscr{I}$. d is no more complete over $\mathscr{E}$. Still, we will consider the metric d instead of a complete metric over $\mathscr{E}$ to prove that the sum function is effectively irreversible. More precisely we will work with the basis given by the intersections of the canonical basis of $\mathscr{I}$ with $\mathscr{E}$: first, it is an effective basis of $\mathscr{E}$ which is an effective Polish subspace of $\mathscr{I}$; second, by Remark 1 the property of being effectively irreversible does not depend on the choice of a particular basis as long as it is effective.

Let $B \subseteq \mathscr{I} \times \mathscr{I}$ be an open set and $(P_0, Q_0) \in B$ with $P_0 \neq Q_0$. Let $\varepsilon > 0$ be such that $d(P_0, Q_0) > \varepsilon$ and $B(P_0, \varepsilon) \times B(Q_0, \varepsilon) \subseteq B$. Let $\delta = \varepsilon/4$ and $U_B = B(P_0, \delta) \times B(Q_0, \delta) \subseteq B$. Observe that U_B can be effectively obtained from B. We now show how a pair $(P_1, Q_1) \in U_B$ can be moved outside U_B, but still inside B, nearly without changing its sum. By the choice of δ, if $(P_1, Q_1) \in U_B$ then $d(P_1, Q_1) > 2\delta$. For $\lambda \in [0, 1]$, define

$$P(\lambda) = \lambda P_1 + (1 - \lambda) Q_1,$$
$$Q(\lambda) = \lambda Q_1 + (1 - \lambda) P_1.$$

Observe that $P(\lambda) + Q(\lambda) = P_1 + Q_1$ and

$$d(P_1, P(\lambda)) = d(Q_1, Q(\lambda)) = (1 - \lambda) d(P_1, Q_1).$$

As $d(P_1, Q_1) > 2\delta$ there exists $\lambda \in (0, 1)$ such that $(1 - \lambda) d(P_1, Q_1) = 2\delta$. One has

$$d(P_0, P(\lambda)) \leq d(P_0, P_1) + d(P_1, P(\lambda)) < 3\delta < \varepsilon$$

and

$$d(P_0, P(\lambda)) \geq d(P_1, P(\lambda)) - d(P_0, P_1) > \delta,$$

and similarly $\delta < d(Q_0, Q(\lambda)) < \varepsilon$ so $(P(\lambda), Q(\lambda)) \in B \setminus \overline{U}_B$.

Observe that the shift-invariant measures $P(\lambda)$ and $Q(\lambda)$ are not ergodic. As the ergodic measures are dense in the set of shift-invariant measures, there exist two sequences P_n, Q_n of ergodic measures converging to $P(\lambda)$ and $Q(\lambda)$ respectively. As $(P(\lambda), Q(\lambda))$ belongs to the open set $B \setminus \overline{U}_B$, we can assume w.l.o.g. that $(P_n, Q_n) \in B \setminus \overline{U}_B$ for all n. The mapping $(P, Q) \mapsto P + Q$ is continuous so $P_n + Q_n$ converge to $P(\lambda) + Q(\lambda) = P_1 + Q_1$. □

As an application of Theorem 5.1, we solve the open question raised in [Hoy11]. V'yugin [V'y97] proved that the ergodic decomposition is not effective when considering infinite combinations of ergodic measures. We prove that it is already non-effective in the finite case.

Corollary 6.1. *There exist two ergodic shift-invariant measures P and Q such that neither P nor Q is computable but $P + Q$ is computable.*

Proof. Apply Theorem 6.1 and Theorem 5.1. □

6.2 Linear operators

As another application of Theorem 5.1, we significantly improve Pour-El and Richards so-called First Main Theorem.

We first reformulate their theorem into an equivalent form. A c.e. closed subset of $X \times Y$ is a closed subset for the product topology which contains a dense computable sequence (x_n, y_n). Pour-El and Richards theorem concerns linear operator $T : X \to Y$ satisfying the following conditions:

1. its domain $\mathscr{D}(T)$ is dense in X,
2. its domain $\mathscr{D}(T)$ contains a computable sequence (e_n) whose linear span is dense in $\mathscr{D}(T)$,
3. its graph is closed.

If T is unbounded, these conditions do not imply that the graph of T is a c.e. closed set, as the sequence $(e_n, T(e_n))$ is not necessarily dense in Graph(T). However we argue here that conditions 2. and 3. can be replaced by the stronger requirement that the graph of T be c.e. closed, giving an equivalent statement. Indeed, if T satisfies the three conditions above then one can consider the restricted operator T' defined by $\mathrm{Graph}(T') = \mathrm{closure}(\{(x, T(x)) : x \in \mathrm{span}(\{e_n : n \in \mathbb{N}\})\})$. As Graph$(T)$ is closed, $\mathrm{Graph}(T') \subseteq \mathrm{Graph}(T)$. T' has a c.e. closed graph: an effective enumeration of the rational linear combinations of $(e_n, T(e_n))$ provides a dense computable sequence. Observe that $\mathscr{D}(T')$ is still a vector subspace of X.

Now we reformulate Pour-El and Richards theorem using the c.e. closed graph condition. This formulation was suggested to the author by Brattka (personal communication).

Theorem 6.2 (Pour-El and Richards [PER89]**).** *Let X and Y be effective Banach spaces. Let $T : X \to Y$ be a linear operator whose graph is a c.e.* closed set *and whose domain is dense in X.*

Then T is unbounded if and only if there exists a computable $x \in X$ such that $T(x)$ is not computable.

We improve their theorem by requiring the graph of T to be an effective G_δ-set instead of a closed set.

Theorem 6.3. *Let X and Y be effective Banach spaces. Let $T : X \to Y$ be a linear operator whose graph is a c.e.* effective G_δ-set *and whose domain is dense in X.*

Then T is unbounded if and only if there exists a computable $x \in X$ such that $T(x)$ is not computable.

Proof. By Proposition 2.1, Graph(T) is an effective Polish subspace of $X \times Y$. We consider the first projection $\pi_1 : \mathrm{Graph}(T) \to X$, which is computable and one-to-one. We show that if T is unbounded then π_1 is effectively irreversible. It will enable us to conclude, as applying Theorem 5.1 will give a non-computable pair $(x, T(x))$ such that x is computable, which implies that $T(x)$ is not computable.

The proof that π_1 is effectively irreversible resembles the proof of Proposition 4.5. Let $B_X \subseteq X$ and $B_Y \subseteq Y$ be basic open sets such that $B := B_X \times B_Y$ intersects Graph(T). Let $U_B = B(x_0, q) \times B(y_0, r)$ intersect Graph(T) and such that $B(x_0, q) \subseteq B_X$, $B(y_0, 2r) \subseteq B_Y$. As Graph(T) is c.e., U_B can be effectively found. Let x be such that $(x, T(x)) \in U_B$. As T is unbounded, for every $\varepsilon > 0$ there exists $a_\varepsilon \in \mathscr{D}(T)$ such that $\|a_\varepsilon\| < \varepsilon$ and $\|T(a_\varepsilon)\| = r$. There exists $\lambda \in \{-1, 1\}$ such that $r \leq d(T(x + \lambda a_\varepsilon), y_0) < 2r$. If ε is small enough then $x + \lambda a_\varepsilon \in B(x_0, q)$. As a result, $(x + \lambda a_\varepsilon, T(x + \lambda a_\varepsilon)) \in B \setminus U_B$ and $x + \lambda a_\varepsilon$ is arbitrarily close to x. □

It would be interesting to have examples of linear operator satisfying conditions of Theorem 6.3 but not of Theorem 6.2.

7 Acknowledgements

The author wishes to thank Peter Gács, Emmanuel Jeandel and Cristóbal Rojas for helpful comments on a draft of the paper.

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