Transmission Eigenvalues in Inverse Scattering Theory

Fioralba Cakoni, Houssem Haddar

Transmission Eigenvalues in Inverse Scattering Theory

Fioralba Cakoni* Houssem Haddar †

January 11, 2012

1 Introduction

The interior transmission problem arises in inverse scattering theory for inhomogeneous media. It is a boundary value problem for a coupled set of equations defined on the support of the scattering object and was first introduced by Colton and Monk [28] and Kirsch [44]. Of particular interest is the eigenvalue problem associated with this boundary value problem, referred to as the transmission eigenvalue problem and, more specifically, the corresponding eigenvalues which are called transmission eigenvalues. The transmission eigenvalue problem is a nonlinear and non-selfadjoint eigenvalue problem that is not covered by the standard theory of eigenvalue problems for elliptic equations. For a long time research on the transmission eigenvalue problem mainly focussed on showing that transmission eigenvalues form at most a discrete set and we refer the reader to the survey paper [31] for the state of the art on this question up to 2007. From a practical point of view the question of discreteness was important to answer, since sampling methods for reconstructing the support of an inhomogeneous medium [9], [46] fail if the interrogating frequency corresponds to a transmission eigenvalue. On the other hand, due to the non-selfadjointness of the transmission eigenvalue problem, the existence of transmission eigenvalues for non-spherically stratified media remained open for more than 20 years until Sylvester and Päivärinta [50] showed the existence of at least one transmission eigenvalue provided that the contrast in the medium is large enough. The story of the existence of transmission eigenvalues was completed by Cakoni, Gintides and Haddar [19] where the

*Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19716 (cakoni@math.udel.edu).

†INRIA Saclay Ile de France/CMAP Ecole Polytechnique, Route de Saclay, 91128 Palaiseau Cedex, France (haddar@cmap.polytechnique.fr).

existence of an infinite set of transmission eigenvalue was proven only under the assumption that the contrast in the medium does not change sign and is bounded away from zero. In addition, estimates on the first transmission eigenvalue were provided. It was then showed by Cakoni, Colton and Haddar [15] that transmission eigenvalues could be determined from the scattering data and since they provide information about material properties of the scattering object can play an important role in a variety of problems in target identification.

Since [50] appeared, the interest in transmission eigenvalues has increased, resulting in a number of important advancements in this area (throughout this paper the reader can find specific references from the vast available literature on the subject). Arguably, the transmission eigenvalue problem is one of today's central research subjects in inverse scattering theory with many open problems and potential applications. This survey aims to present the state of the art of research on the transmission eigenvalue problem focussing on three main topics, namely the discreteness of transmission eigenvalues, the existence of transmission eigenvalues and estimates on transmission eigenvalues, in particular, Faber-Krahn type inequalities. We begin our presentation by showing how transmission eigenvalue problem appears in scattering theory and how transmission eigenvalues are determined from the scattering data. Then we discuss the simple case of a spherically stratified medium where it is possible to obtain explicit expressions for transmission eigenvalues based on the theory of entire functions. In this case it is also possible to obtain a partial solution to the inverse spectral problem for transmission eigenvalues. We then proceed to discuss the general case of non-spherically stratified inhomogeneous media. As representative of the transmission eigenvalue problem we consider the scalar case for two types of problems namely the physical parameters of the inhomogeneous medium are represented by a function appearing only in the lower order term of the partial differential equation, or the physical parameters of the inhomogeneous medium are presented by a (possibly matrix-valued) function in the main differential operator. Each of these problems employs different type of mathematical techniques. We conclude our presentation with a list of open problems that in our opinion merit investigation.

2 Transmission Eigenvalues and the Scattering Problem

To understand how transmission eigenvalues appear in inverse scattering theory we consider the direct scattering problem for an inhomogeneous medium of bounded support.

More specifically, we assume that the support $D \subset \mathbb{R}^d$, $d = 2, 3$ of the inhomogeneous medium is a bounded connected region with piece-wise smooth boundary ∂D. We denote by ν the outward normal vector ν to the boundary ∂D. The physical parameters in the medium are represented by a $d \times d$ matrix valued function A with $L^\infty(D)$ entries and by a bounded function $n \in L^\infty(D)$. From physical consideration we assume that A is a symmetric matrix such that $\overline{\xi} \cdot \Im(A(x))\xi \leq 0$ for all $\xi \in \mathbb{C}^d$ and $\Im(n(x)) \geq 0$ for almost all $x \in D$. The scattering problem for an incident wave u^i which is assumed to satisfy the Helmhotz equation $\Delta u^i + k^2 u^i = 0$ in $\mathbb{R}^d$ (possibly except for a point outside D in the case of point source incident fields) reads: Find the total field $u := u^i + u^s$ that satisfies

$$\Delta u + k^2 u = 0 \qquad \text{in } \mathbb{R}^d \setminus \overline{D} \tag{1}$$

$$\nabla \cdot A(x)\nabla u + k^2 n(x) u = 0 \qquad \text{in } D \tag{2}$$

$$u^+ = u^- \qquad \text{on } \partial D \tag{3}$$

$$\left(\frac{\partial u}{\partial \nu}\right)^+ = \left(\frac{\partial u}{\partial \nu_A}\right)^- \qquad \text{on } \partial D \tag{4}$$

$$\lim_{r \to \infty} r^{\frac{d-1}{2}} \left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0 \tag{5}$$

where $k > 0$ is the wave number, $r = |x|$, u^s is the scattered field and the Sommerfeld radiation condition (5) is assumed to hold uniformly in $\hat{x} = x/|x|$. Here for a generic function f we denote $f^\pm = \lim_{h \to 0} f(x \pm h\nu)$ for $h > 0$ and $x \in \partial D$ and

$$\frac{\partial u}{\partial \nu_A} := \nu \cdot A(x)\nabla u, \quad x \in \partial D.$$

It is well-known that this problem has a unique solution $u \in H^1_{loc}(\mathbb{R}^d)$ provided that $\overline{\xi} \cdot \Re(A(x))\xi \geq \alpha|\xi|^2 > 0$ for all $\xi \in \mathbb{C}^d$ and almost all $x \in D$. The direct scattering problem in $\mathbb{R}^3$ models for example the scattering of time harmonic acoustic waves of frequency ω by an inhomogeneous medium with spatially-varying sound speed and density and $k = \omega/c_0$ where c_0 is the background sound speed. In $\mathbb{R}^2$, (1)-(5) could be considered as the mathematical model of the scattering of time harmonic electromagnetic waves of frequency ω by an infinitely long cylinder such that either the magnetic field or the electric field is polarized parallel to the axis of the cylinder. Here D is the cross section of the cylinder where A and n are related to relative electric permittivity and magnetic permeability in the medium and $k = \omega/\sqrt{\epsilon_0 \mu_0}$ where ϵ_0 and μ_0 are the constant electric permittivity and magnetic permeability of the background, respectively [26].
The transmission eigenvalue problem is related to non-scattering incident fields. Indeed, if u^i is such that $u^s = 0$ then $w := u|_D$ and $v := u^i|_D$ satisfy the following homogenous

problem

$$\nabla \cdot A(x)\nabla w + k^2 n w = 0 \qquad \text{in } D \tag{6}$$

$$\Delta v + k^2 v = 0 \qquad \text{in } D \tag{7}$$

$$w = v \qquad \text{on } \partial D \tag{8}$$

$$\frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \qquad \text{on } \partial D. \tag{9}$$

Conversely, if (6)-(9) has a nontrivial solution w and v and v can be extended outside D as a solution to the Helmholtz equation, then if this extended v is considered as the incident field the corresponding scattered field is $u^s = 0$. As will be seen later in this paper, there are values of k for which under some assumptions on A and n, the homogeneous problem (6)-(9) has non-trivial solutions. The homogeneous problem (6)-(9) is referred to as the *transmission eigenvalue problem*, whereas the values of k for which the transmission eigenvalue problem has nontrivial solutions are called *transmission eigenvalues*. (In next sections we will give a more rigorous definition of the transmission eigenvalue problem and corresponding eigenvalues.) As will be shown in the following sections, under further assumptions on the functions A and n, (6)-(9) satisfies the Fredholm property for $w \in H^1(D)$, $v \in H^1(D)$ if $A \neq I$ and for $w \in L^2(D)$, $v \in L^2(D)$ such that $w - v \in H^2(D)$ if $A = I$.

Even at a transmission eigenvalue, it is not possible in general to construct an incident wave that does not scatter. This is because, in general it is not possible to extend v outside D in such away that the extended v satisfies the Helmholtz equation in all of $\mathbb{R}^d$. Nevertheless, it is already known [27], [32], [58], that solutions to the Helmholtz equation in D can be approximated by entire solutions in appropriate norms. In particular let $\mathcal{X}(D) := H^1(D)$ if $A \neq I$ and $\mathcal{X}(D) := L^2(D)$ if $A = I$. Then if v_g is a Herglotz wave function defined by

$$v_g(x) := \int_\Omega g(d) e^{ikx\cdot d}\, ds(d), \qquad g \in L^2(\Omega), \;\; x \in \mathbb{R}^d, \; d = 2, 3 \tag{10}$$

where Ω is the unit $(d-1)$-sphere $\Omega := \{x \in \mathbb{R}^d : \;\; |x| = 1\}$ and k is a transmission eigenvalue with the corresponding nontrivial solution v, w, then for a given $\epsilon > 0$, there is a v_{g_ϵ} that approximates v with discrepancy ϵ in the $\mathcal{X}(D)$-norm and the scattered field corresponding to this v_{g_ϵ} as incident field is roughly speaking ϵ-small.

The above analysis suggests that it possible to determine the transmission eigenvalues from the scattering data. To fix our ideas let us assume that the incident field is a plane wave given by $u^i := e^{ikx\cdot d}$, where $d \in \Omega$ is the incident direction. The corresponding scattered field has the asymptotic behavior [26]

$$u^s(x) = \frac{e^{ikr}}{r^{\frac{d-1}{2}}} u_\infty(\hat{x}, d, k) + O\left(\frac{1}{r^{\frac{d+1}{2}}}\right) \qquad \text{in } \mathbb{R}^d,\ d = 2, 3. \tag{11}$$

as $r \to \infty$ uniformly in $\hat{x} = x/r$, $r = |x|$ where u_∞ is known as the *far field pattern* which is a function of the observation direction $\hat{x} \in \Omega$ and also depends on the incident direction d and the wave number k. We can now define the *far field operator* $F_k : L^2(\Omega) \to L^2(\Omega)$ by

$$(F_k g)(\hat{x}) := \int_\Omega u_\infty(\hat{x}, d, k) g(d)\, ds(d). \tag{12}$$

Note that the far field operator $F := F_k$ is related to the scattering operator S defined in [48] by $S = I + \frac{ik}{2\pi} F$ in $\mathbb{R}^3$ and by $S = I + \frac{ik}{\sqrt{2\pi k}} F$ in $\mathbb{R}^2$. To characterize the injectivity of the far field operator we first observe that by linearity $(Fg)(\cdot)$ is the far field pattern corresponding to the scattered field due to the Herglotz wave function (10) with kernel g as incident field. Thus the above discussion on non-scattering incident waves together with the fact that the L^2-adjoint F^* of F is given by $(F^* g)(\hat{x}) = \overline{(Fh)(-\hat{x})}$ with $h(d) := \overline{g(-d)}$ yield the following theorem [9], [26]:

Theorem 2.1 *The far field operator $F : L^2(\Omega) \to L^2(\Omega)$ corresponding to the scattering problem (1)-(5) is injective and has dense range if and only if k^2 is not a transmission eigenvalue of (6)-(9) such that the function v of the corresponding nontrivial solution to (6)-(9) has the form of a Herglotz wave function (10).*

Note that the relation between the far field operator and scattering operator says that the far field operator F not being injective is equivalent to the scattering operator S having one as an eigenvalue.

Next we show that it is possible to determine the real transmission eigenvalues from the scattering data. To fix our ideas we consider far field scattering data, i.e. we assume a knowledge of $u_\infty(\hat{x}, d, k)$ for $\hat{x}, d \in \Omega$ and $k \in \mathbb{R}_+$ which implies a knowledge of the far field operator $F := F_k$ for a range of wave numbers k. Thus we can introduce the far field equation

$$(Fg)(\hat{x}) = \Phi_\infty(\hat{x}, z) \tag{13}$$

where $\Phi_\infty(\hat{x}, z)$ is the far field pattern of the fundamental solution $\Phi(x, z)$ of the Helmholz equation given by

$$\Phi(x, z) := \frac{e^{ik|x-z|}}{4\pi|x-z|} \quad \text{in } \mathbb{R}^3 \qquad \text{and} \qquad \Phi(x, z) := \frac{i}{4} H_0^{(1)}(k|x-z|) \quad \text{in } \mathbb{R}^2 \tag{14}$$

and $H_0^{(1)}$ is the Hankel function of order zero. By a linearity argument, using Rellich's lemma and the denseness of the Herglotz wave functions in the space of $\mathcal{X}(D)$-solutions to the Helmholtz equation, it is easy to prove the following result (see e.g. [9]).

Theorem 2.2 *Assume that $z \in D$ and k is not a transmission eigenvalue. Then for any given $\epsilon > 0$ there exists $g_{z,\epsilon}$ such that*

$$\|Fg_{z,\epsilon} - \Phi_\infty(\cdot, z)\|^2_{L^2(\Omega)} < \epsilon$$

and the corresponding Herglotz wave function $v_{g_{z,\epsilon}}$ satisfies

$$\lim_{\epsilon \to 0} \|v_{g_{z,\epsilon}}\|_{\mathcal{X}(D)} = \|v_z\|_{\mathcal{X}(D)}$$

where (w_z, v_z) is the unique solution of the non-homogenous interior transmission problem

$$\nabla \cdot A(x)\nabla w_z + k^2 n w_z = 0 \qquad \text{in } D \tag{15}$$

$$\Delta v_z + k^2 v_z = 0 \qquad \text{in } D \tag{16}$$

$$w_z - v_z = \Phi(\cdot, z) \qquad \text{on } \partial D \tag{17}$$

$$\frac{\partial w_z}{\partial \nu_A} - \frac{\partial v_z}{\partial \nu} = \frac{\partial \Phi(\cdot, z)}{\partial \nu} \qquad \text{on } \partial D. \tag{18}$$

On the other hand, if k is a transmission eigenvalue, again by linearity argument and applying the Fredholm alternative to the interior transmission problem (15)-(18) it is possible to show the following theorem:

Theorem 2.3 *Assume k is a transmission eigenvalue, and for a given $\epsilon > 0$ let $g_{z,\epsilon}$ be such that*

$$\|Fg_{z,\epsilon} - \Phi_\infty(\cdot, z)\|^2_{L^2(\Omega)} \leq \epsilon \tag{19}$$

with $v_{g_{z,\epsilon}}$ the corresponding Herglotz wave function. Then, for all $z \in D$, except for a possibly nowhere dense subset, $\|v_{g_{z,\epsilon}}\|_{\mathcal{X}(D)}$ can not be bounded as $\epsilon \to 0$.

For a proof of Theorem 2.3 for the case of $A = I$ we refer the reader to [15]. Theorem 2.2 and Theorem 2.3, roughly speaking, state that if D is known and $\|v_{g_{z,\epsilon}}\|_{X(D)}$ is plotted against k for a range of wave numbers $[k_0, k_1]$, the transmission eigenvalues should appear as peaks in the graph. We remark that for some special situations (e.g. if D is a disk centered at the origin, $A = I$, $z = 0$ and n constant) $g_{z,\epsilon}$ satisfying (19) may not exist. However it is reasonable to assume that (19) always holds for the noisy far field operator F^δ given by

$$(F^\delta g)(\hat{x}) := \int_\Omega u^\delta_\infty(\hat{x}, d, k) g(d)\, ds(d),$$

where $u^\delta_\infty(\hat{x}, d, k)$ denotes the noisy measurement with noise level $\delta > 0$ (see Appendix in [15]). Nevertheless, in practice, we have access only to the noisy far field operator F_δ. Due to the ill-posedness of the far field equation (note that F is a compact operator), one looks for the Tikhonov regularized solution $g^\delta_{z,\alpha}$ of the far field equation defined as the unique minimizer of the Tikhonov functional [26]

$$\|F^\delta g - \Phi_\infty(\cdot, z)\|^2_{L^2(\Omega)} + \alpha\|g\|^2_{L^2(\Omega)}$$

where the positive number $\alpha := \alpha(\delta)$ is the Tikhonov regularization parameter satisfying $\alpha(\delta) \to 0$ as $\delta \to 0$. In [2] and [3] it is proven for the case of $A = I$ that Theorem 2.2 is also valid if the approximate solution $g_{z,\epsilon}$ is replaced by the regularized solution $g^\delta_{z,\alpha}$ and the noise level tends to zero. We remark that since the proof of such result relies on the validity of the factorization method (i.e if F is normal, see [46] for details), in general for many scattering problems, Theorem 2.2 can only be proven for the approximate solution to the far field equation. On the other hand, Theorem 2.3 remains valid for the regularized solution $g^\delta_{z,\alpha}$ as the noise level $\delta \to 0$ (see [15] for the proof).

3 The Transmission Eigenvalue Problem for Isotropic Media

We start our discussion of the transmission eigenvalue problem with the case of isotropic media, i.e. when $A = I$. The *transmission eigenvalue problem* corresponding to the scattering problem for isotropic media reads: Find $v \in L^2(D)$ and $w \in L^2(D)$ such that $w - v \in H^2(D)$ satisfying

$$\Delta w + k^2 n(x) w = 0 \qquad \text{in } D \tag{20}$$

$$\Delta v + k^2 v = 0 \qquad \text{in } D \tag{21}$$

$$w = v \qquad \text{on } \partial D \tag{22}$$

$$\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad \text{on } \partial D. \tag{23}$$

As will become clear later, the above function spaces provide the appropriate framework for the study of this eigenvalue problem which turns out to be non-selfadjoint. Note that since the difference between two equations in D occurs in the lower order term and only Cauchy data for the difference is available, it is not possible to have any control on the regularity of each field w and v and assuming (20) and (21) in the $L^2(D)$ (distributional) sense is the best one can hope. Let us denote by

$$H_0^2(D) := \left\{ u \in H^2(D) : \text{ such that } u = 0 \text{ and } \frac{\partial u}{\partial \nu} = 0 \text{ on } \partial D \right\}.$$

Definition 3.1 *Values of $k \in \mathbb{C}$ for which (20)-(23) has nontrivial solution $v \in L^2(D)$ and $w \in L^2(D)$ such that $w - v \in H_0^2(D)$ are called* transmission eigenvalues.

Note that if $n(x) \equiv 1$ every $k \in \mathbb{C}$ is a transmission eigenvalues, since in this trivial case there is no inhomogeneity and any incident field does not scatterer.

3.1 Spherically stratified media

To shed light into the structure of the eigenvalue problem (20)-(23), we start our discussion with the special case of a spherically stratified medium where D is a ball of radius a and $n(x) := n(r)$ is spherically stratified. It is possible to obtain explicit formulas for the solution of this problem by separation of variables and using tools from the theory of entire functions. This allows the possibility to obtain sharper results than are currently available for the general non-spherically stratified case. In particular, it is possible to solve the inverse spectral problem for transmission eigenvalues, prove that complex transmission eigenvalues can exist for non-absorbing media and show that real transmission eigenvalues may exist under some conditions for the case of absorbing media, all of which problems are still open in the general case.
Throughout this section we assume that $\Im(n(r)) = 0$ and (unless otherwise specified). Setting $B := \{x \in \mathbb{R}^3 : |x| < a\}$ the transmission eigenvalue problem for spherically stratified medium is:

$$\Delta w + k^2 n(r) w = 0 \qquad \text{in } B \tag{24}$$

$$\Delta v + k^2 v = 0 \qquad \text{in } B \tag{25}$$

$$w = v \qquad \text{in } \partial B \tag{26}$$

$$\frac{\partial w}{\partial r} = \frac{\partial v}{\partial r} \qquad \text{on } \partial B. \tag{27}$$

Let us assume that $n(r) \in C^2[0,\, a]$ (unless otherwise specified). The main concern here is to show the existence of real and complex transmission eigenvalues and solve the inverse spectral problem. To this end, introducing spherical coordinates (r, θ, φ) we look for solutions of (24)-(27) in the form

$$v(r,\theta) = a_\ell j_\ell(kr) P_\ell(\cos\theta), \qquad \text{and} \qquad w(r,\theta) = b_\ell y_\ell(r) P_\ell(\cos\theta)$$

where P_ℓ is Legendre's polynomial, j_ℓ is a spherical Bessel function, a_ℓ and b_ℓ are constants and y_ℓ is a solution of

$$y'' + \frac{2}{r}y' + \left(k^2 n(r) - \frac{\ell(\ell+1)}{r^2}\right) y_\ell = 0$$

for $r > 0$ such that $y_\ell(r)$ behaves like $j_\ell(kr)$ as $r \to 0$, i.e.

$$\lim_{r\to 0} r^{-\ell} y_\ell(r) = \frac{\sqrt{\pi} k^\ell}{2^{\ell+1}\Gamma(\ell + 3/2)}.$$

From [25], pp. 261-264, in particular Theorem 9.9, we can deduce that k is a (possibly complex) transmission eigenvalue if and only if

$$d_\ell(k) = \det \begin{pmatrix} y_\ell(a) & -j_\ell(ka) \\ \\ y_\ell'(a) & -kj_\ell'(ka) \end{pmatrix} = 0. \tag{28}$$

Setting $m := 1 - n$, from [24] (see also [10]) we can represent $y_\ell(r)$ in the form

$$y_\ell(r) = j_\ell(kr) + \int_0^r G(r, s, k) j_\ell(ks) ds \tag{29}$$

where $G(r, s, k)$ satisfies the Goursat problem

$$r^2 \left[\frac{\partial^2 G}{\partial r^2} + \frac{2}{r}\frac{\partial G}{\partial r} + k^2 n(r) G\right] = s^2 \left[\frac{\partial^2 G}{\partial s^2} + \frac{2}{s}\frac{\partial G}{\partial s} + k^2 G\right] \tag{30}$$

$$G(r, r, k) = \frac{k^2}{2r}\int_0^r \rho\, m(\rho) d\rho, \qquad G(r, s, k) = O\left((rs)^{1/2}\right). \tag{31}$$

It is shown in [24] that (30)-(31) can be solved by iteration and the solution G is an even function of k and an entire function of exponential type satisfying

$$G(r, s, k) = \frac{k^2}{2\sqrt{rs}} \int_0^{\sqrt{rs}} \rho\, m(\rho)\, d\rho \left(1 + O(k^2)\right). \tag{32}$$

Hence for fixed $r > 0$, y_ℓ and spherical Bessel functions are entire function of k of finite type and bounded for k on the positive real axis, and thus $d_\ell(k)$ also has this property. Furthermore, by the series expansion of j_ℓ [26], we see that $d_\ell(k)$ is an even function of k and $d_\ell(0) = 0$. Consequently, if $d_\ell(k)$ does not have a countably infinite number of zeros it must be identically zero. It is easy to show now that $d_\ell(k)$ is not identically zero for every ℓ unless $n(r)$ is identically equal to 1. Indeed, assume that $d_\ell(k)$ is identically zero for every non-negative integer ℓ. Noticing that $j_\ell(kr)Y_\ell^m(\hat{x})$ is a Herglotz wave function, it follows from the proof of Theorem 8.16 in [26] that

$$\int_0^a j_\ell(k\rho)y_\ell(\rho)\rho^2 m(\rho)\,d\rho = 0$$

for all k where $m(r) := 1 - n(r)$. Hence, using the Taylor series expansion of $j_\ell(k\rho)$ and (29) we see that

$$\int_0^a \rho^{2\ell+2} m(\rho)\,d\rho = 0 \tag{33}$$

for all non-negative integers ℓ. By Muntz's theorem [35], we now have $m(r) = 0$, i.e. $n(r) = 1$. Note that from (33) it is easy to see that none of the integrals (33) can become zero if $m(r) \geq 0$ or $m(r) \leq 0$ (not identically zero) which implies that in these cases the transmission eigenvalues form a discrete set as a countable union of countably many zeros of $d_\ell(k)$. Nothing can be said about discreteness of transmission eigenvalues if our only assumption is that $n(r)$ is not identically equal to one. However, if B is a ball in $\mathbb{R}^3$, $n \in C^2[0,\, a]$ and $n(a) \neq 1$, transmission eigenvalues form at most discrete set and there exist infinitely many transmission eigenvalues corresponding to spherically symmetric eigenfunctions.

Theorem 3.1 *Assume that $n \in C^2[0,a]$, $\Im(n(r)) = 0$ and either $n(a) \neq 1$ or $n(a) = 1$ and $\frac{1}{a}\int_0^a \sqrt{n(\rho)}d\rho \neq 1$. Then there exists an infinite discrete set of transmission eigenvalues for (24)-(27) with spherically symmetric eigenfunctions. Furthermore the set of all transmission eigenvalues is discrete.*

Proof: To show existence, we restrict ourself to spherically symmetric solutions to (24)-(27), and look for solutions of the form.

$$v(r) = a_0 j_0(kr) \qquad \text{and} \qquad w(r) = b_0 \frac{y(r)}{r}$$

where

$$y'' + k^2 n(r) y = 0, \qquad y(0) = 0, \quad y'(0) = 1.$$

Using the Liouville transformation

$$z(\xi) \;:=\; [n(r)]^{\frac{1}{4}}\, y(r) \quad \text{where} \quad \xi(r) \;:=\; \int_0^r [n(\rho)]^{\frac{1}{2}}\, d\rho$$

we arrive at the following initial value problem for $z(\xi)$

$$z'' + [k^2 - p(\xi)]z = 0\,, \quad z(0) = 0\,, \quad z'(0) = \big[n(0)\big]^{-\frac{1}{4}} \tag{34}$$

where

$$p(\xi) \;:=\; \frac{n''(r)}{4[n(r)]^2} \;-\; \frac{5}{16}\frac{[n'(r)]^2}{[n(r)]^3}.$$

Now exactly in the same way as in [26], [31], by writing (34) as a Volterra integral equation and using the methods of successive approximations, we obtain the following asymptotic behavior for y

$$\begin{aligned} y(r) &= \frac{1}{k\left[n(0)\,n(r)\right]^{1/4}} \sin\left(k\int_0^r \left[n(\rho)\right]^{1/2} d\rho\right) \;+\; \mathcal{O}\left(\frac{1}{k^2}\right) \quad \text{and} \\ y'(r) &= \left[\frac{n(r)}{n(0)}\right]^{1/4} \cos\left(k\int_0^r [n(\rho)]^{1/2}\,d\rho\right) \;+\; \mathcal{O}\left(\frac{1}{k}\right) \end{aligned}$$

uniformly on $[0,a]$. Applying the boundary conditions (26), (27) on ∂B, we see that a nontrivial solution to (24)-(27) exists if and only if

$$d_0(k) = \det\begin{pmatrix} \dfrac{y(a)}{a} & -j_0(ka) \\ \dfrac{d}{dr}\left(\dfrac{y(r)}{r}\right)_{r=a} & -k\,j_0'(ka) \end{pmatrix} \;=\; 0\,. \tag{35}$$

Since $j_0(kr) = \sin kr/kr$, from the above asymptotic behavior of $y(r)$ we have that

$$d_0(k) = \frac{1}{ka^2}\left[A\sin(k\delta a)\,\cos(ka) - B\cos(k\delta a)\,\sin(ka)\right] \;+\; \mathcal{O}\left(\frac{1}{k^2}\right) \tag{36}$$

where

$$\delta = \frac{1}{a}\int_0^a \sqrt{n(\rho)}d\rho, \qquad A = \frac{1}{[n(0)n(a)]^{1/4}}\,, \qquad B = \left[\frac{n(a)}{n(0)}\right]^{1/4}.$$

If $n(a) = 1$, since $\delta \neq 1$ the first term in (36) is a periodic function if δ is rational and almost-periodic (see [31]) if δ is irrational, and in either case takes both positive and negative values. This means that for large enough k, $d_0(k)$ has infinitely many real zeros which proves the existence of infinitely many real transmission eigenvalues. Now if $n(a) \neq 1$ then $A \neq B$ and the above argument holds independent of the value of δ.
Concerning the discreteness of transmission eigenvalues, we first observe that similar asymptotic expression to (36) holds for all the determinants $d_\ell(k)$ [26]. Hence the above argument shows that $d_\ell(k) \neq 0$ and hence they have countably many zeros, which shows that transmission eigenvalues are discrete. □

Next we are interested in the inverse spectral problem for the transmission eigenvalue problem (24)-(27). The question we ask is under what conditions do transmission eigenvalues uniquely determine $n(r)$. This question was partially answered in [52], [53] under restrictive assumptions on $n(r)$ and the nature of the spectrum. The inverse spectral problem for the general case is solved in [10], provided that all transmission eigenvalues are given, which we briefly sketch in the following:

Theorem 3.2 *Assume that* $n \in C^2[0, +\infty)$*,* $\Im(n(r)) = 0$ *and* $n(r) > 1$ *or* $n(r) < 1$ *for* $r < a$*,* $0 < n(r) = 1$ *for* $r > a$*. If* $n(0)$ *is given then* $n(r)$ *is uniquely determined from a knowledge of the transmission eigenvalues and their multiplicity as a zero of* $d_\ell(k)$*.*

Proof: We return to the determinant (28) and observe that $d_\ell(k)$ has the asymptotic behavior [26]

$$d_\ell(k) = \frac{1}{a^2 k\,[n(0)]^{\ell/2+1/4}} \sin k \left(a - \int_0^a [n(r)]^{1/2} dr \right) + O\left(\frac{\ln k}{k^2} \right). \tag{37}$$

We first compute the coefficient $c_{2\ell+2}$ of the term $k^{2\ell+2}$ in its Hadamard factorization expression [35]. A short computation using (28), (29)and the order estimate

$$j_\ell(kr) = \frac{\sqrt{\pi}(kr)^\ell}{2^{\ell+1}\Gamma(\ell+3/2)} \left(1 + O(k^2 r^2)\right) \tag{38}$$

shows that

$$\begin{aligned} c_{2\ell+2} \left[\frac{2^{\ell+1}\Gamma(\ell+3/2)}{\sqrt{\pi} a^{(\ell-1)/2}} \right]^2 &= a \int_0^a \frac{d}{dr} \left(\frac{1}{2\sqrt{rs}} \int_0^{\sqrt{rs}} \rho\, m(\rho)\, d\rho \right)_{r=a} s^\ell\, ds \\ &- \ell \int_0^a \frac{1}{2\sqrt{as}} \int_0^{\sqrt{as}} \rho\, m(\rho)\, d\rho\, s^\ell\, ds + \frac{a^\ell}{2} \int_0^a \rho\, m(\rho)\, d\rho. \end{aligned} \tag{39}$$

After a rather tedious calculation involving a change of variables and interchange of orders of integration, the identity (39) remarkably simplifies to

$$c_{2\ell+2} = \frac{\pi a^2}{2^{\ell+1}\Gamma(\ell+3/2)} \int_0^a \rho^{2\ell+2}\, m(\rho)\, d\rho. \tag{40}$$

We note that $j_\ell(r)$ is odd if ℓ is odd and even if ℓ is even. Hence, since G is an even function of k, we have that $d_\ell(k)$ is an even function of k. Furthermore, since both G and j_ℓ are entire function of k of exponential type, so is $d_\ell(k)$. From the asymptotic behavior of $d_\ell(k)$ for $k \to \infty$, i.e. (37), we see that the rank of $d_\ell(k)$ is one and hence by Hadamard's factorization theorem [35],

$$d_\ell(k) = k^{2\ell+2} e^{a_\ell k + b_\ell} \prod_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \left(1 - \frac{k}{k_{n\ell}}\right) e^{k/k_{n\ell}}$$

where a_ℓ, b_ℓ are constants or, since d_ℓ is even,

$$d_\ell(k) = k^{2\ell+2} c_{2\ell+2} \prod_{n=1}^{\infty} \left(1 - \frac{k^2}{k_{n\ell}^2}\right) \tag{41}$$

where $c_{2\ell+2}$ is a constant given by (40) and $k_{n\ell}$ are zeros in the right half plane (possibly complex). In particular, $k_{n\ell}$ are the (possibly complex) *transmission eigenvalues* in the right half plane. Thus if the transmission eigenvalues are known so is

$$\frac{d_\ell(k)}{c_{2\ell+2}} = k^{2\ell+2} \prod_{n=1}^{\infty} \left(1 - \frac{k^2}{k_{n\ell}^2}\right)$$

as well as from (37) a nonzero constant γ_ℓ independent of k such that

$$\frac{d_\ell(k)}{c_{2\ell+2}} = \frac{\gamma_\ell}{a^2 k} \sin k \left(a - \int_0^a [n(r)]^{1/2}\, dr\right) + O\left(\frac{\ln k}{k^2}\right),$$

i.e.

$$\frac{1}{c_{2\ell+2}\,[n(0)]^{\ell/2+1/4}} = \gamma_\ell.$$

From (40) we now have

$$\int_0^a \rho^{2\ell+2}\, m(\rho)\, d\rho = \frac{\left(2^{\ell+1}\Gamma(\ell+3/2)\right)^2}{[n(0)]^{\ell/2+1/4}\, \gamma_\ell \pi a^2}.$$

If $n(0)$ is given then $m(\rho)$ is uniquely determined by Müntz's theorem [35]. $\square$

It has recently been shown that in the case when $0 < n(r) < 1$ the eigenvalues corresponding to spherically symmetric eigenfunctions, i.e. the zeros of $d_0(kr)$ (together with their multiplicity) uniquely determine $n(r)$ [1]. The main result proven in [1] is stated in the following theorem.

Theorem 3.3 *Assume that $n(r) \in C^1(0,a)$ such that $n'(r) \in L^2(0,a)$, $\Im(n(r)) = 0$ and $\frac{1}{a}\int_0^a \sqrt{n(\rho)}d\rho < 1$. Then $n(r)$ is uniquely determined from a knowledge of k_{n0} and its multiplicity as a zero of $d_0(k)$.*

The argument used in [1] refers back to the classic inverse Sturm-Liouville problem and it breaks down if $n(r) > 1$.

As we have just showed, for a spherically symmetric index of refraction the real and complex transmission eigenvalues uniquely determine the index of refraction up to a normalizing constant. From Theorem 3.1 we also know that real transmission eigenvalues exist. This raises the question as to whether or not complex transmission eigenvalues can exist. The following simple example in $\mathbb{R}^2$ shows that in general complex transmission eigenvalues can exist [10].

Example of existence of complex transmission eigenvalues. Consider the interior transmission problem (20) and (21) where D is a disk of radius one in $\mathbb{R}^2$ and constant index of refraction $n \neq 1$. We will show that if n is sufficiently small there exist complex transmission eigenvalues in this particular case. To this end we note that k is a transmission eigenvalue provided

$$d_0(k) = k\left(J_1(k)J_0(k\sqrt{n}) - \sqrt{n}J_0(k)J_1(k\sqrt{n})\right) = 0.$$

Viewing d_0 as a function of $\sqrt{n}$ we compute

$$d_0'(k) = k\left(kJ_1(k)J_0'(k\sqrt{n}) - J_0(k)J_1(k\sqrt{n}) - k\sqrt{n}J_0(k)J_1'(k\sqrt{n})\right)$$

where differentiation is with respect to $\sqrt{n}$. Hence

$$d_0'(k)|_{\sqrt{n}=1} = k\left(kJ_1(k)J_0'(k) - J_0(k)J_1(k) - kJ_0(k)J_1'(k)\right).$$

But $J_0'(t) = -J_1(t)$ and $\frac{d}{dt}\left(tJ_1(t)\right) = tJ_0(t)$ and hence

$$d_0'(k)|_{\sqrt{n}=1} = -k^2\left(J_1^2(k) + J_0^2(k)\right) \tag{42}$$

i.e.

$$f(k) = \lim_{\sqrt{n}\to 1^+} \frac{d_0(k)}{\sqrt{n}-1} = -k^2\left(J_1^2(k) + J_0^2(k)\right) \tag{43}$$

Since $J_1(k)$ and $J_0(k)$ do not have any common zeros, $f(k)$ is strictly negative for $k \neq 0$ real, i.e. the only zeros of $f(k)$, $k \neq 0$, are complex. Furthermore, $f(k)$ is an even entire function of exponential type that is bounded on the real axis and hence by Hadamard's factorization theorem [35] $f(k)$ has an infinite number of complex zeros. By Hurwitz's theorem in analytic function theory (c.f. [25], p. 213) we can now conclude that for n close enough to one $d_0(k) = 0$ has complex roots, thus establishing the existence of complex transmission eigenvalues for the unit disk and constant $n > 1$ sufficiently small (Note that by Montel's theorem ([25], p. 213) the convergence in (43) is uniform on compact subsets of the complex plane).

A more comprehensive investigation of the existence of complex transmission eigenvalues for spherically stratified media in $\mathbb{R}^2$ and $\mathbb{R}^3$ has been recently initiated in [49]. Based on tools of analytic function theory, the authors has shown that infinitely many complex transmission eigenvalues can exist. We state here the main results of [49] and refer the reader to the paper for the details of proofs.

Theorem 3.4 *Consider the transmission eigenvalue problem (24)-(27) where* $B := \{x \in \mathbb{R}^d : |x| < 1\}$, $d = 2, 3$ *and* $n = n(r) > 0$ *is a positive constant. Then:*

(i) In $\mathbb{R}^2$, *if* $n \neq 1$ *then there exists an infinite number of complex eigenvalues.*

(ii) In $\mathbb{R}^3$, *if* n *is a positive integer not equal to one then all transmission eigenvalues corresponding to spherically symmetric eigenfunctions are real. On the other hand if* n *is a rational positive number* $n = p/q$ *such that either* $q < p < 2q$ *or* $p < q < 2p$ *then there exists an infinite number of complex eigenvalues.*

Note that complex transmission eigenvalues for n rational satisfying the assumptions of Theorem 3.4 (ii) all must lie in a strip parallel to real axis. We remark that in [49] the authors also show the existence of infinitely many transmission eigenvalues in $\mathbb{R}^3$ for some particular cases of inhomogeneous spherically stratified media $n(r)$. The existence of complex eigenvalues indicates that the transmission eigenvalue problem for spherically stratified media is non-selfadjoint. In the coming section we show that this is indeed the case in general.

We end this section by considering the transmission eigenvalue problem for absorbing media in $\mathbb{R}^3$ [11]. When both the scattering obstacle and the background medium are absorbing it is still possible to have real transmission eigenvalues which is easy to see in the case of a spherically stratified medium. In particular, let $B := \{x \in \mathbb{R}^3 : |x| < a\}$ and consider the interior transmission eigenvalue problem

$$\Delta_3 w + k^2 \left(\epsilon_1(r) + i\frac{\gamma_1(r)}{k}\right) w = 0 \qquad \text{in } B \tag{44}$$

$$\Delta_3 v + k^2 \left(\epsilon_0 + i\frac{\gamma_0}{k}\right) v = 0 \qquad \text{in } B \tag{45}$$

$$v = w \qquad \text{on } \partial B \tag{46}$$

$$\frac{\partial v}{\partial r} = \frac{\partial w}{\partial r} \qquad \text{on } \partial B \tag{47}$$

where $\epsilon_1(r)$ and $\gamma_1(r)$ are continuous functions of r in $\overline{B}$ such that $\epsilon_1(a) = \epsilon_0$ and and ϵ_0 and γ_0 are positive constants. We look for a solution of (44)-(47) in the form

$$v(r) = c_1 j_0(k\tilde{n}_0 r) \quad \text{and} \quad w(r) = c_2 \frac{y(r)}{r} \tag{48}$$

where $\tilde{n}_0 := \left(\epsilon_0 + i\frac{\gamma_0}{k}\right)^{1/2}$ (where the branch cut is chosen such that $\tilde{n}_0$ has positive real part), j_0 is a spherical Bessel function of order zero, $y(r)$ is a solution of

$$y'' + k^2 \left(\epsilon_1(r) + i\tfrac{\gamma_1(r)}{k}\right) y = 0 \tag{49}$$

$$y(0) = 0, \quad y'(0) = 1 \tag{50}$$

for $0 < r < a$ and c_1 and c_2 are constants. Then there exist constants c_1 and c_2, not both zero, such that (48) will be a nontrivial solution of (44)-(47) provided that the corresponding $d_0(k)$ given by (35) satisfies $d_0(k) = 0$. We again derive an asymptotic expansion for $y(r)$ for large k to show that for appropriate choices of n_0 and γ_0 there exist an infinite set of positive values of k such that $d_0(k) = 0$ holds.
Following [36] (p. 84, 89), we see that (49) has a fundamental set of solutions $y_1(r)$ and $y_2(r)$ defined for $r \in [a,\, b]$ such that

$$y_j(r) = Y_j(r)\left[1 + O\left(\frac{1}{k}\right)\right] \tag{51}$$

as $k \to \infty$, uniformly for $0 \leq r \leq a$ where

$$Y_j(r) = \exp\left[\beta_{oj} k + \beta_{1j}\right]$$

$$(\beta'_{oj})^2 + \epsilon_1(r) = 0 \qquad \text{and} \qquad 2\beta'_{oj}\beta_{1j} + i\gamma_1(r) + \beta''_{oj} = 0. \tag{52}$$

From (52) we see that, module arbitrary constants,

$$\beta_{0j} = \pm \int_0^r \sqrt{\epsilon_1(\rho)}\, d\rho \qquad \text{and} \qquad \beta_{ij} = \mp \frac{1}{2}\int_0^r \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(r)}} d\rho + \log\left[\epsilon_1(r)\right]^{-1/4}$$

where $j = 1$ corresponds to the upper sign and $j = 2$ corresponds to the lower sign. Substituting back into (51) and using the initial condition (50) we see that

$$y(r) = \frac{1}{ik\left[\epsilon_1(0)\epsilon_1(r)\right]^{1/4}} \sinh\left[ik\int_0^r \sqrt{\epsilon_1(\rho)}\, d\rho - \frac{1}{2}\int_0^r \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(\rho)}} d\rho\right] + O\left(\frac{1}{k^2}\right) \tag{53}$$

as $k \to \infty$. Similarly,

$$j_0(k\tilde{n}_0 r) = \frac{1}{ik\sqrt{\epsilon_0} r} \sinh\left[ik\sqrt{\epsilon_0} r - \frac{1}{2}\frac{\gamma_0}{\sqrt{\epsilon_0}} r\right] + O\left(\frac{1}{k^2}\right) \tag{54}$$

as $k \to \infty$. Using (53), (54), and the fact that these expressions can be differentiated with respect to r, implies that

$$\begin{aligned} d &= \frac{1}{ika^2\left[\epsilon_1(0)\epsilon_0\right]^{1/4}} \sinh\Bigg[ik\sqrt{\epsilon_0} a - ik\int_0^a \sqrt{\epsilon_1(\rho)}\, d\rho \\ &- \frac{1}{2}\frac{\gamma_0 a}{\sqrt{\epsilon_0}} + \frac{1}{2}\int_0^a \frac{\gamma_1(\rho)}{\sqrt{\epsilon_1(\rho)}} d\rho\Bigg] + O\left(\frac{1}{k^2}\right) \end{aligned} \tag{55}$$

as $k \to \infty$.

We now want to use (55) to deduce the existence of transmission eigenvalues. We first note that since j_0 is an even function of its argument, $j_0(k\tilde{n}_0 r)$ is an entire function of k of order one and finite type. By representing $y(r)$ in terms of j_0 via a transformation operator (29) it is seen that $y(r)$ also has this property and hence so does d. Furthermore, d is bounded as $k \to \infty$. For $k < 0$ d has the asymptotic behavior (55) with γ_0 replaced by $-\gamma_0$ and γ_1 replaced by $-\gamma_1$ and hence d is also bounded as $k \to -\infty$. By analyticity k is bounded on any compact subset of the real axis and therefore $d(k)$ is bounded on the real axis. Now assume that there are not an infinity number of (complex) zeros of $d(k)$. Then by Hadamard's factorization theorem $d(k)$ is of the form

$$d(k) = k^m e^{ak+b} \prod_{\ell=1}^{n} \left(1 - \frac{k}{k_\ell}\right) e^{k/k_\ell}$$

for integers m and n and constants a and b. But this contradicts the asymptotic behavior of $d(k)$. Hence $d(k)$ has an infinite number of (complex) zeros, i.e. there exist an infinite number of transmission eigenvalues.

3.2 The existence and discreteness of real transmission eigenvalues, for real contrast of the same sign in D

We now turn our attention to the transmission eigenvalue problem (20)-(23). The main assumption in this section is that $\Im(n) = 0$ and and that the contrast $n-1$ does not change sign and is bounded away from zero inside D. Under this assumption it is now possible to write (20)-(23) as an equivalent eigenvalue problem for $u = w - v \in H_0^2(D)$ as solution of the fourth order equation

$$\left(\Delta + k^2 n\right) \frac{1}{n-1} \left(\Delta + k^2\right) u = 0 \tag{56}$$

which in variational form, after integration by parts, is formulated as finding a function $u \in H_0^2(D)$ such that

$$\int_D \frac{1}{n-1}(\Delta u + k^2 u)(\Delta \overline{v} + k^2 n \overline{v})\, dx = 0 \qquad \text{for all } v \in H_0^2(D). \tag{57}$$

The functions v and w are related to u through

$$v = -\frac{1}{k^2(n-1)}(\Delta u + k^2 u) \qquad \text{and} \qquad w = -\frac{1}{k^2(n-1)}(\Delta u + k^2 n u).$$

In our discussion we must distinguish between the two cases $n > 1$ and $n < 1$. To fix our ideas, we consider in details only the case where $n(x) - 1 \geq \delta > 0$ in D. (A similar analysis can be done for $1-n(x)\geq\delta>0$, see [19], [20]). Let us define

$$n_* = \inf_D(n) \qquad \text{and} \qquad n^* = \sup_D(n).$$

The following result was first obtained in [31] (see also [16]) and provides a Faber-Krahn type inequality for the first transmission eigenvalue.

Theorem 3.5 *Assume that* $1 < n_* \leq n(x) \leq n^* < \infty$. *Then*

$$k_1^2 > \frac{\lambda_1(D)}{n^*} \tag{58}$$

where k_1^2 *is the smallest transmission eigenvalue and* $\lambda_1(D)$ *is the first Dirichlet eigenvalue of* $-\Delta$ *on* D.

Proof: Taking $v=u$ in (57) and using Green's theorem and the zero boundary value for u we obtain that

$$\begin{aligned} 0 &= \int_D \frac{1}{n-1}(\Delta u + k^2 u)(\Delta \overline{u} + k^2 n \overline{u})dx \\ &= \int_D \frac{1}{n-1}\left|(\Delta u + k^2 n u)\right|^2 dx + k^2 \int_D \left(|\nabla u|^2 - k^2 n |u|^2\right) dx. \end{aligned} \tag{59}$$

Since $n-1\geq n_*-1>0$, if

$$\int_D \left(|\nabla u|^2 - k^2 n|u|^2\right) dx \geq 0 \tag{60}$$

then $\Delta u + k^2 n u = 0$ in D which together with the fact $u\in H_0^2(D)$ implies that $u=0$. Consequently we obtain that $w=v=0$, whence k is not a transmission eigenvalue. But,

$$\inf_{u\in H_0^2(D)} \frac{(\nabla u,\nabla u)_{L^2(D)}}{(u,u)_{L^2(D)}} = \inf_{u\in H_0^1(D)} \frac{(\nabla u,\nabla u)_{L^2(D)}}{(u,u)_{L^2(D)}} = \lambda_1(D) \tag{61}$$

where $(\cdot,\cdot)_{L^2(D)}$ denotes the L^2-inner product. Hence we have that

$$\int_D \left(|\nabla u|^2 - k^2 n|u|^2\right) dx \geq \|u\|^2_{L^2(D)} \left(\lambda_1(D) - k^2 n^*\right).$$

Thus, (60) is satisfied whenever $k^2 \leq \dfrac{\lambda_1(D)}{n^*}$. Thus, we have shown that any transmission eigenvalue k (in particular the smallest transmission eigenvalue k_1), satisfies $k^2 > \dfrac{\lambda_1(D)}{n^*}$.
$\square$

Remark 3.1 From Theorem 3.5 it follows that if $1 < n_* \leq n(x) \leq n^* < \infty$ in D and k_1 is the smallest transmission eigenvalue, then $n^* > \dfrac{\lambda_1(D)}{k_1^2}$ which provides a lower bound for $\sup_D(n)$.

To understand the structure of the interior transmission eigenvalue problem we first observe that, setting $k^2 := \tau$, (57) can be written as

$$\mathbb{T}u - \tau\mathbb{T}_1 u + \tau^2\mathbb{T}_2 u = 0, \tag{62}$$

where $\mathbb{T}\colon H_0^2(D) \to H_0^2(D)$ is the bounded, positive definite self-adjoint operator defined by mean of the Riesz representation theorem

$$(\mathbb{T}u, v)_{H^2(D)} = \int_D \frac{1}{n-1}\Delta u\, \Delta \overline{v}\, \mathrm{d}x \qquad \text{for all } u, v \in H_0^2(D),$$

(note that the $H^2(D)$ norm of a field with zero Cauchy data on ∂D is equivalent to the $L^2(D)$ norm of its Laplacian), $\mathbb{T}_1\colon H_0^2(D) \to H_0^2(D)$ is the bounded compact self-adjoint operator defined by mean of the Riesz representation theorem

$$\begin{aligned}
(\mathbb{T}_1 u, v)_{H^2(D)} &= -\int_D \frac{1}{n-1}\left(\Delta u\, \overline{v} + u\, \Delta\overline{v}\right) dx - \int_D \Delta u\, \overline{v}\, dx \\
&= -\int_D \frac{1}{n-1}\left(\Delta u\, \overline{v} + u\, \Delta\overline{v}\right) dx + \int_D \nabla u \cdot \nabla \overline{v}\, dx \qquad \text{for all } u, v \in H_0^2(D)
\end{aligned}$$

and $\mathbb{T}_2 : H_0^2(D) \to H_0^2(D)$ is the bounded compact non-negative self-adjoint operator defined by mean of the Riesz representation theorem

$$(\mathbb{T}_2 u, v)_{H^2(D)} = \int_D \frac{n}{n-1} u\, \overline{v}\, dx \qquad \text{for all } u, v \in H_0^2(D)$$

(compacteness of $\mathbb{T}_1$ and $\mathbb{T}_2$ is a consequence of the compact embedding of $H_0^2(D)$ and $H_0^1(D)$ in $L^2(D)$). Since $\mathbb{T}^{-1}$ exists we have that (62) becomes

$$u - \tau\mathbb{K}_1 u + \tau^2\mathbb{K}_2 u = 0, \tag{63}$$

where the self-adjoint compact operators $\mathbb{K}_1\colon H_0^2(D) \to H_0^2(D)$ and $\mathbb{K}_2\colon H_0^2(D) \to H_0^2(D)$ are given by $\mathbb{K}_1 = \mathbb{T}^{-1/2}\mathbb{T}_1\mathbb{T}^{-1/2}$ and $\mathbb{K}_2 = \mathbb{T}^{-1/2}\mathbb{T}_2\mathbb{T}^{-1/2}$. (Note that if A is a bounded, positive and self-adjoint operator on a Hilbert space U, the operator $A^{1/2}$ is defined by $A^{1/2} = \int_0^\infty \lambda^{1/2} dE_\lambda$ where dE_λ is the spectral measure associated with A). Hence, setting $U := \left(u, \tau\mathbb{K}_2^{1/2} u\right)$, the interior transmission eigenvalue problem becomes the eigenvalue problem

$$\left(\mathbf{K} - \frac{1}{\tau}\mathbf{I}\right) U = 0, \qquad U \in H_0^2(D) \times H_0^2(D)$$

for the compact non-selfadjoint operator $\mathbf{K} : H_0^2(D) \times H_0^2(D) \to H_0^2(D) \times H_0^2(D)$ given by

$$\mathbf{K} := \begin{pmatrix} \mathbb{K}_1 & -\mathbb{K}_2^{1/2} \\ \mathbb{K}_2^{1/2} & 0 \end{pmatrix}.$$

Note that although the operators in each term of the matrix are selfadjoint the matrix operator $\mathbf{K}$ is not. This expression for $\mathbf{K}$ clearly reveals that the transmission eigenvalue problem is non-selfadjoint. However, from the above discussion we obtain a simpler proof of the following result previously proved in [23], [30], [54] (see also [26]) using analytic Fredholm theory.

Theorem 3.6 *The set of real transmission eigenvalues is at most discrete with $+\infty$ as the only (possible) accumulation point. Furthermore, the multiplicity of each transmission eigenvalue is finite.*

The non-selfadjoinness nature of the interior transmission eigenvalue problem calls for new techniques to prove the existence of transmission eigenvalues. For this reason the existence of transmission eigenvalues remained an open problem until Päivärinta and Sylvester showed in [50] that for large enough index of refraction n there exits at least one transmission eigenvalue. The existence of transmission eigenvalues was completely resolved in [19], where the existence of an infinite set of transmission eigenvalues was proven only under the assumption that $n > 1$ or $0 < n < 1$. Here we present the proof in [19]. To this end we return to the variational formulation (57). Using the Riesz representation theorem we now define the bounded linear operators $\mathbb{A}_\tau : H_0^2(D) \to H_0^2(D)$ and $\mathbb{B} : H_0^2(D) \to H_0^2(D)$ by

$$(\mathbb{A}_\tau u, v)_{H^2(D)} = \int_D \frac{1}{n-1} \left[(\Delta u + \tau u)(\Delta \overline{v} + \tau \overline{v}) + \tau^2 u \, \overline{v} \right] \, dx \tag{64}$$

and

$$(\mathbb{B} u, v)_{H^2(D)} = \int_D \nabla u \cdot \nabla \overline{v} \, dx. \tag{65}$$

Obviously, both operators $\mathbb{A}_\tau$ and $\mathbb{B}$ are self-adjoint. Furthermore, since the sesquilinear form $\mathcal{A}_\tau$ is a coercive sesquilinear form on $H_0^2(D) \times H_0^2(D)$, the operator $\mathbb{A}_\tau$ is positive definite and hence invertible. Indeed, since $\frac{1}{n(x)-1} > \frac{1}{n^*-1} = \gamma > 0$ almost everywhere in

D, we have

$$
\begin{aligned}
(\mathbb{A}_\tau u, v)_{H^2(D)} &\geq \gamma\|\Delta u + \tau u\|^2_{L^2} + \tau^2\|u\|^2_{L^2} \\
&\geq \gamma\|\Delta u\|^2_{L^2} - 2\gamma\tau\|\Delta u\|_{L^2}\|u\|_{L^2} + (\gamma+1)\tau^2\|u\|^2_{L^2} \qquad (66) \\
&= \epsilon\left(\tau\|u\|_{L^2} - \frac{\gamma}{\epsilon}\|\Delta u\|_{L^2(D)}\right)^2 + \left(\gamma - \frac{\gamma^2}{\epsilon}\right)\|\Delta u\|^2_{L^2(D)} + (1+\gamma-\epsilon)\tau^2\|u\|^2_{L^2} \\
&\geq \left(\gamma - \frac{\gamma^2}{\epsilon}\right)\|\Delta u\|^2_{L^2(D)} + (1+\gamma-\epsilon)\tau^2\|u\|^2_{L^2}
\end{aligned}
$$

for some $\gamma < \epsilon < \gamma + 1$. Furthermore, since $\nabla u \in H^1_0(D)^2$, using the Poincaré inequality we have that

$$\|\nabla u\|^2_{L^2(D)} \leq \frac{1}{\lambda_1(D)}\|\Delta u\|^2_{L^2(D)} \tag{67}$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D. Hence we can conclude that

$$(\mathbb{A}_\tau u, u)_{H^2(D)} \geq C_\tau\|u\|^2_{H^2(D)}$$

for some positive constant C_τ. We now consider the operator $\mathbb{B}$. By definition $\mathbb{B}$ is a non-negative operator and furthermore, since $H^1_0(D)$ is compactly embedded in $L^2(D)$ and $\nabla u \in H^1_0(D)$, we can conclude that $\mathbb{B} : H^2_0(D) \to H^2_0(D)$ is a compact operator. Finally, it is obvious by definition that the mapping $\tau \to \mathbb{A}_\tau$ is continuous from $(0, +\infty)$ to the set of self-adjoint positive definite operators. In terms of the above operators we can rewrite (57) as

$$(\mathbb{A}_\tau u - \tau\mathbb{B}u, v)_{H^2(D)} = 0 \qquad \text{for all } v \in H^2_0(D), \tag{68}$$

which means that k is a transmission eigenvalue if and only if $\tau := k^2$ is such that the kernel of the operator $\mathbb{A}_\tau u - \tau\mathbb{B}$ is not trivial. In order to analyze the kernel of this operator we consider the auxiliary generalized eigenvalue problems

$$\mathbb{A}_\tau u - \lambda(\tau)\mathbb{B}u = 0 \qquad u \in H^2_0(D). \tag{69}$$

It is known [20] that for a fixed τ there exists an increasing sequence $\{\lambda_j(\tau)\}_{j=1}^\infty$ of positive eigenvalues of the generalized eigenvalue problem (69), such that $\lambda_j(\tau) \to +\infty$ as $j \to +\infty$. Furthermore, these eigenvalues satisfy the min-max principle

$$\lambda_j(\tau) = \min_{W \subset \mathcal{U}_j}\left(\max_{u \in W\setminus\{0\}} \frac{(\mathbb{A}_\tau u, u)}{(\mathbb{B}u, u)}\right) \tag{70}$$

where $\mathcal{U}_j$ denotes the set of all j dimensional subspaces W of $H^2_0(D)$ such that $W \cap \ker(\mathbb{B}) = \{0\}$, which ensures that $\lambda_j(\tau)$ depends continuously on $\tau \in (0, \infty)$.

In particular, a transmission eigenvalue $k > 0$ is such that $\tau := k^2$ solves $\lambda(\tau) - \tau = 0$ where $\lambda(\tau)$ is an eigenvalue corresponding to (69). Thus to prove that transmission eigenvalues exist we use the following theorem (see [20] for the proof).

Theorem 3.7 *Let $\tau \longmapsto \mathbb{A}_\tau$ be a continuous mapping from $]0, \infty[$ to the set of self-adjoint and positive definite bounded linear operators on a Hilbert space $H_0^2(D)$ and let $\mathbb{B}$ be a self-adjoint and non negative compact bounded linear operator on $H_0^2(D)$. We assume that there exists two positive constants $\tau_0 > 0$ and $\tau_1 > 0$ such that*

1. *$\mathbb{A}_{\tau_0} - \tau_0 \mathbb{B}$ is positive on $H_0^2(D)$,*

2. *$\mathbb{A}_{\tau_1} - \tau_1 \mathbb{B}$ is non positive on a m-dimensional subspace W_m of $H_0^2(D)$.*

Then each of the equations $\lambda_j(\tau) = \tau$ for $j = 1, \dots, k$, has at least one solution in $[\tau_0, \tau_1]$ where $\lambda_j(\tau)$ is the j^{th} eigenvalue (counting multiplicity) of the generalized eigenvalue problem (69).

Now we are ready to prove the existence theorem.

Theorem 3.8 *Assume that $1 < n_* \leq n(x) \leq n^* < \infty$. Then, there exist an infinite set of real transmission eigenvalues with $+\infty$ as the only accumulation point.*

Proof: First we recall that from Theorem 3.5 we have that as long as $0 < \tau_0 \leq \lambda_1(D)/n^*$ the operator $\mathbb{A}_{\tau_0} u - \tau_0 \mathbb{B}$ is positive on $H_0^2(D)$, whence the assumption 1. of Theorem 3.7 is satisfied for such τ_0. Next let k_{1,n_*} be the first transmission eigenvalue for the ball B_1 of radius one, i.e. $B_1 := \{x \in \mathbb{R}^d : |x| < 1\}$, $d = 2, 3$, and constant index of refraction n_* (i.e. corresponding to (24)-(27) for $B := B_1$ and $n(r) := n_*$). This transmission eigenvalue is the first zero of

$$W(k) = \det \begin{pmatrix} j_0(k) & j_0(k\sqrt{n_*}) \\ -j_0'(k) & -\sqrt{n_*} j_0'(k\sqrt{n_*}) \end{pmatrix} = 0 \qquad \text{in} \quad \mathbb{R}^3 \tag{71}$$

where j_0 is the spherical Bessel function of order zero, or

$$W(k) = \det \begin{pmatrix} J_0(k) & J_0(k\sqrt{n_*}) \\ -J_0'(k) & -\sqrt{n_*} J_0'(k\sqrt{n_*}) \end{pmatrix} = 0 \qquad \text{in} \quad \mathbb{R}^2 \tag{72}$$

where J_0 is the Bessel function of order zero (if the first zero of the above determinant is not the first transmission eigenvalue, the latter will be a zero of a similar determinant corresponding to higher order Bessel functions or spherical Bessel functions). By a scaling argument, it is obvious that $k_{\epsilon,n_*} := k_{1,n_*}/\epsilon$ is the first transmission eigenvalue corresponding to the ball of radius $\epsilon > 0$ with index of refraction n_*. Now take $\epsilon > 0$ small enough such that D contains $m := m(\epsilon) \geq 1$ disjoint balls $B_\epsilon^1, B_\epsilon^2 \dots B_\epsilon^m$ of radius ϵ, i.e. $\overline{B_\epsilon^j} \subset D$, $j = 1...m$, and $\overline{B_\epsilon^j} \cap \overline{B_\epsilon^i} = \emptyset$ for $j \neq i$. Then $k_{\epsilon,n_*} := k_{1,n_*}/\epsilon$ is the first transmission eigenvalue for each of these balls with index of refraction n_* and let $u^{B_\epsilon^j,n_*} \in H_0^2(B_\epsilon^j)$, $j = 1...m$ be the corresponding eigenfunctions. We have that $u^{B_\epsilon^j,n_*} \in H_0^2(B_\epsilon^j)$ and

$$\int_{B_\epsilon^j} \frac{1}{n_* - 1}(\Delta u^{B_\epsilon^j,n_*} + k_{\epsilon,n_*}^2 u^{B_\epsilon^j,n_*})(\Delta \overline{u}^{B_\epsilon^j,n_*} + k_{\epsilon,n_*}^2 n_* \overline{u}^{B_\epsilon^j,n_*})\, dx = 0. \tag{73}$$

The extension by zero $\tilde{u}^j$ of $u^{B_\epsilon^j,n_*}$ to the whole D is obviously in $H_0^2(D)$ due to the boundary conditions on $\partial B_{\epsilon,n_*}^j$. Furthermore, the vectors $\{\tilde{u}^1, \tilde{u}^2, \dots \tilde{u}^m\}$ are linearly independent and orthogonal in $H_0^2(D)$ since they have disjoint supports and from (73) we have that, for $j = 1...m$,

$$\begin{aligned} 0 &= \int_D \frac{1}{n_* - 1}(\Delta \tilde{u}^j + k_{\epsilon,n_*}^2 \tilde{u}^j)(\Delta \overline{\tilde{u}}^j + k_{\epsilon,n_*}^2 n_* \overline{\tilde{u}}^j)\, dx & (74)\\ &= \int_D \frac{1}{n_* - 1}|\Delta \tilde{u}^j + k_{\epsilon,n_*}^2 \tilde{u}^j|^2\, dx + k_{\epsilon,n_*}^4 \int_D |\tilde{u}^j|^2\, dx - k_{\epsilon,n_*}^2 \int_D |\nabla \tilde{u}^j|^2\, dx. & (75) \end{aligned}$$

Denote by W_m the m-dimensional subspace of $H_0^2(D)$ spanned by $\{\tilde{u}^1, \tilde{u}^2, \dots \tilde{u}^m\}$. Since each $\tilde{u}^j$, $j = 1, ..., m$ satisfies (74) and they have disjoint supports, we have that for $\tau_1 := k_{\epsilon,n_*}^2$ and for every $\tilde{u} \in \mathcal{U}$

$$\begin{aligned} (\mathbb{A}_{\tau_1}\tilde{u} - \tau_1 \mathbb{B}\tilde{u},\, \tilde{u})_{H_0^2(D)} &= \int_D \frac{1}{n - 1}|\Delta \tilde{u} + \tau_1 \tilde{u}|^2\, dx + \tau_1^2 \int_D |\tilde{u}|^2\, dx - \tau_1 \int_D |\nabla \tilde{u}|^2\, dx \\ &\leq \int_D \frac{1}{n_* - 1}|\Delta \tilde{u} + \tau_1 \tilde{u}|^2\, dx + \tau_1^2 \int_D |\tilde{u}|^2\, dx - \tau_1 \int_D |\nabla \tilde{u}|^2\, dx = 0. \end{aligned} \tag{76}$$

This means that assumption 2. of Theorem 3.7 is also satisfied and therefore we can conclude that there are $m(\epsilon)$ transmission eigenvalues (counting multiplicity) inside $[\tau_0,\, k_{\epsilon,n_*}]$. Note that $m(\epsilon)$ and k_{ϵ,n_*} both go to $+\infty$ as $\epsilon \to 0$. Since the multiplicity of each eigenvalue is finite we have shown, by letting $\epsilon \to 0$, that there exists a infinite countable set of transmission eigenvalues that accumulate at ∞. $\square$

In a similar way [19] it is possible to prove the following theorem.

Theorem 3.9 *Assume that $0 < n_* \leq n(x) \leq n^* < 1$. Then, there exist an infinite set of real transmission eigenvalues with $+\infty$ as the only accumulation point.*

The above proof of the existence of transmission eigenvalues provides a framework to obtain lower and upper bounds for the first transmission eigenvalue. To this end denote by $k_1(n, D) > 0$ the first real transmission eigenvalue corresponding to n and D. From the proof of Theorem 3.8 it is easy to see the following monotonicity results for the first transmission eigenvalue (see [19] for the details of the proof).

Theorem 3.10 *Let $n_* = \inf_D(n)$ and $n^* = \sup_D(n)$, and B_1 and B_2 be two balls such that $B_1 \subset D$ and $D \subset B_2$.*

(i) If the index of refraction $n(x)$ satisfies $1 < n_* \leq n(x) \leq n^* < \infty$, then

$$0 < k_1(n^*, B_2) \leq k_1(n^*, D) \leq k_1(n(x), D) \leq k_1(n_*, D) \leq k_1(n_*, B_1). \tag{77}$$

(ii) If the index of refraction $n(x)$ satisfies $0 < n_* \leq n(x) \leq n^* < 1$, then

$$0 < k_1(n_*, B_2) \leq k_1(n_*, D) \leq k_1(n(x), D) \leq k_1(n^*, D) \leq k_1(n^*, B_1). \tag{78}$$

We remark that from the proof of Theorem 3.10 it is easy to see that for a fixed D the monotonicity result $k_j(n^*, D) \leq k_j(n(x), D) \leq k_j(n_*, D)$ holds for all transmission eigenvalues k_j such that $\tau := k_j^2$ is solution of any of $\lambda_j(\tau) - \tau = 0$. Theorem 3.10 shows in particular that for constant index of refraction the first transmission eigenvalue $k_1(n, D)$ as a function of n for D fixed is monotonically increasing if $n > 1$ and is monotonically decreasing if $0 < n < 1$. In fact in [10] it is shown that this monotonicity is strict which leads to the following uniqueness result of the constant index of refraction in terms of the first transmission eigenvalue.

Theorem 3.11 *The constant index of refraction n is uniquely determined from a knowledge of the corresponding smallest transmission eigenvalue $k_1(n, D) > 0$ provided that it is known a priori that either $n > 1$ or $0 < n < 1$.*

Proof: Here, we show the proof for the case of $n > 1$ (see [10] for the case of $0 < n < 1$). Assume two homogeneous media with constant index of refraction n_1 and n_2 such that $1 < n_1 < n_2$, and let $u_1 := w_1 - v_1$, where w_1, v_1 is the nonzero solution of (20)-(23) with $n(x) := n_1$ corresponding to the first transmission eigenvalue $k_1(n_1, D)$. Now, setting $\tau_1 = k_1(n_1, D)$ and after normalizing u_1 such that $\nabla u_1 = 1$, we have

$$\frac{1}{n_1 - 1}\|\Delta u_1 + \tau_1 u_1\|^2_{L^2(D)} + \tau_1^2 \|u_1\|^2_{L^2(D)} = \tau_1 = \lambda(\tau_1, n_1)$$

Furthermore, we have

$$\frac{1}{n_2 - 1}\|\Delta u + \tau u\|^2_{L^2(D)} + \tau^2 \|u\|^2_{L^2(D)} < \frac{1}{n_1 - 1}\|\Delta u + \tau u\|^2_D + \tau^2 \|u\|^2_{L^2(D)}$$

for all $u \in H_0^2(D)$ such that $\|\nabla u\|_D = 1$ and all $\tau > 0$. In particular for $u = u_1$ and $\tau = \tau_1$

$$\frac{1}{n_2 - 1}\|\Delta u_1+\tau_1 u_1\|^2_{L^2(D)}+\tau_1^2\|u_1\|^2_{L^2(D)} < \frac{1}{n_1 - 1}\|\Delta u_1+\tau_1 u_1\|^2_{L^2(D)}+\tau_1^2\|u_1\|^2_{L^2(D)} = \lambda(\tau_1, n_1).$$

But

$$\lambda(\tau_1, n_2) \le \frac{1}{n_2 - 1}\|\Delta u_1 + \tau_1 u_1\|^2_{L^2(D)} + \tau_1^2\|u_1\|^2_{L^2(D)} < \lambda(\tau_1, n_1)$$

and hence for this τ_1 we have a strict inequality, i.e.

$$\lambda(\tau_1, n_2) < \lambda(\tau_1, n_1). \tag{79}$$

Obviously (79) implies the the first zero τ_2 of $\lambda(\tau, n_2) - \tau = 0$ is such that $\tau_2 < \tau_1$ and therefore we have that $k_1(n_2, D) < k_1(n_1, D)$ for the first transmission eigenvalues $k_1(n_1, D)$ and $k_1(n_2, D)$ corresponding to n_1 and n_2, respectively. Hence we have shown that if $n_1 > 1$ and $n_2 > 1$ are such $n_1 \neq n_2$ then $k_1(n_1, D) \neq k_1(n_2, D)$, which proves uniqueness. $\square$

3.3 The case of inhomogeneous media with cavities

Motivated by a recent application of transmission eigenvalues to detect cavities inside dielectric materials [8], we now discuss briefly the structure of transmission eigenvalues for the case of a non-absorbing inhomogeneous medium with cavities, i.e. inhomogeneous medium D with regions $D_0 \subset D$ where the index of refraction is the same as the background medium. The interior transmission problem for inhomogeneous medium with cavities is investigated in [14], [19] and [34], and is also the first attempt to relax the aforementioned assumptions on the contrast. More precisely, inside D we consider a region $D_0 \subset D$ which can possibly be multiply connected such that $\mathbb{R}^d \setminus \overline{D}_0$, $d = 2, 3$ is connected and assume that its boundary ∂D_0 is piece-wise smooth. Here ν denotes the unit outward normal to ∂D and ∂D_0. Now we consider the interior transmission eigenvalue problem (20)-(23) with $n \in L^\infty(D)$ a real valued function such that $n \ge c > 0$, $n = 1$ in D_0 and $n - 1 \ge \tilde{c} > 0$ or $1 - n \ge \tilde{c} > 0$ almost everywhere in $D \setminus \overline{D}_0$. In particular, $1/|n - 1| \in L^\infty(D \setminus \overline{D}_0)$. Following the analytic framework developed in [14], we introduce the Hilbert space

$$V_0(D, D_0, k) := \{u \in H_0^2(D) \text{ such that } \Delta u + k^2 u = 0 \text{ in } D_0\}$$

equipped with the $H^2(D)$ scalar product and look for the solution v and w both in $L^2(D)$ such that $u = w - v$ in $V_0(D, D_0, k)$. It is shown in [14] that (20)-(23), with n satisfying the above assumptions, can be written in the variational form

$$\int_{D\setminus\overline{D}_0} \frac{1}{n-1}\left(\Delta+k^2\right) u \left(\Delta+k^2\right)\bar{\psi}\,dx + k^2 \int_{D\setminus\overline{D}_0} (\Delta u + k^2 u)\,\bar{\psi}\,dx = 0 \qquad (80)$$

for all $\psi \in V_0(D, D_0, k)$. Next let us define the following bounded sesquilinear forms on $V_0(D, D_0, k) \times V_0(D, D_0, k)$:

$$\begin{aligned} \mathcal{A}(u,\psi) &= \pm \int_{D\setminus\overline{D}_0} \frac{1}{n-1}\left(\Delta u\,\Delta\bar{\psi} + \nabla u\cdot\nabla\bar{\psi} + u\,\bar{\psi}\right) dx \\ &+ \int_{D_0} \left(\nabla u\cdot\nabla\bar{\psi} + u\,\bar{\psi}\right) dx \end{aligned} \qquad (81)$$

and

$$\begin{aligned} \mathcal{B}_k(u,\psi) &= \pm k^2 \int_{D\setminus\overline{D}_0} \frac{1}{n-1}\left(u(\Delta\bar{\psi} + k^2\bar{\psi}) + (\Delta u + k^2 n u)\bar{\psi}\right) dx \\ &\mp \int_{D\setminus\overline{D}_0} \frac{1}{n-1}\left(\nabla u\cdot\nabla\bar{\psi} + u\,\bar{\psi}\right) dx - \int_{D_0}\left(\nabla u\cdot\nabla\bar{\psi} + u\,\bar{\psi}\right) dx \end{aligned} \qquad (82)$$

where the upper sign corresponds to the case when $n-1 \geq \tilde{c} > 0$ and the lower sign corresponds to the case when $1-n \geq \tilde{c} > 0$ almost everywhere in $D\setminus\overline{D}_0$. Hence k is a transmission eigenvalue if and only if the homogeneous problem

$$\mathcal{A}(u_0,\psi) + \mathcal{B}_k(u_0,\psi) = 0 \quad \text{for all } \psi \in V_0(D, D_0, k) \qquad (83)$$

has a nonzero solution. Let $A_k : V_0(D, D_0, k) \to V_0(D, D_0, k)$ and B_k be the self-adjoint operators associated with $\mathcal{A}$ and $\mathcal{B}_k$, respectively, by using the Riesz representation theorem. In [14] it is shown that the operator $A_k : V_0(D, D_0, k) \to V_0(D, D_0, k)$ is positive definite, i.e. $A_k^{-1} : V_0(D, D_0, k) \to V_0(D, D_0, k)$ exists, and the operator $B_k : V_0(D, D_0, k) \to V_0(D, D_0, k)$ is compact. Hence we can define the operator $A_k^{-1/2}$ which is also bounded, positive definite and self-adjoint. Thus we have that (83) is equivalent to finding $u \in V_0(D, D_0, k)$ such that

$$u + A_k^{-1/2} B_k A_k^{-1/2} u = 0. \qquad (84)$$

In particular, it is obvious that k is a transmission eigenvalue if and only if the operator

$$I_k + A_k^{-1/2} B_k A_k^{-1/2} : V_0(D, D_0, k) \to V_0(D, D_0, k) \qquad (85)$$

has a nontrivial kernel where I_k is the identity operator on $V_0(D, D_0, k)$. To avoid dealing with function spaces depending on k we introduce the orthogonal projection oprator

P_k from $H_0^2(D)$ onto $V_0(D, D_0, k)$ and the corresponding injection $R_k : V_0(D, D_0, k) \to H_0^2(D)$. Then one easily sees that $I_k + A_k^{-1/2} B_k A_k^{-1/2}$ is injective on $V_0(D, D_0, k)$ if and only if

$$I + R_k A_k^{-1/2} B_k A_k^{-1/2} P_k : H_0^2(D) \to H_0^2(D) \tag{86}$$

is injective. Furthermore as discussed in [14], $T_k := R_k A_k^{-1/2} B_k A_k^{-1/2} P_k : H_0^2(D) \to H_0^2(D)$ is a compact operator and the mapping $k \to R_k A_k^{-1/2} B_k A_k^{-1/2} P_k$ is continuous. Therefore, from the max-min principle for the eigenvalues $\lambda(k)$ of the compact and self-adjoint operator $R_k A_k^{-1/2} B_k A_k^{-1/2} P_k$ we can conclude that $\lambda(k)$ is a continuous function of k. Finally, it is clear that the multiplicity of a transmission eigenvalue is finite since it corresponds to the multiplicity of the eigenvalue $\lambda(k) = -1$. Now the problem is brought into the right framework, similar to the one in Section 3.2, to prove the discreteness and existence of transmission eigenvalues. Using the analytic Fredholm theory [26], it is proven in [14] that real transmission eigenvalues form at most a discrete set with $+\infty$ as the only possible accumulation point. Concerning the existence of transmission eigenvalues, it is now possible to apply a similar procedure as in Section 3.2. In particular, we can use a slightly modified version of Theorem 3.7 (see also Theorem 4.7) to show that each equation $\lambda_j(k)+1 = 0$ has at least one solution, which are transmission eigenvalues, where $\{\lambda_j(k)\}_{j=0}^{\infty}$ is the increasing sequence of eigenvalues of the auxiliary eigenvalue problem

$$(I - \lambda(k) R_k A_k^{-1/2} B_k A_k^{-1/2} P_k) u = 0.$$

Finally we have the following theorem (see [14] and [19] for more details) where we set $n_* := \inf_{D \setminus \overline{D}_0}(n)$, $n^* := \sup_{D \setminus \overline{D}_0}(n)$ and recall that $\lambda_1(D)$ denotes the first Dirichlet eigenvalue for $-\Delta$ on D.

Theorem 3.12 *Let $n \in L^\infty(D)$, $n = 1$ in D_0 and assume that n satisfies either $1 < n_* \leq n(x) \leq n^* < \infty$ or $0 < n_* \leq n(x) \leq n^* < 1$ on $D \setminus \overline{D}_0$. Then the set of real transmission eigenvalues is discrete with no finite accumulation points, and there exist infinitely many transmission eigenvalues accumulating at $+\infty$.*

As byproduct of the proof of Theorem 3.12 it is possible to show the following monotonicity result for the first transmission eigenvalue (see [34], Theorem 2.10). For a fixed D, denote by $k_1(D_0, n)$ the first transmission eigenvalue corresponding to the void D_0 and the index of refraction n.

Theorem 3.13 *If $D_0 \subseteq \tilde{D}_0$ and $n(x) \leq \tilde{n}(x)$ for almost every $x \in D$ then*

(i) $k_1(D_0, \tilde{n}) \leq k_1(\tilde{D}_0, n)$ if $n - 1 \geq \alpha > 0$ and $\tilde{n} - 1 \geq \tilde{\alpha} > 0$

(ii) $k_1(D_0, n) \leq k_1(\tilde{D}_0, \tilde{n})$ *if* $1 - n \geq \beta > 0$ *and* $1 - \tilde{n} \geq \tilde{\beta} > 0$.

The above results are useful in nondestructive testing to detect voids inside inhomogeneous non-absorbing media using transmission eigenvalues [8].

We end this section by remarking that the study of transmission eigenvalue problem in the general case of absorbing media and background has been initiated in [11] where it was proven that the set of transmission eigenvalues on the open right complex half plane is at most discrete provided that the contrast in the real part of the index of refraction does not change sign in D. Furthermore using perturbation theory it is possible to show that if the absorption in the inhomogeneous medium and (possibly) in the background is small enough then there exist a finite number of complex transmission eigenvalues each near a real transmission eigenvalue associated with the corresponding non-absorbing medium and background.

3.4 Discussion

The case of the contrast changing sign inside D. The crucial assumption in the above analysis is that the contrast does not change sign inside D, i.e $n-1$ is either positive or negative and bounded away from zero in D. Although using weighted Sobolev spaces it is possible to consider the case when $n - 1$ goes smoothly to zero at the boundary ∂D [23], [40], [55], the real interest is in investigating the case when $n - 1$ is allowed to change sign inside D. The question of discreteness of transmission eigenvalues in the latter case has been related to the uniqueness of the sound speed for the wave equation with arbitrary source, which is a question that arises in thermo-acoustic imagining [37]. In the general case $n \geq c > 0$ with no assumptions on the sign of $n - 1$, the study of the transmission eigenvalue problem is completely open. However, recently in [57] progress has been made in the study of discreteness of transmission eigenvalues under more relaxed assumptions on the contrast $n - 1$, namely requiring that $n - 1$ or $1 - n$ is positive only in a neighborhood of ∂D. More specifically, the following theorem is proved in [57].

Theorem 3.14 *Suppose that there are real numbers* $m^* \geq m_* > 0$ *and a unit complex number* $e^{i\theta}$ *in the open right half plane such that*

1. $\Re(e^{i\theta}(n(x) - 1)) > m_*$ *in some neighborhood of* ∂D *or that* $n(x)$ *is real on all of* D*, and satisfies* $n(x) - 1 \leq -m_*$ *in some neighborhood of* D.

2. $|n(x) - 1| < m^*$ *in all of* D .

3. $\Re(n(x)) \geq \delta > 0$ *in all of* D.

Then the spectrum of (20)-(23) (i.e the set of transmission eigenvalues) consists of a (possibly empty) discrete set of eigenvalues with finite dimensional generalized eigenspaces. Eigenspaces corresponding to different eigenvalues are linearly independent. The eigenvalues and the generalized eigenspaces depend continuously on n in the $L^\infty(D)$ topology.

In [57], the author uses the concept of upper triangular compact operator to prove the Fredholm property of the transmission eigenvalue problem and employes careful estimates to control solutions to Helmohltz equation inside D by its values in a neighborhood of the boundary in order to show that the resolvent is not empty. The Fredholm property of the transmission eigenvalue problem can also be proven using an integral equation approach [33]. In Section 4.2.1 we present the proof of similar discreteness results for the transmission eigenvalue problems with $A \neq I$ based on a T-coercivity approach.

The location of transmission eigenvalues. Results concerning complex transmission eigenvalues for the problem (20)-(23) are limited to indicating eigenvalue free zones in the complex plane. A first attempt to localize transmission eigenvalues on the complex plane in done in [10]. However to our knowledge the best result on location of transmission eigenvalues is given in [42] where it is shown that almost all transmission eigenvalues k^2 are confined to a parabolic neighborhood of the positive real axis. More specifically the following theorem is proven in [42].

Theorem 3.15 *Assume that D has C^∞ boundary, $n \in C^\infty(\overline{D})$ and $1 < \alpha \leq n \leq \beta$. Then there exists a $0 < \delta < 1$ and $C > 1$ both independent of n (but depending on α and β) such that all transmission eigenvalues $\tau := k^2 \in \mathbb{C}$ with $|\tau| > C$ satisfies $\Re(\tau) > 0$ and $\Im(\tau) \leq C|\tau|^{1-\delta}$.*

We do not include the proof of the above theorem here (and refer the reader to [42]) since the proof employs an approach that is quite different from the analytical framework developed in this article. Note that although the transmission eigenvalue problem (20)-(23) has the structure of quadratic pencils of operators (62), it appears that available results on quadratic pencils [51] are not applicable to the transmission eigenvalue problem due to the incorrect signs of the involved operators. We also remark that some rough estimates on complex eigenvalues for the general case of absorbing media and background are obtained in [11].

We close the first part of this expose on the transmission eigenvalue problem by noting that in [41] the discreteness and existence of transmission eigenvalue are investigated for the case of (20)-(23) where the Laplace operator is replaced by a higher order differential operator with constant coefficient of even order. Such a framework is applicable to the Dirac system and the plate equation.

4 The Transmission Eigenvalue Problem for Anisotropic Media

We continue our discussion of the interior transmission problem by considering in this section the case where $A \neq I$. We recall that the transmission eigenvalue problem now has the form

$$\nabla \cdot A(x)\nabla w + k^2 n w = 0 \qquad \text{in } D \tag{87}$$

$$\Delta v + k^2 v = 0 \qquad \text{in } D \tag{88}$$

$$w = v \qquad \text{on } \partial D \tag{89}$$

$$\frac{\partial w}{\partial \nu_A} = \frac{\partial v}{\partial \nu} \qquad \text{on } \partial D, \tag{90}$$

where we assume that

$$\begin{aligned} A_* &:= \inf_{x\in D} \inf_{\xi \in \mathbb{R}^3, |\xi|=1} (\xi \cdot A(x)\xi) > 0, & & A^* := \sup_{x\in D} \sup_{\xi \in \mathbb{R}^3, |\xi|=1} (\xi \cdot A(x)\xi) < \infty, \\ n_* &:= \inf_{x\in D} n(x) > 0 & \text{and} \quad & n^* := \sup_{x \in D} n(x) < \infty. \end{aligned} \tag{91}$$

The analysis of transmission eigenvalues for this configuration uses different approaches depending on whether $n = 1$ or $n \neq 1$. In particular, the case where $n(x) \equiv 1$, can be brought into a similar form to the problem discuss in Section 3.2 but for vector fields. Hence we first proceed with this case.

4.1 The case $n = 1$

When $n = 1$ after making an appropriate change of unknown functions, we can write (87)-(90) in a similar form as in the case of $A = I$ presented in Section 3.2 (we follow the approach developed in [13]). Letting $N := A^{-1}$, in terms of new vector valued functions

$$\mathbf{w} = A\nabla w, \quad \text{and} \quad \mathbf{v} = \nabla v,$$

the above problem can be written as

$$\nabla(\nabla \cdot \mathbf{w}) + k^2 N \mathbf{w} = 0 \qquad \text{in} \qquad D \tag{92}$$

$$\nabla(\nabla \cdot \mathbf{v}) + k^2 \mathbf{v} = 0 \qquad \text{in} \qquad D \tag{93}$$

$$\nu \cdot \mathbf{w} = \nu \cdot \mathbf{v} \qquad \text{on} \qquad \partial D \tag{94}$$

$$\nabla \cdot \mathbf{w} = \nabla \cdot \mathbf{v} \qquad \text{on} \qquad \partial D. \tag{95}$$

The first two equations (92)-(93) are respectively obtained after taking the gradient of (87)-(88). Problem (92)-(95) has a similar structure as (20)-(23) in the sense that the main operators appearing in (92)-(93) are the same. We therefore can analyze this problem by reformulating it as an eigenvalue problem for the the fourth order partial differential equation assuming that $(N-I)^{-1} \in L^\infty(D)$, which is equivalent to assuming that $(I - A)^{-1} \in L^\infty(D)$ (given the initial hypothesis made on A and since $N - I = A^{-1}(I - A)$). A suitable function space setting is based on

$$\begin{aligned} H(\mathrm{div}\,, D): &= \left\{\mathbf{u} \in (L^2(D))^d : \nabla \cdot \mathbf{u} \in L^2(D)\right\}, \qquad d = 2, 3 \\ H_0(\mathrm{div}\,, D): &= \{\mathbf{u} \in H(\mathrm{div}\,, D) : \nu \cdot \mathbf{u} = 0 \text{ on } \partial D\} \end{aligned}$$

and

$$\begin{aligned} \mathcal{H}(D): &= \left\{\mathbf{u} \in H(\mathrm{div}\,, D) : \nabla \cdot \mathbf{u} \in H^1(D)\right\} \\ \mathcal{H}_0(D): &= \left\{\mathbf{u} \in H_0(\mathrm{div}\,, D) : \nabla \cdot \mathbf{u} \in H^1_0(D)\right\} \end{aligned}$$

equipped with the scalar product $(\mathbf{u}, \mathbf{v})_{\mathcal{H}(D)} := (\mathbf{u}, \mathbf{v})_{L^2(D)} + (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v})_{H^1(D)}$ and corresponding norm $\|\cdot\|_{\mathcal{H}}$.
A solution $\mathbf{w}, \mathbf{v}$ of the interior transmission eigenvalue problem (92)-(95) is defined as $\mathbf{u} \in (L^2(D))^d$ and $\mathbf{v} \in (L^2(D))^d$ satisfying (92)-(93) in the distributional sense and such that $\mathbf{w} - \mathbf{v} \in \mathcal{H}_0(D)$. We therefore consider the following definition.

Definition 4.1 *Transmission eigenvalues corresponding to (92)-(95) are the values of $k > 0$ for which there exist nonzero solutions $\mathbf{w} \in L^2(D)$ and $\mathbf{v} \in L^2(D)$ such that $\mathbf{w} - \mathbf{v}$ is in $\mathcal{H}_0(D)$.*

Setting $\mathbf{u} := \mathbf{w} - \mathbf{v}$, we first observe that $\mathbf{u} \in \mathcal{H}_0(D)$ and

$$(\nabla\nabla \cdot + k^2 N)\,(N - I)^{-1}\,(\nabla\nabla \cdot \mathbf{u} + k^2 \mathbf{u}) = 0 \qquad \text{in} \quad D. \tag{96}$$

The latter can be written in the variational form

$$\int_D (N - I)^{-1} \left(\nabla\nabla \cdot \mathbf{u} + k^2 \mathbf{u}\right) \cdot \left(\nabla\nabla \cdot \overline{\mathbf{v}} + k^2 N \overline{\mathbf{v}}\right)\, dx = 0 \qquad \text{for all } \mathbf{v} \in \mathcal{H}_0(D). \tag{97}$$

Consequently, $k > 0$ is a transmission eigenvalue if and only if there exists a non trivial solution $\mathbf{u} \in \mathcal{H}_0(D)$ of (97). We now sketch the main steps of the proof of discreteness and existence of real transmission eigenvalues highlighting the new aspects of (97). To this end we see that (97) can be written as an operator equation

$$\mathbb{A}_\tau \mathbf{u} - \tau \mathbb{B}\mathbf{u} = 0 \qquad \text{and} \qquad \tilde{\mathbb{A}}_\tau \mathbf{u} - \tau \mathbb{B}\mathbf{u} = 0, \qquad \text{for} \quad \mathbf{u} \in \mathcal{H}_0(D). \tag{98}$$

Here the bounded linear operators $\mathbb{A}_\tau : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$, $\tilde{\mathbb{A}}_\tau : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ and $\mathbb{B} : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ are the operators defined using the Riesz representation theorem for the sesquilinear forms $\mathcal{A}_\tau$, $\tilde{\mathcal{A}}$ and $\mathcal{B}$ defined by

$$\mathcal{A}_\tau(\mathbf{u}, \mathbf{v}) := \left((N - I)^{-1} \left(\nabla\nabla \cdot \mathbf{u} + \tau\mathbf{u}\right), \left(\nabla\nabla \cdot \mathbf{v} + \tau\mathbf{v}\right)\right)_D + \tau^2 \left(\mathbf{u}, \mathbf{v}\right)_D \tag{99}$$

$$\begin{aligned} \tilde{\mathcal{A}}_\tau(\mathbf{u}, \mathbf{v}) \;&:=\; \left(N(I - N)^{-1} \left(\nabla\nabla \cdot \mathbf{u} + \tau\mathbf{u}\right), \left(\nabla\nabla \cdot \mathbf{v} + \tau\mathbf{v}\right)\right)_D \\ &+\; \left(\nabla\nabla \cdot \mathbf{u}, \nabla\nabla \cdot \mathbf{v}\right)_D \end{aligned} \tag{100}$$

and

$$\mathcal{B}(\mathbf{u}, \mathbf{v}) := \left(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}\right)_D, \tag{101}$$

respectively, where $(\cdot, \cdot)_D$ denotes the $L^2(D)$-inner product. Then the following Lemma can be proven and we refer the reader to [13] for the proof (see also (66)).

Lemma 4.1 *The operators $\mathbb{A}_\tau : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$, $\tilde{\mathbb{A}}_\tau : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$, $\tau > 0$ and $\mathbb{B} : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ are self-adjoint. Furthermore, $\mathbb{B}$ is a positive compact operator.*
If $(I - A)^{-1}A$ is a bounded positive definite matrix function on D, then $\mathbb{A}_\tau$ is a positive definite operator and

$$(\mathbb{A}_\tau u - \tau\mathbb{B}u, u)_{\mathcal{H}_0(D)} \geq \alpha\|u\|^2_{\mathcal{H}_0(D)} > 0 \quad \text{for all} \quad 0 < \tau < \lambda_1(D)A_* \quad \text{and} \quad u \in \mathcal{H}_0(D).$$

If $(A - I)^{-1}$ is a bounded positive definite matrix function on D, then $\tilde{\mathbb{A}}_\tau$ is a positive definite operator and

$$\left(\tilde{\mathbb{A}}_\tau u - \tau\mathbb{B}u, u\right)_{\mathcal{H}_0(D)} \geq \alpha\|u\|^2_{\mathcal{H}_0(D)} > 0 \quad \text{for all} \qquad 0 < \tau < \lambda_1(D) \quad \text{and} \quad u \in \mathcal{H}_0(D).$$

Note that the kernel of $\mathbb{B} : \mathcal{H}_0(D) \to \mathcal{H}_0(D)$ is given by

$$\text{Kernel}(\mathbb{B}) = \{\mathbf{u} \in \mathcal{H}_0(D) \quad \text{such that } \mathbf{u} := \text{curl}\,\varphi,\ \varphi \in H(\text{curl}\,, D)\}\,.$$

To carry over the approach of Section 3.2 to our eigenvalue problem (98), we also need to consider the corresponding transmission eigenvalue problems for a ball with constant index of refraction. To this end, we recall that it can be shown by separation of variables (see [21]), that

$$a_0\Delta w + k^2 w = 0 \qquad \text{in} \qquad B \tag{102}$$

$$\Delta v + k^2 v = 0 \qquad \text{in} \qquad B \tag{103}$$

$$w = v \qquad \text{on} \qquad \partial B \tag{104}$$

$$a_0\frac{\partial w}{\partial \nu} = \frac{\partial v}{\partial \nu} \qquad \text{on} \qquad \partial B_R \tag{105}$$

has a countable discrete set of eigenvalues, where $B := B_R \subset \mathbb{R}^d$ is the ball of radius R centered at the origin and $a_0 > 0$ a constant different from one. We now have all the ingredients to proceed with the approach of Section 3.2. Following exactly the lines of the proof of Theorem 3.8 it is now possible to show the existence of infinitely many transmission eigenvalues accumulating at infinity. The discreteness of real transmission eigenvalue can be obtained by using the analytic Fredholm theory as was done in [13] or alternatively following the proof of Theorem 3.6. As a byproduct of the proof we can also obtain estimates for the first transmission eigenvalue corresponding to the anisotropic medium. Let us denote by $k_1(A_*, B)$ and $k_1(A^*, B)$ the first transmission eigenvalue of (102)-(105) with index of refraction $a_0 := A_*$ and $a_0 := 1/A^*$, respectively. Then the following theorem holds.

Theorem 4.1 *Assume that either $A^* < 1$ or $A_* > 1$. Then problem (92)-(95) has an infinite countable set of real transmission eigenvalues with $+\infty$ as the only accumulation point. Furthermore, let $k_1(A(x), D)$ be the first transmission eigenvalue for (92)-(95) and B_1 and B_2 be two balls such that $B_1 \subset D$ and $D \subset B_2$, Then*

$$0 < k_1(A^*, B_2) \le k_1(A^*, D) \le k_1(A(x), D) \le k_1(A_*, D) \le k_1(A_*, B_1), \qquad \textit{if } A^* < 1,$$

$$0 < k_1(A_*, B_2) \le k_1(A_*, D) \le k_1(A(x), D) \le k_1(A^*, D) \le k_1(A^*, B_1), \qquad \textit{if } A_* > 1.$$

Note that A_* is the infimum of the lowest eigenvalue of the matrix A and A^* is the largest eigenvalue of the matrix A. We end this section by noting that we also have the following Faber-Krahn inequality similar to Theorem 3.5:

$$k_1^2(A(x), D) \ge \lambda_1(D)A_*, \text{ if } A^* < 1 \qquad \text{and} \qquad k_1^2(A(x), D) \ge \lambda_1(D), \text{ if } A_* > 1$$

where again $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ in D.

4.2 The case $n \neq 1$

The case $n \neq 1$ is treated in a different way from the two previous cases for $n = 1$ since now it is not possible to obtain a fourth order formulation. In particular in this case, as will be seen soon, the natural variational framework for (87)-(90) is $H^1(D) \times H^1(D)$. Here, we define transmission eigenvalues as follows:

Definition 4.2 *Transmission eigenvalues corresponding to (87)-(90) are the values of $k \in \mathbb{C}$ for which there exist nonzero solutions $w \in H^1(D)$ and $v \in H^1(D)$, where the equations (87) and (88) are satisfied in the distributional sense whereas the boundary conditions (89) and (90) are satisfied in the sense of traces in $H^{1/2}(\partial D)$ and $H^{-1/2}(\partial D)$, respectively.*

This case has been subject of several investigations [6], [12], [21]. Here we present the latest results on existence and discreteness of transmission eigenvalues. In particular, the existence of real transmission eigenvalues is shown only in the cases where the contrasts $A - I$ and $n - 1$ do not change sign in D (see section 4.2.2), whereas the discretness of the set of transmission eigenvalues is shown under less restrictive conditions on the sign of the contrasts using a relatively simple approach known as T-coercivity. The latter is the subject of the discussion in the next section which follows [6].

4.2.1 Discretness of transmission eigenvalues

The goal of this section is to prove discretness of transmission eigenvalues under sign assumptions on the contrasts that hold only in the neighborhood $\mathcal{V}$ of the boundary ∂D (a result of this type is also mentioned in Section 3.4 for the case of $A = I$). To this end we use the T-coercivity approach introduced in [7] and [22]. Following [6], we first observe that $(w, v) \in H^1(D) \times H^1(D)$ satisfies (87)- (88) if and only if $(w, v) \in X(D)$ satisfies the (natural) variational problem

$$a_k((w, v), (w', v')) = 0, \text{ for all } (w', v') \in X(D) \tag{106}$$

where

$$a_k((w, v), (w', v')) := (A\nabla w, \nabla w')_D - (\nabla v, \nabla v')_D - k^2 \left((nw, w')_D - (v, v')_D\right)$$

for all (w, v) and (w, v) in $X(D)$ and

$$X(D) := \{(w, v) \in H^1(D) \times H^1(D) \,|\, w - v \in H^1_0(D)\}.$$

With the help of the Riesz representation theorem, we define the operator $\mathcal{A}_k$ from $X(D)$ to $X(D)$ such that

$$(\mathcal{A}_k(w,v),(w',v'))_{H^1(D)\times H^1(D)} = a_k((w,v),(w',v'))$$

for all $((w,v),(w',v')) \in X(D)\times X(D)$. It is clear that $\mathcal{A}_k$ depends analytically on $k \in \mathbb{C}$. Moreover from the compact embedding of $X(D)$ into $L^2(D)\times L^2(D)$ one easily observes that

$$\mathcal{A}_k - \mathcal{A}_{k'} : X(D) \to X(D)$$

is compact for all k, k' in $\mathbb{C}$. In order to prove discretness of the set of transmission eigenvalues, one only needs to prove the invertibility of $\mathcal{A}_k$ for one k in $\mathbb{C}$. For the latter, it would have been sufficient to prove that a_k is coercive for some k in $\mathbb{C}$. Unfortunately this cannot be true in general, but we can show that a_k is T-coercive which turns out to be sufficient for our purpose. The idea behind the T-coercivity method is to consider an equivalent formulation of (106) where a_k is replaced by a_k^T defined by

$$a_k^T((w,v),(w',v')) := a_k((w,v),T(w',v')), \quad \forall ((w,v),(w',v')) \in X(D)\times X(D), \quad (107)$$

with T being an *ad hoc* isomorphism of $X(D)$. Indeed, $(w,v)\in X(D)$ satisfies

$$a_k((w,v),(w',v')) = 0 \qquad \text{for all} \qquad (w',v')\in X(D)$$

if, and only if, it satisfies $a_k^T((w,v),(w',v')) = 0$ for all $(w',v')\in X(D)$. Assume that T and k are chosen so that a_k^T is coercive. Then using the Lax-Milgram theorem and the fact that T is an isomorphism of $X(D)$, one deduces that $\mathcal{A}_k$ is an isomorphism on $X(D)$. We shall apply this technique to prove the following lemma where here and in the sequel $\mathcal{V}(\partial D)$ denotes a neighborhood of the boundary ∂D inside D. To this end, we set

$$\begin{aligned} A_\star &:= \inf_{x\in\mathcal{V}(\partial D)} \inf_{\xi\in\mathbb{R}^3, |\xi|=1} (\xi\cdot A(x)\xi) > 0, \quad & A^\star &:= \sup_{x\in\mathcal{V}(\partial D)} \sup_{\xi\in\mathbb{R}^3, |\xi|=1} (\xi\cdot A(x)\xi) < \infty, \\ n_\star &:= \inf_{x\in\mathcal{V}(\partial D)} n(x) > 0 & \text{and} \quad n^\star &:= \sup_{x\in\mathcal{V}(\partial D)} n(x) < \infty. \end{aligned} \quad (108)$$

We point out the difference between the $*$-constants in (91) and $\star$-constants in (108) namely in the first set of constants the infimum and suprimum is taken over the entire D whereas in the second set the infimum and suprimum is taken only over the neighborhood $\mathcal{V}$ of ∂D.

Lemma 4.2 *Assume that either $A(x) \le A^\star I < I$ and $n(x) \le n^\star < 1$, or $A(x) \ge A_\star I > I$ and $n(x) \ge n_\star > 1$ almost everywhere on $\mathcal{V}(\partial D)$. Then there exists $k = i\kappa$, with $\kappa\in\mathbb{R}$, such that the operator $\mathcal{A}_k$ is an isomorphism on $X(D)$.*

Proof: We consider first the case when $A(x) \leq A^\star I < I$ and $n(x) \leq n^\star < 1$ almost everywhere on $\mathcal{V}(\partial D)$. Introduce $\chi \in \mathcal{C}^\infty(\overline{D})$ a cut off function equal to 1 in a neighborhood of ∂D, with support in $\mathcal{V}(\partial D) \cap D$ and such that $0 \leq \chi \leq 1$, and consider the isomorphism ($T^2 = I$) of $X(D)$ defined by $T(w, v) = (w - 2\chi v, -v)$. We will prove that $a^T_{i\kappa}$ defined in (107) is coercive for some $\kappa \in \mathbb{R}$. For all $(w, v) \in X(D)$ one has

$$\begin{aligned} \left|a^T_{i\kappa}((w,v),(w,v))\right| &= |(A\nabla w, \nabla w)_D + (\nabla v, \nabla v)_D - 2(A\nabla w, \nabla(\chi v))_D \\ &\quad + \kappa^2\left((nw, w)_D + (v, v)_D - 2(nw, \chi v)_D\right)|\,. \end{aligned} \tag{109}$$

Using Young's inequality, one can write, for all $\alpha > 0$, $\beta > 0$, $\eta > 0$,

$$\begin{aligned} 2\,|(A\nabla w, \nabla(\chi v))_D| &\leq 2\,|(\chi A\nabla w, \nabla v)_{\mathcal{V}}| + 2\,|(A\nabla w, \nabla(\chi)v)_{\mathcal{V}}| \\ &\leq \eta(A\nabla w, \nabla w)_{\mathcal{V}} + \eta^{-1}(A\nabla v, \nabla v)_{\mathcal{V}} \\ &\quad +\alpha(A\nabla w, \nabla w)_{\mathcal{V}} + \alpha^{-1}(A\nabla(\chi)v, \nabla(\chi)v)_{\mathcal{V}} \\ \text{and} \qquad 2\,|(nw, \chi v)_D| &\leq \beta(nw, w)_{\mathcal{V}} + \beta^{-1}(nv, v)_{\mathcal{V}} \end{aligned} \tag{110}$$

where again $(\cdot,\cdot)_{\mathcal{O}}$ for a generic bounded region $\mathcal{O} \subset \mathbb{R}^d$, $d = 2, 3$, denotes the $L^2(\mathcal{O})$-inner product. Substituting (110) into (109), one obtains

$$\begin{aligned} \left|a^T_{i\kappa}((w,v),(w,v))\right| \geq\ & (A\nabla w, \nabla w)_{D\setminus\overline{\mathcal{V}}} + (\nabla v, \nabla v)_{D\setminus\overline{\mathcal{V}}} + \kappa^2\left((nw, w)_{D\setminus\overline{\mathcal{V}}} + (v, v)_{D\setminus\overline{\mathcal{V}}}\right) \\ & +((1-\eta-\alpha)A\nabla w, \nabla w)_{\mathcal{V}} + ((I - \eta^{-1}A)\nabla v, \nabla v)_{\mathcal{V}} \\ & +\kappa^2((1-\beta)nw, w)_{\mathcal{V}} + ((\kappa^2(1-\beta^{-1}n) - \sup_{\mathcal{V}} |\nabla\chi|^2 A^*\alpha^{-1})v, v)_{\mathcal{V}}. \end{aligned}$$

Taking η, α and β such that $A^\star < \eta < 1$, $n^\star < \beta < 1$ and $0 < \alpha < 1 - \eta$, we obtain the coercivity of $a^T_{i\kappa}$ for κ large enough. This gives the desired result for the first case.
The case $A(x) \geq A_\star I > I$ and $n(x) \geq n_\star > 1$ almost everywhere on $\mathcal{V}(\partial D)$ can be treated in a similar way by using $T(w, v) := (w, -v + 2\chi w)$.

□

We therefore we have the following theorem.

Theorem 4.2 *Assume that either $A(x) \leq A^\star I < I$ and $n(x) \leq n^\star < 1$, or $A(x) \geq A_\star I > I$ and $n(x) \geq n_\star > 1$ almost everywhere on $\mathcal{V}(\partial D)$. Then the set of transmission eigenvalues is discrete in $\mathbb{C}$.*

As another direct consequence of Lemma 4.2 and the compact embedding of $X(D)$ into $L^2(D) \times L^2(D)$, we remark that the operator $\mathcal{A}_k : X(D) \to X(D)$ is Fredholm for all $k \in \mathbb{C}$ provided that only $A(x) \leq A^\star I < I$ or $A(x) \geq A_\star I > I$ almost everywhere in

$\mathcal{V}(\partial D)$. Consequently, with a stronger assumption on A, namely assuming that $A - I$ is either positive definite or negative definite in D, one can relax the conditions on n in order to prove discreteness of transmission eigenvalues. To this end, taking $w' = v' = 1$ in (106), we first notice that the transmission eigenvectors (w, v) (i.e. the solution of (87)-(88) corresponding to an eigenvalue k) satisfy $k^2 \int_D (nw - v)dx = 0$. This leads us to introduce the subspace of eigenvectors

$$Y(D) := \left\{ (w, v) \in X(D) \mid \int_D (nw - v)dx = 0 \right\}.$$

Now, suppose $\int_D (n-1)dx \neq 0$. Arguing by contradiction, one can prove the existence of a Poincaré constant $C_P > 0$ (which depends on D and also on n through $Y(D)$) such that

$$\|w\|_D^2 + \|v\|_D^2 \leq C_P(\|\nabla w\|_D^2 + \|\nabla v\|_D^2), \quad \forall (w, v) \in Y(D). \tag{111}$$

Moreover, one can check that $k \neq 0$ is a transmission eigenvalue if and only if there exists a non trivial element $(w, v) \in Y(D)$ such that

$$a_k((w, v), (w', v')) = 0 \text{ for all } (w', v') \in Y(D).$$

Using this new variational formulation and (111) we can now prove the following theorem.

Theorem 4.3 *Suppose $\int_D (n-1)dx \neq 0$ and $A^* < 1$ or $A_* > 1$. Then the set of transmission eigenvalues is discrete in $\mathbb{C}$. Moreover, the nonzero eigenvalue of smallest magnitude k_1 satisfies the Faber-Krahn type estimate*

$$\begin{array}{ll} |k_1|^2 \geq (A_*(1 - \sqrt{A^*}))/(C_P \max(n^*, 1)\,(1 + \sqrt{n^*})) & \textit{if } A^* < 1 \\ |k_1|^2 \geq (1 - 1/\sqrt{A_*})/(C_P \max(n^*, 1)\,(1 + 1/\sqrt{n_*})) & \textit{if } A_* > 1 \end{array}$$

with C_P defined in (111).

Proof: We consider first the case $A^* < 1$. Denote $\lambda(v) := 2\int_D (n-1)v / \int_D (n-1)$ and consider the isomorphism of $Y(D)$ defined by

$$T(w, v) := (w - 2v + \lambda(v), -v + \lambda(v)).$$

Notice that $\lambda(\lambda(v)) = 2\lambda(v)$ so that $T^2 = I$. For all $(w, v) \in Y(D)$, one has

$$\begin{aligned} & \left|a_k^T((w, v), (w, v))\right| \\ = \; & |(A\nabla w, \nabla w)_D + (\nabla v, \nabla v)_D - 2(A\nabla w, \nabla v)_D - k^2\left((nw, w)_D + (v, v)_D - 2(nw, v)_D\right)| \\ \geq \; & (A\nabla w, \nabla w)_D + (\nabla v, \nabla v)_D - 2\left|(A\nabla w, \nabla v)_D\right| - |k|^2\left((nw, w)_D + (v, v)_D + 2\left|(nw, v)_D\right|\right) \\ \geq \; & (1 - \sqrt{A^*})((A\nabla w, \nabla w)_D + (\nabla v, \nabla v)_D) - |k|^2\,(1 + \sqrt{n^*})((nw, w)_D + (v, v)_D). \end{aligned}$$

Consequently, for $k \in \mathbb{C}$ such that

$$|k|^2 < (A_*(1-\sqrt{A^*}))/(C_P \max(n^*,1)\,(1+\sqrt{n^*}))$$

a_k^T is coercive on $Y(D)$. The claim of the theorem follows from the analytic Fredholm theory.
The case $A_* > 1$ can be treated in an analogous way by using the isomorphism T of $Y(D)$ defined by

$$T(w,v) := (w-\lambda(w), -v+2w-\lambda(w)).$$

$\square$

We remark that in particular, if $n^* < 1$ or if $1 < n_*$, then $\int_D (n-1)dx \neq 0$ and Theorem 4.3 proves that the set of interior transmission eigenvalues is discrete which recovers previously known results in [12], [21]. In those cases the Faber-Krahn type estimates can be made more explicit. For instance if $A^* < 1$ and $1 < n_*$, noticing that for $k^2 \in \mathbb{R}$,

$$\begin{aligned}\Re\left[a_k^T((w,v),(w,v))\right] &= (A\nabla(w-v),\nabla(w-v))_D - k^2((n(w-v),(w-v))_D\\ &+((I-A)\nabla v,\nabla v)_D + ((1-n)v,v)_D),\end{aligned}$$

where the isomophism T is defined by $T(w,v) = (w-2v,-v)$, one easily deduce that the first real transmission eigenvalue k_1 such that $k_1 \neq 0$ satisfies

$$k_1^2 \geq (A_*\lambda_1(D)/n^*)$$

where $\lambda_1(D)$ is the first Dirichlet eigenvalue of $-\Delta$ on D which is also proved in [21] using a different technique.
We end this section by giving a result on the location of transmission eigenvalues, again requiring the sign assumption on the contrasts only on a neighborhood of the boundary ∂D.

Theorem 4.4 *Under the hypothesis of Theorem 4.2 there exist two positive constants ρ and δ such that if $k \in \mathbb{C}$ satisfies $|k| > \rho$ and $|\Re(k)| < \delta\,|\Im(k)|$, then k is not a transmission eigenvalue.*

Proof: Here we give the proof only in the case of $A(x) \leq A^\star I < I$ and $n(x) \leq n^\star < 1$ almost everywhere on $\mathcal{V}(\partial D)$. The case of $A(x) \geq A_\star I > I$ and $n(x) \geq n_\star > 1$ almost everywhere on $\mathcal{V}(\partial D)$ can be treated using similar adaptations as in the proof of Lemma 4.2.

Consider again the isomorphism T defined by $T(w,v)=(w-2\chi v,-v)$ where χ is as in the proof of Lemma 4.2 where we already proved that for $\kappa\in\mathbb{R}$ with $|\kappa|$ large enough, the following coercivity property holds

$$\left|a^T_{i\kappa}((w,v),(w,v))\right| \geq C_1(\|w\|^2_{H^1(D)}+\|v\|^2_{H^1(D)})+C_2\kappa^2(\|w\|^2_D+\|v\|^2_D), \tag{112}$$

where the constants $C_1, C_2>0$ are independent of κ. Take now $k=i\kappa e^{i\theta}$ with $\theta\in[-\pi/2;\pi/2]$. One has

$$\left|a^T_{k}((w,v),(w,v))-a^T_{i\kappa}((w,v),(w,v))\right| \leq C_3\left|1-e^{2i\theta}\right|\kappa^2(\|w\|^2_D+\|v\|^2_D), \tag{113}$$

with $C_3>0$ independent of κ. Combining (112) and (113), one finds

$$\begin{aligned}\left|a^T_{k}((w,v),(w,v))\right| &\geq \left|a^T_{i\kappa}((w,v),(w,v))\right|-C_3\kappa^2\left|1-e^{2i\theta}\right|(\|w\|^2_D+\|v\|^2_D)\\ &\geq C_1(\|w\|^2_{H^1(D)}+\|v\|^2_{H^1(D)})+(C_2-C_3\left|1-e^{2i\theta}\right|)\kappa^2(\|w\|^2_D+\|v\|^2_D).\end{aligned}$$

Choosing θ small enough, to have for example $C_3\left|1-e^{2i\theta}\right|\leq C_2/2$, one obtains the desired result. $\square$

As already mentioned in Section 3.4, Theorem 3.15, proven in [42], provides a more precise location of transmission eigenvalues in the case when $A=I$. We also remark that related results on the discreteness of transmission eigenvalues are obtained in [47].

4.2.2 Existence of transmission eigenvalues

We now turn our attention to the existence of real transmission eigenvalues which unfortunately can only be shown under restrictive assumptions on $A-I$ and $n-1$. The proposed approach presented here follows the lines of [21] which, inspired by the original existence proof in the case $A=I$ discussed in Section 3.2, tries to formulate the transmission eigenvalue problem as a problem for the difference $u:=w-v$. However, due to the lack of symmetry, the problem for u is no longer a quadratic eigenvalue problem but it takes the form of a more complicated nonlinear eigenvalue problem as is explained in the following.
Setting $\tau:=k^2$, the transmission eigenvalue problem reads: find $(w,v)\in H^1(D)\times H^1(D)$ that satisfies

$$\nabla\cdot A\nabla w\ +\tau n w\ =\ 0 \quad\text{and}\quad \Delta v+\tau v=0 \text{ in } D\,, \tag{114}$$

$$w=v \quad\text{and}\quad \nu\cdot A\nabla w=\nu\cdot\nabla v \text{ on } \partial D. \tag{115}$$

We first observe that if (w, v) satisfies (87)-(88), subtracting the equation for w (114) from the equation for v (114) we obtain

$$
\begin{aligned}
\nabla \cdot A\nabla u + \tau n u \;&=\; \nabla \cdot (A - I)\nabla v + \tau(n-1)\, v \quad \text{in } D\,, \\
\nu \cdot A\nabla u \;&=\; \nu \cdot (A - I)\nabla v \quad \text{on } \partial D,
\end{aligned}
\tag{116}
$$

where $u := w - v$, and in addition we also have

$$
\begin{aligned}
&\Delta v + \tau v = 0 \quad \text{in } D\,, \\
&u = 0 \quad \text{on } \partial D.
\end{aligned}
\tag{117}
$$

It is easy to verify that (w, v) in $H^1(D) \times H^1(D)$ satisfies (114)-(115) if and only if (u, v) is in $H^1_0(D) \times H^1(D)$ and satisfies (116)-(117). The proof consists in expressing v in terms of u, using (116), and substituting the resulting expression into (117) in order to formulate the eigenvalue problem only in terms of u. In the case $A = I$, i.e. $(A - I) = 0$, this substitution is simple and leads to an explicit expression for the equation satisfied by u. In the current case the substitution requires the inversion of the operator $\nabla \cdot \big[(A - I)\nabla \cdot\big] + \tau(n-1)$ with a Neumann boundary condition. It is then obvious that the case where $(A - I)$ and $(n - 1)$ have the same sign is more problematic since in that case the operator may not be invertible for special values of τ. This is why we only treat the simpler case of $(A - I)$ and $(n - 1)$ having the opposite sign almost everywhere in D.
To this end we see that for given $u \in H^1_0(D)$, the problem (116) for $v \in H^1(D)$ is equivalent to the variational formulation

$$
\int_D \big[(A - I)\nabla v \cdot \nabla\overline{\psi} - \tau\,(n-1)\, v\, \overline{\psi}\big]\, dx \;=\; \int_D \big[A\, \nabla u \cdot \nabla\overline{\psi} - \tau n\, u\, \overline{\psi}\big]\, dx \tag{118}
$$

for all $\psi \in H^1(D)$. The following result concerning the invertibility of the operator associated with (118) can be proven in a standard way using the Lax-Milgram lemma.

Lemma 4.3 *Assume that either $(A_* - 1) > 0$ and $(n^* - 1) < 0$, or $(A^* - 1) < 0$ and $(n_* - 1) > 0$. Then there exists $\delta > 0$ such that for every $u \in H^1_0(D)$ and $\tau \in \mathbb{C}$ with $\Re\tau > -\delta$ there exists a unique solution $v := v_u \in H^1(D)$ of (118). The operator $A_\tau : H^1_0(D) \to H^1(D)$, defined by $u \mapsto v_u$, is bounded and depends analytically on $\tau \in \{z \in \mathbb{C} : \Re(z) > -\delta\}$.*

We now set $v_u := A_\tau u$ and denote by $\mathbb{L}_\tau u \in H^1_0(D)$ the unique Riesz representation of the bounded conjugate-linear functional

$$
\psi \;\longrightarrow\; \int_D \big[\nabla v_u \cdot \nabla\overline{\psi} - \tau\, v_u\, \overline{\psi}\big]\, dx \quad \text{for } \psi \in H^1_0(D)\,,
$$

i.e.

$$(\mathbb{L}_\tau u, \psi)_{H^1(D)} \;=\; \int\limits_D \big[\nabla v_u \cdot \nabla \overline{\psi} - \tau\, v_u\, \overline{\psi}\big]\, dx \quad \text{for } \psi \in H_0^1(D)\,. \tag{119}$$

Obviously, $\mathbb{L}_\tau$ also depends analytically on $\tau \in \{z \in \mathbb{C} : \Re z > -\delta\}$. Now we are able to connect a transmission eigenfunction, i.e. a nontrivial solution (w, v) of (114)-(115), to the kernel of the operator $\mathbb{L}_\tau$.

Theorem 4.5 *(a) Let $(w, v) \in H^1(D) \times H^1(D)$ be a transmission eigenfunction corresponding to some $\tau > 0$. Then $u = v - w \in H_0^1(D)$ satisfies $\mathbb{L}_\tau u = 0$.*

(b) Let $u \in H_0^1(D)$ satisfy $\mathbb{L}_\tau u = 0$ for some $\tau > 0$. Furthermore, let $v = v_u = A_\tau u \in H^1(D)$ be as in Lemma 4.3, i.e. the solution of (118). Then $(w, v) \in H^1(D) \times H^1(D)$ is a transmission eigenfunction where $w = v - u$.

The proof of this theorem is a simple consequence of the observation that the first equation in (117) is equivalent to

$$\int\limits_D \big[\nabla v \cdot \nabla \overline{\psi} - \tau\, v\, \overline{\psi}\big]\, dx \;=\; 0 \quad \text{for all } \psi \in H_0^1(D). \tag{120}$$

The operator $\mathbb{L}_\tau$ plays a similar role as the operator $\mathbb{A}_\tau - \tau\mathbb{B}$ in (68) for the case of $A = I$. The following properties are the main ingredients needed in order to prove the existence of transmission eigenvalues.

Theorem 4.6 *(a) The operator $\mathbb{L}_\tau : H_0^1(D) \to H_0^1(D)$ is selfadjoint for all $\tau \in \mathbb{R}_{\geq 0}$.*

(b) Let $\sigma = 1$ if $(A_* - 1) > 0$ and $(n^* - 1) < 0$, and $\sigma = -1$ if $(A^* - 1) < 0$ and $(n_* - 1) > 0$. Then $\sigma\mathbb{L}_0 : H_0^1(D) \to H_0^1(D)$ is coercive, i.e. $(\sigma\mathbb{L}_0 u, u)_{H^1(D)} \geq c\, \|u\|^2_{H^1(D)}$ for all $u \in H_0^1(D)$ and $c > 0$ independent of u.

(c) $\mathbb{L}_\tau - \mathbb{L}_0$ is compact in $H_0^1(D)$.

(d) There exists at most a countable number of $\tau > 0$ for which $\mathbb{L}_\tau$ fails to be injective with infinity the only possible accumulation point.

Proof: (a) First we show that $\mathbb{L}_\tau$ is selfadjoint for all $\tau \in \mathbb{R}_{\geq 0}$. To this end for every $u_1, u_2 \in H_0^1(D)$ let $v_1 := v_{u_1}$ and $v_2 := v_{u_2}$ be the corresponding solution of (118). Then we have that

$$\begin{aligned}(\mathbb{L}_\tau u_1,\, u_2)_{H^1(D)} \;&=\; \int\limits_D \big[\nabla v_1 \cdot \nabla \overline{u_2} - \tau\, v_1 \overline{u_2}\big]\, dx = \int\limits_D \big[A\, \nabla v_1 \cdot \nabla \overline{u_2} - \tau n\, v_1\, \overline{u_2}\big]\, dx \\ &\quad - \int\limits_D \big[(A - I)\nabla v_1 \cdot \nabla \overline{u_2} - \tau\, (n-1)\, v_1\, \overline{u_2}\big]\, dx\,. \end{aligned} \tag{121}$$

Using now (118) twice, first for $u = u_2$ and the corresponding $v = v_2$ and $\psi = v_1$ and then for $u = u_1$ and the corresponding $v = v_1$ and $\psi = u_2$, yields

$$(\mathbb{L}_\tau u_1,\, u_2)_{H^1(D)} \;=\; \int_D \left[(A-I)\nabla v_1 \cdot \nabla \overline{v_2} - \tau\,(n-1)\,v_1\,\overline{v_2}\right] dx \\ \qquad - \int_D \left[A\,\nabla u_1 \cdot \nabla \overline{u_2} - \tau n\, u_1\, \overline{u_2}\right] dx \tag{122}$$

which is a selfadjoint expression for u_1 and u_2.
(b) Next we show that $\sigma\mathbb{L}_0 : H_0^1(D) \to H_0^1(D)$ is a coercive operator. Using the definition of $\mathbb{L}_0$ in (119) and the fact that $v = v_u = u + w$ we have

$$(\mathbb{L}_0 u,\, u)_{H^1(D)} \;=\; \int_D \nabla v \cdot \nabla \overline{u}\, dx \;=\; \int_D |\nabla u|^2\, dx + \int_D \nabla w \cdot \nabla \overline{u}\, dx\,. \tag{123}$$

Fom (118) for $\tau = 0$ and $\psi = w$ we have that

$$\int_D \nabla w \cdot \nabla \overline{u}\, dx \;=\; \int_D (A-I)\nabla w \cdot \nabla \overline{w}\, dx\,. \tag{124}$$

If $(A_* - 1) > 0$ then we have $\int_D (A-I)\nabla w \cdot \nabla \overline{w}\, dx \geq (A_* - 1)\|\nabla w\|^2_{L^2(D)} \geq 0$ and hence

$$(\mathbb{L}_0 u,\, u)_{H^1(D)} \;\geq\; \int_D |\nabla u|^2\, dx\,.$$

From Poincaré's inequality in $H_0^1(D)$ we have that $\|\nabla u\|_{L^2(D)}$ is an equivalent norm in $H_0^1(D)$ and this proves the coercivity of $\mathbb{L}_0$. If $(A^* - 1) < 0$, from (122) with $u_1 = u_2 = u$ and $\tau = 0$ we have

$$-(\mathbb{L}_0 u,\, u)_{H^1(D)} \;=\; -\int_D (A-I)\nabla v \cdot \nabla \overline{v}\, dx \;+\; \int_D A\,\nabla u \cdot \nabla \overline{u}\, dx \geq A_* \int_D |\nabla u|^2\, dx$$

which proves the coercivity of $-\mathbb{L}_0$ since $A_* > 0$. Part (c) of the theorem follows from the compact embedding of $H_0^1(D)$ into $L^2(D)$.
(d) Since $(\sigma\mathbb{L}_0)^{-1}$ exists and $\tau \mapsto \mathbb{L}_\tau$ is analytic on $\{z \in \mathbb{C} : \Re(z) > -\delta\}$, this follows directly from the analytic Fredholm theory. We remark that this part is also a consequence of the more general result of Theorem 4.3. □

We are now in the position to establish the existence of infinitely many real transmission eigenvalues, i.e. the existence of a sequence of $\tau_j \in \mathbb{R}$, $j \in \mathbb{N}$, and corresponding $u_j \in H_0^1(D)$ such that $u_j \neq 0$ and $\mathbb{L}_{\tau_j} u_j = 0$. Obviously, these $\tau > 0$ are such that the kernel of

$\mathbb{I} - \mathbb{T}_\tau$ is not trivial, where $-\sigma(\sigma\mathbb{L}_0)^{-1/2}(\mathbb{L}_\tau - \mathbb{L}_0)(\sigma\mathbb{L}_0)^{-1/2}$ is compact, which correspond to one being an eigenvalue of the compact self-adjoint operator $\mathbb{T}_\tau$. From the above discussion we conclude that transmission eigenvalues $k > 0$ have finite multiplicity and are such that $\tau := k^2$ are solutions to $\mu_j(\tau) = 1$ where $\{\mu_j(\tau)\}_1^{+\infty}$ is the increasing sequence of the eigenvalues of $\mathbb{T}_\tau$. Note that from max-min principle $\mu_j(\tau)$ depend continuously on τ which the core of the proof the following theorem (see e.g. [50] for the proof)

Theorem 4.7 *Assume that*

(1) there is a $\tau_0 \geq 0$ such that $\sigma\mathbb{L}_{\tau_0}$ is positive on $H_0^1(D)$ and

(2) there is a $\tau_1 > \tau_0$ such that $\sigma\mathbb{L}_{\tau_1}$ is non positive on some m-dimensional subspace W_m of $H_0^1(D)$,

then there are m values of τ in $[\tau_0, \tau_1]$ counting their multiplicity for which $\mathbb{L}_\tau$ fails to be injective.

Using now Theorem 4.7 and adapting the ideas developed in Section 3.2 and Section 4.1, we can prove the main theorem of this section.

Theorem 4.8 *Suppose that the matrix valued function A and the function n are such that either $(A_* - 1) > 0$ and $(n^* - 1) < 0$, or $(A^* - 1) < 0$ and $(n_* - 1) > 0$. Then there exists an infinite sequence of transmission eigenvalues $k_j > 0$ with $+\infty$ as their only accumulation point.*

Proof: We sketch the proof only for the case of $(A_* - 1) > 0$ and $(n^* - 1) < 0$ (i.e. $\sigma = 1$ in Theorem 4.7). First, we recall that the assumption *(1)* of Theorem 4.7 is satisfied with $\tau_0 = 0$ i.e. $(\mathbb{L}_0 u, u)_{H^1(D)} > 0$ for all $u \in H_0^1(D)$ with $u \neq 0$. Next, by definition of $\mathbb{L}_\tau$ and the fact that $v = w + u$ have

$$(\mathbb{L}_\tau u,\, u)_{H^1(D)} = \int_D \left[\nabla v \cdot \nabla \overline{u} - \tau\, v\, \overline{u}\right] dx = \int_D \left[\nabla w \cdot \nabla \overline{u} - \tau\, w\, \overline{u} + |\nabla u|^2 - \tau\, |u|^2\right] dx. \quad (125)$$

We also have that w satisfies

$$\int_D \left[(A - I)\nabla w \cdot \nabla \overline{\psi} - \tau(n-1)\, w\, \overline{\psi}\right] dx \;=\; \int_D \left[\nabla u \cdot \nabla \overline{\psi} - \tau\, u\, \overline{\psi}\right] dx \qquad (126)$$

for all $\psi \in H^1(D)$. Now taking $\psi = w$ in (126) and substituting the result into (125) yields

$$(\mathbb{L}_\tau u,\, u)_{H^1(D)} \;=\; \int_D \left[(A - I)\nabla w \cdot \nabla \overline{w} - \tau\,(n-1)\, |w|^2 + |\nabla u|^2 - \tau\, |u|^2\right] dx\,. \qquad (127)$$

Let now $B_r \subset D$ be an arbitrary ball of radius r included in D and let $\hat{\tau}$ be such that $\hat{\tau} := k_1^2(A_*, n^*, B_r)$ where $k_1(A_*, n^*, B_r)$ is the first transmission eigenvalue corresponding to the ball B_r with constant contrasts $A = A_* I$ and $n = n^*$ (we refer to [21] for the existence of transmission eigenvalues in this case which is again proved by speration of variables and using the asymptotic behavior of Bessel functions). Let $\hat{v}$, $\hat{w}$ be the non-zero solutions to the corresponding homogenous interior transmission problem, i.e the solution of (87)-(90) with $D = B_r$, $A = A_* I$ and $n = n^*$ and set $\hat{u} := \hat{v} - \hat{w} \in H_0^1(B_r)$. We denote the corresponding operator by $\hat{\mathbb{L}}_\tau$. Of course, by construction we have that (127) still holds, i.e. since $\hat{\mathbb{L}}_{\hat{\tau}}\hat{u} = 0$,

$$0 \;=\; \left(\hat{\mathbb{L}}_{\hat{\tau}}\hat{u}, \hat{u}\right)_{H^1(B_r)} \;=\; \int\limits_{B_r} \left[(A_* - 1)|\nabla \hat{w}|^2 - \hat{\tau}\,(n^* - 1)|\hat{w}|^2 + |\nabla \hat{u}|^2 - \hat{\tau}\,|\hat{u}|^2\right] dx\,. \quad (128)$$

Next we denote by $\tilde{u} \in H_0^1(D)$ the extension of $\hat{u} \in H_0^1(B_r)$ by zero to the whole of D and let $\tilde{v} := v_{\tilde{u}}$ be the corresponding solution to (118) and $\tilde{w} := \tilde{v} - \tilde{u}$. In particular $\tilde{w} \in H^1(D)$ satisfies

$$\int\limits_D \left[(A - I)\nabla \tilde{w} \cdot \nabla \overline{\psi} - \hat{\tau}\, p\, \tilde{w}\, \overline{\psi}\right] dx \;=\; \int\limits_D \left[\nabla \tilde{u} \cdot \nabla \overline{\psi} - \hat{\tau}\, \tilde{u}\, \overline{\psi}\right] dx$$

$$= \int\limits_{B_r} \left[\nabla \hat{u} \cdot \nabla \overline{\psi} - \hat{\tau}\, \hat{u}\, \overline{\psi}\right] dx \;=\; \int\limits_{B_r} \left[(A_* - 1)\nabla \hat{w} \cdot \nabla \overline{\psi} - \hat{\tau}\,(n^* - 1)\, \hat{w}\, \overline{\psi}\right] dx \quad (129)$$

for all $\psi \in H^1(D)$. Therefore, for $\psi = \tilde{w}$ we have by the Cauchy-Schwarz inequality,

$$\int\limits_D (A - I)\nabla \tilde{w} \cdot \nabla \overline{\tilde{w}} - \hat{\tau}\,(n - 1)\,|\tilde{w}|^2\, dx \;=\; \int\limits_{B_r} (A_* - 1)\,\nabla \hat{w} \cdot \nabla \overline{\tilde{w}} + \hat{\tau}\,|n^* - 1|\,\hat{w}\,\overline{\tilde{w}}\, dx$$

$$\le \left[\int\limits_{B_r} (A_* - 1)\,|\nabla \hat{w}|^2 + \hat{\tau}\,|n^* - 1|\,|\hat{w}|^2\, dx\right]^{\frac{1}{2}} \left[\int\limits_{B_r} (A_* - 1)\,|\nabla \tilde{w}|^2 + \hat{\tau}\,|n^* - 1|\,|\tilde{w}|^2\, dx\right]^{\frac{1}{2}}$$

$$\le \left[\int\limits_{B_r} (A_* - 1)\,|\nabla \hat{w}|^2 - \hat{\tau}\,(n^* - 1)\,|\hat{w}|^2\, dx\right]^{\frac{1}{2}} \left[\int\limits_D (A - I)\nabla \tilde{w} \cdot \nabla \overline{\tilde{w}} - \hat{\tau}\,(n - 1)\,|\tilde{w}|^2\, dx\right]^{\frac{1}{2}}$$

since $|n - 1| = 1 - n \ge 1 - n^* = |n^* - 1|$ and thus

$$\int\limits_D \left[(A - I)\nabla \tilde{w} \cdot \nabla \overline{\tilde{w}} - \hat{\tau}\,(n - 1)\,|\tilde{w}|^2\right] dx \;\le\; \int\limits_{B_r} \left[(A_* - 1)\,|\nabla \hat{w}|^2 - \hat{\tau}\,(n^* - 1)\,|\hat{w}|^2\right] dx\,.$$

Substituting this into (127) for $\tau = \hat{\tau}$ and $u = \tilde{u}$ yields

$$\begin{aligned}\left(\mathbb{L}_{\hat{\tau}}\tilde{u}, \tilde{u}\right)_{H^1(D)} &= \int_D \left[(A-I)\nabla\tilde{w}\cdot\nabla\overline{\tilde{w}} - \hat{\tau}\,(n-1)\,|\tilde{w}|^2 + |\nabla\tilde{u}|^2 - \hat{\tau}\,|\tilde{u}|^2\right] dx \\ &\leq \int_{B_r} \left[(A_*-1)|\nabla\hat{w}|^2 - \hat{\tau}\,(n^*-1)\,|\hat{w}|^2 + |\nabla\hat{u}|^2 - \hat{\tau}\,|\hat{u}|^2\right] dx \;=\; 0 \qquad (130)\end{aligned}$$

by (128). Hence from Theorem 4.7 we have that there is a transmission eigenvalue $k > 0$, such that in $k^2 \in (0, \hat{\tau}]$. Finally, repeating this argument for balls of arbitrary small radius we can show the existence of infinitely many transmission eigenvalues exactly in the same way as in the proof Theorem 3.8. □

We can also obtain better bounds for the first transmission eigenvalue stated in the following theorem (see [21] for the proof).

Theorem 4.9 *Let $B_R \subset D$ be the largest ball contained in D and $\lambda_1(D)$ the first Dirichlet eigenvalue of $-\Delta$ on D. Furthermore, let $k_1(A(x), n(x), D)$ be the first transmission eigenvalue corresponding to (87)-(90).*

(1) If $(A_* - 1) > 0$ and $(n^* - 1) < 0$ then

$$\lambda_1(D) \;\leq\; k_1^2(A(x), n(x), D) \;\leq\; k_1^2(A_*, n^*, B_R)$$

where $k_1(A_*, n^*, B_R)$ is the first transmission eigenvalue corresponding to the ball B_R with $A = A_* I$ and $n = n^*$.

(2) If $(A^* - 1) < 0$ and $(n_* - 1) > 0$ then

$$\frac{A_*}{n^*}\,\lambda_1(D) \;\leq\; k_1^2(A(x), n(x), D) \;\leq\; k_1^2(A^*, n_*, B_R)$$

where $k_1(A^*, n_*, B_R)$ is the first transmission eigenvalue corresponding to the ball B_R with $A = A^* I$ and $n = n_*$.

We end our discussion in this section by making a few comments on the case when $(A - I)$ and $(n - 1)$ have the same sign. As indicated above, if one follows a similar procedure, then one is faced with the problem that (118) is not solvable for all τ. Thus we are forced to put restrictions on τ, $(A - I)$ and $(n - 1)$ which only allow us to prove the existence of at least one transmission eigenvalue. In particular, skipping the details, we set $\hat{\tau}(r, A_*) := k_1^2(\frac{A_*+1}{2}, 1, B_r)$ (with the notation of Theorem 4.9 for $k_1(\frac{A_*+1}{2}, 1, B_r)$), where the ball B_r of radius r is such that $B_r \subset D$. Then if $(n^* - 1) > 0$ is small enough such that

$$(n^* - 1) \;<\; \frac{\mu(D,n)}{2\,\hat{\tau}(r, A_*)}\,(A_* - 1) \tag{131}$$

where

$$\mu(D,n) \;:=\; \inf_{\psi \in H^1(D),\, \int_D (n-1)\,\psi\,dx = 0} \frac{\|\nabla \psi\|^2_{L^2(D)}}{\|\psi\|^2_{L^2(D)}},$$

then there exists at least one real transmission eigenvalue in the interval $\big(0, k_1(\frac{A_*+1}{2}, 1, B_r)\big]$. In fact, if $(n^* - 1)$ is small enough such that (131) is satisfied for an $r > 0$ such that in D we can fit m balls of radius r, then one can show [21] that there are m real transmission eigenvalues in $\big(0, k_1(\frac{A_*+1}{2}, 1, B_r)\big]$ counting their multiplicity. It is still an open problem to prove the existence of infinitely many real transmission eigenvalues in this case.

5 Conclusions and Open Problems

In this survey we have presented a collection of results on the transmission eigenvalue problem corresponding to scattering by an inhomogeneous medium with emphasis on the derivation of the existence, discreteness and inequalities for transmission eigenvalues. Although we have focused on theoretical results, computational methods for transmission eigenvalues as well as their use in obtaining information on the material properties of inhomogeneous media from scattering data can be found in [13], [16], [18], [29], [33], [38] and [56]. A similar analysis has been done in [11] and [33] for inhomogeneous media containing obstacles inside. The transmission eigenvalue problem has also been investigated for the case of Maxwell's equation where technical complications arise due to the structure of the spaces needed to study these equations (see [17], [18], [21], [20], [19], [34], [39], [45]). The transmission eigenvalue problem associated with the scattering problem for anisotropic linear elasticity has been investigated in [4] and [5]. As previously mentioned, [41] and [43] investigate transmission eigenvalues for higher order operators with constant coefficients.

Despite extensive research and much recent progress on the transmission eigenvalue problem there are still many open questions that call for new ideas. In our opinion some important questions that impact both the theoretical understanding of the transmission eigenvalue problem as well as their application to inverse scattering theory are the following: 1) Do complex transmission eigenvalues exists for general non-absorbing media? 2) Do real transmission eigenvalues exist for absorbing media and absorbing background? 3) Can the existence of real transmission eigenvalues for non-absorbing media be established if the assumptions on the sign of the contrast are weakened? 4) What would the necessary conditions be on the contrasts that guaranty the discreteness of transmission eigenvalues? 5) Can Faber-Krahn type inequalities be established for the higher eigenvalues? 6) Can completeness results be established for transmission eigenfunctions, i.e nonzero solutions to transmission eigenvalue problem corresponding to transmission eigenvalues? (We remark that in [43] the completeness question is positively answered for transmission eigenvalue problem for operators of order higher than 3. The proof breaks down for operators of order two which are the cases considered in this paper and are related to most of the practical problems in scattering theory), 7) Can an inverse spectral problem be developed for the general transmission eigenvalue problem? We also believe that a better understanding of the physical interpretation of transmission eigenvalues and their connection to the wave equation could provide an alternative way of determining transmission eigenvalues from the (possibly time-dependent) scattering data.

Acknowledgement

The research of F.C. was supported in part by the AFOSR under Grant FA 9550-11-1-0189 and NSF Grant DMS-1106972. The paper was written while F.C. was a CNRS visiting researcher at CMAP, Ecole Polytechnique, Palaiseau. The financial support of CNRS is greatly acknowledged. The authors thanks Professor David Colton for his valuable comments on spherically stratified media, in particular for the observation that resulted in an improved version of Theorem 3.1.

References

[1] T. Aktosun, D. Gintides, and V. Papanicolaou. The uniqueness in the inverse problem for transmission eigenvalues for the spherically symmetric variable-speed wave equation. *Inverse Problems*, 27:115004, 2011.

[2] T. Arens, and A. Lechleiter The linear sampling method revisited. *J. Integral Equations and Applications*, 21(2):179–203, 2009.

[3] T. Arens. Why linear sampling method works. *Inverse Problems*, 20:163–173, 2004.

[4] C. Bellis, F. Cakoni, and B. Guzina. Nature of the transmission eigenvalue spectrum for elastic bodies. *IMA J. Appl. Math.* (to appear).

[5] C. Bellis and B. Guzina. On the existence and uniqueness of a solution to the interior transmission problem for piecewise-homogeneous solids. *Journal of Elasticity*, 101:29–57, 2010.

[6] AS. Bonnet-BenDhia, L. Chesnel, and H. Haddar. On the use of t-coercivity to study the interior transmission eigenvalue problem. *C. R. Acad. Sci., Ser. I*, 340, 2011.

[7] AS. Bonnet-BenDhia, P.Ciarlet, and C.Maria Zwölf. Time harmonic wave diffraction problems in materials with sign-shifting coefficients. *J. Comput. Appl. Math*, 234(6):1912–1919, 2010.

[8] F. Cakoni, M. Cayorën and D. Colton. Transmission eigenvalues and the nondestructive testing of dielectrics. *Inverse Problems*, 24, 2008.

[9] F. Cakoni and D. Colton. *Qualitative Methods in Inverse Scattering Theory*. Interaction of Mechanics and Mathematics. Springer-Verlag, Berlin, 2006.

[10] F. Cakoni, D. Colton, and D. Gintides. The interior transmission eigenvalue problem. *SIAM J. Math. Anal.*, 42(6):2912–2921, 2010.

[11] F. Cakoni, D. Colton, and H. Haddar. The interior transmission eigenvalue problem for absorbing media. (submitted).

[12] F. Cakoni, D. Colton, and H. Haddar. The linear sampling method for anisotropic media. *J. Comp. Appl. Math.*, 146:285–299, 2002.

[13] F. Cakoni, D. Colton, and H. Haddar. The computation of lower bounds for the norm of the index of refraction in an anisotropic media. *J. Integral Equations and Applications*, 21(2):203–227, 2009.

[14] F. Cakoni, D. Colton, and H. Haddar. The interior transmission problem for regions with cavities. *SIAM J. Math. Anal.*, 42(1):145–162, 2010.

[15] F. Cakoni, D. Colton, and H. Haddar. On the determination of Dirichlet or transmission eigenvalues from far field data. *C. R. Acad. Sci. Paris*, 348:379–383, 2010.

[16] F. Cakoni, D. Colton, and P. Monk. On the use of transmission eigenvalues to estimate the index of refraction from far field data. *Inverse Problems*, 23:507–522, 2007.

[17] F. Cakoni, D. Colton, and P. Monk. *The Linear Sampling Method in Inverse Electromagnetic Scattering*. CBMS-NSF, SIAM Publications, Philadelphia, 2011.

[18] F. Cakoni, D. Colton, P. Monk, and J. Sun. The inverse electromagnetic scattering problem for anisotropic media. *Inverse Problems*, 26:074004, 2010.

[19] F. Cakoni, D. Gintides, and H. Haddar. The existence of an infinite discrete set of transmission eigenvalues. *SIAM J. Math. Anal.*, 42:237–255, 2010.

[20] F. Cakoni and H. Haddar. On the existence of transmission eigenvalues in an inhomogeneous medium. *Applicable Analysis*, 88(4):475–493, 2008.

[21] F. Cakoni and A. Kirsch. On the interior transmission eigenvalue problem. *Int. Jour. Comp. Sci. Math.*, 3:142–167, 2010.

[22] L. Chesnel. *Ph.D. Thesis*. ENSTA, Pais, France, 2011.

[23] D. Colton, A. Kirsch, and L. Päivärinta. Far-field patterns for acoustic waves in an inhomogeneous medium. *SIAM J. Math. Anal.*, 20:1472–1483, 1989.

[24] D. Colton and R. Kress. The construction of solutions to acoustic scattering problems in a spherically stratified medium II. *Quart. J. Mech. Appl. Math*, 32:53–62, 1972.

[25] D. Colton and R. Kress. *Integral Equation Methods in Scattering Theory*. John Wiley, New York, 1983.

[26] D. Colton and R. Kress. *Inverse Acoustic and Eletromagnetic Scattering Theory*. Springer, New York, 2nd edition, 1998.

[27] D. Colton and R. Kress. On the denseness of Herglotz wave functions and electromagnetic Herglotz pairs in Sobolev spaces. *Math. Methods Applied Sciences*, 24:12891303, 2001.

[28] D. Colton and P. Monk. The inverse scattering problem for time-harmonic acoustic waves in a inhomogeneous medium. *Quart. Jour. Mech. Applied Math.*, 41:97–125, 1988.

[29] D. Colton, P. Monk, and J. Sun. Analytical and computational methods for transmission eigenvalues. *Inverse Problems*, 25:045011, 2010.

[30] D. Colton and L. Päivärinta. Transmission eigenvalues and a problem of Hans Lewy. *Jour. Comp. Applied Math*, 117:91–104, 2000.

[31] D. Colton, L.Päivärinta, and J. Sylvester. The interior transmission problem. *Inverse Problems and Imaging*, 1:13–28, 2007.

[32] D. Colton and B. Sleeman. An approximation property of importance in inverse scattering theory. *Proc. Edinburgh Math. Soc.*, 44:449–454, 2001.

[33] A Cossonnière. *thesis*. CERFACS, Toulouse, France, 2011.

[34] A. Cossonniere and H. Haddar. The electromagnetic interior transmission problem for regions with cavities. *SIAM J. Math. Anal.*, 43(4):1698–1715, 2011.

[35] P.J. Davis. *Interpolation and Approximation*. Blaisdell Publishing Company, New York, 1963.

[36] A. Erdelyi. *Asymptotic Expansions*. Dover, 1956.

[37] D. Finch. Private comunication.

[38] G. Giovanni and H. Haddar. Computing estimates on material properties from transmission eigenvalues. Research Report RR-7729, INRIA, September 2011.

[39] H. Haddar. The interior transmission problem for anisotropic Maxwell's equations and its applications to the inverse problem. *Math. Methods Applied Sciences*, 27:2111–2129, 2004.

[40] K. Hickmann. Interior transmission problem with refractive index having C^2-transition to the background medium. *Applicable Analysis*, to appear.

[41] M. Hitrik, K. Krupchyk, P. Ola, and L. Päivärinta. Transmission eigenvalues for operators with constant coefficients. *SIAM J. Math. Anal.*, 42:2965–2986, 2010.

[42] M. Hitrik, K. Krupchyk, P. Ola, and L. Päivärinta. The interior transmission problem and bounds on transmission eigenvalues. *Math Research Letters*, 18:279–293, 2011.

[43] M. Hitrik, K. Krupchyk, P. Ola, and L. Päivärinta. Transmission eigenvalues for elliptic operators. *SIAM J. Math. Anal.*, to appear.

[44] A. Kirsch. The denseness of the far field patterns for the transmission problem. *IMA J. Appl. Math.*, 37:213–226, 1986.

[45] A. Kirsch. On the existence of transmission eigenvalues. *Inverse Problems and Imaging*, 3:155–172, 2009.

[46] A. Kirsch and N. Grinberg. *The Factorization Method for Inverse Problems*, Oxford University Press, Oxford, 2008.

[47] E. Lakshtanov and B. Vainberg. Ellipticity in the interior transmission problem in anisotropic media. *arXiv:1108.5987v1 [math-ph]*.

[48] P.D. Lax and R.S. Phillips. *Scattering Theory.* Academic Press, New York, London, 1967.

[49] YJ Leung and D. Colton. Complex transmission eigenvalues for spherically stratified media. *SIAM J. Math. Anal.* (to appear).

[50] L.Päivärinta and J. Sylvester. Transmission eigenvalues. *SIAM J. Math. Anal*, 40:738–753, 2008.

[51] A. Markus. *Introduction to the Spectral Theory of Polynomial Operator Pencils.* American Mathematical Society, Providence, RI, 1988.

[52] J. McLaughlin and P. Polyakov. On the uniqueness of a spherically symmetric speed of sound from transmission eigenvalues. *J. Differential Equations*, 107:351–382, 1994.

[53] J. McLaughlin, P. Polyakov, and P. Sacks. Reconstruction of a spherically symmetric speed of sound. *SIAM J. Appl. Math.*, 54:1203–1223, 1994.

[54] B.P. Rynne and B.D. Sleeman. The interior transmission problem and inverse scattering from inhomogeneous media. *SIAM J. Math. Anal.*, 22:1755–1762, 1992.

[55] V. Serov and J. Sylvester. Transmission eigenvalues for degenerate and singular cases. (submitted).

[56] J. Sun. Estimation of transmission eigenvalues and the index of refraction from Cauchy data. *Inverse Problems*, 27:015009, 2011.

[57] J. Sylvester. Discreteness of transmission eigenvalues via upper triangular compact operators. *SIAM J. Math. Anal.* (to appear).

[58] N. Weck. Approximation by Maxwell-Herglotz-fields. *Math. Methods Applied Sciences*, 27:603–621, 2004.