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Abstract: An abstract Lipschitz stability estimate is proved for a class of inverse problems. It is then applied to the inverse Robin problem for the Laplace equation and to the inverse medium problem for the Helmholtz equation.

Key-words: Derivative in the sense of Fréchet, Lipschitz stability, inverse Robin problem, inverse medium problem

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Une remarque sur la stabilité de type Lipschitz pour les problèmes inverses

Résumé : Une résultat abstrait de stabilité Lipschitzienne est montré pour une certaine classe de problèmes inverses. Il est ensuite appliqué au problème inverse de Robin pour le Laplacien et pour un problème inverse de reconstruction de milieu pour l'équation d'Helmholtz

Mots-clés : Différentielle au sens de Fréchet, Stabilité Lipschitzienne, Problème inverse de Robin, Problème inverse de reconstruction de milieu

1 Introduction

The stability issue for inverse problems consists in estimating the impact of some variation of the data on the parameter we want to identify. Such analysis is important because the inverse problems are ill-posed in general, and having a theoretical stability estimate enables us to quantify such ill-posedness. The stability estimates answer the following question: if the distance between two data is $\delta > 0$, what is the distance between the corresponding parameters as a function ϕ of δ, with $\phi(\delta) \to 0$ when $\delta \to 0$? The quantification of ill-posedness is given by the convergence rate of ϕ when δ tends to 0. The stability results that we can collect in the literature are of different types, but we can point out that some assumptions on the parameter are necessary to obtain the function ϕ, for example boundedness of the parameter with respect to an adapted norm. As a result, these stability estimates are in fact conditional stability estimates. In addition, the stronger are these assumptions the better is the function ϕ we obtain. If for example we think of the well known Calderon's inverse conductivity problem, where the parameter is the conductivity and the data is the Dirichlet-to-Neumann map, the function ϕ is a logarithm when the conductivity lies in an infinite dimensional space with some *a priori* bounds on the conductivity (see [1]), but ϕ becomes a linear function when the conductivity lies in a finite dimensional space of dimension N and again with some *a priori* bounds on the conductivity (see [3]). Besides, as expected, the constant of linearity grows exponentially when $N \to +\infty$ (see [15]).

The objective of the present paper is to prove an abstract theorem that provides the same kind of Lipschitz stability estimate as in [3] in a general case where the mapping from parameter to data is nonlinear with respect to appropriate Banach spaces. Basically, such mapping shall be C^1, injective as well as its Fréchet derivative, and the set of parameters shall be a compact and convex subset of a finite dimensional subspace. Note that some similar abstract theorems, with different assumptions, may be found in [19]. To illustrate the interest of our theorem, we apply it on two different inverse problems: the inverse Robin problem for the Laplace equation and the inverse medium problem for the Helmholtz equation. The original idea of the proof of our abstract theorem is introduced in [7] in the particular context of detection from boundary measurements of an obstacle characterized by two degrees of freedom moving in a fluid. Here, we simply adapt the proof of [7] to a general framework that covers a number of interesting situations. Our proof is elementary and avoids the sophisticated arguments that are used in [3, 17] to achieve such result, in particular the arguments related to the quantification of unique continuation. The author is conscious that his stability result is of qualitative rather than quantitative nature, in particular the Lipschitz constant cannot be expressed in terms of the data, since the proof is based on compactness arguments. However, in [3, 17] such Lipschitz constant is not given as an explicit function of the data either (see theorem 2.7 in [3] and theorem 2.4 in [17]), even if the intermediate results of quantification of unique continuation in these papers have their own interest. In particular, the exponential growth of the Lipschitz constant with respect to the dimension N of the space can be proved independently of the way the Lipschitz constant is obtained (see for example [17]).

Our paper is organized as follows. The second section concerns the statement and proof of the abstract theorem. The third one is dedicated to the inverse Robin problem while the fourth one is dedicated to the inverse medium problem.

2 The abstract theorem

The aim of this section is to prove the following abstract theorem.

Theorem 2.1. *Let V and H be two Banach spaces, their norms being denoted $||.||_V$ and $||.||_H$. The norm of $\mathcal{L}(V,H)$ is denoted $|||.|||$. Let U be an open subset of V, and V_N a finite dimensional subspace of V (of dimension N). Let K_N be a compact and convex subset of $V_N \cap U$.*
We consider a (nonlinear) mapping $T : U \to H$ which satisfies the following assumptions.

1. *$T : V_N \cap U \to H$ is injective*
2. *$T : U \to H$ is C^1, that is: T is differentiable in the sense of Fréchet at any point $x \in U$, the Fréchet derivative being denoted $dT_x : V \to H$, and the mapping $x \in U \mapsto dT_x \in \mathcal{L}(V,H)$ is continuous*
3. *For all $x \in V_N \cap U$, the operator $dT_x : V_N \to H$ is injective*

Then there exists a constant $C > 0$ such that

$$\forall x, y \in K_N, \quad ||x-y||_V \leq C\,||T(x)-T(y)||_H.$$

Proof. Let us consider the mapping $\mathcal{T} : (x,h) \in U \times V \mapsto dT_x(h) \in H$. Such mapping $\mathcal{T}$ is continuous. Indeed, for $x, x_0 \in U$ and $h, h_0 \in V$, we have

$$||dT_x(h) - dT_{x_0}(h_0)||_H \leq ||(dT_x - dT_{x_0})(h)||_H + ||dT_{x_0}(h-h_0)||_H$$

$$\leq |||dT_x - dT_{x_0}|||\,||h||_V + |||dT_{x_0}|||\,||h-h_0||_V,$$

and the result follows from the continuity of $x \mapsto dT_x$.
Hence, by the injectivity of dT_x on V_N and the compactness of the set $K_N \times S_N$, where S_N is the unit sphere of V_N, there exists a constant $c > 0$ such that

$$||dT_x(h)||_H \geq c, \quad \forall x \in K_N, \forall h \in S_N,$$

that is

$$||dT_x(h)||_H \geq c\,||h||_V \quad \forall x \in K_N, \forall h \in V_N. \tag{1}$$

Since the mapping $\mathcal{T}$ is uniformly continuous on the compact set $K_N \times S_N$ there exists $\delta > 0$ such that if $x, y \in K_N$ satisfy $||x-y||_V < \delta$ then

$$||(dT_x - dT_y)(h)||_H \leq \frac{c}{2},$$

that is

$$||(dT_x - dT_y)(h)||_H \leq \frac{c}{2}\,||h||_V, \quad \forall h \in V_N. \tag{2}$$

Let us take $x, y \in K_N$ satisfy $||x-y||_V < \delta$. By denoting $h = y - x$ we have using the convexity of K_N and the fact that by the chain rule the function $s \in [0,1] \mapsto T(x+sh) \in H$ is continuously differentiable,

$$T(y) - T(x) = \int_0^1 \frac{d}{ds} T(x+sh)\,ds = \int_0^1 dT_{x+sh}(h)\,ds$$

$$= dT_x(h) + \int_0^1 (dT_{x+sh} - dT_x)(h)\,ds.$$

From (1) and (2), we obtain that if $x, y \in K_N$ satisfy $||x-y||_V < \delta$ then

$$||T(y) - T(x)||_H \geq (c/2)\,||h||_V = (c/2)\,||y-x||_V.$$

Consider now the other case $||x-y|| \geq \delta$. If we denote m the minimum of the continuous map $(x,y) \mapsto ||T(x)-T(y)||_H$ on the compact set $\{(x,y) \in K_N^2,\ ||x-y||_V \geq \delta\}$, we have that $m>0$ because of the injectivity of T on $V_N \cap U$ and

$$||T(x)-T(y)||_H \geq m \geq \frac{m}{d}\,||x-y||_V,$$

where d is the diameter of that compact set. We just have to take $C = \max(2/c, d/m)$ in the statement of the theorem to complete the proof. $\square$

Remark 2.2. *If the mapping $T : V \to H$ is a linear bounded operator, the theorem takes the simple form : if T is injective, for all finite dimensional subspace V_N (of dimension N) of V, there exists a constant $C_N > 0$ such that*

$$||x||_V \leq C_N\,||Tx||_H, \quad \forall x \in V_N, \tag{3}$$

which readily results from the fact that the operator $T : V_N \to T(V_N)$ is invertible with bounded inverse.
Clearly, as soon as T is not onto, there is no constant $C > 0$ such that the above inequality holds for all $x \in V$. Let us illustrate the fact that C_N may increase exponentially when $N \to +\infty$ on the well known case of the Cauchy problem for the Laplace equation. Let Ω be the square $\{(x_1,x_2) \in (0,X)\times(0,X)\} \subset \mathbb{R}^2$ and $\Gamma = \{0\}\times(0,X)$. We consider the following problem: for a pair of data $(g_0,g_1) \in H^{\frac{3}{2}}(\Gamma)\times H^{\frac{1}{2}}(\Gamma)$, find $u \in H^2(\Omega)$ such that

$$\begin{cases} \Delta u = 0 \text{ in } \Omega \\ u|_\Gamma = g_0 \\ \partial_{x_1} u|_\Gamma = g_1. \end{cases} \tag{4}$$

If U is a function in $H^2(\Omega)$ such that $U|_\Gamma = g_0$ and $\partial_{x_1}U|_\Gamma = g_1$ (such function exits following [13], and is of course not unique), and if we define $f = -\Delta U \in L^2(\Omega)$, the change of variable $v = u - U \in H^2(\Omega)$ implies that the problem (4) is equivalent to the problem

$$\begin{cases} \Delta v = f \text{ in } \Omega \\ v|_\Gamma = 0 \\ \partial_{x_1} v|_\Gamma = 0. \end{cases} \tag{5}$$

The problem (5) amounts, for $f \in H$, to find $v \in V$ such that $Tv = f$, with $V = \{v \in H^2(\Omega),\ v|_\Gamma = 0,\ \partial_{x_1}v|_\Gamma = 0\}$, $H = L^2(\Omega)$ and T the Laplace operator, which is bounded from V to H.
In this case, T is injective and has a dense range, but this range is not closed, as can be seen in what follows. We now consider the sequence $(v_n)_{n\in\mathbb{N}}$ of functions in V:

$$v_n(x_1,x_2) = \phi(x_1)e^{nx_1}e^{inx_2} \tag{6}$$

where ϕ is a $C^2(\mathbb{R})$ function such that

$$\begin{cases} \phi(x_1) = 0 \quad x_1 \leq 0 \\ \phi(x_1) \geq 0 \quad 0 \leq x_1 \leq A \\ \phi(x_1) = 1 \quad x_1 \geq A, \end{cases}$$

with $A \in (0,X)$, and we choose $V_N = \mathrm{span}(v_1,...,v_N) \subset V$ as a subspace of dimension N.
After some easy computations we obtain for some constant $C > 0$,

$$||\Delta v_n||_{L^2(\Omega)} \leq \sqrt{\frac{n}{C}}\sqrt{e^{2nA}-1}, \quad ||u_n||_{L^2(\Omega)} \geq \sqrt{\frac{C}{n}}\sqrt{e^{2nX}-e^{2nA}},$$

so that from (3),

$$C_N \geq \frac{||u_N||_{H^2(\Omega)}}{||\Delta u_N||_{L^2(\Omega)}} \geq \frac{||u_N||_{L^2(\Omega)}}{||\Delta u_N||_{L^2(\Omega)}} \geq \frac{C}{N}\frac{\sqrt{e^{2NX}-e^{2NA}}}{\sqrt{e^{2NA}-1}} \sim_{N\to+\infty} \frac{C}{N}e^{N(X-A)}.$$

In the case of nonlinear mapping T, some examples showing exponentially growing constants C_N are presented in [8, 14, 15, 17].

3 Application to the inverse Robin problem

Let us consider a bounded, connected open domain $\Omega \in \mathbb{R}^d$ ($d \geq 2$) with Lipschitz continuous boundary $\partial\Omega$. The boundary $\partial\Omega$ is partitioned into two non-empty open subsets Γ and Γ_0 such that $\partial\Omega = \overline{\Gamma} \cup \overline{\Gamma_0}$ and $\Gamma \cap \Gamma_0 = \emptyset$.
For some $g \in L^2(\Gamma_0)$ with $g \neq 0$ and $\lambda \in L^\infty_+(\Gamma)$, where

$$L^\infty_+(\Gamma) := \{\lambda \in L^\infty(\Gamma),\ \exists m > 0,\ \lambda(x) \geq m \text{ a.e. on } \Gamma\},$$

the forward Robin problem consists in finding $u \in H^1(\Omega)$ such that

$$\left\{\begin{array}{c} \Delta u = 0 \ \text{ in } \ \Omega \\ \partial_\nu u = g \ \text{ on } \ \Gamma_0 \\ \partial_\nu u + \lambda u = 0 \ \text{ on } \ \Gamma, \end{array}\right. \tag{7}$$

where ν is the outward unit normal of Ω.
Problem (7) is clearly equivalent to the following weak formulation: find $u \in H^1(\Omega)$ such that for all $v \in H^1(\Omega)$,

$$\int_\Omega \nabla u \cdot \nabla v\, dx + \int_\Gamma \lambda uv\, ds = \int_{\Gamma_0} gv\, ds, \tag{8}$$

and well-posedness of problem (8) follows from Lax-Milgram's lemma and Poincaré-Friedrichs' inequality, which implies the equivalence between the standard norm of $H^1(\Omega)$ and the norm $||\cdot||$ defined by

$$||u||^2 = \int_\Omega |\nabla u|^2\, dx + \int_\Gamma u^2\, ds.$$

The Robin inverse problem consists in finding the unknown impedance λ on Γ from the measurement of the solution u on Γ_0. Such problem arises for example in non-destructive testing of the corrosion which contaminates an inaccessible part of the boundary from measurements of the potential on the accessible part. We now establish a Lipschitz stability estimate for that problem with the help of the theorem 2.1. The stability issue concerning this problem has been addressed in [5], in which a local Lipschitz stability estimate is proved as well as a global monotone Lipschitz stability estimate. In [2], some logarithmic stability estimate is obtained in 2D assuming a prescribed bound on the impedance λ in some Hölder space, without any monotony assumption. In [12], for higher dimension a stability estimate between Hölder and logarithmic is obtained with very strong regularity assumptions for a similar inverse scattering problem, while for the same problem these assumptions are relaxed in [16] and a logarithmic stability estimate is obtained. Note that the techniques used in [12, 16] for an inverse scattering problem are directly applicable to the present problem for the Laplace equation. Lastly, in [17] a Lipschitz stability estimate is obtained for a piecewise constant impedance, and the Lipschitz constant is proved to grow exponentially with respect to the number of portions on which the impedance is constant. Our objective is now to obtain a Lipschitz stability result which is close to that of [17] but in a slightly more general situation and with a completely different technique. In this view, we apply theorem 2.1 to the Robin inverse problem with $V = L^\infty(\Gamma)$, $H = L^2(\Gamma_0)$, $U = L^\infty_+(\Gamma)$ and the (nonlinear) mapping $T : L^\infty_+(\Gamma) \to L^2(\Gamma_0)$ which maps λ to the trace $u|_{\Gamma_0}$ of u on Γ_0, where u is the solution of problem (7). In order to specify V_N and K_N, we also define the subspace $C_I(\Gamma)$ of $L^\infty(\Gamma)$ as follows. Let I be a given integer and Γ_i, $i = 1, ..., I$, be open subsets of Γ such that $\Gamma_i \cap \Gamma_j = \emptyset$ for $i \neq j$, $\overline{\Gamma} = \cup_{i=1}^I \overline{\Gamma_i}$, and the sets $\partial\Gamma_i$ are negligible in the sense of the Lebesgue surface measure supported by $\partial\Omega$. The subspace $C_I(\Gamma)$ is then a space of piecewise continuous functions λ, precisely for all $i = 1, ..., I$, $\lambda|_{\Gamma_i} \in C^0(\overline{\Gamma_i})$. Lastly, we consider a subspace V_N spanned by some N linearly independent functions in $C_I(\Gamma)$ and K_N any compact and convex subset $V_N \cap L^\infty_+(\Gamma)$.

Remark 3.1. *An example of such set K_N, for $I = N$, is the set of piecewise continuous functions λ defined, for given real numbers m, M such that $m > 0$, by*

$$\lambda = \sum_{n=1}^N \alpha_n \, \lambda_n, \quad m \leq \alpha_n \leq M,$$

where the functions $\lambda_n \in L^\infty(\Gamma)$ are such that $\lambda_n|_{\Gamma_n}$ are positive functions in $C^0(\overline{\Gamma_n})$ and λ_n vanishes outside Γ_n. When the function λ_n coincides with the characteristic function of Γ_n we obtain the case of piecewise constant functions which is analyzed in [17].

By the theorem 2.1 we obtain the following Lipschitz stability result for the inverse Robin problem.

Theorem 3.2. *There exists a constant $C > 0$ such that*

$$\forall \lambda_1, \lambda_2 \in K_N, \quad ||\lambda_1 - \lambda_2||_{L^\infty(\Gamma)} \leq C \, ||u_1 - u_2||_{L^2(\Gamma_0)},$$

where u_1 and u_2 are the solutions of problem (7) associated with λ_1 and λ_2, respectively.

The proof consists in checking that the three assumptions of the theorem 2.1 are satisfied, which is the aim of the three following lemmas.

Lemma 3.3. *The operator $T : C_I(\Gamma) \cap L^\infty_+(\Gamma) \to L^2(\Gamma_0)$ is injective*

Proof. Although well-known (see for example [5]), for readers convenience we recall the proof here. Assume that $u_1|_{\Gamma_0} = u_2|_{\Gamma_0}$ for some λ_1 and λ_2 in $C_I(\Gamma) \cap L^\infty_+(\Gamma)$. Since $\partial_\nu u_1|_{\Gamma_0} = \partial_\nu u_2|_{\Gamma_0} = g$, from unique continuation we have $u_1 = u_2 := u$ in Ω. Then from the Robin condition on Γ, we have $\partial_\nu u + \lambda_1 u = \partial_\nu u + \lambda_2 u = 0$, that is $(\lambda_1 - \lambda_2)u = 0$ on Γ. Assume that for some $i = 1, ..., I$, $\lambda_1 - \lambda_2 \not\equiv 0$ on Γ_i. Since $(\lambda_1 - \lambda_2)|_{\Gamma_i}$ is continuous, u vanishes in an open subset of Γ_i. From the Robin condition, this implies that u and $\partial_\nu u$ both vanish in an open subset of Γ_i, and from unique continuation $u = 0$ in Ω. But this contradicts the fact that $g \not\equiv 0$ on Γ_0. □

Lemma 3.4. *The operator $T : L^\infty_+(\Gamma) \to L^2(\Gamma_0)$ is differentiable and its Fréchet derivative at point $\lambda \in L^\infty_+(\Gamma)$ is the operator $dT_\lambda : L^\infty(\Gamma) \to L^2(\Gamma_0)$ which maps h to $v_h|_{\Gamma_0}$ where v_h is the solution in $H^1(\Omega)$ of problem*

$$\begin{cases} \Delta v_h = 0 \;\text{ in }\; \Omega \\ \partial_\nu v_h = 0 \;\text{ on }\; \Gamma_0 \\ \partial_\nu v_h + \lambda v_h = -h \, u \;\text{ on }\; \Gamma, \end{cases} \tag{9}$$

where u is the solution of problem (7). Moreover, the mapping $\lambda \in L^\infty_+(\Gamma) \mapsto dT_\lambda \in \mathcal{L}(L^\infty(\Gamma), L^2(\Gamma_0))$ is continuous.

Proof. The proof of differentiability of T is well known (see again [5]). But since we have to prove that the mapping from $\lambda \mapsto dT_\lambda$ is continuous, we have to repeat the different steps to construct dT_λ. We recall that $T : L^\infty_+(\Gamma) \to L^2(\Gamma_0)$ is differentiable in the sense of Fréchet at point λ if there exists a linear continuous operator $dT_\lambda : L^\infty(\Gamma) \to L^2(\Gamma_0)$ and a mapping $\varepsilon_\lambda : L^\infty(\Gamma) \to L^2(\Gamma_0)$ which satisfy for small $h \in L^\infty(\Gamma)$,

$$T(\lambda + h) = T(\lambda) + dT_\lambda(h) + ||h||_{L^\infty(\Gamma)}\varepsilon_\lambda(h), \tag{10}$$

with $\varepsilon_\lambda(h) \to 0$ in $L^2(\Gamma_0)$ when $||h||_{L^\infty(\Gamma)} \to 0$.
Let us denote u_h the solution of problem (7) which corresponds to impedance $\lambda + h$ for small h. From (8) we obtain that for small h we have

$$||u_h||_{H^1(\Omega)} \leq C, \tag{11}$$

where C is uniform with respect to h. The function $u_h - u$ solves the problem

$$\left\{\begin{array}{c} \Delta(u_h - u) = 0 \;\text{ in }\; \Omega \\ \partial_\nu(u_h - u) = 0 \;\text{ on }\; \Gamma_0 \\ \partial_\nu(u_h - u) + \lambda(u_h - u) = -h\, u_h \;\text{ on }\; \Gamma \end{array}\right.$$

or equivalently the weak formulation: for all $v \in H^1(\Omega)$,

$$\int_\Omega \nabla(u_h - u)\cdot\nabla v\, dx + \int_\Gamma \lambda(u_h - u)v\, ds = -\int_\Gamma h\, u_h\, v\, ds.$$

This implies that

$$||u_h - u||_{H^1(\Omega)} \leq C\, ||h||_{L^\infty(\Gamma)}||u_h||_{H^1(\Omega)}.$$

By using the estimate (11), it follows that for small h

$$||u_h - u||_{H^1(\Omega)} \leq C\, ||h||_{L^\infty(\Gamma)},$$

where the constant C is uniform with respect to h. Now let us consider $e_h = u_h - u - v_h$, where v_h is given by (9). First, we readily check that the operator $h \in L^\infty(\Gamma) \mapsto v_h|_{\Gamma_0} \in L^2(\Gamma_0)$ is linear continuous. Secondly, the function e_h solves the problem

$$\left\{\begin{array}{c} \Delta e_h = 0 \;\text{ in }\; \Omega \\ \partial_\nu e_h = 0 \;\text{ on }\; \Gamma_0 \\ \partial_\nu e_h + \lambda e_h = -h\,(u_h - u) \;\text{ on }\; \Gamma, \end{array}\right.$$

which implies that

$$||e_h||_{L^2(\Gamma_0)} \leq C\, ||e_h||_{H^1(\Omega)} \leq C\, ||h||_{L^\infty(\Gamma)}||u_h - u||_{H^1(\Omega)} \leq C\, ||h||^2_{L^\infty(\Gamma)},$$

where C is uniform with respect to h, and then T is differentiable at point λ with $dT_\lambda(h) = v_h|_{\Gamma_0}$.
It remains to prove that $\lambda \in L^\infty_+(\Gamma) \mapsto dT_\lambda \in \mathcal{L}(L^\infty(\Gamma), L^2(\Gamma_0))$ is continuous. For we have to consider for small l the functions v_h^l and v_h defined by (9) and associated with $\lambda + l$ and λ in $L^\infty_+(\Gamma)$, respectively. The function $v_h^l - v_h$ solves the problem

$$\left\{\begin{array}{c} \Delta(v_h^l - v_h) = 0 \;\text{ in }\; \Omega \\ \partial_\nu(v_h^l - v_h) = 0 \;\text{ on }\; \Gamma_0 \\ \partial_\nu(v_h^l - v_h) + \lambda(v_h^l - v_h) = -l\, v_h^l - h(u_l - u) \;\text{ on }\; \Gamma. \end{array}\right.$$

This implies that

$$||v_h^l - v_h||_{H^1(\Omega)} \leq C(||l||_{L^\infty(\Gamma)}||v_h^l||_{L^2(\Gamma)} + ||h||_{L^\infty(\Gamma)}||u_l - u||_{L^2(\Gamma)}),$$

and lastly

$$||v_h^l - v_h||_{L^2(\Gamma_0)} \leq C\,||l||_{L^\infty(\Gamma)}||h||_{L^\infty(\Gamma)},$$

where C is uniform with respect to h and l, that is

$$|||dT_{\lambda+l} - dT_\lambda||| \leq C\,||l||_{L^\infty(\Gamma)},$$

where C is uniform with respect to l, which completes the proof. □

Lemma 3.5. *For each $\lambda \in C_I(\Gamma) \cap L^\infty_+(\Gamma)$, the operator $dT_\lambda : C_I(\Gamma) \to L^2(\Gamma_0)$ is injective.*

Proof. Assume that for $h \in C_I(\Gamma)$, we have $v_h|_{\Gamma_0} = 0$, where v_h is defined by (9). Since $(v_h|_{\Gamma_0}, \partial_\nu v_h|_{\Gamma_0}) = (0,0)$, from unique continuation we obtain that $v_h = 0$ in Ω, and then $h\,u = 0$ on Γ. We conclude that $h = 0$, similarly as for the injectivity of T. □

Compared to the theorem 2.4 in [17], here we assume no more regularity for data g and the boundary $\partial\Omega$ than the regularity which is required for well-posedness of the forward problem. Secondly, the set of impedances we consider is a little more general than that of [17], in particular the impedances are not necessarily piecewise constant, but simply piecewise continuous (see remark 3.1). Above all, the technique of proof is very different. While in [17] some refined arguments of quantification of unique continuation based on harmonic analysis are used, here a simple approach based on continuity of the Fréchet derivative of the mapping from the impedance λ to the Dirichlet data $u|_{\Gamma_0}$ is proposed. We should note, however, that some ingredients used in the proof of [17] are useful and interesting by themselves, for example the lower bound of the solution in lemma 4.1.

Remark 3.6. *Our theorem 2.1 could also be applied to the inverse impedance problem for diffraction problems. In particular, it improves the Lipschitz stability estimates obtained in [4] (see theorem 4.3 for a classical impedance boundary condition and theorem 4.5 for a generalized impedance boundary condition) in the sense that the Lipschitz constant in those theorems is in fact uniform with respect to the point.*

4 Application to the inverse medium problem

The scattering of an acoustic wave in an inhomogeneous medium in $\mathbb{R}^3$ is governed by the following system (see [6])

$$\begin{cases} \Delta u + k^2 n(x) u = 0 \text{ in } \mathbb{R}^3 \\ u = u^i + u^s \\ \lim_{R\to+\infty} \int_{\partial B_R} |\partial u^s/\partial r - iku^s|^2 \, ds = 0, \end{cases} \tag{12}$$

where $k > 0$ is the wave number, $n \in L^\infty(\mathbb{R}^3)$ is a (complex) refractive index such that $n(x) = 1$ in $\mathbb{R}^3 \setminus B$ for some open ball B. The data u^i is a smooth function that solves the Helmholtz equation $\Delta u^i + k^2 u^i = 0$ in $\mathbb{R}^3$ and is called the incident field, while u^s and u are the scattered field and total field, respectively. The last equation of the system is the sommerfeld radiation condition.

Classically, the problem (12) is equivalent to the following one in a bounded domain with artificial boundary condition: find $u^s \in H^1(\Omega_R)$ such that

$$\left\{\begin{array}{c} \Delta u^s + k^2 n u^s = k^2(1-n(x))u^i \text{ in } B_R \\ \partial u^s/\partial r = S_R(u^s|_{\partial B_R}) \text{ on } \partial B_R, \end{array}\right. \tag{13}$$

where B_R is an open ball of radius R such that $\overline{B} \subset B_R$, $S_R : H^{1/2}(\partial B_R) \to H^{-1/2}(\partial B_R)$ is the Dirichlet-to-Neumann map, defined for $g \in H^{1/2}(\partial B_R)$ by $S_R g = (\partial u_g/\partial r)|_{\partial B_R}$, where u_g is the solution in $\mathbb{R}^3 \setminus \overline{B_R}$ of the Helmholtz equation satisfying the Sommerfeld radiation condition and the Dirichlet condition $u_g = g$ on ∂B_R. The operator S_R satisfies the inequalities

$$\mathrm{Re}\,\langle S_R g, g\rangle \leq 0 \quad \text{and} \quad \mathrm{Im}\,\langle S_R g, g\rangle \geq 0 \quad \forall g \in H^{\frac{1}{2}}(\partial B_R), \tag{14}$$

where the brackets $\langle \cdot, \cdot \rangle$ denote the duality product between $H^{1/2}(\partial B_R)$ and $H^{-1/2}(\partial B_R)$.
The problem (13) is itself equivalent to the following weak formulation: find $u^s \in H^1(B_R)$ such that

$$\int_{B_R} (\nabla u^s \cdot \nabla \overline{v} - k^2 n\, u^s \overline{v})\, dx - \langle S_R u^s, v\rangle = \int_{B_R} k^2(n-1)u^i \overline{v}\, dx \quad \forall v \in H^1(B_R). \tag{15}$$

It is well known that problem (15) is well-posed as soon as $\mathrm{Im}\,(n(x)) \geq 0$. Indeed, problem (15) is of Fredholm type, so that uniqueness implies existence. Concerning uniqueness, for $u^i = 0$, taking the imaginary part of equation (15) for $v = u^s$, we obtain

$$\mathrm{Im}\,\langle S_R u^s, u^s\rangle = -\int_{B_R} k^2 \mathrm{Im}\,(n(x))\, |u^s|^2\, dx \leq 0,$$

which from theorem 2.12 in [6] implies that $u^s = 0$. In view of (13), from standard regularity results for elliptic equations the solution u^s belongs to $H^2(B_R)$. In addition, it is shown in [6] that u^s has the asymptotic expression

$$u^s(x) = \frac{e^{ikr}}{r} u^\infty(\hat{x}) + \mathcal{O}\left(\frac{1}{r^2}\right), \quad r \to +\infty,$$

uniformly for all directions $\hat{x} = x/r \in S^2$, where $r = |x|$ and S^2 is the unit sphere in $\mathbb{R}^3$, and the far field u^∞ is given by

$$u^\infty(\hat{x}) = \frac{1}{4\pi}\int_{\partial B_R} \left(u^s(y) \frac{\partial e^{-ik\hat{x}\cdot y}}{\partial r(y)} - \frac{\partial u^s}{\partial r}(y) e^{-ik\hat{x}\cdot y}\right) ds(y), \quad \hat{x} \in S^2. \tag{16}$$

The inverse medium problem consists in finding the unknown refractive index n in B from the measurements on S^2 of the far fields $u^\infty(\cdot, d)$ corresponding to the scattered fields $u^s(\cdot, d)$ that are associated with plane waves $u^i(x) = e^{ikx\cdot d}$ in all directions $d \in S^2$.
The stability issue for that problem has been addressed first in [18] with the help of ideas from [1]. Such result was improved in [10], where a logarithmic stability estimate is obtained assuming $1-n$ is bounded in some Sobolev space $H^s(\mathbb{R}^3)$ with $s > 3/2$, the exponent of the logarithm being specified as a function of s. It should be noted that the inverse medium problem is very close to the Calderon's inverse conductivity problem, for which a number of papers concerning the stability issue has been published (see for example a review of them in [20]). We now establish a Lipschitz stability estimate for our inverse medium problem with the help of theorem 2.1. More precisely, we apply the theorem with $V = L^\infty(B)$, $H = L^2(S^2 \times S^2)$, $U = L^\infty_+(B)$, where

$$L^\infty_+(B) := \{n \in L^\infty(B),\ \exists m > 0,\ \mathrm{Im}\,(n(x)) \geq m \text{ a.e. on } B\},$$

and the (nonlinear) operator $T : \{n \in L^\infty(B),\ \mathrm{Im}\,(n(x)) \geq 0 \text{ a.e. on } B\} \to L^2(S^2 \times S^2)$ maps n to the set of far fields $u^\infty(\cdot, d)$ on S^2 for all directions $d \in S^2$, where $u^\infty(\cdot, d)$ corresponds to the scattered fields $u^s(\cdot, d)$ that solves problem (13) with $u^i(x) = e^{ikx\cdot d}$. We then choose any finite dimensional subspace V_N of $L^\infty(B)$, and lastly any compact and convex subset K_N of $V_N \cap U$. We have the following result.

Theorem 4.1. *There exists a constant $C > 0$ such that*

$$\forall n_1, n_2 \in K_N, \quad ||n_1 - n_2||_{L^\infty(B)} \leq C\, ||u_1^\infty - u_2^\infty||_{L^2(S^2\times S^2)},$$

where $u_1^\infty(\hat{x}, d)$ and $u_2^\infty(\hat{x}, d)$, which refer to the refractive indices n_1 and n_2, respectively, are the far fields in the direction $\hat{x}$ of the solutions u_1^s and u_2^s of problem (13) with $u^i = e^{ikx\cdot d}$.

Let us verify the three assumptions of the theorem 2.1 in the three following lemmas.

Lemma 4.2. *The mapping $T : \{n \in L^\infty(B),\ \mathrm{Im}(n(x)) \geq 0 \text{ a.e. on } B\} \to L^2(S^2 \times S^2)$ is injective.*

Proof. The proof is classical and for example done in [6, 11]. It uses the construction in [9] of complex geometrical optics solutions. For readers convenience we give a sketch of it. Assume that n_1 and n_2 produce the same far fields $u_1^\infty(\cdot, d)$ and $u_2^\infty(\cdot, d)$ for all $d \in S^2$. The first step consists in proving that the set of total fields $\{u(\cdot, d),\, d \in S^2\}$ is dense in the closure of the space

$$H(B_R) := \{v \in H^2(B_R), \quad \Delta v + k^2 n v = 0 \quad \text{in} \quad B_R\}$$

endowed with the norm $L^2(B_R)$, which is the lemma 10.4 in [6]. In fact, the lemma is proved for $n \in C^1(\mathbb{R}^3)$ instead of $n \in L^\infty(\mathbb{R}^3)$ and space $C^2(\overline{B_R})$ instead of $H^2(B_R)$ in the definition of $H(B_R)$. However, a careful reading of the proof shows that the lemma still holds for our weaker assumptions.
The second step consists in proving that

$$\int_B (n_1 - n_2) v_1 v_2 \, dx = 0, \quad \forall v_1 \in H_1(B_R), \quad \forall v_2 \in H_2(B_R), \tag{17}$$

where $H_i(B_R)$ is the space $H(B_R)$ for $n = n_i$, $i = 1, 2$.
From the first step, is suffices to prove that

$$\int_B (n_1 - n_2) v_1 u_2(\cdot, d) \, dx = 0, \quad \forall v_1 \in H_1(B_R), \quad \forall d \in S^2.$$

Setting $u = u_1(\cdot, d) - u_2(\cdot, d)$ we have

$$\Delta u + k^2 n_1 u = k^2 (n_2 - n_1) u_2,$$

then

$$(n_1 - n_2) v_1 u_2 = \frac{1}{k^2}(u \Delta v_1 - \Delta u v_1),$$

and by the Green's theorem,

$$\int_B (n_1 - n_2) v_1 u_2 \, dx = \frac{1}{k^2} \int_{\partial B} \left(u \frac{\partial v_1}{\partial r} - \frac{\partial u}{\partial r} v_1 \right) ds.$$

From Rellich's lemma 2.11 in [6] applied to function u and the fact that $u \in H^2(B_R)$, we have that $u|_{\partial B} = 0$ and $(\partial u / \partial r)|_{\partial B} = 0$. We hence obtain

$$\int_B (n_1 - n_2) v_1 u_2 \, dx = 0,$$

which is the required result.
The last step of the proof consists in choosing appropriate functions v_1 and v_2 in (17). From lemma 10.2 in [6], there exists a constant $C > 0$ such that for each $z \in \mathbb{C}^3$ with $z \cdot z = 0$ and $|\mathrm{Re} z| \geq 2k^2||n||_{L^\infty(\mathbb{R}^3)}$, there exists a function $v \in H(B_R)$ which satisfies

$$v(x) = e^{iz\cdot x}(1 + w(x)), \quad ||w||_{L^2(B_R)} \leq \frac{C}{|\mathrm{Re}\, z|}.$$

We choose v_1 and v_2 associated with n_1 and n_2, respectively, and with z_1 and z_2 of the form

$$z_1 = y + \rho a + ib, \quad z_2 = y - \rho a - ib,$$

where $\rho > 0$, $y \in \mathbb{R}^3 \setminus \{0\}$, and $a, b \in \mathbb{R}^3$ are chosen such that $\{y, a, b\}$ is an orthogonal basis in $\mathbb{R}^3$ with $|a| = 1$ and $|b| = \sqrt{|y|^2 + \rho^2}$. We check that

$$z_j \cdot z_j = 0, \quad |\mathrm{Re}\, z_j| \geq \rho.$$

Plugging these v_1 and v_2 in (17) and passing to the limit $\rho \to +\infty$, we obtain

$$\int_B (n_1(x) - n_2(x))\, e^{2iy\cdot x}\, dx = 0 \quad \forall y \in \mathbb{R}^3,$$

which implies $n_1 = n_2$ in B. □

Lemma 4.3. *The mapping $T : L^\infty_+(B) \to L^2(S^2 \times S^2)$ is differentiable and its Fréchet derivative at point $n \in L^\infty_+(B)$ is the operator $dT_n : L^\infty(B) \to L^2(S^2 \times S^2)$ which maps h to the far fields $v_h^\infty(\cdot, d)$ corresponding to the scattered fields $v_h^s(\cdot, d)$ for all incidence directions $d \in S^2$, where $v_h^s(\cdot, d)$ is the solution in $H^1(B_R)$ of problem*

$$\begin{cases} \Delta v_h^s + k^2 n v_h^s = -k^2 h u \text{ in } B_R \\ \partial v_h^s/\partial r = S_R(v_h^s|_{\partial B_R}) \text{ on } \partial B_R, \end{cases} \tag{18}$$

in which $u = u^s + e^{ikx\cdot d}$ and u^s is the solution of problem (13). In problem (18), the function $h \in L^\infty(B)$ has been extended by 0 outside B, without change of notations. Moreover, the mapping $n \in L^\infty_+(B) \mapsto dT_n \in \mathcal{L}(L^\infty(B), L^2(S^2 \times S^2))$ is continuous.

Proof. Let us denote u_h^s the solution of problem (13) which corresponds to the refractive index $n + h$ for small h. First we establish a uniform bound for u_h^s. Taking the imaginary part of (15) and choosing $v = u^s$, we obtain for $f := (n-1)u^i$,

$$k^2 \int_{B_R} \mathrm{Im}(n(x))|u^s|^2\, dx + \mathrm{Im}\,\langle S_R u^s, u^s\rangle = -k^2 \mathrm{Im}\left(\int_{B_R} f\overline{u^s}\, dx\right).$$

Using the second inequality of (14), it follows that for $m = \mathrm{infess}\,(\mathrm{Im}(n))$,

$$||u^s||_{L^2(B_R)} \leq \frac{1}{m}||f||_{L^2(B)} \tag{19}$$

that is

$$||u^s||_{L^2(B_R)} \leq \frac{||n-1||_{L^\infty(B)}}{m}||u^i||_{L^2(B)}$$

It follows that for some $n \in L^\infty_+(B)$ and for small $h \in L^\infty(B_R)$, since $u^i(x) = e^{ikx\cdot d}$

$$||u_h^s||_{L^2(B_R)} \leq C, \tag{20}$$

where the constant C does not depend on h and on $d \in S^2$.
The function $u_h^s - u^s$ solves the problem

$$\begin{cases} \Delta(u_h^s - u^s) + k^2 n(u_h^s - u^s) = -k^2 h u_h \text{ in } B_R \\ \partial(u_h^s - u^s)/\partial r = S_R((u_h^s - u^s)|_{\partial B_R}) \text{ on } \partial B_R, \end{cases}$$

so that the estimate (19) with $f = -hu_h$ implies that for small h,

$$||u_h^s - u^s||_{L^2(B_R)} \leq C\, ||h||_{L^\infty(B)} ||u_h||_{L^2(B)},$$

that is, using the uniform bound (20),

$$||u_h^s - u^s||_{L^2(B_R)} \leq C\, ||h||_{L^\infty(B)},$$

where C is independent of h and d.
We consider now the function $e_h^s = u_h^s - u_h^s - v_h^s$, where v_h^s is the solution of problem (18). First we have to check that the far field associated with v_h^s, denoted v_h^∞, is such that $h \in L^\infty(B) \mapsto v_h^\infty \in L^2(S^2 \times S^2)$ is a bounded operator. In order to use the expression of far field given by (16), we need a bound in $H^2(B_R)$ for v_h^s. Using estimate (19) for $f = -hu$ and then bound (20), we obtain

$$||v_h^s||_{L^2(B_R)} \leq C\, ||h||_{L^\infty(B)} ||u||_{L^2(B)} \leq C\, ||h||_{L^\infty(B)},$$

where C is independent of h and d. By using a weak formulation of (18) we obtain

$$\int_{B_R} |\nabla v_h^s|^2\, dx - k^2 \int_{B_R} n|v_h^s|^2\, dx - \langle S_R v_h^s, v_h^s \rangle = k^2 \int_{B_R} hu\overline{v_h^s}\, dx.$$

Taking the real part of the above equality and using the first inequality of (14), we obtain

$$\int_{B_R} |\nabla v_h^s|^2\, dx \leq k^2 \int_{B_R} \mathrm{Re}(n(x))|v_h^s|^2\, dx + k^2 \mathrm{Re}\left(\int_{B_R} hu\overline{v_h^s}\, dx \right)$$

$$\leq C\, ||v_h^s||^2_{L^2(B_R)} + C\, ||h||_{L^2(B)} ||v_h^s||_{L^2(B)} \leq C\, ||h||^2_{L^\infty(B)},$$

that is

$$||v_h^s||_{H^1(B_R)} \leq C\, ||h||_{L^\infty(B)},$$

where C is independent of h and d. Using the equation $\Delta v_h^s = -k^2 n v_h^s - k^2 hu$ in B_R and the regularity results for elliptic problems, we conclude that

$$||v_h^s||_{H^2(B_R)} \leq C\, ||h||_{L^\infty(B)},$$

where C is independent of h and d. From the far field expression (16) and continuity of traces on ∂B_R, we obtain

$$||v_h^\infty||_{L^2(S^2 \times S^2)} \leq C\, ||h||_{L^\infty(B)},$$

where C is uniform with respect to h. It remains to find a bound for e_h^∞, which is the far field associated with the scattered field e_h^s. The function e_h^s solves the problem

$$\begin{cases} \Delta e_h^s + k^2 n e_h^s = -k^2 h(u_h - u) \text{ in } B_R \\ \partial e_h^s/\partial r = S_R(e_h^s|_{\partial B_R}) \text{ on } \partial B_R. \end{cases}$$

The estimate (19) with $f = -h(u_h - u)$ then implies that for small h,

$$||e_h^s||_{L^2(B_R)} \leq C\, ||h||_{L^\infty(B)} ||u_h - u||_{L^2(B)},$$

hence

$$||e_h^s||_{L^2(B_R)} \leq C\,||h||^2_{L^\infty(B)},$$

where C is independent of h and d. Proceeding for e_h^∞ as for v_h^∞, we obtain that

$$||e_h^\infty||_{L^2(S^2\times S^2)} \leq C\,||h||^2_{L^2(B)},$$

where C is uniform with respect to h. We have proved that $h \mapsto v_h^\infty$ is the Fréchet derivative of T at point n.
Lastly, let us prove that $n \in L^\infty_+(B) \mapsto dT_n \in \mathcal{L}(L^\infty(B), L^2(S^2 \times S^2))$ is a continuous mapping. Let us denote $v_h^{l,s}$, the solution of (18) which corresponds to $n+l$ instead of n, the solution $v_h^{l,s} - v_h^s$ solves the problem

$$\left\{\begin{array}{c} \Delta(v_h^{l,s}-v_h^s)+k^2 n(v_h^{l,s}-v_h^s) = -k^2 l v_h^{l,s} - k^2 h(u_l - u) \ \text{ in } \ B_R \\ \partial(v_h^{l,s}-v_h^s)/\partial r = S_R((v_h^{l,s}-v_h^s)|_{\partial B_R}) \ \text{ on } \ \partial B_R. \end{array}\right.$$

The estimate (19) for $f = -l v_h^{l,s} - h(u_l - u)$ implies that for small h and l,

$$\begin{aligned} ||v_h^{l,s} - v_h^s||_{L^2(B_R)} &\leq C\,(||l||_{L^\infty(B)}||v_h^{l,s}||_{L^2(B)} + ||h||_{L^\infty(B)}||u_l - u||_{L^2(B)}) \\ &\leq C\,||l||_{L^\infty(B)}||h||_{L^\infty(B)}, \end{aligned}$$

where C is uniform with respect to $l, h \in L^\infty(B)$ and $d \in S^2$. Proceeding as above, we arrive at

$$||v_h^{l,\infty} - v_h^\infty||_{L^2(S^2\times S^2)} \leq C\,||l||_{L^\infty(B)}||h||_{L^\infty(B)},$$

where C is uniform with respect to l and h, and lastly

$$|||dT_{n+l} - dT_n||| \leq C\,||l||_{L^\infty(B)},$$

where C is uniform with respect to l, which completes the proof. □

Lemma 4.4. *For each $n \in L^\infty_+(B)$, the operator $dT_n : L^\infty(B) \to L^2(S^2 \times S^2)$ is injective.*

Proof. Assume that the far fields $v_h^\infty(\cdot, d)$ defined in lemma 4.3 vanish for all $d \in S^2$. We have from lemma 4.3

$$\Delta v_h^s + k^2 n v_h^s = -k^2 h\, u(\cdot, d) \quad \text{in} \quad B_R.$$

For any $v \in H(B_R)$, we hence have

$$hv\, u(\cdot, d) = \frac{1}{k^2}(v_h^s \Delta v - \Delta v_h^s v),$$

and by the Green's theorem,

$$\int_B hv\, u(\cdot, d)\, dx = \frac{1}{k^2}\int_{\partial B}\left(v_h^s \frac{\partial v}{\partial r} - \frac{\partial v_h^s}{\partial r} v\right)\, ds.$$

From Rellich's lemma, we have that $v_h^s|_{\partial B} = 0$ and $(\partial v_h^s/\partial r)|_{\partial B} = 0$. We hence obtain

$$\int_B hv\, u(\cdot, d)\, dx = 0, \quad \forall v \in H(B_R), \quad \forall d \in S^2.$$

With the help of the denseness result already used in the proof of lemma 4.2, it follows that

$$\int_B hvw\, dx = 0, \quad \forall v, w \in H(B_R),$$

and we conclude that $h = 0$ as in lemma 4.2. □

Remark 4.5. *We can obtain a slightly more general result than theorem 4.1 by choosing a compact and convex subset $K_N \subset V_N \cap F$, where F is the closed subset $F = \{n \in L^\infty(B),\ \mathrm{Im}\,(n(x)) \geq 0 \text{ a.e. on } B\}$, instead of $K_N \subset V_N \cap L^\infty_+(B)$. In this case the theorem (2.1) is not applicable since F is not an open domain. The proof of 4.1 shall hence be slightly modified. First the technique used to obtain bounds for the solution u^s_h shall be different since there is no lower bound for n: we can then use the abstract approach used in the proof of proposition 5 of [4]. Secondly, the mapping $T : n \in F \mapsto H$ is no more differentiable in F but is is right-differentiable, that is there exists a linear continuous operator $dT_n : L^\infty(B) \to L^2(S^2 \times S^2)$ and a mapping $\varepsilon_n : L^\infty(B) \to L^2(S^2 \times S^2)$ such that for small $h \in L^\infty(B)$ with $n + h \in F$,*

$$T(n+h) = T(n) + dT_n(h) + ||h||_{L^\infty(B)}\varepsilon_n(h),$$

with $\varepsilon_n(h) \to 0$ in $L^2(S^2 \times S^2)$ when $||h||_{L^\infty(B)} \to 0$. Similarly, the mapping $n \in F \mapsto dT_n \in \mathcal{L}(V, H)$ is right-continuous.

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